MORSE INEQUALITIES AND VON NEUMANN II_1-FACTORS

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In this paper it is shown that in the Morse inequalities on a multiply connected manifold, instead of the classical Betti numbers, one can consider real Betti numbers equal to the von Neumann relative dimensions of the homology groups or cohomology groups of the local system obtained from the representation of the fundamental group in a II_1-factor, according to the conjecture of [1]. The reason for this consists in the following. Even in finite-dimensional representations $\chi\colon \pi_1(X) \to GL(N, \mathbb{C})$, the analogues of the Morse inequalities for a function f on a smooth n-dimensional closed manifold X have the form

$$(1) \qquad N m_p(f) \geqslant b_p^\chi,$$

where $m_p(f)$ is the number of critical points of the index p of f and $b_\rho^\chi = b_p^\chi(X)$ is the dimension of the space of p-dimensional homology of X with values in $\mathbb{C}^N$ as in a π_1-module with representation χ. Inequality (1) follows from the fact that its left-hand side is in fact the dimension of the group of chains in the complex providing the necessary homology and constructed from the Morse cell complex defined by f. The analogues of the numbers $\bar{b}_p^\chi = b_p^\chi / N$ can also be defined for $M = \infty$ in the case where $\mathbb{C}^N$ is replaced by a Hilbert space and the representation χ maps π_1 into a II_1-factor.

1. Let $\mathcal{H}$ be a complex Hilbert space, $\mathfrak{A}$ a II_1-factor in $\mathcal{H}$, $\operatorname{tr}_{\mathfrak{A}}$ the faithful normal trace on $\mathfrak{A}$, normalized in the usual way, i.e., $\operatorname{tr}_{\mathfrak{A}} I = 1$ for the identity operator I in $\mathcal{H}$ (see, for example, [2]), and $\dim_{\mathfrak{A}}$ the corresponding relative dimension (defined on closed subspaces L of $\mathcal{H}$ affiliated with $\mathfrak{A}$, i.e., for which the corresponding orthogonal projection P_L belongs to $\mathfrak{A}$, by the formula $\dim_{\mathfrak{A}} L = \operatorname{tr}_{\mathfrak{A}} P_L$). We assume that a representation $\chi\colon \pi_1(X) \to \mathfrak{A}$ is given and Γ_χ is a local system with fiber $\mathcal{H}$ on X, constructed for the representation χ. The homology group $H_p(X; \Gamma_\chi)$ can be identified with some closed subspaces in space of the form $\mathcal{H}^N = \mathcal{H} \otimes \mathbb{C}^N$, affiliated with II_1-factors $\mathfrak{A}_N = \mathfrak{A} \otimes \operatorname{End}(\mathbb{C}^N)$ with trace $\operatorname{tr}_{\mathfrak{A}_N} = \operatorname{tr}_{\mathfrak{A}} \otimes \operatorname{tr}$, where tr is the unsual matrix trace on the algebra $\operatorname{End} \mathbb{C}^N$, $\operatorname{tr} I = N$ (in the definition of homology here and below, instead of the images of differentials, we must take the closures of these images). We denote by $\dim_{\mathfrak{A}}$ the corresponding relative dimension in each of the spaces $\mathcal{H}^N$ and consider the real Betti numbers

$$\bar{b}_p^\chi = \dim_{\mathfrak{A}} H_p(X; \Gamma_\chi),$$

first introduced by Atiyah [3] and Singer [4] and also studied in [5]–[8].

Date: Received 14/JAN/86.
1991 *Mathematics Subject Classification.* Primary 46L35; Secondary 57N65.
UDC 515.164.174.
Translated by J. Szűcs.

Theorem 1. *The following inequalities hold*:

$$\sum_{k=0}^{p}(-1)^k m_{p-k}(f) \geqslant \sum_{k=0}^{p}(-1)^k \bar{b}^{\chi}_{p-k}, \quad p=0,1,\dots,n. \tag{2}$$

The Betti numbers $\bar{b}^{\chi}_p$ can also be obtained by applying the relative dimension $\dim_{\mathfrak{A}}$ to the cohomology group $H^p(X;\Gamma_\chi)$. The group $H^p(X;\Gamma_\chi)$ can also be defined as the cohomology group of the complex of L^2-forms

$$0 \to \Lambda^0_\chi \to \cdots \to \Lambda^p_\chi \xrightarrow{1\otimes d} \Lambda^{p+1}_\chi \to \cdots \to \Lambda^n_\chi \to 0,$$

where Λ^p_χ is the space of L^2-forms of degree p on X with values in a vector bundle with fiber $\mathcal{H}$, constructed from the representation χ (see [4]–[6]).

2. We sketch the proof. To verify (2), it is sufficient to construct a complex

$$0 \to L_n \xrightarrow{d_n} L_{n-1} \xrightarrow{d_{n-1}} L_{n-2} \to \cdots \to L_1 \xrightarrow{d_1} L_0 \to 0, \tag{3}$$

satisfying the following conditions:

a) $L_p = \mathcal{H}^{m_p(f)}$, $p=0,1,\dots,n$.

b) Every mapping d_p is given by a matrix with entries in $\mathfrak{A}$.

c) The homology groups of the complex (3) are isomorphic to $H^p(X,\Gamma_\chi)$, and the isomorphism is given by $\mathfrak{A}$-operators which are restrictions of matrix operators with entries in $\mathfrak{A}$.

The needed complex can be constructed if we calculate the homology of the local system Γ_χ by means of the standard passage from X to the homotopy equivalent CW-complex of Morse, constructed from f and containing exactly $m_p(f)$ cells of dimension p. For this (and for the construction of homology) we need operations over subspaces and their mappings that do not leave the realm of $\mathfrak{A}$-subspaces (i.e., subspaces affiliated with the corresponding factors $\mathfrak{A}_N$) and admissible mappings (obtained by restricting matrix operators with entries in $\mathfrak{A}$). Now we have to take into account that $\dim_{\mathfrak{A}}$ has the same algebraic properties as the ordinary dimension. For example, if a sequence of $\mathfrak{A}$-subspaces and admissible mappings

$$0 \to L' \xrightarrow{d_1} L \xrightarrow{d_2} L'' \to 0$$

is given, which is almost exact in the sense that $\operatorname{Ker} d_1 = 0$, $\overline{\operatorname{Im} d_1} = \operatorname{Ker} d_2$ and $\overline{\operatorname{Im} d_2} = L''$, then

$$\dim_{\mathfrak{A}} L = \dim_{\mathfrak{A}} L' + \dim_{\mathfrak{A}} L''$$

(see [4]–[8]). Therefore, properties a)–c) imply inequalities (2).

Remark 1. Instead of a II_1-factor, one can take any von Neumann algegra $\mathfrak{A}$ with a fixed finite trace $\operatorname{tr}_{\mathfrak{A}}$ such that $\operatorname{tr}_{\mathfrak{A}} I = 1$.

Remark 2. Using the same procedure, one can define the combinatorial Reidemeister torsion $R(X,\chi)$ with value in a II_1-factor induced by a representation χ of the group π_1. The square of the absolute value of the Reidemeister torsion is uniquely determined for unitary χ. Following the scheme of [10], it is also possible to define an analytic torsion.

3. Examples. It is known that the regular represention ρ of a countable group Γ into $l^2(\Gamma)$ possesses a standard normalized trace (see, for example, [7]). If all nontrivial classes of conjugate elements are infinite, then ρ induces a II_1-factor. The same property is enjoyed by all representations $\chi = \rho \otimes \rho'$, where ρ' is a finite-dimensional representation of Γ.

Example 1. The indicated property is shared by all discrete subgroups Γ of the group of motions of an n-dimensional simply connected symmetric space M of negative curvature with compact quotient space $X = M/\Gamma$. Here $\Gamma = \pi_1(X)$ and

$$\bar{b}_p^{\rho}(X) = \int_{V(\Gamma)} \operatorname{tr} G_p(x,x)\, d\sigma(x) = c_p(M) \operatorname{Vol}(x), \tag{4}$$

where $G_p(x,y)$ is the kernel of the orthogonal projection onto the space of L^2-harmonic p-forms on M, $V(\Gamma)$ is the fundamental domain of Γ, and $d\sigma$ is the volume element. Then we always have $c_0 = c_n = 0$. If $M = H^2$, the Lobachevsky space, then $c_1 = c_2 = 0$. Moreover, $c_p(H^{2p}) \neq 0$.

Conjecture. $c_p(M) = 0$ for $2p \neq n$. What are the numbers $\bar{b}_1^{\chi}(X)$ for $X = H^3/\Gamma$ if $\chi = \rho \otimes \rho'$, where p' is finite-dimensional?

Remark. The continuous spectrum of the Laplacian on p-forms of M can contain 0. This property is topologically invariant. Here new and deep topological phenomena appear: invariants lying between the von Neumann torsion of Reidemeister and homology (for example, for $M = H^3$, $p = 1$).

Example 2. Let $\chi = \rho$, X a closed 4-dimensional manifold such that the group $\Gamma = \pi_1(X)$ is infinite but $H_1(X,Z) = \Gamma/[\Gamma,\Gamma] = 0$, and $\mu(X)$ the Euler characteristic of X. In view of the equality $\bar{b}_0^{\chi} = 0$ we obtain $\mu(X) = 2 + b_2 = \bar{b}_2^{\chi} - 2\bar{b}_1^{\chi}$, $\bar{b}_2^{\chi}$, $\bar{b}_2^{\chi} \geqslant b_2 + 2$. If now Y is a manifold of any dimension, homotopy equivalent to $X \setminus \{x_0\}$ on a 3-skeleton, then for the Morse function f on Y we have, by Theorem 1,

$$m_2(f) - m_1(f) + m_0(f) \geqslant \bar{b}_2^{\chi} - \bar{b}_1^{\chi} = b_2 + 2 + \bar{b}_1^{\chi} \geqslant b_2 + 2. \tag{5}$$

Even in the case $\bar{b}_1^{\chi} = 0$, this inequality is better than the classical one by 1.

Remark. It follows from Theorem 1 that $\bar{b}_1^{\chi} \geqslant m_1 - m_2 - 1$, where m_1 is the number of generators of the group $\pi_1(X)$ and m_2 is the number of relations. In particular, in contract to the scalar Laplace–Beltrami operator, the vectorial operator for groups $\pi_1(X)$ which are nearly free always has a large space of square integrable null modes on the universal cover.

Addendum (S. P. Novikov). On Morse-type inequalities for vector fields. At the end of [1], an application of an idea of [9] for obtaining Morse-type inequalities for vector fields $\eta^j(x)$ on a manifold X is discussed. Let a metric g_{ij} be given. Consider the operators

$$d_{t\omega} = d + t\omega, \quad \pm\delta_{t\omega} = * d_{t\omega} *, \quad \Delta_{t\omega} = (d_{t\omega} + \delta_{t\omega})^2$$

for a 1-form $\omega = \omega_i\, dx^i$, $\omega_i(x) = g_{ij}\eta^j(x)$. Then $\Delta_{t\omega} = \Delta + t^2(\omega,\omega) + t\hat{C}(\omega)$, where $\hat{C}$ is a matrix operator depending on ω and the metric. As $t \to \infty$, the null modes of $\operatorname{Ker} \Delta_{t\omega}$, on the spaces of even and odd forms $\Lambda^{\pm}(X)$ are concentrated near points $\mathcal{P}_q$, where $\eta(\mathcal{P}_q) = 0$. In special coordinates near the points $\mathcal{P}_q$, where $g_{ij} = \delta_{ij}$ and $\Gamma^i_{jk}(\mathcal{P}_q) = 0$, the operator can be written (see [1]) in the form

$$\Delta_{t\omega} \sim \Delta_t^{(q)} = -\sum_i \partial_i^2 + C_{ij}C_{ik}x^j x^k + C_{ij}(a^{*i} - a^i)(a^{*j} + a^j). \tag{6}$$

Here $C_{ij} = \partial_j \eta^i(\mathcal{P}_q)$, $\det C_{ij} \neq 0$. The operator a^{*i} is multiplication by dx^i, and a^i is the adjoint operator. At the point $\mathcal{P}_q$, the usual commutation relations of fermions are satisfied for a^{*i} and a^j. In the case of [9], $\omega = dh$ and the matrix $C = (C_{ij})$ is symmetric. Therefore, the Hamiltonian (6) can be diagonalized by an orthogonal transformation. In our case, the selfadjoint operator

$$C_{ij}(a^{*i} - a^i)(a^{*j} + a^j) = \hat{C}$$

acts on the space $\Lambda^*(\mathbb{R}^n)$ of dimension 2^n. A basis in Λ^p can be realized in the form

$$a^{*i_1}a^{*i_2}\cdots a^{*i_p}\Omega, \quad i_1 < \cdots < i_p,\ a^j\Omega = 0.$$

Let there be given a Bogolyubov transformation (an automorphism of the Clifford algebra)

$$a^i = t^i_j b^j + p^i_j b^{*j}, \quad a^{*i} = p^i_j b^j + t^i_j b^{*j},$$

where $b^{*i}b^{*j} = -b^{*j}b^{*i}$ and $b^{*i}b^j + b^j b^{*i} = \delta^{ij}$. Then, as is known, the matrices $U = T - P$ and $V = T + P$ are arbitrary orthogonal matrices from the group $O(n)$, with $T = (t^i_j)$ and $P = (P^i_j)$.

Lemma 1. *Let $\hat{C} = C'_{ij}(b^{*i} - b^i)(b^{*j} + b^j) = \hat{C}^*$, the general real quadratic form of fermions. Then*

$$C' = U^t C V, \quad C = (C_{ij}) \tag{7}$$

(*the "polar decomposition" of the matrix C' in $\mathbb{R}^n$*).

The proof is by direct verification.

Lemma 2. *Let $\det C \neq 0$. If the s_i are the eigenvalues of the matrix $\sqrt{C^t C}$, then the specturm of $\hat{C}$ on $\Lambda^*(\mathbb{R}^n)$ has the form*

$$\lambda_{i_1\cdots i_k} = -\sum_{i=1}^n s_i + s_{i_1} + \cdots + s_{i_k}, \quad i_1 < i_2 < \cdots < i_k. \tag{8}$$

The operator $\Delta^{(q)}_t$ has exactly one null mode. It lies in the space Λ^{even} if $\det C > 0$ and in Λ^{odd} if $\det C < 0$.

Formulas (7) and (8) apparently do not occur in the literature.

Now we can sharpen the Morse-type inequalities of [1], providing upper estimates for the null modes of $\Delta_{t\omega}$. By the conjecture of [1], the numbers of null modes $b_\pm(t)$, $t \to \infty$, do not depend on the metric and are invariants of the pair (X, η), where η is a vector field. Now, by Lemma 2, these numbers can be estimated from above in terms of the numbers $m_\pm(\eta)$ of critical points, where $\eta(\mathcal{P}_q) = 0$, such that $\operatorname{sign}\det C^{(q)} = \pm 1$.

Theorem. *If all critical points are nondegenerate, then*

$$\lim_{t\to\infty} b_\pm(t) \leqslant m_\pm(\eta), \quad b_+ - b_- = \chi(X).$$

The authors are grateful to I. A. Volovich, A. A. Kirillov, Ya. G. Sinai, and L. D. Faddeev for their valuable comments.

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