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The Morse Theory and von Neuman Topological Invariants for Non-Simply-Connected Manifolds[1]

The real Betti numbers $\bar{b}_p(X)$ were invented first by Atiyah (see [1]) for compact non-simply-connected n-dimensional manifolds X through the L_2-cohomology of its universal covering M. More general numbers $\bar{b}_p^\chi$ were defined by Singer (see [2]) for every representation $\chi : \pi_1(X) \to \mathfrak{A}$ where $\mathfrak{A}$ is a von Neuman algebra with finite normalized trace (the Atiyah numbers $\bar{b}_p(X)$ correspond to the regular representation χ of the group $\pi_1(X) = \Gamma$ into the space $l_2(\Gamma)$. It was found (see [3]) that in the classical Morse inequalities on the manifold X one can replace ordinary Betti numbers b_p by the numbers $\bar{b}_p^\chi$ for every χ. In some cases von Neuman-Morse inequalities are stronger than the ordinary Morse inequalities. Sometimes they allow to establish non-triviality of the L_2-cohomology for coverings of non-simply-connected manifolds.

It was established in the works [4, 5] that for $M = H^n$ (Hyperbolic or Lobachevski space) and M–strictly convex domain with Bergman metric we have $\bar{b}_p = 0$ for $p \neq n/2$. In the cases where $\bar{b}_p = 0$ and the spectra of all Laplacians on the spaces of p-forms are separated from zero and or spectrum touches zero, we define a von Neuman analog of the Ray-Singer torsion $\bar{R}$.

The fact that the spectrum of Δ_p touches zero, does not depend on Riemannian metric on X. For $p = 0$ such event takes place (see [6]) if and only if Γ is amenable. For $M = H^{2k+1}$ the spectrum of Δ_p touches zero for $p = k, k+1$ only (see [4]). Also, the power in t of the power-like decay for $t \to \infty$ of the $\theta_p(t) = Tr_\Gamma \exp(-t\Delta_p)$ also does not depend on metric, as well as the power of the power-like asymptotics for $\lambda \to +0$ for the density of states $N_p(\lambda) = Tr_\Gamma E_\lambda(\Delta_p)$ where $E_\lambda(\Delta_p)$ is the spectral projector of the operator Δ_p, and the trace Tr_Γ is defined by the integration of the diagonal of the kernel over the fundamental

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domain, according to [1]. More precisely, by means the Variational Principle analogous to [7], we prove the following

Theorem 1. *Let N_p, θ_p and N', θ' correspond to two different Riemannian metrics on the compact manifold X covered by M. There exists positive nonzero constant C such that*

$$N_p(C^{-1}\lambda) \leq N'_p(\lambda) \leq N_p(C\lambda), C^{-1}\theta_p(Ct) \leq \theta'_p(t) \leq C\theta_p(C^{-1}t).$$

If $\theta_p(t) \leq o(t^{-\epsilon_p})$ for all p for $t \to \infty$ where $\epsilon_p > 0$ for all $p = 0, ..., n$, then the torsion $\bar{R}$ is well-defined.

Conjecture: The estimates $\theta_p(t) = o(t^{-\epsilon_p}), \epsilon_p > 0$, are valid always if $\bar{b}_p = 0$.

For $M = H^3$ we have $\theta_1(t) \sim ct^{-1/2})$ (see [8])[2]. It allows to define $\bar{R}$ for 3-manifolds of the constant negative curvature. However, as Senya Vishik communicated to us privately, in this case $\bar{R} = 0$. Probably, nontrivial von Neumann torsion might appear for the nontrivial representations different from the regular one[3].

References

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[2]Originally this result was extracted by the present author (S.Novikov), who worked some period studying Cosmological Models of the General Relativity in early 1970s, from the old famous work of physicist E.Lifshitz on time evolution of the completely isotropic Friedman Cosmological Model with 3-space sections of constant negative curvature (1946). Misha Shubin replaced it by the quotation to the more recent rigorous mathematical work [8] of Senya Vishik

[3]It turned out that Senya Vishik made mistake in his calculation: John Lott recalculated it later by the suggestion of the present author (S.Novikov) and found out that density of torsion is nonzero indeed $\bar{R} \neq 0$ in this case, so it is proportional to the volume of fundamental domain with nonzero coefficient. M.Gromov and M.Farber proved later that these ,,Novikov-Shubin" power invariants are homotopy invariant. They pointed out that they appear as natural homological quantities for infinite-dimensional complexes.

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