EVOLUTION OF A WHITHAM ZONE IN THE KORTEWEG–DE VRIES THEORY

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Effect of viscosity. We consider the Korteweg–de Vries–Burgers (KdVB) equation with low viscosity $\mu > 0$

(1)
$$u_t + u_{xxx} + uu_x + \mu u_{xx} = 0, \quad |\mu u_{xx}| \ll |u_{xxx}|, |u_{xx}|$$

Under the conditions of (1) the evolution of an oscillatory zone is described in the framework of the Bogolyubov–Whitham averaging method using a set of cnoidal travelling waves of the KdV equation. The averaging of the viscous term over that set leads additionally to a right-hand side

where E(s) and K(s) are complete elliptical integrals (see [2]), in the equation for v_{α} (see [1], p. 264). We note the properties

(4)
$$g_1 \leqslant 0, \quad g_2 \ge 0, \quad g_3 > 0, \quad g(\lambda r) = \lambda^2 g(r)$$

(5)
$$v_1 \leqslant v_2 \leqslant v_3, \quad r_1 \leqslant r_2 \leqslant r_3, \quad v(\lambda r) = \lambda v(r).$$

The present paper is devoted to a numerical study of the evolution of an oscillatory zone in the "step decay" when there is a low viscosity present. The boundary conditions for the KdVB Eq. (1) are the following:

$$u \to A_{\pm}, \quad x \to \pm \infty.$$

According to ideas worked out in [3] and [4], this process is for $\mu = 0$ described by means of the Whitham equation inside the oscillatory zone, $\Delta(t) = [x^-(t), x^+(t)]$, which is joined to the trivial Hopf equation $u_t + uu_x = 0$ at its boundaries x^{\pm} . In particular, we have [see also (12) and (13) below]

(6)
$$u(x^{-},t) = r_3(x^{-},t), \quad u(x^{+},t) = r_1(x^{+},t), \\ r_1(x^{-},t) = r_2(x^{-},t), \quad r_3(x^{+},t) = r_2(x^{+},t).$$

The process was thus for $\mu = 0$ described by the evolution of a multiple-valued function r(x,t), where r = u is outside Δ and $r = \{r_{\alpha}\}$ is inside Δ . This distinguishes the class of problems arising from dispersive hydrodynamics from the

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classical case. Equation (1) possesses a stationary solution $u(\eta) = u(x - Vt, \mu)$ such that

(7)
$$u \to A_{\pm}, \quad x \to \pm \infty$$

if the viscosity is low, the solution $u(x - Vt, \mu)$ in this case oscillates, according to (1), infinitely with a damped amplitude as $x \to -\infty$ and tends to a soliton at the leading edge, $x > x^+$. When $\mu = 0$, the equation for u(x - Vt) has the form $u''(\eta) = \ldots$. Its right side has in the (u, u') phase plane two singularities (a center and a saddle-point) corresponding to the constants A_{\pm} . The separatrix of the saddle-point is closed and corresponds to a soliton; it contains the center inside itself. For fixed A_{\pm} we have for $\mu = 0$ a set of periodic solutions inside the separatrix

(8)
$$u(x,t) = \frac{2a}{s^2} dn^2 \left[\left(\frac{a}{6s^2} \right)^{1/2} (x - Vt), s \right] + \gamma,$$

where $a = r_2 - r_1$, and $\gamma = r_2 + r_1 - r_3$,

(9)
$$3A_{+}A_{-} = 4r_1r_2 - (r_3 - r_2 - r_1)^2$$
, $A_{-} + A_{+} = \frac{2}{3}(r_1 + r_2 + r_3) = 2V$.

For small $\mu > 0$, there is a solution $u(x - Vt, \mu)$ with asymptotic behavior (7), which is described by a spiral in the same phase plane going from the center to the saddle-point; the parameters r_{α} change slowly when x changes; the quantities A_{\pm} are constant in that sense along this solution:

(10)
$$\frac{dA_{\pm}}{dx} = 0.$$

Averaging the solution $u(x-Vt,\mu)$ over the cycle (8), we obtain an exact stationary solution $r_{\alpha}(x-Vt,\mu)$ of Eq. (2), where the following quantities are constants:

(11)
$$3V = \sum_{\alpha} r_{\alpha} = \frac{3}{2}(A_{+} + A_{-}) = \text{const}, \quad A_{+}A_{-} = \text{const}$$

The solution with the properties (11) is found by a single quadrature. Its graph is shown in Fig. 1 (dashed curve). If $A_{\pm} = \mp 1$, we have V = 0, and the leading edge $r_2(x^+) = r_3(x^+)$ is situated at the end point x^+ , where $r_2^+ = r_3^+ = 1/2$, and the trailing edge $r_1 = r_2$ is at $x = -\infty$, where $r_1 - r_2 = -1/2$. The general case can be reduced to this situation by a change in scales and a Galilean transformation: $x \to x + Ct$, $r_{\alpha} \to r_{\alpha} + 0$, and $v_{\alpha} \to v_{\alpha} + C$.

It is rather obvious that in the presence of viscosity $\mu > 0$, the evolutionary process of the multiple-valued function r(x,t), if it is at all correctly defined for the given initial condition (in particular, one does not encounter a hydrodynamic type of inversion), after a sufficiently long time $t \to \infty$ develops from the boundary conditions (7) to the stationary solution described above by (11).

An exact determination of multiple-valued initial conditions r(x,t) for $\mu = 0$ was given in [4], including the evolution of the region $\Delta(t)$, and a numerical realization of that process was given. Here we develop the approach of [4] to the case $\mu \neq 0$. As in [4], the multiple-valued function r(x,t) with the conditions (6) must be once smooth (class C^1), including at the points x^{\pm}, r^{\pm} . Near the end points x^{\pm}, r^{\pm} of the interval $\Delta(t)$, an asymptotic behavior, similar to the asymptotic behavior (10),

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(11) in [4], must hold:

(12)
$$x'' = (a_{+} + b_{+}(r - r^{+}))f(1 - s^{2}) + O((r - r^{+})^{3}),$$
$$x'' = x - x^{+} \leq 0, \quad f(y) = y^{2}[\ln(16/y) + 1/2]$$

(13)
$$x' = a_{-}(r - r^{-})^{2} + b_{-}(r - r^{-})^{3} = o((r - r^{-})^{3}), \quad x' = x - x^{-} \ge 0.$$

The following equations follow from (12) and (13) for the quantities $r^{\pm}(t)$ and $x^{\pm}(t)$:

(14)
$$\dot{r}^{+} = -(r^{+} - r_{1}^{+})^{2}[(12a_{+})^{-1} + 16/45], \quad \dot{x}^{+} = v^{+} = (r_{1}^{+} + 2r^{+})/3;$$

(15)
$$\dot{r}^- = -1/(2a_-), \quad \dot{x}^- = 2r^- = r_3^-.$$

A comparison with (12) and (13) from [4] shows that the presence of viscosity changes only the equation for r^+ . As $t \to \infty$, the quantity a_+ has a finite limit $a_{\infty}^+ \neq \infty$, in contrast with the case $\mu = 0$. Inside the interval $\Delta(t)$ Eq. (2) holds, outside it, the equation $u_t + uu_x = 0$ for u = 4 holds. The geometry of the three sets of characteristics for the quantities $r_{\alpha}(x,t)$ is the same as in [4], including the region close to the zone boundaries $[x^{\pm}(t),t]$ in the (x,t) plane. However, the value of r_{α} is not conserved along the characteristics, which leads to certain numerical complications. The choice of the initial condition for $t = t_0 > 0$ is the following: we use in the region $\Delta(t_0)$ the self-similar Gurevich–Pitaevskii (GP) solution ([1], pp. 275–284) for the inversion of the front of a dispersive shock wave under the conditions $r_3^- < A_-$, $r_1^+ > A_+$. Outside the zone $\Delta(t_0)$ we use the function $r(x, t_0) = u(x)$, which reaches the values A_{\pm} smoothly and monotonically as $x \to \pm \infty$, with the conditions (12), (13), and (6).

We now discuss the limitations on the initial conditions following from our scheme. Up to the "inversion of the front" (t < 0), when there is no oscillatory zone, the condition for the applicability of the Hopf equation,

$$(16) |u_{xxx}| \ll |uu_x|, |\mu u_x| \ll |uu_x|,$$

is satisfied. Let its solution x = ut + P(u) be such that P(u) changes over characteristic distances A. We denote the scale of the characteristic changes of x by B. At the time t_0 the oscillatory zone which is formed at $t \sim 0$ manages to develop locally to the self-similar GP asymptotic form. As time elapses t_0 , one could ignore the viscous term because of (1)

$$|\mu u_{xx}| \ll |u_{xxx}|, |uu_x|.$$

At $t = t_0$ we must have inside $\Delta(t_0)$

(18)
$$\Delta r/t_0 = (r_3^- - r_1^+)/t_0 \gg \mu |g_\alpha(r)| \sim \mu (\Delta r)^2,$$

where the quantities $g_{\alpha}(r)$ are of the order $(\Delta r)^2$ by virtue of their homogeneity (4). It is also necessary that

(19)
$$\Delta r \ll A$$

in the zone where P(u) can be approximated by a cubic parabola ([1], p. 275). In the zone $\Delta(t_0)$ we must fit a large number of periods of the function (8)

(20)
$$k = T^{-1} \sim \frac{\pi}{K(s)} \left(\frac{a}{6s^2}\right)^{1/2}, \quad T \ll \Delta(t_0).$$

Outside $\Delta(t_0)$ we must check the conditions of applicability of the Hopf equation. Apart from (16), we must ensure that the dropped terms do not turn out to be effective outside Δ , when that process takes place. Using the total group of scale transformations of the Hopf equation and of (2), we render them dimensionless in the usual way:

(21)
$$x = Bx', \quad u = Au', \quad t = BA^{-1}t', \quad \mu = B^{-1}\mu'.$$

In the dimensionless variables we have

(22)
$$\Delta'(t'_0) \sim (t'_0)^{3/2}, \quad \Delta r' \sim (t'_0)^{1/2}$$

in the GP regime. Hence it follows that all the conditions (16)–(20) reduce to the inequalities

(23)
$$t'_0 \ll 1, \quad (\mu')^{-2/3} \gg t_0 \gg (A^{1/2}B)^{-7/4}.$$

At this point we drop the prime on the letters and we drop the quantities A and B [they deal with the derivation and drop out of the Hopf equations and (2)]. The new quantity μ is now no longer small. In the new variables, after the Galilean transformation, we have $A_{+} = -1$, $A_{-} = 1$. Outside $\Delta(t_0)$ we choose in the numerical examples the function $x = tu + 3[u - \operatorname{arctanh}(u)]$. The GP solution and the segment Δ have, at time t_0 , the form (see [1], p. 283 for z^{\pm}):¹

(24)
$$r_{\alpha}(x,t_{0}) = (t_{0}-t_{1})^{1/2}l_{\alpha}(r) + r_{0},$$
$$z^{-} \leqslant z \leqslant z^{+}, \quad z = (x-x_{1})\lambda(t_{0}-t_{1})^{-3/2}.$$

The parameters t_1, λ, x , and r_0 are arbitrary. We must choose them from the four matching conditions (see above) for the values $u(x^{\pm})$ and $u^1(x^{\pm})$. The oscillatory zone is thus at the initial time t_0 determined by the function $u(x, t_0)$.

The results of the numerical simulation are illustrated in Figs. 1–4. As tests for the closeness of the given reglme to the final and intermediate asymptotic behavior we took the following quantities:

1) V(r) and $-A_+A_-(r)$ for $\mu = 0.1$ and $\mu = 1$ at large t; one should have V = 0 and $-A_+A_- = 1$ in the exact stationary regime (11).

2) the quantity $v_2(x/t)$ for $\mu = 0.1$ and t = 0.03; 2.7; 11. At the time t = 2.7 the quantity $r^+(t)$ has a maximum; the quantity v_2 is close to the linear $v_2 \simeq xt^{-1}$ at that time. This affects the realization of the intermediate asymptotic behavior: for $\mu = 0$ such a regime was described by the asymptotic behavior as $t \to \infty$ in the step decay problem according to [1] (p. 268). When $\mu = 1$, this regime is not realized as an intermediate regime (we note that in the original variables a low viscosity corresponds to the dimensionless $\mu = 1$).

Therefore, if the evolutionary process in this description is determined over an infinite time, it reaches uniquely the stationary regime (11) as $t \to \infty$. The nature of the intermediate stages of the evolution depends on the initial conditions and on the parameter μ . We used here the as assumption that oscillations indeed occur. The way these oscillations reach the GP asymptotic behavior at the time $t = t_0$ is studied in [4]. Here we assume this time to be the initial time.

Problem. Study the general features of the way in which the oscillatory zone arises from the KdV theory (the viscosity becomes important later on), where all

¹We know from Ref. 3 that $z^- \simeq -1.41$, $z^+ \simeq +0.117$; more precise numerical calculations show that with an accuracy of 4 decimal places $z^- \simeq -\sqrt{2}$. Using a method developed by I.M. Krichever, one can find that solution analytically exactly, as the "averaged finite-zone solution" in the terminology of [5].

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FIGURE 1. Evolution of the multiple-valued function r(x,t) for $\mu = 0.1$. The dashed curve indicates the stationary solution. Here and in Figs. 2 and 4 the numbers of the curves indicate the time.



FIGURE 2. Evolution of r(x, t) for $\mu = 1$.



FIGURE 3. Quantities characterizing the approach to the stationary solution: curve 1: V(x), 2: $(-A_+A_-)$ as a function of μx for $\mu = 0.1$ and t = 11.9. The dashed curves are the corresponding curves for $\mu = 1$ and t = 1.45.

Whitham type equations become effective for the first time. It is so far not clear which classes of initial conditions are realizable for the Whitham equation.

L. P. Pitaevskii has informed us that very recently he and A. V. Gurevich have independently obtained Eq. (2) and the stationary regime (11).



FIGURE 4. Evolution of $v_2(x,t)$ for $\mu = 0.1$. The time t = 2.7 corresponds to the maximum of $r^+(t)$ (the arrow in Fig. 1).

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