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The Liouville form of averaged Poisson brackets[(1)]

S.P. Novikov and A.Ya. Mal'tsev

Poisson brackets of hydrodynamic type (H.T.) are defined [1] for vector fields $u^i(x)$ with values in a manifold M with local coordinates $(u^1, ..., u^N)$. They have the geometric form (let $x \in S^1$)

$$\{u^p(x), u^q(y)\} = g^{pq}(u(x))\delta'(x-y) + b_k^{pq}(u(x))u_x^k\delta(x-y).$$

The theory of such brackets is closely related to Riemannian geometry and has been successfully used to integrate H.T. systems (see the survey [2]). Theorem 1 of §6 of that survey requires a correction. The Liouville form of a H.T. bracket is defined by a tensor field $\gamma^{pq}(u)$ on M such that

$$g^{pq} = \gamma^{pq} + \gamma^{qp}, \quad b_k^{pq} = \partial\gamma^{pq}/\partial u^k.$$

Strongly Liouville coordinates are coordinates in which a bracket is Liouville and satisfies the Jacobi identity after restricting the tensor γ^{pq} to any hypersurface that is affine in the coordinates $(u^1, ..., u^N)$.

The averaging method of Whitham (the non-linear WKB method) considers the evolutionary system

$$\varphi_t = K(\varphi, \varphi_x, ..., \varphi_{x...x}), \tag{1}$$

equipped with a family of exact quasiperiodic solutions

$$\varphi = \Phi(k_1x + \omega_1 t + \eta_{01}, ..., k_m x + \omega_m t + \eta_{0m}; u^1, ..., u^N) \tag{2}$$

on invariant m-tori. Here (u) are parameters and $\Phi(\eta_1, ..., \eta_m; u)$ is a function that is 2π-periodic in each of the arguments η_j; $k_j(u)$, $\omega_j(u)$ are known functions; initial phases η_{0j} are arbitrary. The parameters (u) run through the manifold M.

The Hamiltonian version of the method, developed by Novikov and Dubrovin [1], assumes the following:

1) The system (1) is Hamiltonian with respect to a local translation-invariant Poisson bracket

$$\{\varphi(x), \varphi(y)\}_0 = \sum_{l=0}^{r} B_l(\varphi(x), \varphi_x(x), ..., \varphi^{(m_l)}(x))\delta^{(l)}(x-y)$$

[(1)] Corrigendum to the survey [2] of B.A. Dubrovin and S.P. Novikov.

with a local Hamiltonian $H\{\varphi\}$. N involutory integrals $I_1, \ldots, I_N$ are given, $I_p = \int j_p dx$, $\{I_p, I_q\} = 0$. Their densities j_p depend on the field $\varphi(x)$ and a finite number of derivatives at the point x; the first k of them annihilate the bracket.

2) Hamiltonian flows generated by the integrals I_p must leave the family of exact solutions (2) invariant and generate a linear dependency of the angles η_j on the "time variables". More precisely, the orbits of these flows on the $(N+m)$-dimensional family of quasiperiodic functions of x of the form $\Phi(kx+\eta_0; u)$ are tori T^m, which form an $(N-m)$-parameter non-degenerate family of finite-dimensional completely integrable Hamiltonian systems with respect to the restriction of the Poisson bracket $\{\cdot,\cdot\}_0$. Non-degeneracy means that rank $(\partial k_j/\partial u^p) = m$. In particular, we have $N-m \geqslant m$, since all the k_j belong to a (non-local) annihilator of the Poisson bracket $\{\cdot,\cdot\}_0$ on the space of quasiperiodic functions.

3) The averages of the densities $\overline{j_k}$ along tori of this family must coincide with the parameters u^k (this is a choice of coordinates we call "physical"). A full set of local coordinates on M must be given by the variables $(k_1, \ldots, k_m, J_1, \ldots, J_m, a^1, \ldots, a^k)$, where $k_j = 2\pi/T_j$ are the reciprocals of the "quasiperiods" T_j, and $(a^1, \ldots, a^k)$ are the averaged densities $a^q = \overline{j_q} = u^q$, $q = 1, \ldots, k$, of the integrals I_p that belong to the annihilator of the original Poisson bracket $\{\cdot,\cdot\}_0$. Here $N = 2m+k$.

Under these conditions we state the corrected *principle of conservation of Hamiltonian structure under averaging*: all averaged Hamiltonian systems generated by the integrals I_p are systems of H.T., Hamiltonian with respect to the H.T. Poisson bracket with the same Hamiltonians $\widetilde{I}_p = \int u^p(X)dX$, where $u^p = \overline{j_p}$ (that is, the densities are averaged). These averaged densities commute. The H.T. Poisson bracket is such that

a) $g^{pq} = \text{const}$, $b^{pq}_k = 0$ (in the variables k, J, a).

b) $g^{pq} = \gamma^{pq}+\gamma^{qp}$, $b^{pq}_k = \partial\gamma^{pq}/\partial u^k$, so that in the physical variables (u^p) the bracket is a Liouville one.

The expressions for the quantities γ^{pq}, g^{pq} in [2], §6 are correct, but the proof of the Jacobi identity is absent. Not all of the necessary conditions (see above) are stated there, though they appear, in principle, already in [1]. The statement of Theorem 1 in §6 of [2] concerning the "strong Liouville" property is, in general, not true. (However, the Poisson bracket of an ideal compressible fluid in physical variables p, ρ, s has this property; see [2].) This principle has been verified in all the examples studied (by various particular methods: based on the results of Whitham and Hayes for non-degenerate Lagrangian systems; on their degenerate analogue for the Gardner–Zakharov–Faddeev bracket in the case of KdV and its perturbations (here $m = 1$; it was done by M. Pavlov in his diploma thesis circa 1987); for both local brackets in the case of KdV as far back as 1982, using the theory of finite-zone potentials and the Flaschka–Forest–McLaughlin methods). In the case of the Leonard–Magri bracket, the locality of the annihilator condition (see above) is not satisfied, but the result still holds.

At present, we have proved a theorem stating that if what precedes part (a) of the Principle (see above) has been established, then part (b) holds. Thus in all the important examples analyzed it is true that in the physical variables the bracket has a Liouville form. In the general case the Principle (see above) requires a complete proof.

References

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