MORSE INEQUALITY AND VON NEUMANN ALGEBRAS

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Let $\bar{b}_p^\chi$ denote the p-dimensional real Betti number (see [1]) of a compact not simply connected n-dimensional manifold X, assigned to a representation $\chi : \pi_1(X) \to \mathfrak{A}$ where $\mathfrak{A}$ is a von Neumann algebra with finitely-normalized trace ($\mathrm{tr}_{\mathfrak{A}} I = 1$).

Theorem 1. *If f is a Morse function on X, m_p is the number of its index p critical points, then*

$$\sum_{k=0}^{p}(-1)^k m_{p-k} \geq \sum_{k=0}^{p}(-1)^k \bar{b}_{p-k}^\chi \ (p = 0, 1, \ldots, n). \tag{1}$$

These inequalities, which generalize the classical Morse inequalities, eventually are more precise than their classical analogs, even in the simplest case of the regular representation χ. For instance, this is the case if $p = 2$ and X is homotopically equivalent to a manifold $Y \setminus y_0$ on the 3-skeleton, where Y is a compact 4-manifold, and also $H_1(X, \mathbb{Z}) = 0$, and the group $\pi_1(X)$ is infinite. Inequalities (1) imply that $\bar{b}_1^\chi \geq m_1 - m_2 - 1$ where m_1 is the number of generators of $\pi_1(X)$, m_2 is the number of defining relations. It follows that for the groups $\pi_1(X)$ close to free groups, there is an infinite-dimensional space of quadratically integrable harmonic 1-forms on the universal covering of X. If $X = M/\Gamma$ where M is a symmetric space, χ is the regular representation of $\Gamma = \pi_1(X)$, and the continuous spectrum of the Laplace operators in forms on M is separated from 0, then a von Neumann analog of the Ray–Singer torsion can be defined, such that In $R(X, \chi) = c(M)\, Vol X$. The event of non-separateness of the spectrum to 0 does not depend on the metrics on X. Some new deep topological phenomena appear here.

References

[1] Singer I.M. Some remarks on operator theory and index theory//Lect. Notes in Math. 1977. V. 575. P. 128137.