ON THE EQUATION $[L, A] = \varepsilon \cdot 1$

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We shall investigate the equation $[L, A] = \varepsilon \cdot 1$ for the second order Schrödinger operator $L = -\partial_x^2 + u$ and some odd-order operator A using the ideas originated from soliton theory [**?**] (after the recent papers [**?**]–[**?**]).

The first nontrivial case is

$$L = -\partial_x^2 + u, \quad A = -4\partial_x^3 + 6u\partial_x + 3u', \tag{1}$$

$$[L, A] = \varepsilon \cdot 1 \leftrightarrow u''' = 6uu' + \varepsilon. \tag{2}$$

Consider the operator $\tilde{L}$ and Eq. (3):

$$\tilde{L} = L - (\lambda + \varepsilon t) \cdot 1, \quad \partial \tilde{L}/\partial t = [\tilde{L}, A]. \tag{3}$$

Equation (2) is obviously the stationary equation for (3). As normally in the soliton theory [**?**]–[**?**] we shall present Eq. (3) in the form

$$\left[\frac{\partial}{\partial t} - \Lambda, \frac{\partial}{\partial x} - Q\right] = 0, \quad \lambda \to \lambda + \varepsilon t, \tag{4}$$

where the matrices (Λ, Q) depend polynomially on $\lambda + \varepsilon t$ and $(u(x,t), u'(x,t), \dots)$. In the case of Eqs. (1) and (2) we have

$$\Lambda = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \begin{array}{l} a = -u', \quad b = 2u + 4\lambda, \\ c = -4\lambda^2 + 2\lambda u + 2u^2 - u'', \end{array} \tag{5}$$

$$Q = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}, \quad \lambda \to \lambda + \varepsilon t. \tag{6}$$

The stationary problem (2) we shall obtain from (4) by replacing

$$\frac{\partial}{\partial t} \to \varepsilon \frac{\partial}{\partial \lambda}, \quad t = 0.$$

So Eq. (2) is equivalent to the Lax-type equations:

$$[\varepsilon \partial_x - \Lambda, \partial_x - Q] = 0, \tag{7}$$

$$\partial \Lambda / \partial_x = [Q, \Lambda] + \varepsilon \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

We have to find the common solution of two linear equations (9) and (10):

$$\varepsilon \frac{\partial \psi}{\partial \lambda} = \Lambda \psi, \tag{9}$$

$$\frac{\partial \psi}{\partial x} = Q\psi. \tag{10}$$

For $\varepsilon = 0$ we have the situation standard in the "finite-zoned" periodic theory [**?**]–[**?**]. Our main idea in studying (2) is to introduce the Riemann surfaces analogous to Riemann surfaces in the "finite-zoned" periodic theory.

STEP 1. Introduce the matrix U such that $U\Lambda U^{-1} =$ diagonal, $U^{-1}\tilde{\psi} = \psi$, and we find U^{-1}:

$$U^{-1} = \begin{pmatrix} 1 & 1 \\ \delta & f \end{pmatrix}, \quad U\Lambda U^{-1} = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & -\sqrt{R} \end{pmatrix}, \tag{11}$$

$$\delta = -(a - \sqrt{-\det\Lambda})/b, \quad f = -(a + \sqrt{-\det\Lambda})/b,$$

$$R(\lambda) = -\det\Lambda = a^2 + bc = -16\lambda^3 - 4C\lambda - \mathcal{D}. \tag{12}$$

From (2) and (12) we obtain the new useful form of this equation:

$$\mathcal{D}' = 2\varepsilon u, \quad C = \varepsilon x + C_0, \quad C_0 = \text{const}, \tag{13}$$

$$\mathcal{D} = 2u^3 + (\varepsilon x + C_0)u - (u')^2, \quad C = u'' - 3u^2, \tag{14}$$

$$R'(\lambda) = -\varepsilon b, \quad (\mathcal{D}''/2\varepsilon)^2 = R(-\mathcal{D}'/4\varepsilon) = R(-u/2). \tag{15}$$

For Eqs. (9) and (10) we have the "gauge" transformations

$$\varepsilon\tilde{\psi}_\lambda = [-\varepsilon U(U^{-1})_\lambda + U\Lambda U^{-1}]\tilde{\psi} = (\Lambda_0 + \Lambda_1^*\varepsilon + \Lambda_1^d\varepsilon)\tilde{\psi}, \tag{16}$$

$$\tilde{\psi}_x = [-U(U^{-1})_x + UQU^{-1}]\tilde{\psi} = (Q_0 + \varepsilon Q_1^*)\tilde{\psi}, \tag{17}$$

$$\Lambda_0 = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & -\sqrt{R} \end{pmatrix}, \quad \Lambda_1^* = \frac{1}{f-\delta}\begin{pmatrix} 0 & f_\lambda \\ -\delta_\lambda & 0 \end{pmatrix},$$

$$\Lambda_1^d = \begin{pmatrix} 2u'/b\sqrt{R} & 0 \\ 0 & -2u'/b\sqrt{R} \end{pmatrix} + t_\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} \sqrt{R}/b & 0 \\ 0 & -\sqrt{R}/b \end{pmatrix} + t_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1^* = \begin{pmatrix} 0 & -b/4R \\ b/4R & 0 \end{pmatrix},$$

$$t = -\frac{1}{2}\ln(f-\delta), \quad f - \delta = \det U^{-1}.$$

The quantities (δ, f) satisfy the "almost Riccati" equations:

$$\begin{aligned} \delta' + \delta^2 - (u-\lambda) &= O(\varepsilon) = -\varepsilon b/4R, \\ f' + f^2 - (u-\lambda) &= O(\varepsilon) = \varepsilon b/4R, \end{aligned} \tag{18}$$

(because the matrix εQ_1^* is equal to (19)):

$$\varepsilon Q_1^* = \begin{pmatrix} 0 & f' + f^2 - (u-\lambda) \\ -\delta' - \delta^2 + (u-\lambda) & 0 \end{pmatrix}. \tag{19}$$

Important remark. The matrix $\tilde{\psi}$ should be 1-valued matrix function on the Riemann surface Γ: $\mu^2 = R(\lambda)$.

STEP 2. We solve Eq. (16) semiclassically in the standard form (see [**?**])[1]

$$\begin{gathered} \tilde{\psi} = R^{1/4}(1 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots)\exp\left\{\frac{1}{\varepsilon}[B_0 + \varepsilon B_1 + \dots]\right\} V, \\ V_\lambda = 0, \quad V = V(x), \end{gathered} \tag{20}$$

such that:

a) All matrixes $A_j(\lambda)$ are antidiagonal and algebraic functions on Γ.
b) All matrixes $B_j(\Lambda)$ are diagonal and all differential forms $d_\lambda B_j$ are algebraic on Γ.

[1] Volowitch pointed out the author the factor $V(x)$ is important for the second equation in x (see Vladimirov and Volowitch, 1984, Trieste IC/84/128).

After the elementary calculations we obtain

$$(21)\qquad \begin{gathered} B_{0\lambda} = \Lambda_0, \quad B_{1\lambda} = \Lambda_1^d, \quad B_{q\lambda} = \Lambda_1^* A_{q-1}, \quad q \geqslant 2, \\ A_q = (\operatorname{ad} \Lambda_0)^{-1}(H_q), \quad \operatorname{ad} \Lambda_0(Y) = [\Lambda_0, Y], \\ H_1 = -\Lambda_1^*, \quad H_2 = A_{1\lambda} + [A_1, \Lambda_1^d], \quad q \geqslant 3, \end{gathered}$$

$$(22)\qquad H_q = A_{q-1,\lambda} + [A_{q-1}, \Lambda_1^d] + \sum_{j=1}^{q-2} A_j \Lambda^* A_{q-1-j}.$$

From (21) and (22) we deduce the important conclusion: Matrix $\tilde{\psi}$ has the asymptotic (23) for $\lambda \to \infty$

$$(23)\qquad \begin{gathered} R^{1/4}\tilde{\psi}_{11} = (\text{const}) k \exp\left\{\frac{8}{5\varepsilon}k^5 + k\left(x + \frac{C_0}{\varepsilon}\right)\right\}\left(1 + \frac{\mathcal{D}}{4\varepsilon}k^{-1} + O(k^{-2})\right), \\ k = (-\lambda)^{1/2}. \end{gathered}$$

Step 3. Our important conjecture is that the diagonal matrix-function $R^{-1/4}\tilde{\psi}_{11}^{(1)}(\lambda, x) = \exp\{B_0/\varepsilon + B_1\}$ may be found as the first 1-*valued* approximation in ε on the surface Γ for the function $\tilde{\psi}$ satisfying (16). We have the differential 1-form $\Omega = d_\lambda \ln \tilde{\psi}_{11}^{(1)}\Omega = B_{0\lambda}/\varepsilon + B_{1\lambda} = (\sqrt{R}/\varepsilon + u'/(u + 2\lambda)\sqrt{R} - (1/2)d_\lambda \ln t)$. Its analytical properties are:

(a) Ω has exactly 2 poles on Γ in the points (γ, ∞):

(1) $\gamma = (-u/2, +)$, $\operatorname{res}_\gamma \Omega = +1$, $\operatorname{Ord}_\gamma \Omega = 1$.

(2) ∞, $\operatorname{res}_\infty \Omega = -1$, $\operatorname{Ord}_\gamma \Omega = 6$.

(b) $\exp(\int^\lambda \Omega) = R^{-1/4}\tilde{\psi}_{11}^{(1)}$ has the asymptotic (23) for $\lambda \to \infty$.

(c) All periods of Ω along the cycles on the surface Γ belong to $(2\pi i) \times \mathbb{Z}$.

We have therefore the special case of the so-called "Baker–Akhiezer functions" in the "finite-zoned" periodic theory (see [**?**, **?**, **?**]), but Riemann surface Γ depends on x "slowly" (g_2, g_3 depend on x). We have:

$$(24)\qquad \begin{gathered} \mathcal{D}/4\varepsilon = F(x, g_2, g_3) = -\zeta(x + A \mid g_2, g_3), \quad g_2, g_3\text{—fixed}, \\ A = -x_0 + \frac{8}{5\varepsilon}U_6/U_2. \end{gathered}$$

We may deduce now from (12) and (13) that $g_2 = -(\varepsilon x + C_0)$, $4g_3 = -\mathcal{D}$,

$$U_{2j} = \oint_b \omega_{2j}, \quad \oint_a \omega_{2j} = 0, \quad \omega_{2j} \sim d(k^{2j-1}), \quad \lambda \to \infty.$$

Main result. *Equation* (**??**) *is the transcendent equation for definition of* $\mathcal{D}$.

After finding the solution of (21) we define:

$$(25)\qquad 2\varepsilon u(x) = d\mathcal{D}/dx.$$

Step 4. Only some very special solution of (2) is important in the "string" theory:[2]

$$(26)\qquad \begin{gathered} u(x) \sim u_0 = \sqrt{-\varepsilon x/3}, \\ \mathcal{D}(x) \sim \mathcal{D}_0 = -4u_0^3. \end{gathered}$$

[2]Asymptotics for the general solution of (2) for $|x| \to \infty$, $x \in c$ has been investigated in 1913 by Boutroux (see remark in the Appendix).

Using the form (15) of (2), we shall see that for $x \to -\infty$ the Riemann surface Γ will have the form (see Fig. 1)

$$\mu^2 = R(\lambda, x), \quad R_0 = -16\lambda^3 - 4\varepsilon x\lambda - \mathcal{D}_0 = -16(\lambda + u_0/2)^2(\lambda - u_0),$$
$$R = R_0 - (\mathcal{D} - \mathcal{D}_0), \quad \tau = x^2 u_0 > 0,$$
$$\tilde{\mathcal{D}} = \mathcal{D}/\mathcal{D}_0 = 1 + \sum_{n\geqslant 1} p_n \tau^{-n} = 1 + p \pmod{O(\tau^{-\infty})},$$
$$\tilde{u} = u/u_0 = 1 + \sum_{n\geqslant 1} q_n \tau^{-n} = 1 + q \pmod{O(\tau^{-\infty})},$$
$$p_1 = (16\tau)^{-1}, \quad q_1 = -(24\tau)^{-1},$$
$$\mathcal{D} - \mathcal{D}_0 \sim +\varepsilon/12x,$$
$$u - u_0 \sim -1/24x^2, \quad x \to -\infty.$$

FIGURE 1

For the important solution (26) the surface Γ is such that only one branching point is real and two are complex (near the point $-u_0/2$). The pole $(-u/2)$ of $\tilde{\psi}$ is such that $-u/2 > -u_0/2$ on the real line. The exact position of the pole $(-u/2)$ may be found only from the exact solution. We shall compare the important solution (26) with the solutions (24) and (25) in the subsequent paper. *Our conjecture is that* (**??**) *and* (**??**) *describe exactly the important solution* (**??**).[3]

The general case $L = -\partial_x + u$, $A = \sum_{j=0}^{n} c_j A_{n-j}$ may be investigated exactly in the same way. The case of higher order operator is also clear (but less effective) if the orders of L and A are relatively prime (see [**?**]). For the operators of nonrelatively prime order see [**?**] and [**?**]. This case is more complicated.

APPENDIX (B. A. Dubrovin, S. P. Novikov)

We shall discuss here the "nonlinear semiclassics" for studying the equation $u''' = 6uu' + \varepsilon$ called "Whitham method".

Theorem. [4] *The solution may be found as a formal decomposition in* ε

(A1) $$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots,$$

such that

(A2) $$u_0 = 3\mathcal{P}\left(4g_2/5\varepsilon + \frac{i(\omega - \omega')}{\pi\varepsilon} + \frac{S_0(\omega + \omega')}{\pi\varepsilon} \;\Big|\; g_2, g_3\right),$$
$$-g_2 = \varepsilon x + g_2^0, \quad g_3 = 2g_2 f - 5c/2(\omega + \omega'), \quad f = \eta/\omega,$$

g_2^0, c, S_0 *being arbitrary constants,* $\omega + \omega' \in R$*,* ω, ω' *being periods.*

Proof. As normally in Whitham method, we start from the family of exact solutions, $u = \varphi(k(x - x_0); g_2, g_3)$ of the equation with $\varepsilon = 0$, such that the function

(A3) $$\varphi(\tau; g_2, g_3) = 2\mathcal{P}\left(\frac{\omega + \omega'}{\pi}\tau\right)$$

[3] P. Grinewitch pointed out to the author that this conjecture needs some corrections. Important result was obtained by Krichever.

[4] For $|x| \to \infty$ formula (A2) imply the old results of ... (1913).

is 2π-periodic in τ. The first term of the asymptotic decomposition (A1) has the form

$$u_0 = \varphi\left(\frac{S(X)}{\varepsilon}; g_2(X), g_3(X)\right), \quad X = \varepsilon x. \tag{A4}$$

$$a \cdot b = 1$$
$$a, b \in H_1(\Gamma)$$
$$a = [-\infty, \lambda_3], \ = [\lambda_1, \lambda_3]$$
$$\lambda_1 \lambda_2 \in C, \ \lambda_3 \in R$$

FIGURE 2

The averaged equations have the form

$$dg_2/dX = -1, \quad d\bar{g}_3/dX = -f(g_2, \bar{g}_3). \tag{A5}$$

For the phase we have

$$\frac{dS}{dX} = k = -\frac{\pi}{\omega + \omega'}. \tag{A6}$$

Lemma. *The averaged equations* (**??**) *may be presented in the "Flashka–McLaughlin form"* (see Fig. 2)

$$-\frac{\partial w}{\partial X} = \frac{\partial p}{\partial \lambda}, \quad \oint_a dp(\lambda) = 0, \quad R(\lambda) = -4(\lambda^3 - g_2\lambda - g_3), \tag{A7}$$

$w = \sqrt{R(\lambda)}$, *$p(\lambda)$—"quasimomentum", i.e., $p(z) = (\zeta(z) - fz)$ and $\lambda = \mathcal{P}(z)$, $w = 2i\mathcal{P}'(z)$ on the surface Γ, $\mathcal{D} = -4g_3$.*

The proof may be deduced from the Lax representation (3) of Eq. (2). □

Corollary 1. *The quantity*

$$2ic = \oint_a w\, d\lambda = \int_0^{2(\omega+\omega')} (\mathcal{P}'(z))^2\, dz, \quad c \in R \tag{A8}$$

is the integral of (**??**).

From the value of this quantity we deduce the relations (A2).

Corollary 2. *The solution of* (**??**) *has the form*

$$S = \frac{1}{2}\oint_b w\, d\lambda + S_0, \quad i\oint_b dp(\lambda) = \frac{\pi}{\omega + \omega'} = k. \tag{A9}$$

We may see here that the result of Whitham-type method is in the good agreement with the results of more exact methods, developed in the main part of the paper for the special solution: phase S is the same as (23) in the leading term $\tilde{\psi}^{(1)}$ and the value of S (A9) is the averaged form of Eq. (24) in the main part of the paper.

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