QUANTIZATION OF FINITE-GAP POTENTIALS AND NONLINEAR QUASICLASSICAL APPROXIMATION IN NONPERTURBATIVE STRING THEORY

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In recent articles [1]–[3], and ending with [4], there has been discovered a remarkable circumstance resulting from the combinatorial Kazakov–Migdal–Rostov approach in the continuous limit. In cases of nonperturbative conformal string theories interacting with a "two-dimensional gravitation" according to Polyakov's scheme, as well as some other ones, when the central charge is $c < 1$, there appears a simple system of equations for the renormgroup, i.e., a set of Laks type equations with certain ordinary differential operators in x:

$$\frac{\partial L}{\partial t_k} = [L_1 A_k]. \tag{1}$$

Equations (1) are studied for the following boundary conditions:

$$[L, A] = \varepsilon \cdot 1, \tag{2}$$

where ε is a quantum constant significant for our method.

Equations of type (1), (2) for $\varepsilon = 0$ have well-known finite-gap and multisoliton solutions; they are completely integrable Hamiltonian systems and can be exactly solved with θ-functions on Riemann surfaces (see [5]–[7]).

Definition. Equation (2) is called a quantization of finite-gap potentials.

The simplest case is where we have a second-order scalar operator $L = -\partial_x^2 + u$, and A is an operator of odd degree. All such operators A are well-known in the theory of the Korteweg–de Vries (KdV) equation. In absolutely the simplest case

$$L = -\partial_x^2 + u, \quad A = -4\partial_x^3 + 6u\,\partial_x + 3u' \tag{3}$$

the study of Eq. (2) is a rather complicated task, and from the naive point of view it is unsolvable (it is a Painlevé type equation of the first kind). In this article, for a smsll parameter ε, we describe a nonlinearly quasiclassical approach to the construction of a (possibly exact) solution of Eq. (2) based on a continuation of ideas of the theory of finite-gap integration. We first describe the entire method in the simplest partial case (3); generalizations to other cases will become obvious.

Painlevé equation I has been studied earlier in [15]–[17]. Elliptic functions appeared in the classic work [15] published in 1913, where asymptotics were studied as $|x| \to \infty$, $x \in \mathbb{C}$, The study of asymptotics (as $|x| \to \infty$) was significantly advanced in [16]. Krichever in [17] obtained qualitative results on solutions of the Painlevé equation I. It should be emphasized that our methods for studying asymptotics

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(as $\varepsilon \to 0$) are not entirely rigorous in the “physical region” of interest and have to agree with results in [15]–[17] for $|x| \to \infty$. Of physical interest is a singular solution which $u(x) \sim \sqrt{-\varepsilon x/3}$ as $x \to -\infty$ (only on the real axis). Our hypothesis guarantees that the method described in the main part of this article has precisely such solution. To avoid confusion, we emphasize that we use a term “quasiclassical approximation” in its literal sense only on the real axis.

1. Some Results from the Theory of Solitons. General Relations

Equation (2) is a stationary equation for a Laks type system with an operator

$$\tilde{L} = L - (\lambda + \varepsilon t) \cdot 1, \quad \partial\tilde{L}/\partial t = [\tilde{L}, A]. \tag{4}$$

Using the method cited in [5], we replace the standard Laks representation (4) by a representation of the following form, where according to (4) λ is replaced by $\lambda + \varepsilon t$:

$$\left[\frac{\partial}{\partial t} - \Lambda, \frac{\partial}{\partial x} - Q\right] = 0, \tag{5}$$

Furthermore, matrices Λ and Q for operators (3) have the following form:

$$\Lambda = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -u' & 2u + 4\lambda \\ 2u^2 - u'' + 2\lambda u - 4\lambda^2 & u' \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}. \tag{6}$$

We obtain stationary equation (2) by substituting $\partial_t \to \varepsilon\partial_\lambda$, after which we obtain the following Laks type pair for ordinary equations (2):

$$[\varepsilon\partial_\lambda - \Lambda, \partial_x - Q] = 0, \quad \Lambda_x = [Q, \Lambda] + \varepsilon \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{7}$$

$$\varepsilon\frac{\partial\Psi}{\partial\lambda} = \Lambda\Psi, \quad \frac{\partial\Psi}{\partial x} = Q\Psi. \tag{8}$$

Equation (7) is a condition of compatibility of system (8). Equation (4) implies that it is impossible to find solutions of stationary system (2) with the usual boundary conditions on classes of initial functions $u(x)$ and a sensible spectral problem for the Schrödinger operator L, since Eq. (4) shows that the eigenvalues of the operator L (if they are well-defined) move with a constant velocity. Representations of type (7) have been used for self-similar solutions in works by Flaschka, Newell, Its, Novokshenov, and others, but the method described in this article is completely different (it is possible that a combination of it with methods described by those authors will also be beneficial in the theory of auto-modeling solutions).

We will try to explicitly use Riemann surfaces, taking into account the polynomial nature of matrices Λ and Q with respect to λ and the smallness of the parameter ε. We carry out a gauge transformation of Eqs. (8) by diagonalizing the matrix Λ to the zeroth order of ε as follows. Let $\Psi = U^{-1}\widetilde{\Psi}$, where

$$U^{-1} = \begin{pmatrix} 1 & 1 \\ \delta & \chi \end{pmatrix}, \qquad \chi = -(a + \sqrt{R})/b, \quad \delta = -(a - \sqrt{R})/b, \quad R = -\det\Lambda = a^2 + bc. \tag{9}$$

Transformation (9) turns system (8) into

$$\varepsilon\widetilde{\Psi}_\lambda = \left[\Lambda_0 + \varepsilon\Lambda_1^d + \varepsilon\Lambda_1^* + \varepsilon t_\lambda \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\widetilde{\Psi} = [-\varepsilon U(U^{-1})_\lambda + U\Lambda U^{-1}]\widetilde{\Psi}, \tag{10}$$

$$\tilde{\psi}_x = \left[Q_0 + \varepsilon Q_1^* + t_x \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\widetilde{\Psi} = [-U(U^{-1})_x + UQU^{-1}]\widetilde{\Psi}, \tag{10$'$}$$

where

$$\Lambda_0 = U\Lambda U^{-1} = \operatorname{diag}(\sqrt{R}, -\sqrt{R}), \quad Q_0 = \operatorname{diag}(\sqrt{R}/b, -\sqrt{R}/b),$$
$$\Lambda_1^* = \begin{pmatrix}0 & \chi_\lambda(\chi-\delta)\\ -\delta_\lambda/(\chi-\delta) & 0\end{pmatrix}, \quad Q_1^* = \begin{pmatrix}0 & -b/(4R)\\ b/(4R) & 0\end{pmatrix},$$
$$\Lambda_1^d = \operatorname{diag}(2u'/(b\cdot\sqrt{R}), -2u'/(b\cdot\sqrt{R})),$$
$$t = -1/2(\chi-\delta), \quad \det U^{-1} = \chi - \delta.$$

Equation (2) for operators (3) has the following form:

$$u''' = 6uu' + \varepsilon. \tag{11}$$

Defining a function $D = 2uu'' - 4u^3 - (u')^2$, $C = u'' - 3u^2$, Eqs. (11) imply that

$$C = \varepsilon x + C_0, \quad C_0 = \text{const}, \quad D' = 2\varepsilon u, \quad R' = -\varepsilon b, \tag{12}$$

where $R = -\det\Lambda$ has the form

$$R(\lambda, x) = -16\lambda^3 - 4\lambda(\varepsilon x + C_0) - D. \tag{13}$$

Riemann surface Γ is defined by the following equation:

$$\det(\Lambda + \mu\cdot 1) = \mu^2 - R(\lambda, x) = 0. \tag{14}$$

We see that points of bifurcation on the surface Γ move "slowly" by virtue of (11) and (12), which allows us to apply a nonlinear quasiclassical approximation of Whitham type; however, we will derive more precise results. After a gauge transformation, the condition that Eqs. (10$'$) are diagonal in x implies that functions χ and δ for $\varepsilon = 0$ are independent solutions of the Riccati equation. If $\varepsilon \neq 0$ then there are nondiagonal elements of the matrix Q_2^*:

$$\begin{aligned} \chi' + \chi^2 - (u-\lambda) &= -\varepsilon b/(4R),\\ \underset{.}{'} + \delta^2 - (u-\lambda) &= \varepsilon b/(4R). \end{aligned} \tag{15}$$

It is useful to remember that representation (6) is obtained in a basis $C(x,y,\lambda)$, $S(x,y,\lambda)$ of solutions of the Schrödinger equation $(L_y - \lambda)\Psi = 0$ in a variable y, where the matrix of solutions is unitary at a point x:

$$C_{yy} = (u-\lambda)C, \quad S_{yy} = (u-\lambda)S,$$
$$\begin{pmatrix}C & S\\ C' & S'\end{pmatrix}_{y=s} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The Blokh basis has a form $\Psi^+ = C + \chi(x)S$, $\Psi^+ = C + \delta(x)S$ for $\varepsilon = 0$.

The equation $(L_y - \lambda)\Psi = 0$ does not change under a change in basis in the space of solutions independent of y.

Representation (9) corresponds to a change to a Blokh basis for $\varepsilon = 0$.

After gauge transformation (9) in variables (x,λ) which does not depend on y, the function $\widetilde{\Psi}(x,y,\lambda)$ still satisfies the Schrödinger equation in y. As $\varepsilon \to 0$, it has to become a Blokh function meromorphic on the surface Γ with an essentially singular point at $\lambda = \infty$. As usual, we also carry out the following gauge transformation (defined and single-valued on a surface $\widehat{\Gamma}$ which is a four-sheeted covering of the surface Γ, and in whose definition functions $b^{1/2}$ and $R^{1/4}$ are single-valued):

$$\widetilde{\Psi} = \widetilde{\widetilde{\Psi}}/\sqrt{\chi - \delta}, \quad 1/(\chi - \delta) = R^{1/5} b^{1/2}.$$

Then $\widetilde{\widetilde{\Psi}}$ satisfies an equation in which all matrices have nonzero traces:

$$\varepsilon \widetilde{\widetilde{\Psi}}_\lambda = (\Lambda_0 + \varepsilon \Lambda_1^d + \varepsilon \Lambda_1^*) \widetilde{\widetilde{\Psi}}, \quad \widetilde{\widetilde{\Psi}}_x = (Q_0 + \varepsilon Q_1^*) \widetilde{\widetilde{\Psi}}. \tag{16}$$

By our construction we should look for a matrix function $\widetilde{\widetilde{\Psi}}$ which is meromorphic on the covering $\widehat{\Gamma}$ of the surface Γ with a singular point of exponential type for $\lambda \to \infty$.

We use the quasiclassical approximation to determine the type of singularity. From now on we assume that $y = 0$, and focus only on the x dependence.

2. Linear and Nonlinear Quasiclassical Approximations

Relations (12) and (18) for the deformation of Riemann surfaces can be used to obtain an averaged equation of Whitman type. Let $X = \varepsilon x$. Eq. (12) then implies that

$$dD/dX = 2u.$$

Averaging this equation over the period of a one-gap potential for a small $\varepsilon \to 0$, we obtain a "drift" of the Weierstrass function, where $-g_2 = \varepsilon x + C_0$, $-g_3 = D/4$:

$$d\bar{D}/dX = 2\bar{u}(X, \bar{D}). \tag{17}$$

Equation (17) is the desired Whitham type equation for this case. Using the homogeneity of the function $\bar{u}(X, \bar{D})$, we see that Eq. (17) can be easily integrated (the function $\bar{u}$ is defined, for example, in [8, 10]; in the Supplement we give more details about this approximation).

We will later compare this result with a more precise method. Now we describe a "linear" quasiclassical approximation for system (8) and use the fact that the dependence on the variable λ is known for matrices Λ and Q. After applying gauge transformations to obtain an equation of type (16), we first look for $\widetilde{\widetilde{\Psi}}$ as a series of the perturbation theory using the rules of quantum mechanics (see, for example, [9]). We look for $\widetilde{\widetilde{\Psi}}$ in the form of a series in ε by using a "naive quasiclassical approximation"[1] as follows:

$$\widetilde{\widetilde{\Psi}} = (1 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots) \exp\{(B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \dots)/\varepsilon\}; \tag{18}$$

where matrices B_j are diagonal for all $j \geqslant 0$ and matrices A_q have zero diagonal elements for all $q \geqslant 1$. Equation (16) is easily solvable in λ as a series in ε. The solution has the following form (an operator $\operatorname{ad} \Lambda_0(H) = [\Lambda_0, H]$ is invertible for

[1] I. A. Volovich showed us that in order to perform a substitution into the equation for x we should look for $\widetilde{\widetilde{\Psi}}$ in the form $\widetilde{\widetilde{\Psi}} \to \widetilde{\widetilde{\Psi}} \cdot V$, where $V_\lambda = 0$, $V = V(x)$ (see [14]). However, this does not influence results in this article.

matrices with zero diagonal elements):

$$
\begin{aligned}
&B_{0\lambda} = \Lambda_0, \quad B_{1\lambda} = \Lambda_1^d, \quad B_{q\lambda} = \Lambda_1^* A_{q-0}, \quad q \geqslant 2,\\
&A_q = (\operatorname{ad}\Lambda_0)^{-1}(H_q), \quad H_1 = -\Lambda_1^*, \quad H_2 = A_{1\lambda} + [A_1, \Lambda_1^d],\\
&H_q = A_{q-1,\lambda} + [A_{q-1}, \Lambda_1^d] + \sum_{j=1}^{q-2} A_j \Lambda_1^* A_{q-j-1}.
\end{aligned}
\tag{19}
$$

Equations (19) imply that all $A_j, B_{q,\lambda}$ are algebraic on Γ. Furthermore, as $\lambda \to \infty$, the quantity A_1 has an order $\lambda^{-5/2}$, and the quantity B_1 has an order $\lambda^{-3/2}$. This order grows for quantities B_j and A_j as j grows. An initial approximation of $\tilde{\tilde{\Psi}}$ has the following form:

$$
\tilde{\tilde{\Psi}}_0 = \exp\left\{ \frac{1}{\varepsilon} \int^{\lambda} \sqrt{R}\, d\lambda \right\}, \quad \tilde{\Psi}_0 = \frac{1}{\sqrt{\chi - \delta}} \tilde{\tilde{\Psi}}_0. \tag{20}
$$

Let $ik = \lambda^{1/2}$. Quantity (20) after an expansion into series as $\lambda \to \infty$ takes the following form:

$$
\tilde{\tilde{\Psi}}_0 = \exp\{+8k^5/(5\varepsilon) + k(x + c_0/\varepsilon)\}(1 + D/(4\varepsilon) + O(k^{-2})). \tag{21}
$$

The above statements about orders with respect to $\lambda^{-1/2}$ imply that the remaining quantities A_j and B_j for $j \geqslant 1$ have the same asymptotics. Therefore, as $k \to \infty$, we have

$$
\sqrt{\chi - \delta}\, \tilde{\Psi} = \tilde{\tilde{\Psi}} = \exp\{+8k^5/(5\varepsilon) + k(x + C_0/\varepsilon)\}(1 + Dk^{-1}/(4\varepsilon) + O(k^{-2})). \tag{22}
$$

Now we study "nonnaive classical approximation" for $\tilde{\tilde{\Psi}}$. Equation (22) coincides with the standard asymptotics of the Baker–Akhiezer function (the Blokh solution of the one-gap operator L) when we recall relation (12), which relates the first term of the asymptotics in k^{-1} with the potential u. Now recall that Eqs. (12) hold in precisely this case, as seen from Eqs. (2) and (11). Furthermore, there is a new term $(+8k^5/(5\varepsilon))$ in the exponent. The main reason that the naive quasiclassical approximation already deviates from the actual function in first approximation is due to the fact that function (20) is not single-valued even at first approximation. A "nonnaive" quasiclassical approximation in parameter ε on a Riemann surface Γ consists of substituting Eq. (20) by a function $\tilde{\psi}_0$ which is single-valued on $\widehat{\Gamma}$, which is four-sheeted over Γ, and writing the series for $\tilde{\tilde{\Psi}}$ in ε as a summation such that every sequence of ε-approximations of the function $\tilde{\tilde{\Psi}}$ consists of functions Φ_n single-valued on $\widehat{\Gamma}$. To determine such first approximation of Φ_1, we write a naive quasiclassical function B using the following term in ε:

$$
\begin{aligned}
&\tilde{\tilde{\Psi}}_1 = \exp\left\{ \frac{1}{\varepsilon} \int \sqrt{R}\, d\lambda + \frac{\varepsilon u'\, d\lambda}{(u + 2\lambda)\sqrt{R}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},\\
&R = -16\lambda^3 - 4(\varepsilon x + C_0)\lambda - D = -4(4\lambda^3 - g_2\lambda - g_3).
\end{aligned}
\tag{23}
$$

Equation (23) shows that the second term in the exponent is an integral over $\varepsilon^{-1}B = \varepsilon^{-1}B_0 + B_1\lambda$ of a quantity which has two poles on the surface Γ at points $\lambda = -u/2$ (on both sheets)

$$
\gamma_+ = (-u/2, +), \quad \gamma_- = (-u/2, -) \tag{24}
$$

with residues of the form $u'/(\pm\sqrt{R(-u/2)}\cdot 2) = \pm 1/2$ (see below). The quantity $\tilde{\tilde{\Psi}}_1$ is single-valued on the Riemann surface $\widehat{\Gamma}$, which is a four-sheeted covering of Γ (according to functions $R^{1/4}, b^{1/2}$). The first "nonnaive" quasiclassical approximation of the function $\tilde{\tilde{\Psi}}$ has the form (24), where the exponent is replaced by a Baker–Akhiezer type function $\Phi(x,\mathcal{P})$ defined on Γ with the following analytic properties (where condition 2) is somewhat unusual:

1) $\Phi(x,\mathcal{P})$ is single-valued and meromorphic on Γ with one essentially singular point at $\lambda=\infty$, where Φ has asymptotics (23), but squared, i.e.,

$$\Phi = \exp\{2\cdot(+8k^5/(5\varepsilon) + k(x + C_0/\varepsilon)(1 + Dk^{-1}/(4\varepsilon) + O(k^{-2}));$$

2) $\Omega = (\ln\Phi)_\lambda$ has two simple poles with residues ± 1 at points (24). Indeed, we have

$$\operatorname{Res}\Omega = u'/\sqrt{R(-u/2)}, \quad \lambda = -u/2.$$

Since $R(\lambda) = -\det\Lambda = a^2 + bc$, condition $\lambda = -u/2$ is equivalent to $b = 0$ and $a = -u'$, so therefore the residues are equal to ± 1 on two sheets of the surface Γ. Both poles move with x and are always located on the opposite sheets of Γ.

Proposition. *In general, analytic conditions* (1), (2) *define a Baker–Akhiezer type function* $\Phi(x,\mathcal{P})$*, which is single-valued on* Γ*, which in turn defines the first nonnaive quasiclassical approximation of the form* (23) *as follows*:

$$\tilde{\tilde{\Phi}}_1 = \begin{pmatrix} \Phi^{1/2}(\mathcal{P}_+) & 0 \\ 0 & \Phi^{1/2}(\mathcal{P}_-) \end{pmatrix}, \quad \mathcal{P}_+ = (\lambda,+),\ \mathcal{P}_- = (\lambda,-), \tag{25}$$

where the exact value of the root is defined below. Quantity (25) *satisfies an equation*

$$\varepsilon\tilde{\tilde{\Phi}}_{1,\lambda} = (\Lambda_0 + \varepsilon\Lambda_1^d + \varepsilon\Lambda_1^*)\tilde{\tilde{\Phi}}_1 + O(\varepsilon^2).$$

The quantity $\tilde{\tilde{\Phi}}_1$ *is single-valued on a surface* $\widehat{\Gamma}$ *which corresponds to a four-sheeted covering of* Γ*, which in turn corresponds to both factors of a function*

$$\sqrt{\chi-\delta} = (-2\sqrt{R}/b)^{1/2},$$

i.e., to functions $b^{1/2}$ *and* $R^{1/4}$.

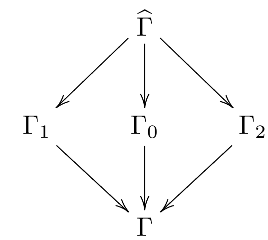
This statement has already been proven.

We now write the above analytic properties in a different way. Consider a function $\tilde{\tilde{\Psi}}_1 b^{-1/2}$. We see that the main term in asymptotics (23) does not change. As for poles (24), after a multiplication by $b^{-1/2}$ one of them (either γ_+ or γ_-) disappears, but a new one appears at $\lambda=\infty$:

$$\tilde{\tilde{\Psi}}_1 \cdot b^{-1/2} = \exp\left\{\frac{1}{\varepsilon}\int\sqrt{R}\,d\lambda + \frac{2\varepsilon u'\,d\lambda}{b\sqrt{R}} - \frac{\varepsilon b_\lambda\,du}{2b}\right\}. \tag{26}$$

Lemma. *A differential* $\Omega' = (\sqrt{R}/\varepsilon + 2u'/(b\sqrt{R}) - b_\lambda/(2b))d\lambda$ *on a surface* Γ *has two poles with residues* ± 1 (*at points* γ_+ *and* $\lambda=\infty$).

We have the following diagram of coverings:



The function $b^{-1/2}$ is single-valued on the surface Γ_1, the function $R^{1/4}$ is single-valued on Γ_2, and the function $R^{1/4}b^{-1/2}$ is single-valued on Γ_0. Each factor is separately single-valued on $\widehat{\Gamma}$. A function $\Psi_1 b^{-1/2}$ can be defined as a single-valued one on Γ, since our Baker–Akhiezer type functions do not have zeros or poles at points λ_j, $R(\lambda_j) = 0$ and $\lambda_4 = \infty$. Furthermore, $\tilde{\tilde{\Psi}}_1 b^{-1/2}$ has the usual analytic properties:

1) it has asymptotics (26) as $\lambda \to \infty$ on Γ;
2) it is single-valued on Γ;
3) the poles of its logarithmic derivative are at points $\lambda = -u/2$ (on the same sheet) and $\lambda = \infty$ with residues ± 1.

This is the usual Baker–Akhiezer function on Γ. The different aspects of the case where one of the poles of the logarithmic derivative is at a point $\lambda = \infty$ have already been studied (see [11]).

3. Quasiclassical and (More) Exact Solution of the Main Equations $[L, A] = \varepsilon \cdot 1$

We "freeze" the Riemann surface Γ and apply the standard approach to obtain the following equations [where the phase incorporates new terms as dictated by asymptotics (26) and (22)]:

$$u_{\mathrm{sc}} = -2\partial_x^2 \log\theta(\bar{A}x + \bar{B} \mid g_2, \bar{g}_3) + k = 2\wp(x + 8/(5\varepsilon)\cdot U_6/U_2 \mid g_2, \bar{g}_3) \tag{27}$$

where $\bar{g}_3$ is the average of g_3 over the period of variable x and $\bar{g}_3 = \bar{g}_3(X)$ (the subscript "sc" indicates that the function u_{sc} is "quasiclassical"). We have the following equation

$$D/(4\varepsilon) = -\zeta(x + 8/(5\varepsilon)\cdot U_6/U_2 \mid g_2, g_3). \tag{28}$$

We now regard (28) to be an equation for a quantity $D(x, X)$, where $x = \varepsilon x$ [since constants g_2 and g_3 and Weierstrass functions, i.e., coefficients $R/4$, are equal to εx and D according to Eqs. (12) and (13)], $\omega_{2j} \sim dk^{2j-1}$), $\lambda \to \infty$,

$$A = A(g_2, g_3, \varepsilon), \quad B = B(g_2, g_3, \varepsilon),$$
$$\bar{A} = (g_2, \bar{g}_3, \varepsilon), \quad \bar{B} = B(g_2, \bar{g}_3, \varepsilon),$$
$$k = k(g_2, \bar{g}_3), \quad U_{2j} = \oint_b \omega_{2j}, \quad \oint_a \omega_{2j} = 0.$$

After obtaining $D(x, X)$ from (28), we define an "exact," or "more exact" solution $u_{\mathrm{ex}}(x)$ as follows:

$$2\varepsilon u_{\mathrm{ex}}(x) = \frac{dD}{dx} = \frac{\partial D}{\partial x} + \varepsilon\frac{\partial D}{\partial X}. \tag{29}$$

It can also be checked that $\bar{g}_3 = -\bar{D}/4$ can be obtained from (17).

Proposition. *Eq.* (29) *gives* $u_{\mathrm{ex}}(x)$ *as the exact solution of an equation* $[L, A] = \varepsilon\cdot 1$ *for any* ε, $u_{\mathrm{ex}}(x) \sim \sqrt{-\varepsilon x/3}$.

The above statement is proven as follows.

The above corrections do not influence either the position of poles Ω' or the terms appearing in the asymptotics for $\lambda \to \infty$, including the term $D/(16\varepsilon)$ of k^{-1}. Equations (12) are exact. Therefore, the first quasiclassical approximation for $\tilde{\tilde{\Psi}}$ which is single-valued on Γ can give an exact equation for $u(x)$. The above arguments are intuitive; a rigorous proof follows.

Physically interesting solutions have asymptotics

$$u \sim \left(-\frac{\varepsilon x}{3}\right)^{1/2}, \quad x \to -\infty. \tag{30}$$

Solutions with asymptotics (30) satisfy

$$u \sim u_0 = \left(-\frac{\varepsilon x}{3}\right)^{1/2}, \quad D \sim D_0 = -4u_0^3, \quad x \to -\infty. \tag{31}$$

We rewrite initial equation (11) in a form which explicitly contains the Riemann surface Γ. Choosing D as the unknown function, we easily obtain a general identity, or Eq. (11) in a new form:

$$\left(\frac{D''}{2\varepsilon}\right)^2 = R(-u/2) = \widetilde{R}(u) = 2\left(\frac{D'}{2\varepsilon}\right)^3 + 2(\varepsilon x + C_0)\left(\frac{D'}{2\varepsilon}\right) - D, \quad u = D'/2\varepsilon, \tag{32}$$

where $\widetilde{R}(\lambda) = 2\lambda^3 + 2(\varepsilon x + C_0)\lambda - D$.

Thus, we have (letting $\lambda = -D'/4\varepsilon$)

$$\left(\frac{D''}{2\varepsilon}\right)^2 = R\left(-\frac{D'}{4\varepsilon}\right)$$

By applying a translation $x \to x + C_0/\varepsilon$ we can get rid of the constant C_0. The roots of the polynomial $R_0 = R(\lambda, x, D_0)$ have the following form:

$$\lambda_0 = u_0, \quad \lambda_1 = -u_0/2, \quad \lambda_2 = u_0/2. \tag{33}$$

Defining variables $\tilde{D} = D/D_0$, $\tilde{u} = u/u_0$, we obtain the following equation:

$$\begin{aligned} \frac{d\tilde{D}}{dx} &= \frac{3}{2x}(\tilde{u} - \tilde{D}), \\ \frac{d\tilde{u}}{dx} &= -\frac{u}{2x} + u_0^{1/2}\sqrt{2\tilde{u}^3 - 6\tilde{u} + 4\tilde{D}}. \end{aligned} \tag{34}$$

A solution $u(x), D(x)$ which asymptotically tends to (u_0, D_0) as $x \to -\infty$ satisfies $\tilde{u} \to 1$, $\tilde{D} \to 1$. Let $\tilde{u} = 1 + q$, $\tilde{D} = 1 + p$. Since $q \to 0$ and $p \to 0$ as $x \to -\infty$, a simple qualitative analysis of Eq. (34) leads to

$$\begin{gathered} p = \frac{1}{16x^2u_0}(1 + O(\tau^{-1})), \quad q = -\frac{1}{24x^2u_0}(1 + O(\tau^{-1})); \\ \tau = x^2u_0, \quad p = \sum_{n\geqslant 1} p_n\tau^{-n} + O(\tau^{-\infty}), \quad q = \sum_{n\geqslant 1} q_n\tau^{-n} + O(\tau^{-\infty}); \\ \Delta D = D - D_0 \approx \varepsilon/(12x); \quad \Delta u = u - u_0 \approx -1/(24x^2). \end{gathered} \tag{35}$$

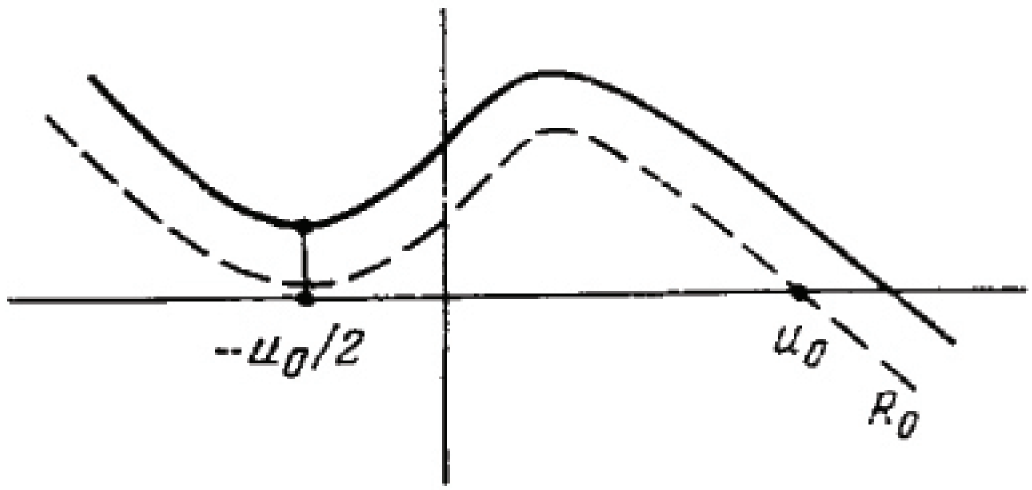


FIGURE 1

We see that $p > 0$ for large $|x|$. Therefore, the graph of the Riemann surface has the form illustrated in Fig. 1 [the dashed line shows a polynomial $R_0 = R(\varepsilon x, \lambda, D_0)$]. The two roots of a polynomial $R(\varepsilon x, \lambda, D)$ are complex, since $p > 0$. The local minimum of the polynomial $R(\varepsilon x, \lambda, D)$ with respect to λ is at a point $-u_0/2$ for each given x. If we know quantities ΔD and Δu, we can easily compare this situation with one appearing in exact analytic method of the soliton theory described above.

Conclusion. To obtain a physically meaningful solution $u(x)$ with a root asymptotic for $x \to -\infty$, we have to fix a real Riemann surface with two complex roots near $-u_0/2$ and a real root near u_0 with corrections cited above. The root $-u/2$ of the Baker–Akhiezer function should be positioned to the left of the point $-u_0/2$, where $\Delta u \sim -x_0^2/24$. The minimum of the curve $R(\lambda)$ for $x = x_0$ should be at the point $-u_0/2$ (where x_0 is large and negative) and at a height $-\Delta D \sim -\varepsilon/(12x_0)$, according to Eq. (35). The pole of the Baker–Akhiezer function should be at $\lambda = \infty$.

Remark. The method cited in this article can be applied without changes to equations of the form

$$[L, A] = \varepsilon \cdot 1, \quad A = \sum_{j=0}^{n} c_j A_{n-j},$$

where A_{n-j} are operators of odd order known in soliton theory, $L = -\partial_x^2 + u$, and c_j are arbitrary constants.

This study can also be carried out for operators L and A of any order by using certain aspects of the method described in [7]. If the orders of operators L and A are not mutually prime then the problem becomes more difficult (see [12, 13]).

Supplement (by B. A. Dubrovin and S. P. Novikov)

4. Nonlinear Quasiclassical Approximation for Equation $[L, A] = \varepsilon \cdot 1$

Proposition. *The main term of a formal asymptotic expansion of an equation* $u''' = 6uu' + \varepsilon$ *has the form* $u = u_0 + \varepsilon u_1 + \dots$, *where*

$$u_0 = 2\wp\left(\frac{4g_2}{5\varepsilon} + \frac{i(\omega - \omega')}{\pi\varepsilon} + \frac{\delta_0(\omega+\omega')}{\pi\varepsilon}\bigg|_{g_2,g_3}\right),$$
$$-g_2 = \varepsilon x + g_2^0, \quad 3g_3 = 2g_2 f - 5c/(2(\omega+\omega')), \quad g_2^3 - 27g_3^2 < 0,$$
$$-k = \pi(\omega+\omega'), \quad f = (\eta+\eta')/(\omega+\omega'), \tag{S.1}$$

and g_2^0, c, S_0 *are real constants. Here* $w + w'$ *is a real period of a real surface* Γ *of type* 1, $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$, $\zeta'(z) = \wp(z)$. *Averaged equation (17) takes the form*

$$dg_2/dg_3 = f(g_2, g_3). \tag{S.2}$$

If $\omega \to \alpha\omega$, $w' \to \alpha\omega'$, *then* $g_2 \to \alpha^{-4}g_2$, $g_3 \to \alpha^{-6}g_3$, $\eta \to \alpha^{-1}\eta$, $f \to \alpha^{-2}f$.

We sketch the proof. We choose solutions (11) for $\varepsilon = 0$ of the form

$$u = \varphi(k(x - x_0); g_2, g_3),$$

where the function $\varphi(\tau; g_2, g_3)$ is periodic with a period 2π in τ,

$$\varphi(\tau; g_2, g_3) = 2\wp\left(\frac{\omega+\omega'}{\pi}\tau \;\Big|\; g_2, g_3\right). \tag{S.3}$$

As usual [18], we look for the main term of the asymptotic expansion in the form

$$u_0(x) = \varphi\left(\frac{S(X)}{\varepsilon}; g_2(X), g_3(X)\right). \tag{S.4}$$

The averaged equations have the form

$$dg_3/dX = -1, \quad dg_3/dX = -f(g_2, g_3). \tag{S.5}$$

The phase $S(X)$ is determined by an equation

$$\frac{dS}{dX} = k = -\frac{\pi}{\omega+\omega'}. \tag{S.6}$$

Lemma. *The averaged equations can be written in the "Flaschka–MacLaughlin–Krichever form"*[2]

$$-\frac{\partial w}{\partial X} = \frac{\partial p}{\partial \lambda}, \quad X = \varepsilon x, \tag{S.7}$$

where

$$w = \sqrt{R(\lambda)}, \quad g_2 = -\varepsilon x + g_2^0, \quad -4g_3 = D,$$
$$-ip(z) = \zeta(z) - fz, \quad \lambda = \wp(z), \quad w = 2i\wp(z), \tag{S.8}$$

$$\oint_a dp(\lambda) = 0, \quad R(\lambda) = -4(4\lambda^3 - g_2\lambda - g) = -16\lambda^3 - 4(\varepsilon x + C_0)\lambda - D. \tag{S.9}$$

[2]Results of this theory are cited in [18].

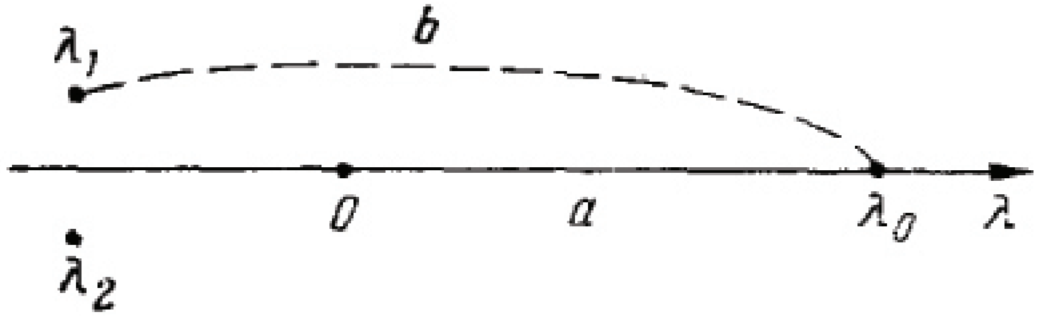


FIGURE 2

The proof follows the standard scheme. The lemma implies the following results.

1. A quantity

$$2ic = \oint_a w\,d\lambda = 2i\int_0^{2(\omega+\omega')} (\wp'(z))^2\,dz \tag{S.10}$$

is an integral of averaged equations (S.5), since $\oint_a dp(\lambda) = 0$ (c is real).

The following calculation gives result (S.1):

$$\oint_a w\,d\lambda = \int_0^{2(\omega-\omega')} w\,d\lambda = \frac{8\pi g_2}{5(\omega+\omega')} + \frac{2i(\omega-\omega')}{\omega+\omega'}c. \tag{S.11}$$

2. The solution of Eq. (S.6) has the following form:

$$S = \frac{1}{2}\oint_b w\,d\lambda + S_0, \tag{S.12}$$

since

$$i\oint_b dp(\lambda) = \frac{\pi}{\omega+\omega'} = -k. \tag{S.13}$$

We calculate integral (S.12) as in (S.11), obtaining the phase of the solution (S.1). We choose a- and b-cycles on Γ according to the desired class of curves (see Fig. 1). The bifurcation points in the complex λ-plane are indicated in Fig. 2.

The basis a-cycle is a preimage in Γ of a half-line $[-\infty, \lambda_0] = a \in H_1(\Gamma)$. The basis b-cycle connects the bifurcation points λ_1 and λ_2.

All the necessary phase properties of (S.12) and (S.1) follow from the requirement that the solution u_0 is real.

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