THE SEMICLASSICAL ELECTRON IN A MAGNETIC FIELD AND LATTICE
SOME PROBLEMS OF LOW DIMENSIONAL "PERIODIC" TOPOLOGY

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1. Hamiltonian Systems and Foliations

For any *Phase space*, i.e. a manifold M with the *Poisson structure*, we have a skew-symmetric Poisson Tensor h^{ij} with 2 upper indices in any local coordinates (x^i), such that the *Poisson Bracket* $\{f,g\} = h^{ij} f_i g_j$ satisfies the Jacoby identity. Here f_i means a partial derivative of the function $f(x)$ on the manifold and the standard summation rule is used for tensor indices. The Poisson bracket is well-defined for multivalued functions as well (i.e. closed 1-forms $df = f_i\, dx^i$, $dg = g_j\, dx^j$).

An Annihilator (Casimir) for the Poisson bracket contains all (perhaps well-defined only locally) C^∞-functions on M such that $\{f,g\} = 0$ for any function g. So the annihilator is a sheaf on M. It determines some integrable foliation A in any domain where the rank of the matrix h is constant. This matrix may be reduced to a constant in such a domain.

A Hamiltonian system determined by the Hamiltonian H is such that $f_t = \{f, H\}$ for any (perhaps locally well-defined) function f. Here H may not be a function, but a multivalued function, i.e. dH may be a closed 1-form on the manifold M.

For obvious reasons any trajectory $x(t)$ belongs to the intersection of 2 foliations:

$$x(t) \subset (dH = 0) \cap A.$$

In particular, in the case of the so-called *Symplectic manifolds* a Poisson Tensor is nondegenerate $\det(h^{ij}) \neq 0$ and foliation A is trivial. We are coming to the codimension 1 foliations with Morse-type singularities determined by the closed 1-form $dH = 0$. Such foliations were studied by this author in early 80s (see [N1, N2]) and later by the author's pupils in Moscow (see [N3]). An analogue of Morse theory was constructed for the critical points of multivalued functions H on finite-dimensional manifolds (Novikov, Farber, Sikorav, Pazitnov). A nice conjecture by the author about the "quasiperiodic structure" of this foliation was proved (see the papers of Zorich, Le Tu and Alaniya in [N3]; this problem is still open for the analogous foliations given by 2 and more equations). This quasiperiodic structure may be nontrivial even in the case of the closed 2-dimensional manifolds M.

Topological theory of generic Hamiltonian systems on surfaces was constructed by Katok, Hubbard and others in 70s. Topological invariants of such foliations come from Poincare maps along trajectories crossing any closed transversal curve. These maps preserve some length element on the circle and have a finite number of

Date: Submitted: October 1994.

the 1-st kind of discontinuity points. In particular, Hubbard studied foliations determined by the real part of the square root from the generic holomorphic quadratic differential on the Riemann surface with some complex structure. Such foliations present, more or less, all generic topological types.

However, nobody has studied a special class of foliations determined by the real part of the meromorphic or even by the holomorphic (1-st kind) 1-form for the genus $g \geqslant 2$.

In the case $g = 2$ any algebraic curve is hyperelliptic

$$M\colon (y^2 = P_5(x)).$$

Our foliations present the simplest natural generalization of the straight-line flows on the torus. They are given by the effective algebraic formula:

$$\mathrm{Re}\left[\frac{(ax+b)dx}{\sqrt{P_5(x)}}\right] = 0.$$

In some special nongeneric cases (for example, if all coefficients are real in the hyperelliptic case) topological invariants of these foliations may probably be calculated analytically. To study their small perturbations we have to use analytical perturbation theory. In general we have to use a computer to find them. Until now almost nothing has been done for this very concrete class.

Recently Zorich found some nonstandard ergodic properties of the generic Hamiltonian foliations on surfaces.

We shall see later in section 3 that for our goals some very nongeneric class is important:

Let some closed 2-dimensional submanifold in the 3-torus be given (the so-called *Fermi surface* M^2 *in the space of quasimomenta* T^3); a closed form ω on M^2 is by definition a restriction of the constant 1-form on T^3, determined by an *external magnetic field*.

Our leaves will coincide with the semiclassical (or adiabatic) trajectories of the quantum electron in the Lattice and the external magnetic field in the space of quasimomenta. Some beautiful problems of *Periodic Topology* appear here (see section 3 below).

2. Semiclassics for the Linear ODEs and Foliations

Let me demonstrate here how the foliations determined by the real part of the meromorphic 1-form may appear in the problems of semiclassical analysis for linear ODE systems.

Consider some linear ODE system

$$\epsilon\Psi_t = \Lambda(t,\epsilon)\Psi, \quad \epsilon \to 0.$$

Here Λ is an $(n \times n)$-matrix, whose elements depend on the variable t as a rational function:

$$\Lambda = \Lambda_0(t) + \Lambda_1(t)\epsilon + \dots$$

We may start from the case when all $\Lambda_i = 0$ for $i \geqslant 1$ (as in the papers of the author and P. Grinevich in the theory of the so-called "String equation"; see [N4, GN]).

How is a semiclassical approximation constructed?

Semiclassical Formulas make sense only for systems which are diagonal in the zero order:

$$\Lambda_0(t) = \mathrm{diag}[\lambda_1(t), \dots, \lambda_n(t)]$$

Therefore the first step is:

STEP 1. Diagonalization of the system in the zero order.

After the substitution $\Psi = U(t)\Phi$ we have a gauge transformation for the system:

$$\epsilon\Phi_t = \bar{\Lambda}\Phi = [U^{-1}\Lambda U - \epsilon U^{-1}U_t]\Phi.$$

Taking $U(t)$ such that the matrix $U^{-1}\Lambda_0 U$ is diagonal we are approaching the desired form. However *our matrices* $U(t), \bar{\Lambda}(t)$ *live on the Riemann surface* $\Gamma\ \det[\Lambda_0(t) - y] = 0$.

For traceless 2×2-matrices Λ this Riemann surface has the form $y^2 = \det[\Lambda_0(t)]$.

STEP 2. Formal semiclassical series on the Riemann surface.

We may choose the following expression such that it formally solves the ODE system above:

$$\Phi_{\mathrm{sc}} = \Big(1 + \sum_{i\geqslant 1} \epsilon^i A_i\Big) \exp\Big\{\epsilon^{-1}B_{-1} + B_0 + \sum_{i\geqslant 1}\epsilon^i B_i\Big\}.$$

Here all A_i are offdiagonal matrices whose entries are algebraic functions on the surface Γ, all B_i are the diagonal matrices whose entries are integrals from the algebraic (meromorphic) 1-forms on Γ,

$$B_{-1} = \mathrm{diag}\left\{\pm\int_{t_0}^{t}\sqrt{\det[\Lambda_0(t)]}\,dt\right\}.$$

Our formal series lives in fact on the surface which is a branching covering over the surface Γ. Only the first 2 terms (corresponding to the powers $\epsilon^{-1}, \epsilon^0$ in the exponent) are really important.

STEP 3. Semiclassical asymptotics for the exact solutions. Consider the oriented foliation

$$\mathrm{Re}[\Omega] = 0, \quad \Omega = \sqrt{\det[\Lambda_0(t)]}\,dt$$

on the surface Γ.

The semiclassical formula above gives right asymptotics for the exact solutions of the linear ODE system for $\epsilon \to 0$ *along the path of integration for* B_0 *if this path is transversal to our foliation in the positive direction where the real part of the integral increases.*

For the linear ODE systems of order n we use the analogous foliations $\mathrm{Re}[\lambda_i(t)] = 0$ for all eigenvalues of the matrix $\Lambda_0(t)$.

3. Semiclassical (or Adiabatic) Motion of the Quantum Electron in the Periodic Lattice and Weak Magnetic Field

In the absence of a magnetic field, quantum states of an electron in the lattice Γ correspond to "Bloch waves" $\psi_n(x,p)$. These functions satisfy to the stationary Schroedinger equation in the variable x (we ignore all dimensional units here)

$$H\psi_n = E_n\psi, \quad H = -\Delta + V(x), \quad V(x+\Gamma) = V(x).$$

Let the lattice Γ be generated by the basic vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in the x-space R^3. It is important for our goals to emphasize that this basis is orthonormal in the euclidean metric in 3-space and therefore it is "better" than any other basis obtained from it by the group $SL_3(Z)$. Only the action of the group $SL_3(Z)\cap O_3(R)$ leads to the equivalent bases in our problem. Corresponding translations we denote by T_1, T_2, T_3. We have

$$T_i H = H T_i.$$

We may choose the basic eigenfunctions $\psi_n(x,p)$ such that

$$T_j\psi_n = \exp\{2\pi i p_j\}\psi_n, \quad j = 1,2,3,$$

for the real vector p. By definition, this vector is well-defined modulo a dual lattice in p-space: (p_1, p_2, p_3) is equivalent to $(p_1+n_1, p_2+n_2, p_3+n_3)$ for the integers n_j.

Therefore p is a point of the torus T^3. People call p a "*Quasimomentum*". The *dispersion relation* for the energy $E_n(p)$, which is an eigenvalue of the Hamiltonian operator with $B = 0$, is well-defined as a real function on the torus T^3. For the generic operators in the 3-dimensional case we may have an equality $E_n(p) = E_m(p)$, $n \neq m$, for the isolated points p only. Therefore, in the generic case there are no such points on the important chosen surface—the so-called *Fermi surface* $M^2\colon \{E = E_0\}$. This energy level E_0 (*Fermi Level*) is an intrinsic invariant of a metal, depending on the number of free electrons in it. It may have a very complicated topology, for example, for some "noble metals" such as gold and others (see in the book of A. Abrikosov [A], for example).

Now consider this system in an external relatively "weak" homogeneous and constant (i.e. constant in space and time) magnetic field B, which does not deform the lattice itself.

It is very difficult to study this system (see, for example, my survey article [N5] for the Schroedinger operator in the magnetic field and lattice, translated into English by AMS in 1985). In the case where all magnetic fluxes are irrational numbers, no exact magnetic analogue of Bloch waves exists. Nobody knows whether any convenient picture exists or not, which might be built on the base of the exact eigenfunctions here. Chern classes for the dispersion relations based on magnetic Bloch functions for the generic 2-dimensional Schroedinger operator in the external magnetic field with rational flux through an elementary cell in the lattice were studied for the first time in the work [N6]. It was rediscovered later by physicists in connection with the Integral Quantum Hall effect.

Remark: Alan Connes conjectured after the author's talk at the Geometry Conference (Tel-Aviv University, December 1993) that all topological invariants of the adiabatic electron in the magnetic field (see below) can be expressed naturally through the language of noncommutative C^*-algebras. It should be point out that in fact the language of von Neumann algebras looks very poor for our problems; probably, only a few of the interesting physical quantities could be treated naturally through them. In the cases where the author knew such treatment was possible, it was only a hard mathematical foundation and extension of the understanding, already reached, using standard geometry and topology in the more special cases (for example, in the case of the integral Quantum Hall effect). For the problems below, not one of our "adiabatic" topological quantities has yet been explained through quantum C^*-algebras. It would be interesting to do this, but probably very difficult.

Many years ago quantum solid state physicists started to use the semiclassical or adiabatic one-zone picture in practical studies. In particular, physicists from the Kharkov–Moscow school (for example, I. Lifshitz, M. Azbel, M. Kaganov) actively used it. This picture leads to interesting topological problems. The author began thinking about it in the early 80s (see, for example [N1]). Some topological problems about this picture, posed in the early 80s, have now been solved by the author's pupils (see below). Let us describe this picture.

We shall now work with one value of n only and forget this index. In the semiclassical one-zone adiabatic approach people use the dispersion relation $E(p)$ as a classical Hamiltonian function in the phase space $R^3 \times T^3$ with the canonical coordinates (x, p) such that $\{p_i, p_j\} = 0$, in particular. In the external magnetic field $B(x)$ we use the same Hamiltonian $E(p)$, but change the Poisson brackets

$$\{p_i, p_j\} = eB_{ij}(x).$$

In a case of interest to physicists, we have a field B independent of x and t. Therefore, the components of the quasimomenta give us a closed subalgebra of Poisson brackets. In this small phase space T^3 we have a Poisson bracket with nontrivial annihilator A and a Hamiltonian $E(p)$. "*Symplectic leaves*" $A = \text{const}$ are the planes, orthogonal to the magnetic field $B = (b_{23}, b_{31}, b_{12})$ as a vector in the euclidean 3-space R^3, which is a universal covering over the torus T^3. We have, in fact, a multivalued annihilator function on the torus T^3. The levels of the Hamiltonian are Fermi surfaces $M^2 \subset T^3$ in the space of the quasimomenta. Therefore, the trajectories in p-space are the sections of Fermi surfaces by the planes orthogonal to the magnetic field.

We may consider this picture as a Hamiltonian foliation of the Fermi surface. This foliation is obviously very special and nongeneric between the Hamiltonian foliations of the surfaces. We shall use the word "generic" now inside this special subclass only.

Geometrically, this foliation is obtained by the family of parallel plane sections of the periodic surface in the universal covering euclidean p-space R^3. The *periodic surface* is given by the equation $E(p) = 0$. The function $E(p)$ here is periodic in all 3 variables p_1, p_2, p_3. Its analytical structure is unknown. There are many different solid media with complicated Fermi surfaces. So we may consider all this class as well: nobody knows any additional physical restrictions.

Which nontrivial topological properties may have a family of parallel plane sections of the generic periodic surface in R^3?

As possibilities we may have the following types of trajectories:

(1) *isolated point* (a critical point of our foliation),
(2) *closed curve in* R^3 (a curve homotopic to zero in T^3),
(3) *periodic nonclosed curve in* R^3 (a curve, closed in T^3, but nonhomotopic to zero),
(4) *nonperiodic nonclosed curve in* R^3 (such a curve in T^3 has a closure which is generically 2-dimensional; physicists call it an "*Open Trajectory*"). It definitely exists for some media.

In case 4, we may ask about the asymptotics of this trajectory in the space R^3 for the time $t \to \pm\infty$. The author's very first conjecture (see [N1]) was that there exists generically an asymptotic direction, which is the same for both signs $+\infty$ and $-\infty$ in time. In fact, the topology of this picture proved to be much more interesting than he expected originally.

The *Topological closure* of an open trajectory we define as a minimal compact 2-manifold, with possibly several components of the boundary (which is a part of the Fermi surface), containing our trajectory, such that each component of the boundary is a closed trajectory, homotopic to zero in the torus T^3. A genus of this surface is by definition a *genus of an open* trajectory.

For any trajectory with genus equal to g there exist a closed 2-manifold N^2 of the genus $g(N^2) = \mathrm{g}$ with the curve $y(t) \subset N^2$ and a continuous mapping (singular bordism, which is an immersion, not the imbedding in general) $f\colon N^2 \to T^3$ such that $f(y(t)) = x(t)$. Obviously we have $g(N^2) \leqslant g(M^2)$ where M^2 is the Fermi surface.

In the most interesting case of *rank* 3 the image of the map $\pi_1(M^2) \to \pi_1(T^3) = Z^3$ has a rank equal to 3. A family of 1-forms with constant coefficients $\omega_0 + \epsilon\omega_1$ we call a *perturbation* of the form ω_0 for all small enough values of the parameter ϵ.

The following results were obtained in fact by Zorich in 1984 and by Dynnikov in 1993:

Theorem 1. *For any Fermi surface $M^2 \subset T^3$ of rank equal to* 3 *and any small perturbation of the rational* 1*-form with constant coefficients with Morse type singularities only on the surface M^2, the genus of the open trajectory is less than or equal to* 1 (see [Z]).

Theorem 2. *Let some generic periodic function-dispersion relation E on the space of Quasimomenta be given, which determines nondegenerate Fermi-surfaces for $E_0 < E < E_1$. Let also the constant magnetic field be given with Morse singularities only on the Fermi surface $E = E_0$, such that there exists an open trajectory in it with genus equal to $g \geqslant 2$ (and this number is maximal for all trajectories on this Fermi surface).*

Any Fermi surface $E = E_1$ close enough to the original surface $E = E_0$ may have open trajectories with genus no more than $g - 1$ (see [D]).

(Let me to point out that there are no such theorems in the papers by Zorich and Dynnikov. They formulated as a main theorem only the Corollary below. The author extracted these theorems from their proofs.)

The proof of the Theorem 1 (Zorich) is purely topological (see below).

The proof of Theorem 2 (Dynnikov) also uses some metric arguments. It is much more complicated then the proof of Theorem 1. Let me present here the idea of the proof of Theorem 1:

We may think, without any loss of generality, that $\omega_0 = dp_3$ because this form is rational. Let also dp_3 be a Morse form on the Fermi surface $E = E_0$. All the leaves of the foliation $dp_3 = 0$ of the Fermi surface are compact. Nonsingular leaves $p_3 = \text{const}$ are equal to the disjoint union of the circles in the tori $[(E = E_0) \cap T^2] \subset T^3$. We take a minimal number of the values $p_3 = c_i$, $i = 1, 2, \dots$ such that all these leaves are nonsingular and divide the Fermi surface into cylinder like pieces or elementary "pants" W_k, whose 2 boundary leaves ∂W_k contain exactly 3 circles $\partial W_k = S_1 \cap S_2 \cap S_3$ (there is only 1 critical point inside). So we conclude that 2 of them (say S_1, S_2) belong to the same torus T^2, $p_3 = \text{const}$. They are nonselfintersecting and pairwise nonintersecting closed curves in this torus.

Therefore, only 2 cases are possible: the domain between them in the torus T^2 is equal to the cylinder or one of them bounds a disk in T^2. In both cases, we conclude that at least one of the curves S_1, S_2, S_3 is homotopic to zero in T^2 (in the first case it will be the third curve S_3, whose homology class in the torus is equal to the difference $S_1 - S_2$).

Now we cut the Fermi surface along all these curves (one for each of the elementary pants) homotopic to zero in the torus. It is easy to prove that all connected pieces of the Fermi surface will have a genus equal to 1 after cutting.

Adding any closed small perturbation to the form $\omega_0 = dp_3$, we observe that for any compact leaf homotopic to zero in T^3 there exists a compact leaf of the perturbed foliation homotopic to zero and close to the original one. Therefore, cutting a Fermi surface along this perturbed leaf, we get its decomposition on the genus 1 pieces bounded by the circles homotopic to zero. Any noncompact leaf does not cross these closed curves (because they are also leaves) and belongs to the required piece with genus equal to 1). Theorem 1 is proved.

As already mentioned, the proof of Theorem 2 is much more complicated. Dynnikov and Tsarev constructed examples of open orbits with any genus (to appear). The author does not know whether these examples are completely nongeneric or not (i.e. do they appear in any generic finite dimensional family of Fermi surfaces for a fixed magnetic field or not). For a fixed Fermi surface their dependence on a magnetic field might be point set theoretically complicated. It would be natural to study this numerically in order to formulate good conjectures.

As a *corollary* we conclude that any generic open trajectory belongs to some strip of finite width in the universal covering space of quasimomenta R^3. The proof of this is the following:

For any map $f\colon T^2 \to T^3$, of the 2-torus in the 3-torus monomorphic on the fundamental group $\pi_1(T^2)$, the covering map $R^2 \to R^3$ belongs completely to the domain containing points whose distance from a fixed 2-plane is bounded. This plane in R^3 is determined by the image of the fundamental group of the 2-torus in the 3-lattice. Therefore this plane is based on the integral sublattice

$$L_f = f_*(\pi_1(T^2)) = Z^2 \subset Z^3.$$

The section of this domain by the plane orthogonal to the magnetic field (i.e. by the plane $d\omega = 0$) is a strip in the plane with finite width. This proves our corollary.

It is not difficult to prove that this trajectory passes this strip in one (oriented) direction.

The same is true for the both asymptotics of time $t \to \pm\infty$.

The basis in the 2-sublattice L_f is well-defined modulo $SL_2(Z)$-transformations. It means in fact that a basic element in the second exterior power is fixed

$$l_f \in \Lambda^2(Z^2) \subset \Lambda^2(Z^3) = H_2(T^3), \quad l_f = f_*(\mu).$$

The element μ here is an oriented fundamental homology class of the 2-torus. A full description of the element l_f needs 3 integers (its coordinates in $H_2(T^3)$ in the original basis of the second exterior power of 3-lattice). These 3 integers are relatively prime (if nonzero), because it is easy to prove that we have in fact an imbedding of the 2-torus in the 3-torus. Any connected oriented codimension 1 submanifold has a homology class which is trivial or indivisible in the group $H_2(T^3, Z) = Z^3$. If the image of the fundamental group is Z^3, we have at least one such torus. Their number is always no more then the genus of Fermi surface.

Two different 2-tori in T^3, constructed by Zorich and Dynnikov, do not intersect each other (it is not difficult to extract this from their constructions). We conclude from this that they determine the same element in the group $H_2(T^3)$, because otherwise their intersection would be nontrivial homologically. *All open trajectories* *determine the same indivisible element in the group* $H_2(T^3) = Z^3$ *depending on the Fermi surface and generic magnetic field only. This is the most interesting topological invariant in this picture.*

It looks like the plane $L_f \subset Z^3$ should be "more or less" parallel to the original plane given by the magnetic field or close to it. However, for irrational magnetic fields it cannot be the same plane. We are not able to estimate how far it can be from this original plane.

An additional topological invariant of the open trajectory is its so-called "rotation number" on the 2-torus, but this invariant depends on the magnetic field as a continuous (locally nonconstant) function. The topology of the torical splitting of the Fermi surface and its integer-valued invariants axe well-defined in the generic case and ae locally constant. Therefore we may ask the following fundamental question:

Which kind of observable quantities in the conductivity of normal metals may correspond to the integer-valued topological invariants of generic open orbits in the external homogeneous, not very strong, magnetic field?

The author would like to point out that each trajectory determines some "*adiabatic wave function*"

$$\phi(x,t) = \psi(x, p(t))$$

associated with the 2-torus found by the theorems above. The observable quantities should be extracted from this wave function. However, a very deep understanding of solid state physics of concrete media is necessary to discover which kind of quantity might directly correspond to the second homology class of 2-torus in T^3—the "topological closure" of the trajectory (or of the adiabatic wave function).

Another problem is to analyze the quasiperiodic structure of the leaves in the 2-plane orthogonal to the magnetic field, which definitely exists as a corollary from the very general results on such foliations. What kind of "quasicrystals" do we get in this plane? Is it possible to extract something from it for the physical quantities?

Remark (added in December 1994): After some analysis and discussions of this problem in the author's seminar in Moscow, his student Malcev (a physicist) pointed out that in the presence of an open orbit with average direction η in the space of quasimomenta, we have some special properties for conductivity in a relatively strong magnetic field:

In the plane, orthogonal to this field, conductivity has a nonzero limit for $B \to \infty$ only in the direction orthogonal to the vector η.

This fact may be extracted from the material of the textbook by L. Landau and E. Lifshitz, vol. 10, where calculations were performed using the semiclassical (adiabatic) picture, described above.

Discussing this remark on the basis of Theorems 1 and 2, we take into account the following properties of the direction η:

it generically exists.

What is more important, *it belongs to some integral plane in the lattice* $Z^3 = H_1(T^3, Z) \subset R^3$ *in the space of quasimomenta. The last plane is locally rigid under variation of the direction of the magnetic field.*

Therefore, we come to the following

Conclusion. *All* (*small enough*) *variations of the magnetic field lead to the vectors* η *in the same integral* 2*-plane in the* 3*-lattice. This property may be tested directly* *by experiment under appropriate conditions, depending on the actual metal with complicated Fermi surface.*

For example, some “noble metals” (such as gold) have a very complicated Fermi surface. However, some serious work should be done to understanding to which real physical parameters (temperature, magnetic field and so on) this effect may correspond.

I thank my friend and colleague L. Falkovski for consultations in solid state physics.

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