

NEW IDEAS IN ALGEBRAIC TOPOLOGY
(*K*-THEORY AND ITS APPLICATIONS)

S.P. NOVIKOV

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INTRODUCTION

In recent years there has been a widespread development in topology of the so-called *generalized homology theories*. Of these perhaps the most striking are *K*-theory and the *bordism* and *cobordism* theories. The term *homology theory* is used here, because these objects, often very different in their geometric meaning,

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share many of the properties of ordinary homology and cohomology, the analogy being extremely useful in solving concrete problems. The K -functor, which arose in algebraic geometry in the well-known work of Grothendieck, has been successfully applied by Atiyah and Hirzebruch to differential topology and has led quickly to the solution of a number of delicate problems.

Among the results obtained strictly with the help of K -theory the work of Atiyah and Singer on the problem of the index of elliptic operators and of Adams on vector fields on spheres and the Whitehead J -homomorphism are outstanding. More or less influenced by the K -functor other functors have appeared, with importance for topology — the J -functor, bordism theories and Milnor's microbundle k -functor. These have thrown new light on old results and have led to some new ones. Note, for example, the results of Milnor, Mazur, Hirsch, Novikov and others on the problem of the relation between smooth and combinatorial manifolds, based on Milnor's k_{PL} -functor, and the theorems of Browder and Novikov on the tangent bundles of manifolds of the same homotopy type, successfully treated with the help of Atiyah's J -functor. Particularly interesting applications of bordism theory have been obtained by a number of authors (Conner and Floyd, Brown and Peterson, Lashof and Rothenberg).

In this survey I shall try roughly to describe this work, though the account will be very far from being complete. In order to describe recent results I shall of course have to devote a large part of the survey to presenting material that is more or less classical (and is in any case no longer new). This material is collected in the first two chapters¹.

CHAPTER I. CLASSICAL CONCEPTS AND RESULTS

To begin with we recall the very well-known concepts of fibre bundle, vector bundle, etc. We shall not give strict definitions, but confine ourselves to the intuitive ideas.

§ 1. THE CONCEPT OF A FIBRE BUNDLE

Let X be a space, the *base*, and to each point $x \in X$ let there be associated a space F_x , the *fibre*, such that in a good topology the set $\bigcup_x F_x = E$, the *space*, is projected continuously onto X , each point of the fibre F_x being mapped to the corresponding point x . To each path $g: I \rightarrow X$, where I is the interval from 0 to 1, there corresponds a map $\lambda(g): F_{x_0} \rightarrow F_{x_1}$, where $x_0 = g(0)$ and $x_1 = g(1)$. The map $\lambda(g)$ has to depend continuously on the path g and satisfy the following conditions:

- a) $\lambda(g^{-1}) = \lambda(g)^{-1}$, where g^{-1} is the inverse path.²
- b) $\lambda(fg) = \lambda(f)\lambda(g)$, where fg denotes the composition of paths.

If we suppose that all the fibres F_x are homeomorphic and that, for each closed path g , $\lambda(g)$ is a homeomorphism belonging to some subgroup G of the group of homeomorphisms of the "standard fibre" F , then the group G is called the *structure group of the bundle*. These concepts *standard fibre* and *structure group* can be defined rigorously. Thus, the concept of fibre bundle includes the "space" E ,

¹The distribution of the literature over the various sections is indicated on p. 22.

²If the maps λ are not homeomorphisms, but only homotopy equivalences, then a) has to be weakened by postulating only that $\lambda(e) = 1$, where e is the constant path.

the “base” X , the projection $p: E \rightarrow X$, the “fibre” F , homeomorphic to $p^{-1}(x)$, $x \in X$, and the group G .

We illustrate this by examples.

Example 1. The line R is projected onto the circle S^1 , consisting of all complex numbers $|z| = 1$, where $p: \phi \rightarrow e^{i\phi}$.

Here $E = R$, $X = S^1$, F is the integers, G the group of “translations” of the integers $g_n: m \rightarrow m + n$.

Example 2. The sphere is projected onto the real projective space RP^n so that a single point $x \in RP^n$ is a pair of points of the sphere — vectors α and $-\alpha$ if the sphere is given by the equation $\sum x_i^2 = 1$. Here $E = S^n$, $X = RP^n$, the fibre F is a pair of points and G is a group of order 2.

Example 3. The Möbius band Q is a skew product with base the circle $X = S^1$, fibre the interval from -1 to 1 and $E = Q$. The group G is of order 2, because the fibre — the interval — when taken round a contour — the base — is mapped to itself by reflection with respect to zero.

Example 4. If H is a Lie group and G a closed subgroup, then we obtain a fibre bundle by setting $X = H/G$, $E = H$ and $G = G$, with p the natural projection $H \rightarrow H/G$ onto the space of cosets. For example, if H is the group of rotations of three-dimensional space SO_3 and G the group of rotations of a plane, then H/G is the sphere S^2 and we have a fibration $p: SO_3 \rightarrow S^2$ with fibre $S^1 = SO_2$. There are a great number of fibre bundles of this type.

Example 5. A Riemannian metric induces on a closed manifold the concept of the parallel “transport” of a vector along a path. This shows that the tangent vectors form a fibre bundle (the “tangent bundle”), the base being the manifold itself and the fibre all the tangent vectors at a point (a Euclidean space). The group G is SO_n if there are no paths changing orientation (“the manifold is oriented”) and O_n otherwise.

Such a fibre bundle, with fibre R^n and group $G = O_n$ or SO_n , we shall call a “real vector bundle”.

Example 6. If a manifold M^n is smoothly embedded in a manifold W^{n+k} , then a neighbourhood E of it in W^{n+k} may be fibred by normal balls. The neighbourhood is then also a fibre bundle with fibre D^k (a ball) or R^k , group SO_k or O_k and base M^n . This also is a vector bundle (“the normal bundle”).

The concept of a “complex vector bundle” with fibre C^n and group U_n or SU_n is introduced in a similar way.

It is useful to relate to any fibre bundle its “associated” principal bundle, a principal bundle being one in which the fibre coincides with the group G and all the transformations are right translations of the group.

A principal bundle can be described as follows: the group G acts without fixed points on the space E , the base X is the set of orbits E/G and the projection $E \rightarrow E/G$ is the natural one.

The fact is that an arbitrary fibre bundle with arbitrary fibre may be uniquely defined by the choice of the maps $\lambda(f) \in G$ for the closed paths f from a single point. One can therefore by this same choice construct a principal fibre bundle with the same base X , but with fibre G , a transformation of the fibre being replaced by

the translation of G induced by the same element $\lambda(f)$. One can also change one fibre into another if the same group acts on it.

§ 2. A GENERAL DESCRIPTION OF FIBRE BUNDLES

1. Bundle maps. If two bundles $\eta_1 = (E_1, X_1, F, G)$ and $\eta_2 = (E_2, X_2, F, G)$ are given with common group and fibre, then a map $E_1 \rightarrow E_2$ is said to be a *bundle map* if it sends fibres into fibres homomorphically and if it commutes with the action of G on F . The bundle map induces a map $X_1 \rightarrow X_2$.

2. Equivalence of bundles. Two bundles with the same base X , fibre F and group G are said to be *equal* if there is a bundle map of the one to the other such that the induced map of the base is the identity.

3. Induced bundles. If one has a bundle $\eta = (E, X, F, G)$, a space Y and a map $f: Y \rightarrow X$, then over Y there is a unique bundle “induced by f ” with the same fibre and group and with base Y , mapped to the first bundle η and including on the base the map $f: Y \rightarrow X$. This bundle is denoted by $f^*\eta$. That is, to a bundle over X there uniquely corresponds a bundle over Y (bundles are mapped contravariantly just as functions are).

4. Examples.

Example 1. If a bundle over X is equivalent to a “trivial” bundle, that is, is such that $G = e$ and $E = X \times F$, then the same is true of any bundle induced by it — this implies that an induced bundle $f^*\eta$ cannot be more complicated than the original one.

Example 2. Let M^k be a manifold embedded in R^{n+k} . Consider the manifold $G_{k,n}$ of k -planes in R^{n+k} passing through the origin. Over $G_{k,n}$ (as base) there is a fibre bundle η with fibre R^k : to be precise, the fibre over $x \in G_{k,n}$ is the plane x itself, of dimension k . The group G here is O_k . One can translate the tangent plane at a point $m \in M^k$ to the origin. We get a map $f: M^k \rightarrow G_{k,n}$; the tangent bundle to M^k , clearly, is $f^*\eta$, where η is the bundle over $G_{k,n}$.

One may assume here that n is very large. There is an important classifying

Theorem. *Every fibre bundle with finite-dimensional base X and group O_k is induced by a map $X \rightarrow G_{k,n}$, unique up to homotopy, provided n is sufficiently large.*

The set of fibre bundles over X with group O_k is therefore identical with the set of homotopy classes of maps of X to $G_{k,n}$ (denoted by $\pi(X, G_{k,n})$).

Because of this theorem the bundle η constructed in Example 2 is said to be “universal”. Such universal bundles can also be constructed in a similar way $G = SO_k$ (one has to take oriented k -planes in R^{n+k}) and for $G = U_k$ (complex k -planes in C^{n+k}). It is easy to construct similar universal bundles for an arbitrary Lie group G .

The standard notation for the base of a universal bundle for a group G is BG . For example $G_{k,n} = BO_k$ for n infinitely large.

In the sequel we shall only be concerned with vector bundles, real or complex.

§ 3. OPERATIONS ON FIBRE BUNDLES

1. **Sum (Whitney).** Let η_1 and η_2 be two vector bundles with bases X_1 and X_2 . Then over $X_1 \times X_2$ there is a bundle $\eta_1 \times \eta_2$, whose fibre is the direct product of the fibres of η_1 and η_2 . If $X_1 = X_2 = X$, then there is a “diagonal map” $\Delta: X \rightarrow X \times X$, where $\Delta(x) = (x, x)$. Set

$$\eta_1 \oplus \eta_2 = \Delta^*(\eta_1 \times \eta_2).$$

We get a bundle $\eta_1 \oplus \eta_2$ over X , the “sum” of the bundles η_1 and η_2 with common base X .

2. **Product (tensor).** Let η_1 and η_2 be two bundles with common base X . Then over $X \times X$ there is a bundle $\eta_1 \hat{\otimes} \eta_2$, whose fibre is the tensor product of the fibres (over R , if they are real, or over C if they are complex bundles). Set, as before,

$$\eta_1 \otimes \eta_2 = \Delta^*(\eta_1 \hat{\otimes} \eta_2);$$

$\eta_1 \otimes \eta_2$ is a bundle with the same base space X .

It is easy to prove bilinearity:

$$\eta_1 \otimes (\eta_2 \oplus \eta_3) = (\eta_1 \otimes \eta_2) \oplus (\eta_1 \otimes \eta_3).$$

The unit for tensor multiplication is the trivial (line) bundle with one-dimensional fibre R or C .

3. **Representation of the structure group.** Let G be the structure group of the bundle η with fibre F and base X and for simplicity let $h: G \rightarrow O_N$ or $h: G \rightarrow U_N$ be a faithful orthogonal or unitary representation. By the method indicated above one can “change” the fibre F to R_N or C_N by means of the representation h . We get a bundle denoted by h . The following cases are the most interesting:

- 1) h is the natural inclusion $O_N \subset U_N$ (“complexification”, denoted by c).
- 2) h is the natural inclusion $U_N \subset O_{2N}$ (“realization”, denoted by r).
- 3) If any $h: G \subset O_N$ or $G \subset U_N$ is given, we can take its “exterior power” $\bigwedge^i h$, $0 \leq i \leq N$, where $\bigwedge^1 h = h$ and $\bigwedge^n h = \bigwedge^0 h = e$. For example, differential forms are “sections” of the exterior powers of the (co-) tangent bundle.

Note that $cr\eta = \eta \oplus \bar{\eta}$ and $rc\eta = \eta \oplus \eta$, where $\bar{\eta}$ denotes the complex conjugate bundle to η .

4) If two representations h_1 and h_2 are given, then we can form their sum and product, which we shall denote simply by the symbols for addition (with positive integral coefficients) and multiplication. All these operations enable us to construct new bundles with the same base.

CHAPTER II. CHARACTERISTIC CLASSES AND COBORDISMS

§ 4. THE COHOMOLOGICAL INVARIANTS OF A FIBRE BUNDLE. THE CHARACTERISTIC CLASSES OF STIEFEL–WHITNEY, PONTRYAGIN AND CHERN

To each real fibre bundle there are associated a collection of invariants, the Stiefel and Pontryagin classes. Let η be a bundle with fibre R^n and base X . Classes $W_i \in H^i(X, Z_2)$ are defined, with $W_0 = 1 \in H^0(X, Z_2)$ and $W_i = 0$, $i > n$. They are called the Stiefel–Whitney classes. There are also Pontryagin classes $p_i \in H^{4i}(X, Z)$ and the Euler–Poincaré class $\chi \in H^n(X, Z)$ if $n = 2k$. Moreover, $p_k = \chi^2$ for $n = 2k$ and $p_i = 0$ for $i > k$. If the classes p_i and χ are taken mod 2, then they are expressible in terms of the Stiefel classes; to be

precise, $p_i = W_{2i}^2 \bmod 2$ and $\chi = W_n \bmod 2$. We form the ‘‘Stiefel polynomial’’ $W = 1 + W_1 + W_2 + \dots$ and the ‘‘Pontryagin polynomial’’ $P = 1 + p_2 + p_4 + \dots$.

In a similar way one can associate to a complex bundle ζ , with fibre C^n , Chern classes $c_i \in H^{2i}(X, Z)$ and a Chern polynomial $C = 1 + c_1 + c_2 + \dots$. Note that $c_i \bmod 2$ is equal to $W_{2i}(r\zeta)$, where r is the operation of making the bundle ζ real, and $c_{2i}(c\eta) = (-1)^i p_i(\eta)$, where c is the operation of complexifying the bundle η .

Properties of these classes are:

- a) Bundle maps map classes to classes.
- b) The Whitney formulae $W(\eta_1 \oplus \eta_2) = W(\eta_1)W(\eta_2)$.

$$C(\zeta_1 \oplus \zeta_2) = C(\zeta_1)C(\zeta_2),$$

up to elements of order 2: $P(\eta_1 \oplus \eta_2) = P(\eta_1)P(\eta_2)$.

c) Let us factorize the Chern polynomial $C(\zeta)$ formally: $C = \prod_{i=1}^n (1 + b_i)$, $\dim b_i = 2$, and form the series $\text{ch } \zeta = \sum_{i=1}^n e^{b_i} \in H^*(X, Q)$, where Q is the rational numbers. Clearly $\text{ch } \zeta$ is expressible in terms only of the elementary symmetric functions of the b_i which are the $c_k(\zeta)$. Therefore $\text{ch } \zeta = \sum_{k \geq 0} \text{ch}^k \zeta$ meaningful in the cohomology ring. Similarly, for a real bundle η , let

$$\text{ch } \eta = \text{ch } c\eta = \sum_{k=0}^{\infty} \text{ch}^{2k} c\eta, \quad \text{ch}^{2k+1} c\eta = 0.$$

$\text{ch } \zeta$ is called the ‘‘Chern character’’. It has the following properties:

$$\text{ch } \zeta_1 \oplus \zeta_2 = \text{ch } \zeta_1 + \text{ch } \zeta_2,$$

$$\text{ch } \zeta_1 \otimes \zeta_2 = \text{ch } \zeta_1 \text{ch } \zeta_2.$$

d) For the Euler class χ we have $\chi(\eta_1 \oplus \eta_2) = \chi(\eta_1)\chi(\eta_2)$. If η is the tangent bundle of a smooth closed manifold M^n , then the scalar product $(\chi(\eta), [M^n])$ is equal to the Euler characteristic of M^n . If M^k is a complex manifold, then $\chi(r\eta) = c_{\frac{k}{2}}(\eta)$.

e) If η is a complex U_n -bundle with base X and $r\eta$ its real form, then $P(r\eta) = \prod_{i=1}^n (1 + b_i^2)$, where $c(\eta) = \prod_{i=1}^n (1 + b_i)$, $\dim b_i = 2$.

For example $p_1(r\eta) = c_1^2 - 2c_2$.

It has already been remarked that $W(r\eta) = C(\eta) \bmod 2$.

Example 1. The natural one-dimensional normal bundle η_1 of RP^n in RP^{n+1} (the ‘‘Möbius band’’) has Stiefel polynomial $W(\eta_1) = 1 + x$, where x is the basis element of the group $H^1(RP^n, Z_2)$. Similarly the one-dimensional complex normal bundle ζ_1 of CP^n in CP^{n+1} (the ‘‘complex Möbius band’’) has Chern polynomial $C(\zeta_1) = 1 + x$, where x is the basis element of the group $H^2(CP^n, Z) = Z$.

Example 2. If $\tau(M)$ is the tangent bundle of the manifold (complex if it is complex), then for $M = RP^n$ or CP^n we have the formulae

$$\tau(RP^n) \oplus 1_R = \eta_1 \oplus \dots \oplus \eta_1 \quad (n+1 \text{ terms}),$$

$$\tau(CP^n) \oplus 1_C = \zeta_1 \oplus \dots \oplus \zeta_1 \quad (n+1 \text{ terms}),$$

where 1_R and 1_C are the one-dimensional trivial bundles. Therefore $W(RP^n) = (1 + x)^{n+1}$, $x \in H^1(RP^n, Z_2)$ and $C(CP^n) = (1 + x)^{n+1}$, $x \in H^2(CP^n, Z)$. By property e) it is easy to deduce that $P(CP^n) = (1 + x^2)^{n+1}$; here $W(M)$, $C(M)$, $P(M)$ denote the polynomials of the manifold, that is, the polynomials W , C , P of the tangent bundle. In particular $C(CP^1) = C(S^2) = 1 + 2x$, $C(CP^2) = 1 + 3x + 3x^2$ and $P(CP^2) = 1 + 3x^2$, $\text{ch } \tau(CP^n) = (n+1)e^x - 1$.

§ 5. THE CHARACTERISTIC NUMBERS OF PONTRYAGIN, CHERN AND STIEFEL.
COBORDISMS

If M^n is a closed manifold, one can consider polynomials of degree n in the Stiefel and Pontryagin classes and take their scalar products with the fundamental cycle of the manifold. We get numbers (or numbers mod 2 for Stiefel classes). If M is a complex or quasicomplex manifold, one can do the same with the Chern classes of its tangent bundle. We get Stiefel, Pontryagin and Chern numbers. The following important theorem holds:

Theorem of Pontryagin–Thom. 1) *A manifold is the boundary of a compact manifold with boundary if and only if its Stiefel numbers are zero.*

2) *An oriented manifold is the boundary of an oriented compact manifold with boundary if and only if its Pontryagin and Stiefel numbers are zero.*³

The corresponding theorem (Milnor, Novikov) for quasicomplex manifolds has a more complicated formulation, since one first has to define what it means to “be a boundary”, but after this it runs analogously, involving the Chern numbers.

One defines the “cobordism rings” $\Omega^* = \sum \Omega^i$ roughly as follows: the “sum” of two manifolds of some class or other (orientable, quasicomplex etc.) is defined by forming their disjoint union and the “product” by forming their direct product; one also defines what it means to “be a boundary” for each such class. The correctness of the definition has to be verified. In this way there arise the “cobordism rings” Ω^* with the operations of addition and multiplication. Such a ring arises naturally for each of the classical series of Lie groups $\{O_n\}$, $\{SO_n\}$, $\{U_n\}$, $\{SU_n\}$, $\{Sp_n\}$, $\{Spin_n\}$ and the unit group. The following types of cobordism occur:

- $N = \Omega_O^*$ — non-oriented manifolds,
- $\Omega = \Omega_{SO}^*$ — oriented manifolds,
- Ω_U^* — quasicomplex manifolds (a complex structure on the stable tangent bundle),
- Ω_{SU}^* — special quasicomplex manifolds (the first Chern class c_1 is equal to zero),
- Ω_{Spin}^* — spinor manifolds,
- Ω_{Sp}^* — symplectic manifolds,
- Ω_e^* — the Pontryagin framed manifolds, Ω_e^i being isomorphic to the stable homotopy group of spheres of index i , $\Omega_e^i \approx \pi_{N+i}(S^N)$.

The following results on cobordism rings are known.

1°. Any type of cobordism is completely defined by characteristic numbers after factoring by torsion. After tensoring with the field of rational numbers all the cobordism rings become polynomial rings (theorems arising from results of Cartan–Serre in homotopy theory and Thom’s work on cobordism).

2°. Ω_O^* is a polynomial algebra over the field Z_2 (Thom). $\Omega_{SO}^* \otimes Z_2$ is a subalgebra of Ω_O^* ; Ω_{SO}^* does not contain elements of order 4 or elements of odd order; the quotient of Ω_{SO}^* by 2-torsion is a polynomial ring over the integers (Rokhlin, Wall, Averbukh, Milnor).

3°. Ω_U^* is a polynomial ring. $\Omega_{Sp}^* \otimes K$ and $\Omega_{Spin}^* \otimes K$ are polynomial algebras, where the characteristic of the field K is not equal to 2. (Milnor, Novikov).

³Statement 2) was finally proved only at the end of the 50’s by Milnor, Rokhlin, Averbukh and Wall.

4°. $\Omega_{SU}^* \otimes K$ is a polynomial algebra if the characteristic of the field K is not equal to 2. The ring Ω_{SU}^* has no elements of odd order. The whole subgroup $\Omega_{SU}^{\text{odd}} = \sum \Omega_{SU}^{2k+1}$ belongs to the ideal generated by $\Omega_{SU}^1 = Z_2$, that is, $\Omega_{SU}^{2k+1} = \Omega_{SU}^1 \Omega_{SU}^{2k}$ (Novikov). It seems that these are all the results on cobordisms that have been known for some time (several years, at least).

To these results it is useful to add the following: for a complex (real) manifold to represent a polynomial generator of the ring Ω_U^* or Ω_{SO}^* it is necessary and sufficient that (in the complex case) the components $\text{ch}^n \eta$ of the tangent bundle should be such that

$$|(n! \text{ch}^n \eta, [M^n])| = \begin{cases} 1, & n+1 \neq p^i, \\ p, & n+1 = p^i, \end{cases}$$

where p is an arbitrary prime number. For example, for $M^n = CP^n$ we have $(n! \text{ch}^n \eta, [CP^n]) = n+1$, from which it follows that CP^n is a polynomial generator only for dimensions of the form $p-1$, where p is prime. In particular, CP^1 , CP^2 , CP^4 , and CP^6 are generators, while CP^3 , CP^5 , CP^7 and CP^8 are not. Milnor has exhibited a system of “genuine” geometrical generators. For Ω_{SO}^* the situation is analogous, but one is then only concerned with CP^{2k} .

§ 6. THE HIRZEBRUCH GENERA. THEOREMS OF RIEMANN–ROCH TYPE

The simplest and oldest invariants of a complex Kähler (in particular, algebraic) manifold are the “dimensions of the holomorphic forms of rank Q ”, denoted by $h^{q,0}$. An important invariant is the “arithmetic genus” $\chi = \sum (-1)^q h^{q,0}$. Apart from ordinary forms one can also speak of forms with values in an algebraic vector bundle ζ over M^n ; we then obtain

$$\chi(M^n, \zeta) = \sum (-1)^q h^{q,0}(\zeta).$$

Another invariant (in the real case) is the “signature” of a manifold of dimension $4k$, this being simply the signature of the quadratic form $(x^2, [M^{4k}])$, where $x \in H^{2k}(M^{4k}, R)$. This signature is denoted by $\tau(M^{4k})$.

We set:

1) $L = \sum_{i \geq 0} L_i = \prod_j \frac{\text{tg } h b_j}{b_j}$, $\dim b_i = 2$ and $P = \prod_j (1 + b_j^2)$. Here $L_i = L_i(p_1, \dots, p_i)$ and $\dim L_i = 4i$.

2) $T = \sum_{i \geq 0} T_i = \prod_j \frac{b_j}{1 - e^{-b_j}}$, $T_i = T_i(c_1, \dots, c_i)$, $\dim T_i = 2i$, $C = \prod_j (1 + b_j)$.

3) $A = \sum_{i \geq 0} A_i = \prod_j \frac{b_j}{\sin h \frac{b_j}{2}}$, $A_i = A_i(p_1, \dots, p_i)$, $\dim A_i = 4i$.

Note that $T = e^{\frac{1}{2} \sum b_j} A$ and also that

$$\begin{aligned} L_1 &= \frac{1}{3} p_1, & L_2 &= \frac{1}{45} (7p_2 - p_1^2), \\ T_1 &= \frac{1}{2} C_1, & T_2 &= \frac{1}{12} (C_2 - C_1^2), \\ A_1 &= \frac{1}{24} p_1, & A_2 &= \frac{p_1^2 - 4L_1}{32 \cdot 28}. \end{aligned}$$

Hirzebruch proved the following facts which are the basis of cobordism theory:

$$(L_k(p_1, \dots, p_k), [M^{4k}]) = \tau(M^{4k}) \quad (\text{real case}) \quad -$$

for $k=1$ this becomes the theorem of Rokhlin and Thom; $(\text{ch } \zeta T(M^n), [M^n]) = \chi(M^n, \zeta)$, where ζ is an algebraic bundle and M^n is an algebraic manifold, — for

$n = 1$ this becomes the classical theorem of Riemann–Roch and for $n = 2$ the formulae of Noether and Kodaira.

The following generalizations of these Hirzebruch formulae turn out to be extremely important:

1°. If $W_1 = W_2 = 0$, for a real manifold M^{4k} , then $(A_k, [M^{4k}])$ is an integer. For odd k this number is divisible by 2. For $k = 1$ this reduces to the theorem of Rokhlin.

2°. For a quasi complex manifold M^n the Todd genus $(T_n, [M^n])$ is an integer (Borel–Hirzebruch, Milnor).

Note that in the algebraic case the first integrality theorem was known, when $c_1 = 0$, since $T = e^{c_1/2}A$.

§ 7. BOTT PERIODICITY

As has already been said, the set of G -bundles with base X is identical with the set of homotopy classes $\pi(X, BG)$, where BG is the base of the universal bundle. Let $G = O_n, U_n$ or Sp_n . How many bundles are there with base S^k ? The set of these bundles is $\pi(S^k, BG) = \pi_k(BG)$. By a theorem of Cartan–Serre we know that $\pi_k(BG) \otimes Q$ is determined by cohomological invariants (Pontryagin and Chern classes). However, we do not know the torsion. For example $\pi_2(BSO_n) = Z_2$ for $n > 2$, $\pi_4(BU_2) = Z_2$.

How are we to compute fully the homotopy groups of the spaces BG ? The theorems of Bott completely compute $\pi_i(BO_n)$ for $i < n$, $\pi_i(BU_n)$ for $i < 2n$, and $\pi_i(BSp_n)$ for $i < 4n$.

Theorem. *There exists a canonical isomorphism between the sets*

- (1) $\pi(X, BU_n)$ and $\pi(E^2 X, BU_n)$ for $\dim X \leq 2n - 2$,
- (2) $\pi(X, BO_n)$ and $\pi(E^8 X, BO_n)$ for $\dim X \leq n - 9$,
- (3) $\pi(X, BSp_n)$ and $\pi(E^8 X, BSp_n)$ for $\dim X \leq 4n - 8$.

For $x = S^k$ these are isomorphisms of cohomology groups. EX here denotes the suspension of the space X , and $E^k X = EE^{k-1} X$.

For $X = S^k$ the Chern character of the bundle generating the group $\pi_{2k}(BU_n)$ is equal to $\text{ch}^k \eta = x$, where x is the basis element of the group $H^{2k}(S^{2k}, Z) \subset H^{2k}(S^{2k}, Q)$. For $X = S^{4k}$ the Chern character of the bundle generating the group $\pi_{4k}(BO_n)$ is equal to $\text{ch}^{2k} \eta = a_k x$, where x is as in the complex case and

$$a_k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 2, & \text{if } k \text{ is odd.} \end{cases}$$

Here, of course, the dimension of the sphere is supposed to satisfy the conditions of Bott's theorem. Using the relation between the Chern classes and the Chern character we get the fact that the Chern class of a complex bundle with base S^{2k} is divisible by $(k-1)!$, while the Pontryagin class of a real bundle over S^{4k} is divisible by $a_k(2k-1)!$.

One can also add to Bott's theorem information on the first dimensions not satisfying the "stability" conditions:

1) $\pi_{2k+1}(BU_k) = Z_k!$; 2) the kernel of the map $\pi_n(BO_n) \rightarrow \pi_n(BO_{n+1})$ is always Z_2 for odd $n+1 \neq 2, 4, 8$; from this there follows at once the nonparallelizability of the spheres S^k for $k \neq 1, 3, 7$ and the nonexistence of finite-dimensional division algebras over R , except for the dimensions 2, 4, 8. (This has also been

proved by Adams with other arguments; as a corollary of Bott's Theorem it was proved by Milnor and Kervaire.)

§ 8. THOM COMPLEXES

Let X be the base of a fibre bundle η with fibre R^n . The following space will be called the Thom complex T_η : $T_\eta = E_\eta/A_\eta$, where E_η is the bundle space of η and A_η is the subspace consisting of all vectors of length ≥ 1 in each fibre.

The Thom isomorphism is defined to be a certain map

$$\phi: H^i(X) \rightarrow H^{n+i}(T_\eta), \quad i \geq 0$$

(cohomology over Z_2 for O_n -bundles and over Z for SO_n -bundles).

Note that there is a natural inclusion

$$j: X \subset T_\eta.$$

Example. $X = BG$ and η the universal G -bundle, where G is a subgroup of O_n . Then η has R^n as fibre. For example $G = O_n, SO_n, U_n \subset O_{2n}, SU_n \subset O_{2n}, Sp_n \subset O_{4n}, e \subset O_n$. There is an inclusion $Spin_n \subset O_n$. The space T_η is denoted in this case by MG . For $G = e \subset O_n$ we have $Me = S^n$.

Thom's Theorem (for $G = e$, Pontryagin). *The groups $\pi_{n+k}(MG)$ are isomorphic to the cobordism groups Ω_G^k for large n and $G = O_n, SO_n, U_{n/2}, SU_{n/2}, Sp_{n/4}, e$.*⁴

On the basis of this fact cobordisms can be computed. Note that the ring structure of the cobordisms has a natural homotopy interpretation in terms of the Thom complex MG (Milnor, Novikov).

§ 9. NOTES ON THE INVARIANCE OF THE CLASSES

The Stiefel classes can be defined in terms of the Thom complex. Let

$$W_i(\eta) = \phi^{-1}Sq^i\phi(1),$$

where $\phi: H^k(X) \rightarrow H^{k+n}(T_\eta)$. This formula is easily verifiable for the universal bundle. If it is applied to the tangent (or normal) bundle η of a smooth manifold M^n , it makes it possible to compute the Stiefel classes of the manifold in terms of the cohomological invariants of the manifold — the Steenrod squares and the cohomology ring (the formulae of Thom and Wu). In a similar way it is sometimes possible to compute the Pontryagin classes up to some modulus or other (for example, $p_i \bmod 3$). However, unlike the Stiefel classes, the Pontryagin classes (rational, integral) are not invariants of homotopy type (Dold). Moreover, except for Hirzebruch's formula, $(L_k, [M^{4k}]) = \tau(M^{4k})$, there are no "rational" homotopy-invariant relations for simply-connected manifolds (we shall say more about this later). On the basis of cobordisms and Hirzebruch's formula about L_k , Thom, Rokhlin and Shvarts have proved the combinatorial invariance of the rational classes p_k ; the class $L_k(M^{4k+1})$ is even a homotopy invariant (Appendix). A fundamental problem is the topological meaning of the rational Pontryagin classes (we shall say more about this later).

⁴In Thom's work this is formulated only for $G = O, SO$. For $G = e$ this theorem is contained in older work of Pontryagin. In the remaining cases the proof is similar.

CHAPTER III. GENERALIZED COHOMOLOGIES. THE *K*-FUNCTOR AND THE THEORY OF BORDISMS. MICROBUNDLES.

§ 10. GENERALIZED COHOMOLOGIES. EXAMPLES.

We first recall what is meant in general by cohomology theories. The H^* are defined by the following properties:

1°. “Naturality” (functoriality) and homotopy invariance — to a homotopy class of maps of complexes $X \rightarrow Y$ there corresponds a homomorphism $H^*(Y) \rightarrow H^*(X)$, composition of maps corresponds to composition of homomorphisms and the identity map to the identity homomorphism.

2°. “Factorization” $H^*(X, Y) = \tilde{H}^*(X/Y)$, $Y \subset X$.

3°. “Exact sequence of a pair”.

4°. “Normalization” — one has to prescribe the cohomology of a point. (Absolute cohomology groups $H^i(X)$ should be considered as the corresponding $H^i(X \cup P, P)$.)⁵ It is important, in particular, that for a point $X = P$ we have $H^i(P) = 0$, $i \neq 0$; $H^0(P) = G$ is called the “coefficient group”.

Note the important property of cohomology that it is a “representable functor”, that is

$$H^i(X, Y; G) = \pi(X/Y, K(G, i)),$$

where $K(G, i)$ is an Eilenberg–MacLane complex for which $\pi_i(K(G, i)) = G$ and $\pi_j(K(G, i)) = 0$, $i \neq j$.

In topology it is well-known how these general statements are used in solving concrete problems. Homotopy theory, however, makes it possible to construct an unlimited number of algebraic functors of a similar kind, possessing all the above properties with the exception of the normalization property (among them also their representability as homotopy classes of maps).

If the normalization condition is added, then by a theorem of Eilenberg and Steenrod we get nothing other than ordinary cohomology.

A “generalized cohomology” is a representable functor for which the cohomology of a point is non-trivial in arbitrary dimensions. One can also define “generalized homologies”.

A “cohomology operation” is a mapping of such a functor into itself or into another similar functor (not always natural with respect to additive homomorphisms).

Even for ordinary cohomology there is a very large number of “cohomology operations”, for example the Pontryagin powers and the Steenrod powers. In the study of these operations and their applications their representability is very important. It seems that the first generalized cohomologies were introduced in particular cases by Atiyah and Hirzebruch in the solution of concrete problems, while the general interrelations were pointed out by Whitehead.

Consider a typical functor of this type. Let $F = \{X_n, f_n\}$ be a spectrum of spaces and maps $f_n: X_n \rightarrow \Omega X_{n+1}$ or $EX_n \rightarrow X_{n+1}$, where Ω is the Serre operation of taking loops on X_{n+1} .

Let K be an arbitrary complex. Maps

$$f_{n*}: \pi(K, X_n) \rightarrow \pi(K, \Omega X_{n+1}) \approx \pi(EK, X_{n+1}).$$

⁵Here and in what follows we shall be concerned with spaces with base point; in $X \cup P$ the base point is P .

are defined. Let $H_F^i(K, L) = \text{dir lim}_{n \rightarrow \infty} \pi(E^{n-i}K/L, X_n)$ (the limit being taken over f_{n*}). The groups $H_F^i(K, L)$ are defined for $-\infty < i < \infty$. We also set $H_F^i(K) = H_F^i(K \cup P, P)$, where P is a point.

What is the cohomology of a point in this case?

By definition

$$H_F^i(P) = H_F^i(P \cup P_1, P_1) = \text{dir lim}_{n \rightarrow \infty} \pi_{n+i}(X_n).$$

In all the “good” cases we have “stability”, that is for large n $\pi_{n+i}(X_n)$ only depends on i .

Thus, in every such case the computation of the cohomology of a point is the solution of a problem (as a rule a difficult one) in homotopy theory.

We set $H_F^*(K, L) = \sum H_F^i(K, L)$.

In many cases in such “generalized cohomologies” one has also the structure of a graded ring. This has not yet been axiomatized.

In most cases we have to deal with generalized cohomology rings.

There is an easily proved fact which sometimes makes it possible to compute “generalized cohomologies” if they are known for a point.

There exists a spectral sequence $\{E_r, d_r\}$, where $E_r = \sum_{p,q} E_r^{p,q}$, $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ and $E_2^{p,q} = H^p(K, L; H_F^q(P))$ (in the usual sense) and $\sum_{p+q=n} E_\infty^{p,q}$ is related to $H_F^n(K, L)$.

Example 1. $X_n = K(G, n)$, $\Omega X_n = X_{n-1}$; we get the ordinary cohomology theory. Here cohomology is trivial for negative dimensions.

Example 2. a) $X_{2n} = BU$, $X_{2n+1} = U$ ($U = \bigcup_n U_n$). Bott periodicity gives $X_n = \Omega X_{n+1}$. Let $H_F^i(K, L) = K_C^i(K, L)$, $K_C^* = \sum K_C^i$.

b) $X_{8n} = BO$, $X_{8n-1} = O$ ($O = \bigcup_n O_n$).

Then $X_{8n+4} = BSp$ and $X_{8n+3} = Sp$ ($Sp = \bigcup_n Sp_n$) (Bott periodicity). Let $H_F^i(K, L) = K_R^i(K, L)$, $K_R^* = \sum K_R^i$. The last two cohomology theories are called the K -theories (real and complex). We shall give here an alternative definition (Grothendieck).

Denote by $L(Y)$ the set of all vector bundles with base Y and by $L(Y)$ the free abelian group (with composition sign $+$) generated by $L(Y)$. Let

$$\eta_1 + \eta_2 = \eta_1 \oplus \eta_2, \quad \eta_1, \eta_2 \in L(Y).$$

The quotient group of $L(Y)$ by this equivalence relation is denoted by $K(Y)$, or sometimes in the real case by $K_R(Y)$ and in the complex case by $K_C(Y)$. K_R and K_C are rings with respect to \otimes . There is a dimension homomorphism

$$(\text{dim}): K_\Lambda(Y) \rightarrow Z, \quad \Lambda = R, C.$$

Let $K_\Lambda^0 = \text{Ker}(\text{dim})$. Let $K_\Lambda^{-i}(K, L) = K_\Lambda^0(E^i K/L)$, $\bar{K}_\Lambda^* = \sum K_\Lambda^i$, $i < 0$. By Bott periodicity $K_C^0(E^2 Y) = K_C^0(Y)$ and $K_R^0(E^8 Y) = K_R^0(Y)$, and therefore the definition extends to positive numbers $i > 0$, by periodicity.

Now let $K_\Lambda^* = \sum K_\Lambda^i$, $-\infty < i < \infty$. It can be shown that this new definition of K_R^* , K_C^* coincides with the original one.

For the form of the functor at a point it is sufficient by Bott periodicity to consider the structure of the ring only for the sum $\sum_{i \leq 0} K^i = \bar{K}^*$. We give the following generators and relations.

a) \bar{K}_C^* is generated by $u \in K_C^{-2}$ and $1 \in K_C^0$. The ring is a polynomial ring over u .

b) \bar{K}_R^* is generated by $1 \in K_R^0$, $u \in K_R^{-1}$, $v \in K_R^{-4}$, $w \in K_R^{-8}$; the relations are $2u = 0$, $u^3 = 0$, $uv = 0$, $v^2 = 4w$.

“Chern characters” defined by

$$\begin{aligned} \text{ch}: K_C(K, L) &\rightarrow H^*(K, L; Q), \\ \text{ch}: K_R(K, L) &\rightarrow H^*(K, L; Q), \end{aligned}$$

are ring homomorphisms. A new formulation of Bott periodicity is:

$$\begin{aligned} K_C(X \times S^2) &= K_C(X) \otimes K_C(S^2), \\ K_R(X \times S^8) &= K_R(X) \otimes K_R(S^8), \end{aligned}$$

An important fact is that there is a Thom isomorphism for the complex K -functor. Let η be a complex vector bundle with base X and T_η its Thom space. Then there is a Thom isomorphism:

$$\phi_K: K_C(X) \rightarrow K_C^0(X).$$

By contrast with ordinary cohomology, however, there are in this case many “Thom isomorphisms”. This has important consequences later on.

Example 3.

$$\begin{aligned} \{X_n = MO_n\}, & \quad \{X_{4n} = MSp_n\}, \\ \{X_n = MSO_n\}, & \quad \{X_n = S^n = Me_n\}, \\ \{X_{2n} = MU_n\}, & \quad e \subset O_n. \\ \{X_{2n} = MSU_n\}. & \end{aligned}$$

We call the corresponding cohomologies *cobordisms*. They admit also a good geometric interpretation, and there are very useful homology theories dual to them — *bordisms*.

For let us set $H_i^F(K, L) = \pi_{n+i}(X^n \# K/L)$, where $\#$ is the operation of multiplication followed by identifying to a point the coordinate axes. In particular, when $X_n = S^n$, we get the “stable” homotopy groups of the space K/L . For a manifold bordisms and cobordisms are related by a duality law of Poincaré type. The role of the “coefficient group”, that is the ring of bordisms of a point, is here played by the ring Ω_G , of especially good structure when $G = O_n$ or U_n .

Example 4. Let $H_n = (\Omega^n S^n)^0$. Then inclusions $H_n \subset H_{n+1}$ and $O_n \subset H_n$ are defined, since H_n consists of maps $S^n \rightarrow S^n$ of degree $+1$ preserving base point. H_n is an H -space.

Let⁶ $BH_n = (\widehat{\Omega^{n-1} S^n})$. Then (natural) maps $BO_n \rightarrow BH_n$ are defined; let $H = \bigcup_n H_n$; also let $BH = \lim_n BH_n$. Then a natural map $BO \rightarrow BH$ is defined. Let $BH = X_0$ and $X_{-i} = \Omega^i BH$. The groups $\pi_{i+1}(BH)$ are isomorphic to the stable homotopy groups of spheres of index i .

Let $\Pi^{-i}(K, L) = \pi(E^i K/L, BH)$, $i \geq 0$. It is easy to verify that $\Pi^{-i}(K, L)$ is a group for all i . A natural map

$$J: K_R^{-i}(K, L) \rightarrow \Pi^{-i}(K, L)$$

⁶The sign $\widehat{}$ on top here denotes the universal covering space.

is defined. The image JK_R^{-i} is called the J -functor $J(K, L)$. For $K = S^0$, $L = P$ we have

$$\begin{aligned} K_R^{-i}(K, L) &= \pi_i(BO) \approx \pi_{i-1}(O), \\ \Pi^{-i}(K, L) &= \pi_i(BH) = \pi_{N+i-1}(S^N), \end{aligned}$$

N large.

We have the classical J -homomorphism of Whitehead:

$$J: \pi_{i-1}(O_N) \rightarrow \pi_{N+i-1}(S_N).$$

The J -functor is very important in many problems of differential topology.

In this example all the properties of generalized cohomologies are present except that the spectrum X_n is only defined for $n \leq 0$ and therefore, generally speaking, only the cohomology of negative degree is defined. This is more frequently, though less efficaciously, the case in homotopy theory. We give another example of the same type.

Example 5. Consider “Milnor’s microbundles” in the combinatorial sense. A complex K lies piecewise-linearly in a complex $L \supset K$. To each (small) neighbourhood $U \subset K$, there exists a neighbourhood $V \subset L$ such that $V \cap K = U$ and V is combinatorially equivalent to $U \times R^n$ so that a “projection” $p: V \rightarrow U$ is given. On the intersection of two such neighbourhoods U_1 and U_2 the projections $p_{U_1}: V_1 \rightarrow U_1$ and $p_{U_2}: V_2 \rightarrow U_2$ agree on $V_1 \cap V_2$ (but the “dimension” of the fibres is not supposed to be unique). Two microbundles $\eta_1: K \subset L_1$ and $\eta_2: K \subset L_2$ are to be regarded equal if there are neighbourhoods $Q_i \subset L_i$ of the complex K in L_i combinatorially equivalent and preserving the “fibre structure” round K .

For microbundles there is a “Whitney sum”, a “structure group” PL_n and a universal bundle with base BPL_n . There are also inclusions $O_n \subset PL_n$ and $PL_n \subset PL_{n+1}$. Let $PL = \text{dir lim}_{n \rightarrow \infty} PL_n$ and define BPL analogously. A map $BO \rightarrow BPL$ is defined. Using the Whitney sum \oplus , one can introduce a “Grothendieck group” of microbundles $k_{PL}(K, L)$ and $k_{PL} = Z + k_{PL}^0(K, L)$. Analogously one introduces $k_{PL}^{-i}(K, L) = k_{PL}^0(E^i K/L)$. There is the Milnor homomorphism

$$j: K_R(K, L) \rightarrow k_{PL}(K, L);$$

there are also the concepts of “tangent” and “stable normal” microbundle of a combinatorial manifold.

The following facts are known:

- a) A combinatorial manifold is smoothable if and only if its “tangent bundle” $\eta(M^n) \in k_{PL}(M^n)$ belongs to the image of j (Milnor).
- b) The relative groups $\pi_i(PL, O)$ are isomorphic to Milnor’s groups of differentiable structures on spheres Γ^i (Mazur, Hirsch). In particular, the image of $\pi_7(0) = Z$ is divisible by 7 in the group $\pi_7(PL) = k_{PL}^0(S^8)$ (Kervaire). From this it can easily be deduced that for a suitably chosen manifold the homomorphism $j: K_R(M^n) \rightarrow k_{PL}(M^n)$ has kernel isomorphic to Z_7 . From this in its turn it follows that there exists smooth combinatorially identical manifolds with differing tangent bundles — and even differing 7-torsion in the Pontryagin class p_2 (Milnor). In this way the torsion of the Pontryagin classes is seen not to be a combinatorial invariant.
- c) Homotopy type and the tangent bundle (or its invariants, the rational Pontryagin classes) determine a combinatorial manifold modulo “a finite number of possibilities” if $\pi_1 = 0$ and the dimension is ≥ 5 (Novikov).

One may consider the corresponding J -functor. A homomorphism of functors

$$\begin{array}{ccc} k_{PL}^{-i}(K, L) & \xrightarrow{J_{PL}} & \Pi^{-i}(K, L). \\ & \swarrow j & \nearrow J \\ & K_R^{-i}(K, L) & \end{array}$$

is defined. This triangle is commutative, that is $J = J_{PL} \circ j$. If $L = P$, $K = S^0$, then $k_{PL}^{-i}(K, L) = \pi_i(BPL)$. We have

$$\begin{array}{ccc} \pi_{i-1}(PL) & \xrightarrow{J_{PL}} & \pi_{N+i-1}(S^N), \\ & \swarrow j & \nearrow J \\ & \pi_{i-1}(0) & \end{array}$$

where N is large. Note that the map j is a monomorphism for $K = S^0$, $L = P$ (Adams). One can consider a natural “fibration”

$$p: BH \rightarrow X_{SO}$$

with fibre BSO . There corresponds the exact sequence

$$K_R^{-1}(M^n) \xrightarrow{J} \Pi^{-1}(M^n) \xrightarrow{\kappa} \Gamma_{SO}(M^n) \xrightarrow{\delta} K_R^0(M^n) \xrightarrow{J} \Pi^0(M^n).$$

Here M^n is a closed simply-connected smooth manifold, and $\Gamma_{SO}(M^n)$ denotes $\pi(M^n, \Omega X_{SO})$. We denote by $\pi^+(M^n)$ the group of homotopy classes of maps of M^n onto itself of degree $+1$ preserving the stable “tangent” element. $-n \in K_R^0(M^n)$, where $n \in K_R^0$ is the stable “normal” element. Then the group $\pi^+(M^n)$ acts somehow on the groups which enter here. Novikov’s result on the diffeomorphism problem can be formulated as follows: for $n \neq 4k + 2$ the set of manifolds, having the same homotopy type and tangent bundle as a given manifold M^n , is the set of orbits of the group $\pi^+(M^n)$ on $\text{Im } \kappa = \Pi^{-1}(M^n)/JK_R^{-1}(M^n)$, if these manifolds are identified modulo the group of differential structures on the sphere $\theta^n(\partial\pi)$. For $n = 4k + 2$ the additive Arf-invariant $\phi: \text{Im } \kappa \rightarrow Z_2$ is defined on the group $\text{Im } \kappa$ and in place of $\text{Im } \kappa$ one has to take $\text{Ker } \phi$. The role of the group $\text{Ker } J = \text{Im } \delta$ will be indicated in Chap. IV, §12. There is an analogous result in the piecewise linear case with k_{PL}^{-1} replacing K_R^{-1} and BPL replacing BSO . It is interesting to note that although this result is equivalent in form to the old one yet in other (non-stable) problems, concerning the type of n -dimensional knots to be precise, the corresponding statement only approximates the correct one, as J. Levine has shown for embeddings⁷ of the sphere S^n in S^{n+k} . This result of Levine’s is not yet published; the possibility of interpreting the answer as an approximation has also been pointed out to me by A.S. Shvarts. It is interesting to note that in the example of the homotopy type of S^n , where the diffeomorphism classification was earlier obtained by Milnor and Kervaire, the group structure arises on account of the fact that $\pi^+(M^n) = 1$, if we look at things from the point of view of the general theorem. The group $\Gamma_{SO}(M^n)$ brings together a number of problems on diffeomorphisms — the subgroup $\text{Im } \kappa \subset \Gamma_{SO}$, with the problem of normal (tangent)

⁷Here in place of H_n one has to take G_n , consisting of the maps $S^{n-1} \rightarrow S^{n-1}$ of degree $+1$ without fixed point.

bundles (cf. Chap. IV, §12), being related to the set $\{n + \text{Ker } J\} \subset K_R^0(M^n)$, $\text{Ker } J = \text{Im } \delta = \Gamma_{SO}/\text{Im } \kappa$.

CHAPTER IV. SOME APPLICATIONS OF THE K - AND J -FUNCTORS AND BORDISM THEORIES

§ 11. STRICT APPLICATION OF K -THEORY

Atiyah and Hirzebruch have proved a number of theorems generalizing to the case of differential manifolds Grothendieck's form of the Riemann–Roch theorem.

Consider two manifolds $M_1^{n_1}$ and $M_2^{n_2}$, $n_1 - n_2 = 8k$, and a map $f: M_1 \rightarrow M_2$. We shall suppose that the manifolds are oriented. Fix (if possible) elements $c_1 \in H^2(M_1, \mathbb{Z})$ and $c'_1 \in H^2(M_2, \mathbb{Z})$ such that $c_1 \bmod 2 = w_2$ and $c'_1 \bmod 2 = w_2$. Let $f^*c'_1 = c_1$. Denote by \bar{f}_* the map $Df_*D: H^*(M_1) \rightarrow H^*(M_2)$, D being the Poincaré duality operator. Let $\zeta \in K_R(M_1^{n_1})$. Then we have the following fact (“the Riemann–Roch theorem”):

There exists an additive map $f_!: K_R(M_1) \rightarrow K_R(M_2)$ such that

$$\bar{f}_*(\text{ch } \zeta A(M_1)) - \text{ch } f_!\zeta A(M_2).$$

There is an analogous theorem for maps of quasicomplex manifolds, but here one can dispense with the condition $n_1 - n_2 = 8k$, while the T -genus replaces the A -genus. One should note that an important part in the theorems is played by, first, Bott periodicity in terms of K -theory and, secondly, the Thom isomorphism of K -theory. The situation is that in K -theory there is a Thom isomorphism in the following cases:

a) for the complex K -functor

$$\phi_K: K_C(X) \rightarrow K_C^0(T_\eta),$$

where η is a U_N -bundle over X ,

b) for the real K -functor

$$\phi_K: K_R(X) \rightarrow K_R(T_\eta),$$

where η is a $Spin$ -bundle over X ($w_1 = w_2 = 0$).

However, this time there are many Thom isomorphisms. One should notice that the Thom isomorphism may be chosen such that the following conditions hold:

$$\begin{aligned} \text{ch}(\phi_K \alpha) &= T(\eta)\phi(\text{ch } \alpha) \quad (\text{case a}), \\ \text{ch}(\phi_K \alpha) &= A(\eta)\phi(\text{ch } \alpha) \quad (\text{case b}), \end{aligned}$$

where ϕ_K is the chosen Thom isomorphism in K -theory and ϕ is the standard Thom isomorphism in ordinary cohomology. One should notice that it is sufficient to construct these Thom isomorphisms for universal bundles, depending only on the invariants of the Lie groups and their representations.

In this context the T - and A -genera therefore arise out of the commutativity law for the Chern character with the (chosen) Thom isomorphism. Other “universal” Thom isomorphisms may lead to other “multiplicative genera” and “Riemann–Roch theorems”. The integrality of the A -genus is obtained here (as in the algebraic Grothendieck theorem) if M_2 is taken to be a point. Of course, these theorems are not given here in their general form, nor do we give a number of interesting corollaries.

Another important result has been obtained in a neighbouring field by Atiyah and Hirzebruch. This concerns the “index” of an elliptic operator on a manifold without boundary. To an elliptic operator (defined and taking its values on the sections of bundles F_1 and F_2 over X^{2l}) there correspond a “symbol” $\sigma: \pi^*F_1 \xrightarrow{\approx} \pi^*F_2$, where $\pi: S_X \rightarrow X$ is the natural fibration of the manifold of unit tangent (co-)vectors over X and X is a Riemannian manifold. The index of the operator depends in fact only on the homotopy class of the symbol and is trivial if the isomorphism σ induces an isomorphism $(\pi\sigma): F_1 \rightarrow F_2$ of fibre bundles over X (Vol’pert, Dynin). In an elegant way one constructs an “invariant” $\alpha(\sigma) \in K_R^0(T_\eta)$, where η is the tangent bundle on X and the index is a homomorphism

$$I: K_R^0(T_\eta) \rightarrow Z.$$

Moreover, by a theorem of Cartan–Serre the index depends only on $\text{ch } \alpha(\sigma) \in H^*(T_\eta, Q)$. One constructs a “special Thom isomorphism”

$$\phi_K^{-1}: K_R^0(T_\eta) \otimes Q \rightarrow K_R(X) \otimes Q,$$

and the index depends only on $\text{ch}(\phi_K^{-1}\alpha(\sigma))$.

The situation is therefore reduced to pairs consisting of a manifold X and a vector bundle ζ over X with a special operator (“the Hirzebruch index”), whose Chern character is easily computed — such operators give a “complete” set for the index problem.

Moreover, an important theorem, the “intrinsic homological invariance of the index, has been proved in a convenient cobordism form, and the solution of the problem is easily accomplished on the basis of Thom’s theory. The final formula is

$$I(\sigma) = ((\phi^{-1} \text{ch } \alpha(\sigma))T(c\eta), [X]).$$

For example, the A -genus is the index of the “Dirac operator” of a spinor structure.

Particularly interesting results have also been obtained by Atiyah in the theory of smooth embeddings. A striking example is the application of these general theorems to the embeddings of the complex projective spaces CP^n , which cannot be embedded in spaces of dimension less than (for example) $2(n - \beta(n))$, where $\beta(n)$ is the number of 1’s in the binary representation of n . Interesting links have also been found between the K_C -functor, the cohomology of finite groups and the “representation rings”. Indeed, in the most recent papers cohomology operations in the K -functor are used, though not quite explicitly.

§ 12. SIMULTANEOUS APPLICATIONS OF THE K - AND J -FUNCTORS.
COHOMOLOGY OPERATION IN K -THEORY

We state to begin with some “general” theorems on the connection of the K - and J -functors with topological problems. It is easy to see that the normal bundle η of a smooth closed manifold X has a Thom complex T_η with spherical fundamental cycle (the Thom complex is “reduced”), while the Thom complex of a trivial bundle of dimension N over X is the bouquet of a sphere S^N and the suspension $E^N X$, that is, it is “coreduced”. Moreover for any two elements $\alpha_1, \alpha_2 \in K(X)$ possessing either of these properties simultaneously $J(\alpha_1) = J(\alpha_2)$, where $J: K_R(X) \rightarrow J(X)$, that is, these properties are J -invariant. Atiyah noticed the essential fact that the Thom complexes of an element $\alpha + N$, $\alpha \in K_R^0$ and $(n - \alpha) + N_1$, where n is the representative in $K_R^0(X)$ of the normal bundle, are “ S -dual” to each other; he proved that the “reducibility” of the Thom complex of the element α is equivalent

to the statement that this element is equal to the normal bundle in the J -functor, that is, the set of “reducible” elements of $K_R^0(X)$ is exactly $J^{-1}J(n)$, where $n \in K_R^0(X)$ is the normal bundle. Similarly for the “tangent” bundle $(-n) \in K_R^0(X)$. For simply-connected odd-dimensional manifolds of dimension ≥ 5 Novikov and Browder proved the converse “realization theorem”, namely that every element $(-n_i) \in J^{-1}J(-n)$ is the “tangent bundle” of some manifold X_i^k of homotopy type X^k ; for even k the formulation is more complicated; it is final only for $k \equiv 0 \pmod 4$, when the “tangent bundles” consist of all the elements $-n_i \in J^{-1}J(-n)$, satisfying the “Hirzebruch condition” $(L_q, [X^{4q}]) = \tau(X^{4q})$.

It is interesting to note that for a non-simply-connected manifold of dimension $4k + 1$ this theorem is not true by the formula for the rational class $L_k(M^{4k+1})$ given in the Appendix.

From the finiteness of the J -functor and from these theorems it is clear that for simply-connected manifolds one can vary the tangent bundle (and the Pontryagin classes) very freely within a given boundary type. There are analogous theorems for combinatorial manifolds also — one has to consider k_{PL}^0 , in place of K_R^0 and J_{PL} in place of J . A combinatorial manifold has the homotopy type of a smooth manifold if (under analogous homotopy restrictions) its “normal microbundle” $n \in k_{PL}^0(X)$ is such that there is in the set $J_{PL}^{-1}J_{PL}(n)$ an element of the image $jK_R^0(X) \subset k_{PL}^0(X)$, if the dimension is odd (for even dimensions it is, as before, more complicated). For example for $M^6 = S^2 \times S^4$ the “tangent” elements are all $\tau_i \in K_R^0(M^6)$ such that $w_2 = 0$ and $p_1 = 48\lambda u$, where λ is an arbitrary integer and u is the basis element of $H^4(M^6) = \mathbb{Z}$. There is a family of manifolds M^6 homotopically equivalent to $S^2 \times S^4$ and with class $p_1 = 48\lambda u$. It is interesting that it follows from the latest paper of Novikov that the class $p_1(M^6)$ is topologically invariant; we get a proof of the difference of homeomorphism and homotopy type of closed simply-connected manifolds. For $n > 3$ no example of this, even non-simply-connected, was known.

Finally, the following lemma due to Adams and Atiyah turns out to be very important in further applications: if $K_R^0(RP^n) = J^0(RP^k)$ for all dimensions, then on all spheres S^n there are exactly $\rho(n) - 1$ linearly independent vector fields and no more, where $\rho(n)$ is defined as follows: if $n + 1 = 2^b(2a + 1)$, $b = c + 4d$, $0 \leq c \leq 3$, then $\rho(n) = 2^c + 8d$. Note that this number of fields was already known classically and had been proved in a number of cases. In particular, Toda had also noted this lemma in another context, and had computed the number of fields where this was possible, enabling him to compute the homotopy groups of spheres and the classical J -homomorphism on this foundation (for example for $k \leq 19$ and $n \leq 2^{11}$). Adams has solved the j -functor problem completely in this case for all k . We give the basic outline of his method.

It had already been observed (Grothendieck, Atiyah) that there were “operations” in K -theory, related to the exterior powers. They are denoted by λ_i and possess the property

$$\lambda_i(x + y) = \sum_{j+l=i} \lambda_j(x)\lambda_l(y).$$

However, these operations are non-additive and it is therefore difficult to apply them. Adams was led to introduce operations $\psi_\Lambda^i: K_\Lambda(X) \rightarrow K_\Lambda(X)$, (expressed in terms of the λ_i) which were ring homomorphisms and such that $\psi_\Lambda^i \psi_\Lambda^j = \psi_\Lambda^{ij}$, where $\psi_\Lambda^0 = (\dim)$, ψ_C^{-1} — is complex conjugation and $\psi_R^{-1} = \psi_R^1 = 1$. Moreover, $\psi_\Lambda^i(x) = x^i$, if x is a one-dimensional Λ -bundle. Note that $\text{ch}^n \psi_C^k \eta = k^n \text{ch}^n \eta$.

These operations are remarkable from the point of view of the usefulness of their application. It is rather easy to compute the K_C -functor with its operations for CP^n and CP^n/CP^{n-k-1} , rather more difficult to compute the K_R -functor with its operations for RP^n and RP^{n-k-1} . Adams did this and the desired result about $J(RP^k)$ followed almost immediately from the answer. From this it also follows that the classical J -homomorphism

$$J \otimes Z_2: \pi_{i-1}(O_N) \otimes Z_2 \rightarrow \pi_{N+i-1}(S^N) \otimes Z_2$$

is always a monomorphism, and this implies the topological invariance of the tangent bundle of a sphere S^i for $i \equiv 1, 2 \pmod 8$ (for $i \not\equiv 1, 2 \pmod 8$ the invariance was known). In a number of cases (for example for $X = S^{4n}$) Adams has succeeded by other methods in giving an upper bound on the order of the J -functor: to be precise, if $x \in K_C(X)$ then $J(k^N(\phi_C^K - 1)x) = 0$ for large N and for all k . This gives a complete or almost complete answer for S^{4n} . To obtain similar estimates in the general case is an interesting problem.

The introduction of the operations ψ^k made it possible to introduce new “characteristic classes” into K -theory: let $x \in K_R^0(X)$ and let the Thom complex of the bundle $\eta = x + N$ possess a Thom isomorphism $\psi_K: K_R(X) \rightarrow K_R^0(T_\eta)$. Let $\rho_l(x) = \phi_K^{-1}\psi_R^l\phi_K(1)$ by analogy with the Stiefel classes. Since ϕ_K is not uniquely determined in K -theory these classes ρ_l are not uniquely defined but the degree of indeterminacy is easily computed. These classes are very useful for estimating the order of $J^0(X)$ from below (Adams, Bott).

§ 13. BORDISM THEORY

In Chapters I and II we have already spoken of cobordism groups and rings and have pointed out the general cohomological and homological theories connected with them (the first cohomological approach here, as in K -theory, was made by Atiyah). This theory was developed by Conner and Floyd in connection with problems on the fixed points of transformations; recently the theory has been used to obtain a number of other results, among which one should note the work of Brown–Peterson on the relations between the Stiefel–Whitney classes of closed manifolds and also the results on the groups Ω_{SU}^i , obtained simultaneously by a number of authors (Conner–Floyd, Brown–Peterson, Lashof–Rothenberg). In particular in the middle dimension Brown–Peterson and Lashof–Rothenberg have with the help of the group Ω_{SU}^{8k+2} solved the well-known problem about the Arf-invariant of Kervarie–Milnor in the theory of differential structures on spheres.

a) Brown and Peterson study the following ideal:

$$\lambda_i(x + y) = \sum_{j+l=i} \lambda_j(x)\lambda_l(y),$$

where $\tau_{M^n}: M^n \rightarrow BO$ is the classifying map for the (stable) tangent bundle of M^n . By using O -bordism theory it has been proved that this ideal consists only of those elements “trivially” belonging to it purely algebraically, according to the formulae of Thom and Wu. Analogous results, more complicated to formulate, have been obtained for the ideal $I_n(SO, 2)$. In the proof a “right” action of the Steenrod algebra in the category of complexes and bundles is introduced and studied; there is an elegant treatment of the Thom–Wu formulae and an isomorphism of cohomology theories

$$J \otimes Z_2: \pi_{i-1}(O_N) \otimes Z_2 \rightarrow \pi_{N+i-1}(S^N) \otimes Z_2$$

is constructed, $N_*(X)$ being the bordism group; it is sufficient to verify this isomorphism for points only. It is then applied to $X = K(Z_2, m)$ and this gives the result.

b) In the work of Conner and Floyd U -bordism theory is applied to the study of the number of fixed points of involutions of a quasicomplex manifold onto itself. More precisely, they study a quasicomplex involution $T: M^{2n} \rightarrow M^{2n}$ such that its set of fixed points decomposes into a union $V = M^0 \cup M^2 \cup \dots \cup M^{2n-2}$ of quasicomplex submanifolds in whose normal spaces “reflection” (in the form of non-degeneracy) occurs. Cutting out an invariant small neighbourhood B of V from M^{2n} and setting

$$N = (M \setminus B)/T,$$

we see that

$$\partial N = \partial B/T$$

and that N is quasicomplex.

Consider now the “bordism” group $U_*(RP^\infty)$ to which we relate the group $\sum_{i,k} H_k(RP^\infty, \Omega_U^i)$. The ring Ω_U^* is therefore a polynomial ring over Z , having one generator in each dimension $2m$, this generator being CP^1 when $m = 1$. Thus,

$$I_n(O, 2) = \bigcap_{M^n} \text{Ker } \tau_{M^n}^*,$$

while the “geometrical” generator is $(RP^{2k-1} \times M^{2l}, f)$, where M^{2l} is the generator of Ω_U^{2l} and f is the projection $RP^{2k-1} \times M^{2l} \rightarrow RP^\infty$. The original involution determines a “relation”

$$\partial(N, F), F: N \rightarrow RP^\infty,$$

where N was defined above and $F: N \rightarrow RP^\infty$ is obtained by factorizing with respect to T the map $M \setminus B \rightarrow S^\infty$, which commutes with the involutions. Thus, there is a correspondence between involutions on arbitrary manifolds M^{2n} and relations between the basis elements in $U_*(RP^\infty)$, arising by pullback from the group $\sum H_K(RP^\infty, \Omega_U^*)$ related to $U_*(RP^\infty)$.

There is an elegant proof of the following relation:

$$2(RP^{2k-1} \otimes 1) = RP^{2k-3} \otimes CP^1,$$

where RP^{2i-1} are the basis cycles in RP^∞ and 1 and CP^1 are the generators of Ω_U^0 and Ω_U^2 (in fact, all other relations follow from this one).

Since CP^1 is a polynomial generator in Ω_U^* , that is, the powers $(CP^1)^m$ are irreducible in Ω_U^* , it follows that if the fixed points of an involution are isolated, for example, for a complex manifold M^{2n} (the dimension is real, the involution quasicomplex) then their number is divisible by 2^n , since $\partial(N, F)$ in this case is

$$\sum RP^{2n-1} \otimes 1 = \lambda RP^{2n-1} \otimes 1,$$

and λ must be divisible by 2^n since

$$\lambda RP^{2n-1} \otimes 1 = 0.$$

This is the main result, but if one wishes to give a more general formulation, concerned not only with the zero-dimensional case, then in the case when the normal bundles of the manifolds $M^{2i} \subset M^{2n}$ are trivial it can be expressed as follows:

Let $[M^{2i}]$ denote the class of M^{2i} in Ω_U^{2i} and let x be the element $\frac{CP^1}{2} \in \Omega_U^* \otimes Q$. Then the element

$$\frac{1}{2}(x^{n-1}[M^0] + x^{n-2}[M^2] + \dots + [M^{2n-2}])$$

is “integral” in $\Omega_U^* \otimes Q$, that is, belongs to Ω_U^* .

c) Another beautiful application of the general theory of U - and SU -bordisms is the final computation of the 2-torsion in the ring Ω_{SU}^* (Conner–Ployd, Lashof–Rothenberg, Brown–Peterson). Here connections have successfully been found between the U - and SU -bordisms of different objects. From their results it follows, for example, that all the 2-torsion in Ω_{SU}^k is $\Omega_{SU}^1 \Omega_{SU}^{k-1}$; for even k this was not known and the old methods led to serious difficulties. As has already been pointed out, these methods of studying the ring Ω_{SU}^* led to the solution of the Arf-invariant problem in the middle dimension.

The most important unsolved problems of this theory are the study of the multiplicative structure of the ring Ω_{SU}^* and also of the ring Ω_{Spin}^* on which little is known is “modulo 2”. It would be useful to compute “bordisms” and “cobordisms” for a much wider class of spaces, as this would widen greatly the possibilities of application. Note, for example that the U_* -theory is contained as a direct component of the K_C -theory.

The large number of interconnections introduced into topology by the new outlook on homology theory is apparently such that it would be impossible to describe beforehand the circle of problems that will be solved in this region even in the relatively near future.

APPENDIX

THE HIRZEBRUCH FORMULA AND COVERINGS

Novikov has found an analogue to the Hirzebruch formula, relating the Pontryagin classes to the fundamental group. Let M^{4k+n} be a smooth (or PL) manifold and let $x \in H_{4k}(M^{4k+n}, Z)$ be an irreducible element, such that $Dx = y_1, y_2, \dots, y_n \text{ mod Tor}$, $y_i \in H^1$, D being Poincaré duality. Consider a covering $p: \hat{M} \rightarrow M^{4k+n}$, for which there are paths γ , covered by closed by paths, such that $(\gamma, y_i) = 0$ ($i = 1, \dots, n$). Let $\hat{x} \in H_{4k}(\hat{M}, Z)$ be an element such that $p_*\hat{x} = x$, \hat{x} being invariant with respect to the monodromy group of the covering. It is unique up to an additive algebraic restriction on this element. Let $\tau(\hat{x})$ be the signature of the quadratic form (y^2, x) , $y \in H^{2k}(\hat{M}, R)$, the non-degenerate part of which is finite-dimensional.

For $n = 1$, and also for $n = 2$ provided that $H_{2k+1}(\hat{M}, R)$ is finite-dimensional, it has been proved that $(L_K(M^{4k+n}), x) = \tau(\hat{x})$, which already in these cases leads to a number of corollaries and also has application to the problem of the topological invariance of the Pontryagin classes even for simply-connected objects.

Note in proof. The author has recently completed the proof of the topological invariance of the rational Pontryagin classes (see [46]).

Translator’s note. In [46] Novikov states that the topological invariance is an easy consequence of the following fundamental lemma of which he sketches the proof:

Lemma. *Suppose that the Cartesian product $M^{4k} \times R^m$ has an arbitrary smooth structure, turning the product into an open smooth manifold W , M^{4k} being a compact closed simply-connected manifold. Then $(L_k(W), [M^{4k}] \otimes 1) = \tau(M^{4k})$, where the $L_k(W)$ are the Hirzebruch polynomials for the manifold W and $\tau(M^{4k})$ is the signature of the manifold M^{4k} .*

SOME POINTERS TO THE LITERATURE

The literature is distributed over the various sections in the following manner:

- §§1–3 — [1],
- §4 — [2], [15],
- §5 — [3]–[13],
- §6 — [14], [15],
- §7 — [16]–[21],
- §8 — [3], [6], [11], [12], [28].
- §9 — [2], [22]–[24], [33],
- §10 — [25]–[33], [35], [40],
- §11 — [29], [34]–[38],
- §12 — [28], [33], [35], [39]–[41],
- §13 — [42]–[45],
- Appendix — [39], [46].

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