THE TOPOLOGY SUMMER INSTITUTE

(Seattle, U.S.A., 1963)

S.P. Novikov

A Topology Summer Institute of the American Mathematical Society was held in Seattle in the summer of 1963. The most prominent topologists of the U.S.A., England and other countries of the West took part in its work. The theme was algebraic, differential, and combinatorial topology. There were no Soviet specialists at Seattle, and this survey is based on material of the Institute, kindly sent to us by the American Mathematical Society, consisting of the following papers.

1. F. Adams, Cohomology operations.
2. M. Atiyah, Index of elliptic operators on compact manifolds.
3. M. Atiyah, Index theorem for manifolds with boundary.
4. M. Atiyah and R. Bott, Periodicity theorem for complex vector bundles.
5. M. Barratt, Homotopy operations and homotopy groups.
6. E. Brown and F. Peterson, Relations between Stiefel-Whitney classes.
7. E. Brown, Abstract homotopy theory.
8. A. Dold, Half-exact functors and cohomology.
9. P. Conner and E. Floyd, Cobordism theories (intrinsic homology theories).
10. W. Massey and F. Peterson, Cohomology algebra of fibre bundles whose fibres are completely non-homologous to zero.
11. J. Munkres, Higher obstructions to smoothing.
12. J. Milnor and M. Hirsch, An unusual involution on S^7.
13. R. Palais and S. Smale, Generalized Morse theory.
14. E. Spanier, Higher order operations.
15. J. Stasheff, Classification theorem for fibre spaces.
16. P. Hilton, Spectral sequence for factoring maps.
17. M. Hirsch, Immersions and imbeddings of projective spaces.
18. F. Hirzebruch, Lectures on K-theory.

It should be noted that these mimeographed notes are apparently expositions of lectures given by the authors during the Institute. There is no evidence that all the authors presented their lectures in written form, and it is possible that not all the interesting reports were sent to us.

In my opinion the most interesting reports are those of Adams (1) and

Atiyah (2, 3) in which recent brilliant results of the authors are reflected. On the whole we note the great specific weight of the reports connected with generalized cohomology theories and their applications (1, 2, 3, 4, 6, 9, 18).

Among the papers of the Institute there was also a collection of problems listed by the members as being, in the opinion of the author of each of them, the most interesting problems of contemporary topology. At the very beginning of this collection are the statements of seven well-known classical conjectures. (Only two of them were formulated in recent years, namely Problems 1 and 7).

Below we give a translation of this collection of problems. This list of current problems may be of interest to our readers, since in it they can easily get a view of the whole range of interests of the world topological community and predict the topics of papers that will appear in print in the near future. Perhaps the newest in spirit and most important groups of problems are connected with the hopes of further advances in the classical problems of topology, by carrying farther the analogy between ordinary and generalized cohomology and using the deeper information contained in the latter, and also with the hope of taking the analogy between smooth and combinatorial topology farther by using microbundles, cobordism and other concepts appearing in combinatorial topology by analogy with differential topology. We note the strengthening of interest in non-traditional topics of topology, for example infinite-dimensional manifolds, commutative vector fields, and others.

A number of problems concern the usual topics of topology: the action of the Steenrod operations on the cohomology of various spaces, H-spaces, homotopy groups of spheres, homology of fibre bundles, characteristic classes, cobordism of classical groups, embedding problems, smooth structures and quite a number of others.

Finally I give a commentary and, in individual cases, solutions of some of these problems.

PROBLEMS IN DIFFERENTIAL AND ALGEBRAIC TOPOLOGY
PART I

SEVEN CLASSICAL PROBLEMS

1. Let M^3 be a three-dimensional homology sphere with $\pi_1 \neq 0$. Is the second suspension of M^3 homeomorphic to the sphere S^5?

2. Is the simple homotopy type a topological invariant?

3. Can the rational Pontryagin classes be defined as topological invariants?

4. Hauptvermutung: If two PL-manifolds (i.e. formal manifolds in Whitehead's terminology) are homeomorphic, does it follow that they are PL-homeomorphic?

5. Can manifolds be triangulated?

6. The Poincaré conjecture in dimensions 3 and 4.

7. Annulus conjecture: Is the region bounded by two locally flat n-spheres in $(n + 1)$-space necessarily homeomorphic to $S^n \times I(0,1)$?

J. Milnor.

Remark on Problem 7. Stallings gave a proof when the two spheres are combinatorially imbedded. Indeed it seems that when $n \geqslant 5$ the homeomorphism must be piecewise linear (PL-homeomorphism).

IMMERSION AND EMBEDDING

1. Let $M^n \subset R^{n+k}$ be a smooth compact manifold. Does the stable normal bundle admit O_{k-1} as structure group? Equivalently, if M is embedded in R^{n+k}, can it be immersed in R^{n+k-1}?

A. Dold

2. Let $N(k, n)$ denote the set of smooth knots $S^k \subset R^{n+k}$. Does there exist a stability theorem $N(k, n) \approx N(k+1, n)$ for large k? (The conjecture of such a theorem arises in connection with Kervaire's results on the groups of knots $N(k, 2)$.)

A. Dold

3. Let i_n be a generator of the group $\pi_n(S^n)$ and γ_n a generator of $\pi_{n+3}(S^n)$. If $n = 13$ then $[i_n, \gamma_n] = 0$. If this were proved for $n > 13$, $n \equiv 5 \bmod 8$, it would follow that the element $\Delta_* \gamma_n \in \pi_{n+2}(SO_n)$, where Δ_* is the homeomorphism in the exact sequence of the tangent bundle of $(n-1)$-spheres over S^n, is fibre-homotopically equivalent to the trivial bundle of $(n-1)$-spheres over S^{n+3}. Since the bundle is stably trivial, it can be realized as a normal bundle of an exotic $(n+3)$-sphere in R^{2n+3}, so that in particular $\theta_{8n} \neq 0$.

Hsiang and Szczarba

4. What classes in $H_2(S^2 \times S^2)$ are represented by smoothly imbedded 2-spheres?

Conjecture 1: classes $px + qy$, where p or q is 0 or 1 (pessimist)

Conjecture 2: classes $px + qy$, where p and q are relatively prime (optimist).

C. T. C. Wall

5. To find an embedding theorem using an hypothesis of the following general type: M^m is covered by a small number of embedded m-discs.

6. Let $x \in H_2(M^4)$ be a primitive element, where M^4 is a simply connected manifold, smooth and closed. Consider the embedded 2-spheres with simply connected complements realizing x. Are they diffeotopic?

C. T. C. Wall

7. The map F: $\pi_{l-1}(SO_{m-l}) \times \pi_r(S^l) \to \pi_{r-1}(SO_{m-r})$ is defined as follows. Construct a D^{m-l}-bundle over S^l with characteristic class $\alpha \in \pi_{l-1}(SO_{m-l})$. Map S^r into the bundle according to the homotopy class $\zeta \in \pi_r(S^l)$. According to Haefliger, if $r \geqslant l$, $m > 2r + l + 1$, we can approximate to this map by a unique (up to diffeotopy) imbedding. Choose its normal bundle $F(\alpha, \zeta) \in \pi_{r-1}(SO_{m-r})$. Give a homotopy theory interpretation of F.

C. T. C. Wall

8. Does there exist for each integer $n \geqslant 1$ a compact $(n+1)$-manifold without boundary in which each compact n-manifold can be embedded? This is a separate problem for each of the categories: topological, piecewise linear, smooth. (For $n = 1$ it is trivial, for $n = 2$ it is known (Stiefel's thesis).)

N. Steenrod

9. A finite group G operates smoothly on a smooth compact manifold M of dimension m which is $(r-1)$-connected with $r - 1 \geqslant 0$. Find a proper submanifold N on which G still operates, such that N is $(s-1)$-connected with $s - 1 \geqslant 0$. This is sometimes possible, sometimes not. (The condition $n/2 \geqslant r \geqslant s$ seems sufficient. Is it possible to find a first obstruction for $n/2 = r = s$?)

J.F. Adams and M. Atiyah

10. Compute the Kervaire-Milnor-Haefliger groups.

C. T. C. Wall

11. Let $X(t)$ be a smooth curve in affine space R^n; the osculating space of order p for given t is the linear envelope in R^n of the vectors $X'(t), \dots, X^{(p)}(t)$, all concentrated at $X(t)$. If $M \to R^n$ is a smooth map of a manifold M, then the osculating space of order p at a point $x \in M$ is the linear envelope of all osculating spaces of order p for all curves on M through x. The map is called affinely non-singular of order p at x if the osculating space at x has the maximal possible dimension.

It is known that the real projective plane can be immersed in R^{11} without affine singularities. But according to a theorem of E.A. Feldman it cannot be in R^{10}.

PROBLEM: To classify the affine non-singular immersions of a manifold in affine space up to homotopy which preserves non-singularity.

CONJECTURE: M can be immersed in R^m without affine singularities of order p if there exists a trivial bundle j such that

$$\tau(M) \oplus O^2\tau(M) \oplus \dots \oplus O^p\tau(M) \oplus j$$

is a trivial SO_m-bundle over M, where τ is the tangent bundle and O^i is the i-fold symmetric tensore power. (The converse is known).

W. Pohl

12. Is a diffeomorphism $S^n \to S^n$, which can be extended to the disc D^{n+1}, diffeotopic to the identity?

COMBINATORIAL AND SMOOTH STRUCTURES ON MANIFOLDS

1. Does a combinatorial (locally flat) submanifold have a normal microbundle?

C. T. C. Wall

2. Let X^{4k} be a finite complex satisfying the Poincaré duality theorem. Assume that there exists an oriented vector bundle ξ over X such that the Thom complex $T(\xi)$ has a spherical fundamental class. Does there exist a vector bundle η fibre-homotopically equivalent to ξ (i.e. their associated sphere bundles are fibre homotopically equivalent) such that the Hirzebruch formula for the classes 'dual' to the Pontryagin classes of η gives the index of X?

If so, then X has the homotopy type of a smooth closed $4k$-manifold according to results of Browder and Novikov.

C. T. C. Wall

3. Let θ^n act on manifolds (identified under a diffeomorphism of degree +1) by means of the connected sum. What determines this action? Does it depend on more than the tangential homotopy class of the manifold? Analogous question on the action of the subgroup $\theta^n(\partial\pi)$.

W. Browder

4. We define the homeomorphism

$$\varnothing\colon \pi_m(SO_k)\otimes\Gamma_{k+1}\longrightarrow\Gamma_{m+k+1}$$

as follows: First we note that an element of Γ^{k+1} can be represented by a diffeormorphism $R^k\to R^k$ with compact carrier and conversely. Let $f\colon R^m\to SO_k$ be a smooth map equal to unity outside a compact set, representing the element $\{f\}\in\pi_m(SO_k)$. Let $g\colon R^k\to R^k$ be a diffeomorphism with compact carrier representing the element $\{g\}$ of Γ_{k+1}. We define $h\colon R^{m+k}\to R^{m+k}$ by setting

$$h(x,\ y)=(x,\ f(x)^{-1}g^{-1}(f(x)\circ g(y))),$$
$$x\in R^m,\ y\in R^k.$$

It is easy to see that h has a compact carrier. Let $\varnothing(\{f\}\otimes\{g\})$ be the element of Γ_{m+k+1} represented by the diffeomorphism h.

Is the homeomorphism $\varnothing$ non-trivial for any m and k? If so then there exists an $(m+k+1)$-manifold M and an exotic sphere Σ of the same dimension such that M is diffeomorphic to the connected sum of M and Σ.

J.R. Munkres

5. Given a smooth manifold M; which elements λ of the group $H_m(M,\ \partial M;\ \Gamma^{n-m})$ are realized as obstructions to smoothing? Each zero-dimensional homology class is so realized; indeed, each class represented by a smoothly embedded submanifold with trivial normal bundle is so realized.

J.R. Munkres

6. Let $\pi_1,\ \dots,\ \pi_q$ be a finite sequence of finitely generated abelian groups. Does there exist a finite-dimensional smooth closed manifold M such that $\pi_i(M)=\pi_i$ for $i\leqslant q$. This problem seems especially interesting if only one of the groups π_i is different from zero. For example, if the non-zero group is $\pi_3=Z$, then one can make q very great taking as M the exceptional group E. But does there exist a bound for the values of q for which

the problem can be solved?

I. James

7. Milnor asserts that $H^{11}(B_F, Z_3) = Z_3$, where B_F is a stable classifying space for homotopy equivalences of a sphere. The usual methods of constructing fibre homotopy invariants do not give this characteristic class. How can it be constructed?

8. Groups of *PL*-automorphisms.

Let Σ be a fixed finite simplicial complex (for example, the boundary of a simplex) and let G be the group of all piecewise linear homeomorphisms of Σ. Then G can be topologized in three different ways, leading to three different spaces G_1, G_2, G_3.

PROBLEM. Are the natural maps $G_3 \to G_2 \to G_1$ homotopy equivalences between these three spaces?

First topology (Stasheff). Let G_1 be G with the compact open topology.

Second topology (Wall). For each integer k let $G^{(k)}$ be the subspace of G_1 consisting of those *PL*-homeomorphisms of Σ that are simplicial with respect to piecewise linear subdivisions of Σ having not more than k vertices. Now we topologize G as the direct limit of $G^{(k)}$ and we call the result G_2.

Third topology (Milnor). It seems that one can introduce on G the structure of an infinite simplicial complex G_3 with the fine topology so that a map $f: K \to G$ is piecewise linear if and only if the associated map $(x, y) \to (x, f(x)y)$ of $K \times \Sigma$ on itself is piecewise linear. (Here K is an arbitrary simplicial complex). Moreover, it seems that this simplicial structure on G is essentially unique.

It is conjectured that G_1, G_2 and G_3 are topological groups.

J. Milnor

9. Let (Σ^{n+k}, Σ^n) be a standard combinatorial embedding of a sphere Σ^n in a sphere Σ^{n+k}. What are the isotopy classes of *PL*-homeomorphisms of Σ^{n+k} on itself relative to the identity map of Σ^n?

R. Lashof

10. Let $(\Sigma^{n+k}, \gamma^k\Sigma^n)$ be the same pair as in 9, where $\gamma^k(\Sigma^n)$ is a normal microbundle of Σ^n in Σ^{n+k}. Two pairs $(\Sigma^{n+k}, \gamma_1\Sigma^n)$ and $(\Sigma^{n+k}, \gamma_2^k\Sigma^n)$ are considered equivalent if there exists a *PL*-homeomorphism $(\Sigma^{n+k}, \Sigma^n) \to (\Sigma^{n+k}, \Sigma^n)$ which is a homeomorphism of bundles $\gamma_1^k \to \gamma_2^k$ on a neighbourhood of the zero cross section of $E(\gamma_1)$. What are the equivalence classes?

R. Lashof and M. Rothenberg

VECTOR FIELDS, INFINITE-DIMENSIONAL MANIFOLDS, ETC.

1. We define the rank r of a smooth manifold M as the maximal number of vector fields $V_1, \ldots, V_r$ on M such that

a) the fields are everywhere linearly independent,

b) the Poisson brackets $[V_i, V_j]$ are all zero.

PROBLEMS. Is the rank of S^3 equal to unity? More generally, let M be a compact Riemannian manifold such that all two-dimensional curvatures are positive. Is the rank of M at most one? If M and N are compact, is it true that $rk(M \times N) = rkM + rkN$? (In the non-compact case a counterexample is $S^2 \times R$).

J. Milnor

2. Let G be a Lie group operating on real linear spaces V, W of the same dimension. When does there exist a G-map $f\colon V \to W$ that is non-linear but maps $V \setminus O$ on $W \setminus O$ with degree k? (The application is to the study of J-homeomorphism).

J.F. Adams

3. Adams and Walker proved the existence of a cross-section of Stiefel's complex fiberings $V_{n,m}$ on a sphere in certain dimensions. Can one give algebraic formulae determining these cross-sections? (Such formulae are known in the real case). Analogous question is the case of quaternions.

J.F. Adams

4. Is the group of unitary transformations of infinite-dimensional Hilbert space H contractible in the topology induced by the norm? (It is contractible in the topology of pointwise convergence).

R.S. Palais

5. We call a Hilbert manifold M stable if $M \times H$ is diffeomorphic to M. Clearly $M \times H$ is stable. What Hilbert manifolds are stable? Are they all stable?

R.S. Palais

COBORDISM AND CHARACTERISTIC CLASSES

1. Let M^n be a closed smooth orientable manifold of class C^∞ and let $\tau_M\colon M \to BSO$ be its tangent bundle. Set $I_n(SO, p^r) = \bigcap_{M^n} \operatorname{Ker} \tau_M^*$, where $\tau_M^*\colon H^k(BSO, Z_{p^r}) \to H^k(M, Z_{p^r})$. Compute $I_n(SO, p^r)$.

E. Brown and F. Peterson

2. Let $\Psi\colon \pi_{N\times 4k+2}(S^N) \to Z_2$ be the Arf invariant defined by Kervaire. Is Ψ zero?

The following remark can be used in this problem. A homeomorphism $\Psi\colon \pi_{N+8k+2}(S^n) \to Z_2$ can be "let out" over $\Omega_{8k+2}^{\mathrm{Spin}}$: Let $n = 4k + 1$. There exists a secondary cohomology operation

$$\Phi\colon H^n(X) \cap \operatorname{Ker} Sq^{n-1} \to H^{2n}(X)/Sq^2H + Sq^1H,$$

corresponding to the relation

$$Sq^{n+1} = Sq^2Sq^{n-1} + Sq^1(Sq^{n-1}Sq^1) = 0$$

on $H^n(X)$.

If $\Phi(x)$ and $\Phi(y)$ are defined, then $\Phi(x + y)$ is defined and

$\Phi(x + y) = \Phi(x) + \Phi(y) + xy$. In the case when M is a simply connected closed $2n$-manifold and $w_2 = 0$, $\Phi: H^n(M^n) \to H^{2n}(M^n) = Z_2$ is always defined. Let $\Psi(M)$ be the Arf invariant of this form. One can show that Ψ induces a homeomorphism $\Omega_{8k+2}^{\mathrm{Spin}} \longrightarrow Z_2$, which coincides with Kervaire's on the image of $\pi_{N+2n}(S^N) \longrightarrow \Omega_{2n}^{\mathrm{Spin}}$. Is Ψ zero? Is the composite $\Omega_{2n}^{su} \longrightarrow \Omega_{2n}^{\mathrm{Spin}} \longrightarrow Z_2$ zero? (Perhaps Ω^{su} is easier to compute than Ω^{Spin}).

E. Brown

3. Find an effective means of computing all the relations on G-characteristic numbers of a manifold, where G is one of the classical groups.

A. Liulevicius

4. Compute the differentials of the Adams spectral sequence ($p = 2$) for the Thom spectra MSU and MSp. In both cases it is already known that the differentials d_2 and d_3 are not zero.

A. Liulevicius

5. Find the structure of the cobordism rings as modules over stable homotopy groups of spheres. The structure of the module for MSp seems especially interesting.

A. Liulevicius

6. Let M be a smooth manifold and S^r a smoothly embedded sphere with trivial normal bundle. Consider a spherical modification of M, shrinking S^r. If M also contains S^{r+1} whose normal bundle has characteristic class 1 (when this makes sense), then the modification does not depend on the coordinates in the normal bundle of S^r. (These coordinates can vary, deforming S^r on S^{r+1}).

Suppose that this holds, and that S^r represents an element $\alpha \in \pi_r(M)$ such that $p\alpha = 0$. Compare this modification with the one produced over the element $q\alpha$, where q and p are mutually prime. They both kill the same cyclic subgroup of $\pi_r(M)$. Are they identical?

A. Wallace

7. Let T^n be a homotopy sphere bounding M^{n+1}, where M^{n+1} is not parallelizable, but is for example multiply connected. How can one compute the class of T^n in $\theta^n/\theta^n(\partial\pi) = \mathrm{Coker}\, J$?

C. T. C. Wall

8. Find a simply connected spinor 4-manifold with positive definite quadratic form.

C. T. C. Wall

9. Consider an orientable (or not) CW-manifold, i.e. a CW-complex X or pair (X, Y) satisfying Poincaré or Lefschetz duality. Set $Y_1 \sim Y_2$ if there exists X containing the disjoint union $Y_1 \bigcup Y_2$, and the pair $(X, Y_1 \bigcup Y_2)$ satisfies the Poincaré or Lefschetz duality law. Compute

these 'cobordism groups'.

C. T. C. Wall

10. Is the quotient of a combinatorial cobordism ring Ω by its torsion a polynomial ring?

C. T. C. Wall

11. Browder and Liulevicius showed that non-orientable cobordisms are determined by their characteristic numbers mod 2. Give a geometric interpretation of these characteristic classes of PL-manifolds mod 2 and compute $H^r(BPL, Z_2)$.

P. Lashof

HOMOTOPY THEORY

1. W.D. Barcus (Quart. J. Math. Oxford 12 (1961), 268-282) obtained some results on the stable homotopy groups of Eilenberg-MacLane complexes (stability in the sense of S-theory). What can be said about the stable (in this sense) homotopy groups of other spaces whose ordinary homotopy groups are known? For example, the classical groups?

I. James

2. Let X and Y be complexes of the same n-type for every n. Must X and Y have the same homotopy type? This is true if the groups $\pi_i(X)$ are finite for every n, and is false if $\pi_i(X)$ are infinitely generated (J.F. Adams and G. Walker, An example in homotopy theory, Proc. Cambridge Phil. Soc. 60 (1964), 699-700). Is it true if $\pi_n(X)$ is finitely generated for every n? In particular, consider the case when X and Y are H-spaces.

J.F. Adams

H-SPACES

1. Does the presence of p-torsion in the cohomology of a H-space G imply the non-commutativity of $H_*(G, Z_p)$? This is established for Lie groups except $H_*(E_8, Z_5)$.

W. Browder

2. Find operations in *Ext* preserving the total degree, commuting with the differentials of the Adams spectral sequence, and coinciding with the operations in homotopies in the E_∞-term.

A. Liulevicius

3. Do non-simply-connected finite-dimensional topological manifolds have the same homotopy type as CW-complexes? (The answer is yes for the simply connected case.)

C. T. C. Wall

4. What can be said about the action of the Steenrod algebra mod p on cohomology mod p of topological groups?

E. Thomas

5. What can be said about the action of Steenrod squares on non-primitively generated cohomologies mod 2 of finite dimensional H-spaces (examples E_6, E_7, E_8). For the primitively generated case see Thomas, Ann. of Math. 77 (1963), 306-317.)

E. Thomas

6. Find new examples of manifolds that are H-spaces. The known examples are Lie groups, S^7, RP^7 and combinations. From the results of Browder (Aarhus Topology Colloquium, 1962), it follows that finite-dimensional H-spaces have the homotopy type of manifolds (except perhaps the case $n = 4k + 2$).

Consider, in particular, sphere bundles over spheres. (See J.F. Adams, H-spaces with few cells, Topology 1 (1962), 67-72), I. James, On sphere-bundles over spheres, Comm. Math. Helv. 35 (1961), 126-135.) For example, an S^7-bundle over S^{11} or S^{15}, or the Stiefel manifold of 2-frames in quaternion three- or two-dimensional space are candidates. Consider also an S^3-bundle over S^7.

I. James

HOMOLOGY AND FIBERING

1. What is the connection between the torsions in the homologies of the fibres, base, space? For example, if $H^*(B)$ $[H^*(F)]$ have no torsion and $H^*(E) = 0$, then does it follow that $H^*(F)$ $[H^*(B)]$ has no p-torsion elements of order $> p$ for any prime p?. Assume that E, B, F are connected. This is equivalent to $E_2 = E_\infty$ in the Bokshtein spectral sequence.

For example, the loop space of a sphere ΩS^n has no torsion and $\Omega^2 S^n$ has no elements of order p^2. The loops on a Lie group have no torsion and the group itself (simply connected) has no elements of order p^2.

W. Browder

2. Let S_n be the symmetric group of degree n, and A_n the n-fold join of a space A with itself on which S_n acts naturally, permuting the coordinates. What invariants of A are needed to determine the homology of the orbit space A_n/S_n?

I. James

3. Let π be a finite group of automorphisms of a finitely generated group G and let m be a positive integer. Does there exist a finite connected complex such that $H_m(K, Z) = G$, $H_q(K) = 0$, $q \neq m$ and π acts as a group of automorphisms of K inducing the above automorphisms of $G = H_m(K)$? Another problem is obtained by requiring that π acts freely on K. (In this case the Lefschetz fixed point formula gives a limitation: for each

$\alpha \in \pi$, $\alpha \neq 1$, we must assume that the trace of the action of π in $G \oplus Q$ is equal to $(-1)^{n+1}$.)

N. Steenrod

4. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre space. Consider cohomology mod q (where q is a prime). Then $H^*(E)$ is a graded commutative algebra over the Steenrod algebra A (in the sense of Steenrod) and also over $R = H^*(B)/\text{Ker } p^*$ (in the usual sense). Let $H^*(F) = U(X)$ be a free graded commutative algebra over A generated by an A-submodule $X \subset H^*(F)$ and let the elements of X be transgressive.

The problem is to give a sufficient condition for $H^*(E) = U_R(N)$ to be a free graded $(A_q - R)$-algebra generated by an $(A_q - R)$-submodule $N \subset H^*(E)$. Massey and Peterson have proved this for $q = 2$.

W. Massey and F. Peterson

PROBLEMS IN DIFFERENTIAL AND ALGEBRAIC TOPOLOGY
PART II

IMMERSIONS AND EMBEDDINGS (Continued)

13. It is easy to define L-equivalence (or cobordism) classes of embeddings of a k-dimensional manifold in a given manifold M and similarly for immersions. Compute these groups. Also for groups of L-equivalence classes of an immersed manifold (considering only the image and not the mapping).

M. Hirsch

14. Can each parallelizable manifold M^n be embedded in an $(n + [\frac{n+1}{2}])$-dimensional space?

M. Hirsch

15. (Weak version of problem 12). Let $\varnothing$: $S^n \to S^n$ be a diffeomorphism preserving orientation. Let ρ: $S^n \to S^n$ be the reflection in a hyperplane passing through the origin in R^{n+1}. Is the diffeomorphism $(\rho\varnothing)^2$ smoothly isotopic to the identity? (It extends to a diffeomorphism of the ball.)

COMBINATORIAL AND SMOOTH STRUCTURES (Continued)

11. Let M be a triangulated topological manifold. Does M have a subdivision for which each simplex is flat in sone coordinate system? Does this condition imply that M is a combinatorial manifold?

M. Hirsch

12. Let M be a topological or combinatorial manifold with Euler characteristic zero. Does the tangent microbundle decompose into a sum $\varepsilon_1 \oplus \eta$, where ε_1 is trivial?

M. Hirsch

13. Do there exist two-dimensional complexes of the same homotopy type, but different simple homotopy type?

M. Hirsch

14. Is Kervaire's unsmoothable 10-dimensional manifold a boundary?

M. Hirsch

15. Let γ be a closed simple polygonal curve in the regular neighbourhood N of a two-dimensional complex K^2 in an orientable 3-manifold. Find a condition for γ to deform isotopically in N into $N \setminus K^2$.

This is more interesting when $\partial M = S^2$ contains all the vertices of M and also all the triangles having a least 2 edges in M, and K^2 is the complement of the open star of ∂M in the first barycentric subdivision.

M.G. Barratt

16. The preceding type of manifold (Problem 15) is the cylinder of a simplicial map f of S^2 on K^2 and f has very good properties (for example, $f^{-1}(x)$ contains 2, 3 or 4 points). Choose any of these properties and classify the pairs (K^2, f) having it.

M.G. Barratt

17. Let the complex K be dominated (semi-homotopy-equivalence) by a subset of n-dimensional euclidean space. Question: Does K have the homotopy type of a subcomplex of euclidean space of the same dimension?

T. Ganea

18. Let the Euler characteristic of the complex K be zero. Does there exist a map f: $K \to K$ without fixed points and homotopic to the identity? The same question for topological manifolds.

M. Hirsch

19. For fixed n, is there some k such that $\pi_r(PL_k) = \pi_r(PL)$ for $r \geqslant n$?

R. Lashof

20. The previous problem leads to the question: Can Σ^n be embedded in Σ^{n+k} with non-trivial normal microbundle? (Σ^n and Σ^{n+k} are combinatorial spheres.)

R. Lashof

HOMOTOPY THEORY (Continued)

3. Can one say something general about the homomorphism J: $\pi_q(SO_n) \to \pi_{n+q}(S^n)$ on the Samelson product?

M.G. Barratt

4. Analogous question about the homeomorphism ∂: $\pi_{g+1}(BG) \to \pi_g(G)$ on composition elements.

M.G. Barratt

5. Let ν_n be a generator of the group $\pi_{n+3}(S^n)$, $n \geqslant 5$. Is $P_n = [i_n, \nu_n]$ a periodic expression mod 8 as a function of n? Is it always zero for $n \equiv 5$, 7 mod 8? Perhaps it is either zero or an element of order 12 for large n?

M.G. Barratt

6. Let K_p denote the stable p-components of the homotopy groups of spheres (of dimension > 0). This is a graded commutative ring.

Which of the following are true?

(a) K_p is commutative as a non-graded ring.

(b) K_p is nilpotent (for each element is nilpotent).

(c) A product of even-dimensional elements is trivial in K_2.

(d) The cokernel $K_p/\mathrm{Im}J$ is a Z_{p^f}-module for some $f = f(p)$. (The image of J is not an ideal in K_p.)

M.G. Barratt

7. Investigate the phenomenon of periodicity in $\pi_{2n+k}(U(n))$, $1 \leqslant k \leqslant 2n$. It is related to the order of a line bundle over complex projective space in the Grothendieck group J (vector bundles with respect to fibre homotopy equivalence)?

R. Rothenberg

VECTOR FIELDS, ETC. (Continued)

6. If the tangent bundle P^n admits a q-field over $P^k \subset P^n$, does it then admit a 'linear' q-field? (Solved affirmatively by Adams for $k = n$.)

M. Hirsch

7. It is known that if M and M' are homeomorphic and M has a k-field for $k < (n-1)/2$, then M' also has a k-field. Is this true if M and M' are homotopically equivalent?

M. Hirsch

COBORDISM AND CHARACTERISTIC CLASSES (Continued)

12. Is the following true?: If M and M' are compact smooth manifolds and if there exists a homotopy equivalence $M \simeq M'$ agreeing with the tangent bundle, then $M{*}M \setminus \Delta \approx M'{*}M' \setminus \Delta'$, where $*$ denotes the symmetric square and Δ and Δ' are diagonals?

Similarly, if there is an isomorphism $H^*(M) \to H^*(M')$ taking characteristic classes to characteristic classes, then $H^*(M{*}M \setminus \Delta)$ $H^*(M'{*}M' \setminus \Delta')$.

J.R. Munkres

K-THEORY

1. Characteristic classes in K-theory θ_k or ρ_k seem to be the strongest known fibre-homotopy invariants of spinor $8n$-bundles. Can one show that they determine the Stiefel-Whitney classes (W_4, W_5, ...)? Can one give a formula for W_4, W_5, ... as functions of classes θ_k or ρ_k?

J.F. Adams

2. Can one define in K-theory an operation with the formal properties of Sq^i?

A. Dold

3. a) Let G be a compact Lie group and H a closed subgroup. Compute $K^*(G/H)$ and $KO^*(G/H)$.

b) Is the following conjectures true?

Let G be a compact Lie group and let $H_1(G, Z)$ have no torsion. Let U be a closed connected subgroup of maximal rank. Then the natural homeomorphism $R(U) \to K^*(G/U)$ is an epimorphism (see Atiyah-Hirzebruch, Vector bundles and homogeneous spaces, Amer. Math. Soc. Symp. Pure Math. 3 (1961), 7-38).

F. Hirzebruch

4. Let M^{4k} be a compact orientable manifold. Let α be in the kernel of the homeomorphism $R(SO_{4k}) \to R(SO_{4k-1})$. Following the Atiyah-Singer index theorem, we can introduce the rational number

$$I(M, \alpha) = \frac{\operatorname{ch} \alpha}{X_1 \circ \ldots \circ X_{2k}} \prod_{i=1}^{2k} \frac{\frac{x_i}{2}}{\operatorname{sh} \frac{X_i}{2}} [M^{4k}],$$

which is an integer, as the index of an elliptic operator. $I(M, \alpha)$ can be expressed in terms of Pontryagin numbers and the Euler characteristic. Can one thus obtain all relations on the Pontryagin numbers? Analogous problem for weak quasicomplex manifolds, where we can use the Riemann-Roch formula.

F. Hirzebruch

5. Let α be a function that assigns an integer to each partition $[k_1, \ldots, k_n]$ of the number k. For which functions does there exist a connected n-dimensional projective algebraic manifold X whose Chern numbers are given by the formula

$$C_{k_1} \circ \cdots \circ C_{k_n} [X] = \sigma([k_1, \ldots, k_n]).$$

Similar problem for complex, quasicomplex, weakly quasicomplex manifolds.

F. Hirzebruch

REMARKS BY S.P. NOVIKOV
PART I

IMMERSIONS AND IMBEDDINGS

1. It is well known that $CP^2 (n = 4, k = 3)$ can be embedded in R^7, but not immersed in R^6, which disproves the author's conjecture. There are also other counterexamples. I think that under some limitations the conjecture makes sense, though not in so simple a form.

2. No such stability theorem for fixed codimension is known to me, but Kervaire's result concerns the supply of fundamental groups of complements of knots. I think that the situation is quite complicated, at least for $k \geqslant 3$.

3. The conjecture is interesting from the point of view of the stable homotopy groups of spheres. An ordinary sphere S^n embedded in R^{n+k} for $k > (n/2) + 1$ has a trivial normal bundle. L.N. Ivanovskii and I have shown that a certain exotic sphere S^{16} when imbedded in R^{28} and R^{29} necessarily has a non-trivial normal bundle; this follows from the results of smooth topology and the theory of non-stable homotopy groups of spheres (Toda's tables). It seems that this is the first such example. The hypothetical series mentioned by the authors of the problem would give essential information about the Milnor groups.

4. The conjecture arises from the Rokhlin-Milnor-Kervaire theorem that the doubled diagonal 2Δ in $S^2 \times S^2$ is not realized by a smooth sphere. This question is very interesting, but I do not see any approaches.

5. If the number of discs is two, then M^m is a sphere (perhaps in another smoothing) and I do not know a single general theorem on smooth embedding better than Haefliger's embedding in R^{m+m} for $k \geqslant (m/2) + 1$ exclusively in terms of this number of discs. But perhaps one can say more about combinatorial embeddings.

8. For $n = 2$ this theorem follows from the fact that each surface is either a sphere plus some handles or a projective plane plus handles or a Klein bottle plus handles. Therefore it is sufficient to solve the problem for these three manifolds. I should be very surprised if this problem had an affirmative solution for $n \geqslant 4$, but for $n = 3$ interesting connections may come to light.

10. $P^{m,k}$ consists of smooth embeddings of S^m in S^{m+k} relative to diffeomorphisms of pairs (S^m, S^{m+k}) preserving orientation. The group operation is connected with the sum $\#$. For example, for $k > (m/2) + 1$ these groups are trivial (Haefliger). The results of Haefliger on the group $P^{4k-1,\ 2k+1} = Z$ for $k \geqslant 1$ are interesting.

11. In the conjecture it is necessary to introduce a modification, since for $p = 1$ the condition $r(M) \oplus j$ = trivial bundle is far from necessary for immersion. Probably one does not need the bundle j to be trivial.

12. One of the very well known problems of differential topology. The point is that all diffeomorphisms (up to diffeomorphisms extendible on the disc) have been classified by Milnor, Smale, Kervaire, Wall, Cerf, and form a finite abelian group Γ^{n+1}.

COMBINATORIAL AND SMOOTH STRUCTURES ON MANIFOLDS

1. If the codimension is at least 3, then all combinatorial embeddings are locally flat (Zeemann). The question is of great interest, since the presence of a microbundle allows one to work with combinatorial manifolds by analogy with smooth ones.

2. At the end I shall give a negative solution of this problem, and so in the Browder-Novikov theorem the Hirzebruch formula cannot be satisfied preserving the fibre homotopy type of the bundle, e.g. for $k = 3$ (dimension 12).

3. The action of the group θ^n on orientable manifolds can have fixed points, as was first proved by me (for $n = 9$, group $\theta^n/\theta^n(\partial\pi)$) and by Tamura (for $n = 7$, group $\theta^7(\partial\pi)$). For $\theta^n(\partial\pi)$ it is clear that this action depends on the Pontryagin classes and hence on the tangent bundle. For $\theta^n/\theta^n(\partial\pi)$ I do not know of examples where this depends on more than homotopy equivalence; I have heard that the author of the problem has been occupied with these questions, but I do not yet know his results.

4. An affirmative solution of this problem will be given below; the homeomorphism $\varnothing$ is non-trivial for example if $m = 1$, $k = 7$. I remark that the solution of this problem is contained in my note in Dokl. Akad. Nauk SSSR 148 (1963), 32-35, in somewhat different terms.

5. Below I give without proof an example of a π-manifold M^n and a class $\lambda \in H_i(M,\Gamma^{n-i})$ not realized as an obstruction to smoothing a diffeomorphism. It seems that this is the first known example of this kind.

6. If the dimension of the manifold M^n is not bounded above (as a function $n(q) < 2q + 1$), then it is easy by Morse's reconstructions to construct a π-manifold M^n with a given collection of abelian groups $\pi_i = \pi_i(M^n)$, $i \leqslant q$, $n \geqslant 2q + 1$, so that the problem trivially has an affirmative answer.

8. Usually for all purposes in combinatorial topology the Milnor topology (No. 3) is used. If it were proved that the maps $G_3 \to G_2 \to G_1$ induce isomorphisms of homotopy groups, then for all finite-dimensional problems this would be sufficient. Incidentally these problems (including also the uniqueness of the simplicial structure on G) are of very great interest.

VECTOR FIELDS, INFINITE-DIMENSIONAL MANIFOLDS, ETC.

1. A collection of k commuting independent fields on M^n determines the action of R^k on M^n without singularities, i.e. a k-dimensional fibering of a special kind. Below it will be proved that the rank of S^3 is 1 (V.I. Arnol'd), and also some other results.

2. If such a G-map f: $V \to W$ of degree k on $V \setminus O \to W \setminus O$ exists, where $G = SO_n$, $W = V = R^N$ and G operates on R^N according to two inequivalent representations α_1, α_2: $SO_n \to SO_N$, then there exists a number $l = l(k)$ such that the SO_N-bundles $\alpha_i u_n \oplus \ldots \oplus \alpha_i u_n$ (of k^l summands), $i = 1, 2$, constructed from the universal SO_N-bundle u_n (with base $G_{n,q}$, $q \gg n$) by taking the representation α_1 and the k^l-fold sum with itself, turn out to be J-equivalent (fibre homotopically equivalent) according to a theorem of Adams. Similarly for $G = U(n)$.

3. In the real case constructing the mentioned cross-sections of the fibering $V_{n,m}$ on the sphere is equivalent to constructing independent vector fields. All these fields have been known for a very long time and they are assembled from infinitesimal rotations of spheres. (That this is really the maximum possible number of fields was first proved for all spheres by Adams, but was proved earlier by Toda for spheres of the first two or three thousand dimensions.)

4. The group $U(n)$ clearly contracts among all unitary transformations of Hilbert space, but the general question is apparently difficult (for the topology induced by the norm).

COBORDISM AND CHARACTERISTIC CLASSES

1. For $p > 2$ this question is equivalent to the following: By what numbers of the form p^r are the Pontryagin classes (or expressions in them) for n-dimensional manifolds divisible? For example, for $n = 4$ the class p_1 is always divisible by 3 and this is the only relation. A paper of the author is devoted to the case $p = 2$, $r = 1$. I remark that a very interesting case is $I_n(\text{Spin}, 2^r)$, i.e. manifolds for which $W_2 = 0$.

2. This homeomorphism Ψ of Kervaire is the only blank patch in the connection of Milnor's groups θ^n with the stable homotopy classes of spheres.

For dimensions of the form $8k + 2$ the idea is proposed of extending Ψ on the group $\Omega^{\text{Spin}}_{8k+2}$ of spinor cobordisms and proving the triviality of the extended Kervaire homeomorphism or, for example, of the composite

$$\Omega^{\text{Su}}_{8k+2} \longrightarrow \Omega^{\text{Spin}}_{8k+2} \longrightarrow Z_2.$$

Of course, the study of the ring Ω^{Spin} seems easier than the stable homotopy groups of spheres, but all the same it is a difficult and unsolved problem (and likewise for Ω^{su}); Moreover, the extension of Ψ to Ω^{Spin} introduces new difficulties in the proof of its triviality even for $k = 2$, where the Kervaire homeomorphism Ψ on $\pi_{N+10}(S^N)$ is easier to investigate. Just such a definition of $\Psi/\Omega^{\text{Spin}}$ and the proof of the triviality of the extended Ψ for $k = 2$ can be found in my note in Dokl. Akad. Nauk SSSR 153 (1963), 1005-1008, = Soviet Math. Dokl. 4 (1963) 1768-1772, (in connection with an application to another problem), where these difficulties can be seen.

4. In connection with this problem I should like to remark that in the important case MSpin the conjecture that all the Adams differentials are trivial has not been refuted.

7. If M^{n+1} is parallelizable, then the class $T = \partial M$ in $\theta^n/\theta^n(\partial\pi)$ is trivial by definition.

One may always assume that M is simply connected. Already for $n = 9$ it cannot be considered two-connected and this is connected with the nontriviality of the homeomorphism $\theta^n \to \Omega^{\text{Spin}}$ into spinor cobordisms. (For $n = 9$ this image is Z_2 and the corresponding Milnor sphere does not bound a two-connected film M^{n+1}.) For odd n the image of $\theta^n \to \Omega^{\text{Spin}}_n$ is a direct sum of groups of the form Z_2 as I have proved. Therefore the film M^{n+1} can be considered two-connected if the class T^n in $\theta^n/\theta^n(\partial\pi)$ is divisible by 2. The case of three-connected manifolds reduces to the two-connected case. Further nothing is known here, although this question has interesting connections.

8. If $W_2(M^4) = 0$, the signature of the quadratic form on $H_2(M^4)$ is divisible by 16 (Rokhlin). But in all known examples of manifolds there are both positive and negative squares in this form.

9. This definition of cobordism may be of interest. This question may lead to interesting connections.

11. The result of Browder and Liulevicius has not yet appeared in print.

H-SPACES

1. Clearly if there is no p-torsion, then the algebra $H_*(G, Z)$ is exterior. Its non-commutativity is quite a rough reflection of the homotopy non-commutativity of the H-space itself, and for all groups with p-torsion this takes place, except $G = E_8$, $p = 5$.

3. Of course, if the manifolds were triangulizable the question would lapse.

6. There is a substantial theory of H-spaces in which there are many results about finite-dimensional H-spaces. For example, Adams was the first to show that of such spheres there are only S^1, S^3, S^7; Browder showed that the definition of H-space and finite dimensionality imply a number of deep facts. But there are scarcely any examples, except Lie groups, S^7 and products (in the simply connected case). The author announces some 'candidates'.

PART II

IMMERSION AND EMBEDDING (Continued)

13. According to a well-known paper of Thom, the L-equivalence classes for manifolds imbedded in M^n are naturally isomorphic to $\pi(M^n, MSO_{n-k})$. But this is a group only for $k < n - k - 1$ or for $M^n = S^n$. For immersions nothing of the kind is known and such a problem could be interesting if one succeeded in giving a 'good' solution.

14. In a well-known paper of the author of the problem it was proved that a parallelizable manifold M^n can be immersed in R^{n+1}. For embeddings the situation is incomparably more complicated, and apparently a better conjecture than this problem does not exist.

15. The conjecture consists of the following: The square of any diffeomorphism of degree -1 is diffeotopic to the identity. (It is always extendible over the disc.)

COMBINATORIAL AND SMOOTH STRUCTURES (Continued)

13. For the three-dimensional case (already for lens manifolds) the concept of simple homotopy type in the sense of Whitehead differs from homotopy

type. (For simply connected complexes they coincide.)

14. This manifold of Kervaire is not the boundary of any manifold for which $W_1 = W_2 = 0$.

19. By definition $\pi_i(PL) = \operatorname{dir}\lim_{k\to\infty} \pi_i(PL_k)$. In combinatorial topology of manifolds the group PL plays the same role as the group SO in smooth topology. But, as distinct from SO, here no stability theorems are known, although it seems to me that it has not been disproved that the groups $\pi_i(PL_k)$ become stable exactly like the groups $\pi_i(SO_k)$ for $i < k + 1$.

HOMOTOPY THEORY (Continued)

5. The Whitehead product $[i_n, \nu_n]$ is discussed in Problem 3 on immersion and imbedding in Part I.

6. Points (a) and (c) are implausible and probably spring from insufficient information. Points (b) and (d) are very interesting known qualitative problems.

VECTOR FIELDS ETC. (Continued)

6. A discussion in this direction can be found in Problem 3 on vector fields, etc., Part I.

7. This concerns a well-known theorem of Haefliger and the author. It is asked whether this theorem has any topological significance or only homotopy theory significance. This problem is closely connected with the properties of the K- and J-functors of manifolds, as shown by a result of Browder and Novikov.

K-THEORY

1. The classes ρ_k are constructed from the Adams operations in K-theory like the W_k are constructed from the Steenrod operations. In some cases the ρ_k determine the order of the J-functor and it seems very plausible that the W_i can be expressed in terms of the collection ρ_k for Spin-bundles (since otherwise the ρ_k are not defined).

3. $K^*(G/H)$ denotes the complex K-functor, KO^* the real one, $R(G)$ is the ring of unitary representations of the group G. A discussion of these questions can be found in the paper cited.

4. All relations among the Pontryagin numbers are known 'in principle', but whether or not they can all be interpreted as the integrity of indices of elliptic operators (of 'universal' type) would be very interesting to know.

5. For disconnected, projective and other manifolds this problem has been solved (Milnor).

SOLUTIONS OF SOME OF THE PROBLEMS
(of Wall, Munkres and Milnor)

PROBLEM 2. Combinatorial and smooth structures. (My solution.)

Consider the 12-dimensional Milnor manifold M_0^{12} which is 5-connected, parallelizable, and its boundary is a Milnor sphere. The index $I(M_0^{12})$ is equal to 8. Fill ∂M_0^{12} with a disc. We obtain a combinatorial manifold M^{12} with Pontryagin number $p_3 = \frac{8.3^3.5.7}{62}$. This manifold is not homotopically equivalent to a smooth one, since $I(M^{12})$ is a homotopy invariant and $\pi_6(BSO) = 0$. We shall prove that the trivial bundle over the manifold M^{12} satisfies all the conditions except the Hirzebruch formula. Consider its Thom complex $T_N = E^N M^{12} \vee S^N$. Clearly $E^N M^{12}$ is homotopically equivalent to $D^{N+12} \bigcup_f (S^{N+6} \vee \ldots \vee S^{N+6})$, where f: $S^{N+11} \longrightarrow S^{N+6} \vee \ldots \vee S^{N+6}$. Since $\pi_{N+5}(S^N) = 0$, we obtain the result that $E^N M^{12}$ is homotopically equivalent to $S^{N+12} \vee (S^{N+6} \vee \ldots \vee S^{N+6})$ and the cycle $[T_N]$ is spherical.

We remark that this general lemma holds: If a smooth or combinatorial manifold M^n is an almost π-manifold then the normal bundle $\alpha(M^n) \in \hat{k}_{PL}(M^n)$ belongs to the kernel of the J-homeomorphism J_{PL}: $\widetilde{k}_{PL}(M^n) \to \widetilde{J}_{PL}(M^n)$. Therefore the above argument applies to all Milnor manifolds $M^{4(2k+1)}$, $k \geqslant 1$, with index $I = 8$.

PROBLEM 4. Combinatorial and smooth structures. (My solution).

One can define a general homeomorphism

$$\lambda_i^k\colon\ \pi_i(\operatorname{diff} S^k) \longrightarrow \pi_0(\operatorname{diff} S^{i+k}),$$

and also an inclusion homeomorphism q: $\pi_i(SO_k) \to \pi_i(\operatorname{diff} S^k)$. We also recall the Milnor homeomorphisms:

$$T\colon\ \pi_0(\operatorname{diff} S^k) \longrightarrow \Gamma^{k+1},$$

$$S\colon\ \Gamma^{k+1} \longrightarrow G_{k+1} = \pi_{N+k+1}(S^N) \bmod J.$$

Let $\alpha \in \pi_0(\operatorname{diff} S^k)$, then this formula holds

$$ST\lambda_i^k[\alpha q(h)\alpha^{-1}] = [ST(\alpha) \circ J(h)] \bmod J,$$

where $h \in \pi_i(SO)_k$, and it is easy to show that

$$\varnothing\,(h \otimes T(\alpha)) = T\lambda_i^k[\alpha q(h)\alpha^{-1} - q(h)]$$

and $T\lambda_i^k[q(h)] = 0$. Thus, we have

$$S\varnothing\,(h \otimes T(\alpha)) = [J(h) \circ ST(\alpha)] \bmod J.$$

The last formula enables us to extract from tables the non-triviality of $\varnothing$, e.g. for $k = 7$, $m = 1$. (In connection with these results see my note in Dokl. Akad. Nauk SSSR 148 (1963), 32-35 = Soviet Math. Dokl. 4 (1963), 27-31, but the complete derivation of all the formulae will appear in Izv. Akad. Nauk SSSR.)

PROBLEM 5. Combinatorial and smooth structures. (One of the questions of this group is answered.)

We construct an example of a manifold M^n and a cohomology class $\lambda \in H_i(M^n, \Gamma^{n-i})$ that is not realized as such an obstruction. Such examples were not known previously: I do not give the proof because of its complexity. It is an indirect consequence of my paper in Izv. Akad. Nauk SSSR ser. math.

28 (1964), 361-474, §10, Theorem 10.2, whose proof by the way is left there incomplete. Let $n = 12$ and let M^n be a π-manifold (or a π-manifold on a 4-dimensional skeleton) such that $H_2 = H_4 = H_8 = H_{10} = Z$ and $H_i(M^n) = 0$ $(i \neq 2, 4, 8, 10)$, and $Sq^2(u)$ is non-trivial, where u is the basis element of the group $H^2(M^{12}, Z_2) = Z_2$.

I claim that the element $\lambda \in H_4(M^{12}, \Gamma^8 = Z_2)$ is not realized as an obstruction to smoothing, where $\lambda \neq 0$.

The reader can easily construct a manifold with these properties. We remark that the element λ is not realized as a submanifold $M^4 \subset M^{12}$ with trivial normal bundle, but is realized as a submanifold M^4 with $W_2(M^4) \neq 0$, for example $M^4 = CP^2$.

PROBLEM 1. Vector fields, infinite-dimensional manifolds, etc. (Solution of one of the problems, communicated to me by V.I. Arnol'd).

THEOREM. *Let M^n be a closed manifold. If the group $\pi_1(M^n)$ is finite or $\pi_2(M^n) \neq 0$ then the rank of M^n in the sense of Milnor is $\leqslant n - 2$.*

COROLLARY. Rank $S^3 = 1$, rank $M^3 = 1$ *if the universal covering space is not contractible,* rank $S^2 \times T^{n-2} = n - 2$ *where T^{n-2} is the $(n-2)$-dimensional torus.*

PROOF. Suppose, by contradiction, that there is on M^n a system of $n - 1$ commuting (independent) fields. Then R^{n-1} acts on M^n without singularities and determines on M^n a fibering of codimension 1. S.P. Novikov proved that under the conditions of the theorem for any fibering of codimension 1 there exists a non-zero element α of the group π_1 of some fibre, on one side α is not a limit cycle, and after an arbitrary displacement to this side α is homotopic to zero on the corresponding fibre (Dokl. Akad. Nauk SSSR 155 (1964) 1010-1013, 157 (1964), 788-790).

For simplicity let $n = 3$. The above element on the fibre is covered on R^{n-1} by a straight line, a one-parameter subgroup determining a dynamical system on M^n. But on the adjacent fibre the rotation of this field along the displaced cycle will be equal to 1, since it is almost equal to the tangent to the cycle. Therefore the field must have a singularity on the singular cell bounded by it on the fibre. Contradiction.

REMARKS. On some problems of this collection, added in proof.

1. Immersion and embedding, Problem 4. V.A. Rokhlin has just completely solved this problem. The answer is more optimistic than Wall's 'optimist'.

2. Immersion and embedding, Problem 10. As was proved by S.P. Novikov, all the groups $P^{4k-1,q}$ are infinite for $3 \leqslant q \leqslant 2k + 1$. This follows very simply from J-functors, the Browder-Novikov theorem on normal bundles applied to $S^{q-1} \times S^{4k}$, and Pontryagin classes. Conjecture: $P^{4k-1} \equiv Z$ mod finite groups, $P^{l,q}$ is finite for $l \neq 4k - 1$ and $q \geqslant 3$.

3. Vector fields, infinite dimensional manifolds, Problem 1. As was proved by S.P. Novikov, from Theorem 3 of Novikov's note in Dokl. Acad. Nauk SSSR 157 (1964), 788-790, and from a theorem of Sacksteder it follows that if $rk\, M^n = n - 1$, then either M^n is a vector bundle whose fibre is a torus T^k and whose base is the torus T^{n-k} up to homotopy type, or else there exists an invariant torus $T^{n-1} = M^n$.

4. Cobordism and characteristic classes, Problem 2. E. Brown, the author of the problem, together with Peterson, and also Lashof and Rothenberg have realized the proposed idea and established the triviality of the Arf

invariant for $n = 8k + 2$, A parallel study of the torsion structure of the groups Ω^{SU} was carried out by Conner and Floyd.

5. Cobordism and characteristic classes, Problem 8. V.A. Rokhlin has found such a manifold. Its two-dimensional Betti number is 16.

BIBLIOGRAPHICAL NOTES

It would be quite difficult to compile a full bibliography of papers on the subjects of this Institute. I shall indicate some sources (as far as possible in the Russian language) that can be used as an introduction to the topological materials used or cited by the authors of the problems.

1. *Rassloennye prostranstra* (Fibre spaces), Izdat. Inost. Lit., Moscow 1958, with translations of papers of J. Serre, Homologie singulière des espaces fibrés, Ann. of Math. 54 (1951), 425-505, Groupes d'homotopie et classes de groupes abéliens, Ann. of Math. 58 (1953), 258-294, A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. 57 (1953), 115-207, La cohomologie mod 2 de certains espaces homogènes, Comm. Math. Helvetici 27 (1953), 165-197, A Borel and J. Serre, Groupes de Lie et puissances réduites de Steenrod, Amer. J. Math 75 (1953), 409-448, (on homotopy theory) and R. Thom. Quelques propriétés globales des variétés différentiables, Comm. Math. Helvetici 28 (1954), 17-86, (on cobordism).

2. Proceedings of the International Congress of Mathematicians at Edinburgh, 1958, with lectures of Hirzebruch (on Chern classes and numbers) and of Thom (on the problem of smoothing combinatorial manifolds).

3. *Gor'kovskii seminar po gomotopicheskoi topologii, tezisy dokladov* (Proceedings of the Gor'ky seminar on homotopy topology), with expositions of the theory of fibre bundles, characteristic classes and the basic theory of K-functors.

4. The translation journal 'Matematika' with Milnor's lectures on characteristic classes (translated in 1960) and works of Adams on vector fields on the sphere (1964). There are also translations of many other papers (of Cartan, Milnor, Smale, Adams, Atiyah and others).

5. V.A. Rokhlin's survey article on cobordism, Intrinsic homology theories, Uspekhi Mat. Nauk 14, no. 4, (1959), 3-20, = Amer. Math. Soc. Transl. 30 (1963), 255-271, and the papers of S.P. Novikov, Homotopy properties of Thom complexes, Mat. Sb. 57 (1962), 407-442, and Homotopically equivalent smooth manifolds I, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 365-474. Smale's survey article 'A survey of some recent developments in differential topology', Bull. Amer. Math. Soc. 69 (1963), 131-145, = Uspekhi Mat. Nauk 19, no. 1, (1964), 125-138, on embedding problems and finite dimensional Morse theory. S.P. Novikov's survey article on differential topology before 1962 in the collection *Algebra, Topologiya* (Algebra, Topology), Vsesoyuz. Inst. Nauch, i Tekh Inf., Moscow, 1963.

6. Milnor's course on 'Morse theory', with an exposition of Bott's theorems will appear in a Russian translation in the very near future.

THE STATE OF AFFAIRS IN THE SEVEN CLASSICAL PROBLEMS OF TOPOLOGY

PROBLEM 1. Nothing is known in any example.

PROBLEM 2. The simple homotopy type is a combinatorial concept which coincides for simply connected polyhedra with ordinary homotopy type. But if $\pi_1 \neq 0$, it is different for dimension $n \geqslant 3$ (for $n = 3$, the lens spaces are examples). For $n = 3$ the Hauptvermutung was proved by Moise in 1950-1951. Therefore for $n = 3$ the problem is solved. Nothing more is known here.

PROBLEM 3. Rational Pontryagin classes are combinatorial invariants (Thom, Rokhlin-Shvarts, 1957). Progress with this problem has been made very recently by S.P. Novikov (Dokl. Akad. Nauk SSSR). A number of other problems reduce to this one.

PROBLEM 4. For spheres and closed cells this was proved by Smale and Wallace in 1960 for dimensions $\geqslant 6$. If the Betti numbers b_{4i} are zero for $0 < 4i < n$, $n \geqslant 5$, where the manifold is simply connected and of dimension n, then S.P. Novikov proved the finiteness of the number of *PL*-manifolds homeomorphic to a given one. For $n = 3$ the Hauptvermutung was proved by Moise (see Problem 2). For complexes of dimension $\geqslant 5$, the Hauptvermutung was disproved by Milnor in 1961.

PROBLEM 5. Proved for $n = 3$ (Moise, 1951). Nothing more is known here.

PROBLEM 6. For $n \geqslant 5$ this conjecture was proved, as is well known, by Stallings, Smale, Wallace and Zeeman (1960).

PROBLEM 7. The result in the remark on Problem 7 was published by A.V. Chernavskii, Dokl. Akad. Nauk SSSR 158 (1964), 62-65. For the smooth case it follows from the work of Smale. Nothing more is known here.

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