S.P. Novikov - MATH 740

Spring 2014

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Contents

Lecture 1. Introduction: Textbooks and General Remarks. Local coordinates: What is Cartesian system of coordinates. Examples.

Textbooks:
Manfredo P. do Carmo. *Riemannian Geometry.*
Victor Guillemin, Alan Pollack. *Differential Topology.*

Additional Literature:
J. Milnor. *Morse Theory.*
S.P. Novikov, I.A. Taimanov. *Modern Geometric Structures And Fields.*
B.A. Dubrovin, A.T. Fomenko, S.P. Novikov. *Modern Geometry - Methods and Applications: Parts I, II.*

Differential Manifolds, definition, maps, submanifolds.
Language of general topology is necessary: spaces, continuous maps, homeomorphisms, compactness, metric and Hausdorff spaces.
Basic Tools from multivariable calculus: **Implicit Functions**, **Approximations** and **Transversality**. Theorems will be stated (but without proof).
Knowledge of **Linear Algebra** is necessary.
In Riemannian Geometry some theorems from ODE courses will be needed.
Our theory is C^∞: - manifolds, maps, ... Why?
The **"physical" metric** in the 4-space-time is **NOT RIEMANNIAN**. Why do we need Riemannian metrics?
Concerning Differential Topology: Why do we need Approximations and Transversality?
What is MANIFOLD?

Definition: manifold is a Hausdorff (or metric) space locally homeomorphic to an open domain in $\mathbb{R}^n$.

$\mathbb{R}^n$ is an n-manifold.

Open domain $U \subset \mathbb{R}^n$ is a manifold.

What is a coordinate system?

a) Every coordinate is a continuous function on the space X

$$f : X \to \mathbb{R}$$

b) Collection of functions $f_1, \dots, f_n$

$$f_j : X \to \mathbb{R}$$

gives a coordinate system if for every point $x \in X$ we have

$$\vec{f}(x) = \vec{f}(y) \quad \to \quad x = y$$

where $\vec{f}(x) = (f_1(x), \dots, f_n(x))$.

Map $x \to \vec{f}(x)$ gives homeomorphism of X into some open domain $U \subset \mathbb{R}^n$.

Examples.

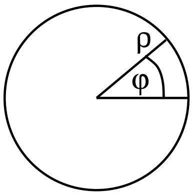
1) $X = \mathbb{R}^2, \;\; f_1 = x, \;\; f_2 = y$.

2) $X = \mathbb{R}^2, \;\; f_1 = \rho = \sqrt{x^2 + y^2}, \;\; f_2 = \varphi$.

a) ρ is **not** a coordinate in $\mathbb{R}^2$

but ρ is a coordinate in $\mathbb{R}^2 \backslash 0 \; = \; X'$

b) φ is **not** a coordinate in $\mathbb{R}^2 \backslash 0$ because φ is **not** a function, it is multivalued.



"Cartesian Coordinates" in $\mathbb{R}^n$

$$\mathbb{R}^n \; \leftrightarrow \; (x^1, \dots, x^n)$$

points $\to$ one-to-one with n-tuples $(x^1, \dots, x^n)$.

"Cartesian Coordinates" in open set $U \subset \mathbb{R}^n$.

Manifold $=$ Metric space M^n locally homeomorphic to open domains in $\mathbb{R}^n$ $\leftrightarrow$ for every point $x \in M^n$ there exists an open set $x \in U \subset M^n$ such that local coordinates (Cartesian) are given in U

$$U \; \to \; \mathbb{R}^n$$

$$x \; \to \; \big(x^1(x), \dots, x^n(x)\big) \; = \; \vec{x}(x)$$

$x^j : U \to \mathbb{R}$ are continuous functions (one-valued!)

$$\vec{x}(x) = \vec{x}(y) \quad \leftrightarrow \quad x = y$$

Lecture 2. Manifolds and Atlases.

Manifolds: = Hausdorff (or metric) spaces such that they are "locally euclidean": for every $x \in M$ there exists an open set $x \in U$ with **homeomorphism**

$$\varphi_U \;:\; U \to \mathbb{R}^n = (x^1, \ldots, x^n) \qquad (\text{so } U \subset \mathbb{R}^n)$$

The set U represents a "Chart" in the "Atlas" on M. "Local coordinates" in U (near x)

$$x \;\to\; \mathbb{R}^n \;\underset{x^i}{\longrightarrow}\; \mathbb{R}$$

are continuous functions in U and

$$\vec{x}(x) \;=\; \vec{x}(y) \;\leftrightarrow\; x \;=\; y$$

for any two points x, y in U.

For a given "Atlas" $\{U_\alpha\}$, covering M, we can introduce "Transition Maps" in the intersections $U \cap V$, where $U = U_\alpha$, $V = U_\beta$. Thus, for any $x \in U \cap V$ we can use the local coordinates $(x^1, \ldots, x^n)$ ($\vec{x}_\alpha$ in U) or $(y^1, \ldots, y^n)$ ($\vec{x}_\beta$ in V). The functions $x^i(y^1, \ldots, y^n)$ and $y^k(x^1, \ldots, x^n)$ represent maps of euclidean domains

$$x^i(\boldsymbol{y}),\;\; i = 1, \ldots, n, \qquad\quad y^k(\boldsymbol{x}),\;\; k = 1, \ldots, n$$

Definition. Manifold M is C^∞ if all the functions $x^i(\boldsymbol{y})$ (given by Atlas for every pair U, V) are C^∞.
Statement. In the C^∞ Atlas all Jacobians $\det|\partial x^i/\partial y^k|$ are **non-zero**.

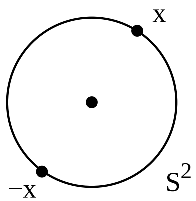
Proof. Since $\vec{x}(\vec{y})$ and $\vec{y}(\vec{x})$ are both C^∞ we have $\det|\partial x^i/\partial y^k| \neq 0$.

$$\sum_k \frac{\partial x^i}{\partial y^k}\frac{\partial y^k}{\partial x^j} \;=\; \delta^i_j$$

Summation agreement: we do not write $\sum$, so, in our notations

$$\frac{\partial x^i}{\partial y^k}\frac{\partial y^k}{\partial x^j} \;=\; \delta^i_j$$

Definition: Oriented Atlas is such that all $\det|\partial x^i/\partial y^k| \;>\; 0$
Oriented manifold: = there exists an oriented Atlas.
Examples:



$n = 2$:
$\mathbb{S}^2$ is oriented manifold.
$\mathbb{RP}^2$ is NOT.
$(x, -x)$ **is one point in** $\mathbb{RP}^2$.

Definitions.

1) C^∞ - function in C^∞ - manifold M with given **Atlas of Charts**: $M \to \mathbb{R}$ is C^∞ in every **Chart**.

2) C^∞ - map

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ (U_\alpha) & & (V_\beta) \end{array}$$

is C^∞ in every Chart. In other words, for every pair U_α, V_β the corresponding functions $y^k_\beta(x^1_\alpha, \ldots, x^n_\alpha)$ are C^∞ in $F^{-1}(V_\beta) \cap U_\alpha$.

Rank of the map $F : M \to N$ at the point $x \in M$:

$$\mathrm{rk}_x F = \mathrm{rk}\left(\frac{\partial y^k_\beta}{\partial x^i_\alpha}\right)\Bigg|_x \quad \text{where} \quad \begin{array}{ccc} U_\alpha & \xrightarrow{F} & V_\beta \\ (x) & & (y) \end{array}$$

Statement. Rank of the map at the point x does not depend on the choice of Atlas.

Proof. Let U_α, V_β be Euclidean domains giving the charts of the manifolds M and N and the functions $\vec{y}(\vec{x})$ be represented by the map

$$\begin{array}{ccc} U_\alpha & \xrightarrow{F} & V_\beta \\ (x) & & (y) \end{array}$$

Consider two other domains W_α and X_β with coordinates $\tilde{x}$ and $\tilde{y}$ representing two other charts containing the points x and $F(x)$ respectively. Let us consider the functions $\tilde{y}(\tilde{x}) = \tilde{y}(y(x(\tilde{x})))$ defined by the map

$$\begin{array}{ccccccc} W_\alpha & \to & U_\alpha & \xrightarrow{F} & V_\beta & \to & X_\beta \\ (\tilde{x}) & & (x) & & (y) & & (\tilde{y}) \end{array}$$

We have

$$\frac{\partial \tilde{y}^i}{\partial \tilde{x}^j} = \frac{\partial \tilde{y}^i}{\partial y^k}\frac{\partial y^k}{\partial x^s}\frac{\partial x^s}{\partial \tilde{x}^j} ,$$

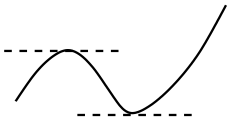
where $\mathrm{rk}\,|\partial \tilde{y}^i / \partial y^k| \neq 0, \ \mathrm{rk}\,|\partial x^s / \partial \tilde{x}^j| \neq 0$.

Conclusion

$$\mathrm{rk}\left|\frac{\partial \tilde{y}^i}{\partial \tilde{x}^j}\right| \;=\; \mathrm{rk}\left|\frac{\partial y^i}{\partial x^j}\right|$$

Statement is proved.

Special Case: $M \xrightarrow{F} \mathbb{R}$ (function).



$$\mathrm{rk}_x f \;=\; 1 \;\;-\;\; \text{regular point } \;(\nabla f|_x \neq 0)$$

$$\mathrm{rk}_x f \;=\; 0 \;\;-\;\; \text{critical point } \;(\nabla f|_x = 0)$$

Example: $f \;=\; x^2+y^2$: $(x, y) \neq (0, 0)$ - regular point, $(0, 0)$ - critical point.

Statement. Every local coordinate x^i_α in the Atlas of Charts for C^∞ - manifold $M \;=\; \cup U_\alpha$ is such that all the points in the Chart U_α are **regular** for $x^i_\alpha : U_\alpha \to \mathbb{R}$.

Proof. In the coordinate system $\vec{x}_\alpha$ in the Chart U_α we have $\nabla x^i_\alpha = (0, \ldots, 1, 0, \ldots, 0) \;\neq\; 0$.

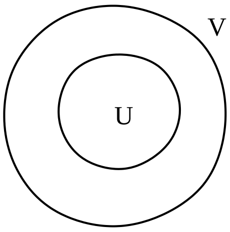
Statement is proved.

"Good Double Atlas" (GDA):

$$M \;=\; \cup_\alpha U_\alpha \;=\; \cup_\alpha V_\alpha$$

where $U_\alpha \subset V_\alpha$ and there are common coordinates $\vec{x}_\alpha$ for every U_α, V_α such that the corresponding domains U_α, V_α in the Euclidean space are defined by the relations:

$$U_\alpha : \;\sum_j (x^j_\alpha)^2 \;<\; 1 \;\;, \qquad V_\alpha : \;\sum_j (x^j_\alpha)^2 \;<\; 2 \;\;.$$



Lemma. For every compact manifold M there exists a GDA.

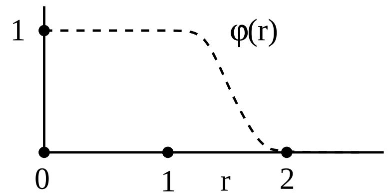
Proof (for compact M). For every point $x \in M$ we can obviously choose "small balls" U_x, V_x, $x \in U_x \subset V_x$ with the required local coordinates.

After that we chose a **finite cover** of M which gives the required GDA.

Lemma is proved.

Choose a C^∞ - function $\varphi(r)$, $r^2 = \sum_j |x^j|^2$, such that:

$\varphi \equiv 0$ for $r \leq 0$,
$\varphi \equiv 1$ for $r \leq 1$,
$\varphi \equiv 0$ for $r \geq 2$,
$\varphi' < 0$ for $1 < r < 2$.



Consider C^∞ - functions on the manifold M with GDA:

$$\tilde{x}^j_\alpha \;=\; \varphi(r_\alpha)\, x^j_\alpha \;\; , \quad r^2_\alpha \;=\; \sum_j (x^j_\alpha)^2$$

We have: $\tilde{x}^j_\alpha = x^j_\alpha$ for $r_\alpha \leq 1$, and $\tilde{x}^j_\alpha = 0$ for $r_\alpha \geq 2$ (i.e. outside V_α).

Theorem. Let a compact manifold M^n with GDA (finite) be given

$$M^n \;=\; U_1 \cup \ldots \cup U_Q \;=\; V_1 \cup \ldots \cup V_Q$$

Consider the coordinates $\vec{x}_\alpha$ ($r_\alpha < 1$ in U_α and $r_\alpha < 2$ in V_α) and the corresponding functions $\varphi(r_\alpha)$ and $\tilde{x}^i_\alpha$ on M^n.

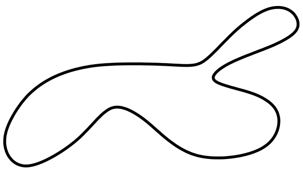
Then the map $F : M^n \to \mathbb{R}^N$, $N = Q(n+1)$:

$$F(x) \;=\; \left[\tilde{x}^1_1, \ldots, \tilde{x}^n_1, \varphi(r_1), \;\ldots\;, \tilde{x}^1_Q, \ldots, \tilde{x}^n_Q, \varphi(r_Q)\right]$$

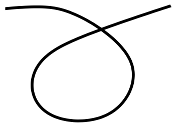
is a C^∞ **imbedding** (nondegenerate).

Terminology:

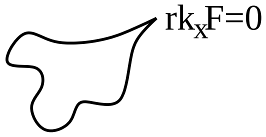
Imbedding: $=$ $\mathrm{rk}_x F \;=\; n$ at every point x and $F(x) = F(y) \;\to\; x = y$.
Immersion: $=$ $\mathrm{rk}_x F \;=\; n \;=\; \dim M$ at every point $x \in M$.



imbedding



$M^1 \to R^2$
immersion



Proof of the **Theorem**.

I. Let $x \in U_\alpha$, then $\mathrm{rk}\, F = n$ because

$$\tilde{x}^i_\alpha = x^i_\alpha \;\text{ for }\; x \in U_\alpha \;, \quad F \;=\; \left[\ldots\,,\, \tilde{x}^1_\alpha, \ldots, \tilde{x}^n_\alpha, \,\ldots\right] \;, \quad \text{and rk} \left|\frac{\partial \tilde{x}^i_\alpha}{\partial x^j_\alpha}\right| \;=\; n$$

(unit matrix in U_α).

II. Suppose $x \neq y$. Can we still have $F(x) = F(y)$?

a) Let $x, y \in U_\alpha$ then $F(x) \neq F(y)$ by the same reason as in I.

b) Let $x \in U_\alpha,\ y \in V_\alpha \quad (r_\alpha > 1)$. Then $\varphi(r_\alpha)|_x = 1,\ \varphi(r_\alpha)|_y < 1$, so $F(x) \neq F(y)$,

$$F \ = \ \big[\dots\,,\, \tilde{x}^1_\alpha, \dots, \tilde{x}^n_\alpha,\, \varphi(r_\alpha),\, \dots\big]$$

c) Let $x \in U_\alpha$, and y is outside V_α. Then $\varphi(r_\alpha)|_x = 1,\ \varphi(r_\alpha)|_y = 0$, so $F(x) \neq F(y)$.

Theorem is proved.

Lecture 3. C^∞-manifolds, Atlases, Charts. Especially good Atlases. Implicit functions and Inversion. Imbedding of Compact Manifolds in $\mathbb{R}^N$. Immersions.

Partition of Unity.

Let

$$\psi_\alpha(x) \ = \ \varphi(r_\alpha) \bigg/ \sum_{\beta=1}^{Q} \varphi(r_\beta)$$

We can easily see the that all the functions $\psi_\alpha(x)$ represent C^∞ - functions on M with the following properties

$$\psi_\alpha(x) \ \geq \ 0 \quad , \qquad \sum_\alpha \psi_\alpha(x) \ \equiv \ 1$$

Example of Application.

What is an Integral on a Manifold?

$$I \ = \ \int_M f(x)\, d^n\sigma$$

($d^n\sigma$ represents some measure on M).

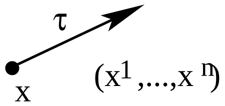
We can write

$$I \ = \ \int_M f(x) \sum_\alpha \psi_\alpha(x)\ d^n\sigma \ = \ \sum_\alpha \int_{V_\alpha} f(\vec{x}_\alpha)\, \psi_\alpha(\vec{x}_\alpha)\, d^n\sigma$$

We can see now that I is represented as a sum of ordinary integrals over local domains where we can also put in local coordinates: $d^n\sigma = g_\alpha(\vec{x}_\alpha)\, dx^1_\alpha \ldots dx^n_\alpha$.

Tangent vector on a C^∞ - manifold $M = \cup_\alpha (U_\alpha, \vec{x}_\alpha)$:

a) Vector τ is attached to a point $x \in M$.



b) Vector τ is characterized by "components" $(\tau^1, \ldots, \tau^n)$ in the given system of local coordinates.

c) Basic vectors e_i in the coordinate system $\vec{x}$ are identified with the operators $\partial/\partial x^i$:

$$e_i \leftrightarrow \frac{\partial}{\partial x^i} \ , \qquad \tau = \tau^i \, e_i$$

d) The vector τ is identified with the differential operator $\tau^i \, \partial/\partial x^i$:

$$\tau \leftrightarrow \tau^i \frac{\partial}{\partial x^i} \ ,$$

so τ acts on the functions $f(x)$ at the point x by the formula:

$$\tau(f) = \tau^i \left. \frac{\partial f}{\partial x^i} \right|_x$$

(derivative along the vector τ).

For a smooth curve $\vec{x}(t) = (x^1(t), \ldots, x^n(t))$ the vector

$$\tau = \frac{d\vec{x}}{dt} = \left(\dot{x}^1, \ldots, \dot{x}^n\right)\big|_{x=x(t_0)}$$

represents the "speed of particle" at the point t_0 (nothing to do with **"relativistic speed"** in the Special Relativity).

Change of Coordinates:

Let us make a non-degenerate change of coordinates

$$x^i = x^i(y^1, \ldots, y^n) \ , \qquad i = 1, \ldots, n \ ,$$

such that we can write $f(\vec{x}) = f(\vec{x}(\vec{y}))$ for any smooth function $f(\vec{x})$ near the point $x \in M$. We say that the sets $(\tau^1, \ldots, \tau^n)$ and $(\tau'^1, \ldots, \tau'^n)$ represent the components of the same vector τ in the coordinate systems $\{x^i\}$ and $\{y^i\}$ respectively if we have for any $f(x)$ at the point x:

$$\tau^i \frac{\partial f}{\partial x^i} = \tau'^j \frac{\partial f}{\partial y^j}$$

By definition, we can write

$$\tau^i \frac{\partial f}{\partial x^i} \;=\; \tau'^j \frac{\partial x^i}{\partial y^j} \frac{\partial f}{\partial x^i} \;,$$

so we come to the **conclusion**

$$\tau^i \;=\; \tau'^j \frac{\partial x^i}{\partial y^j}$$

(summation over j is assumed).

In the same way, for the inverse transformation $\vec{y} = \vec{y}(\vec{x})$ we can write

$$\tau'^j \;=\; \tau^i \frac{\partial y^j}{\partial x^i}$$

where

$$\frac{\partial x^i}{\partial y^j} \frac{\partial y^j}{\partial x^k} \;=\; \delta^i_k$$

Every C^∞ - map is given **locally** by its linear part and the smaller terms

$$y^i \;=\; F^i(x^1, \ldots, x^n) \;=\; \text{Const} \;+\; \sum_{j=1}^{n} A^i_j\, x^j \;+\; O(||x||^2)$$

(near $\vec{x} = 0$).

Inversion of Map (C^∞).

Let us have a map

$$F : \underset{(x)}{\mathbb{R}^n} \;\to\; \underset{(y)}{\mathbb{R}^n} \quad , \qquad x_0 \to y_0$$

given in coordinate form by the functions $y^i \;=\; y^i(x^1, \ldots, x^n)$.

If we have the relation

$$\det \left| \frac{\partial y^i}{\partial x^k} \right| \Bigg|_{x_0} \;\neq\; 0$$

then there exist open sets $V \ni x_0$, $U \ni y_0$ and a map

$$G : \underset{(y)}{\mathbb{R}^n} \;\to\; \underset{(x)}{\mathbb{R}^n} \quad , \qquad y_0 \to x_0 \;,$$

defined in the set U, such that for $x \in V,\ y \in U$ we have the relations

$$G\left(F(x)\right) \;\equiv\; x \quad , \qquad F\left(G(y)\right) \;\equiv\; y$$

Naturally, we have in this case

$$\left.\frac{\partial x^i}{\partial y^j}\right|_{y_0} \left.\frac{\partial y^j}{\partial x^k}\right|_{x_0} \;=\; \delta^i_k \quad , \qquad \text{i.e.} \qquad \left.\left(\frac{\partial x}{\partial y}\right)\right|_{y_0} \left.\left(\frac{\partial y}{\partial x}\right)\right|_{x_0} \;=\; I$$

Implicit Functions.

Let us have a coordinate system $(y^1, \dots, y^{n+k})$ in $\mathbb{R}^{n+k}$ and a system of k equations $z^1 = 0,\ \dots,\ z^k = 0,\ z^i = z^i(y^1, \dots, y^{n+k})$ with the **Condition**:

$$\mathrm{rk}_{y_0}\left(\frac{\partial z^i}{\partial y^j}\right) \;=\; k$$

(maximal rank).

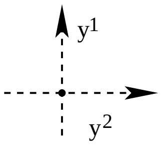
Statement.

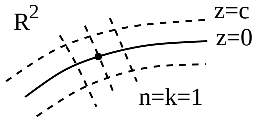
Let us assume that under the above conditions we have the relation

$$\det T \;\equiv\; \det\left.\left(\frac{\partial z^i}{\partial y^j}\right)\right|_{y_0} \;\neq\; 0 \;, \qquad j = n+1, \dots, n+k \;,$$

Jacobi Matrix

$$\left.\left\{ \frac{\partial z^i}{\partial y^1}, \dots, \frac{\partial z^i}{\partial y^n}, \underbrace{\frac{\partial z^i}{\partial y^{n+1}}, \dots, \frac{\partial z^i}{\partial y^{n+k}}}_{\text{nonzero determinant is here}} \right\}\right|_{y_0}$$





Then:

1) There exists an open set $U \ni y_0 \in \mathbb{R}^{n+k}$ near the point y_0, where the values $(y^1, \dots, y^n,\ z^1, \dots, z^k)$ represent a coordinate system;

2) The change

$$\left(y^1, \dots, y^{n+k}\right) \;\xrightarrow{F}\; \left(y^1, \dots, y^n,\ z^1, \dots, z^k\right)$$

is C^∞ and nondegenerate.

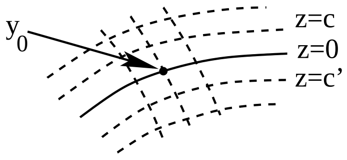
Proof.

Easy to see that the Jacobian Matrix of the transformation F can be written in the form:

$$J = \begin{pmatrix} 1 & 0 & \dots & 0 & \partial z^1/\partial y^1 & \dots & \partial z^k/\partial y^1 \\ 0 & 1 & \dots & 0 & \partial z^1/\partial y^2 & \dots & \partial z^k/\partial y^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \partial z^1/\partial y^n & \dots & \partial z^k/\partial y^n \\ 0 & 0 & \dots & 0 & \partial z^1/\partial y^{n+1} & \dots & \partial z^k/\partial y^{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \partial z^1/\partial y^{n+k} & \dots & \partial z^k/\partial y^{n+k} \end{pmatrix}$$

We immediately get then $\det J_{y_0} = \det T \neq 0$. So, we get our statement from the Inversion Theorem.

Statement is proved.



Under the above conditions, the **"Implicit Function Theorem"** states that on the submanifold, given by the relations $z^1 = 0$, $\dots$, $z^k = 0$, the values $(y^{n+1}, \dots, y^{n+k})$ can be locally expressed as explicit functions of the coordinates $(y^1, \dots, y^n)$:

$$y^{n+1} = \varphi_1 (y^1, \dots, y^n) \; , \; \dots \; , \quad y^{n+k} = \varphi_k (y^1, \dots, y^n)$$

The Implicit Function Theorem can be considered as a corollary of the Statement formulated above. Indeed, we have a "local coordinate system"

$$\left(y^1, \dots, y^n, \, z^1, \dots, z^n \right)$$

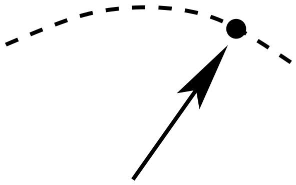
near the values $\vec{z} = 0$ in the domain U near the point y_0.

We can then write in this domain for the inverse coordinate transformation:

$$y^{n+1} = \varphi_1 (y^1, \dots, y^n, \, z^1, \dots, z^k) \; , \; \dots \; , \quad y^{n+k} = \varphi_k (y^1, \dots, y^n, \, z^1, \dots, z^k)$$

Putting now $\vec{z} = 0$ we get immediately the statement of the Implicit Function Theorem.

Another corollary:



$$\operatorname{rk}\left(\frac{\partial y^i}{\partial x^q}\right)_{x_0} = n$$

Every imbedding of manifold

$$\begin{array}{ccc} M^n & \to & M^{n+k} \\ (x) & & (y) \end{array}$$

locally can be given by k **nondegenerate** equations $z^1 = 0,\ \ldots,\ z^k = 0$.

Proof. Consider the imbedding $M^n \to M^{n+k}$, given by smooth functions $y^i = \varphi_i\,(x^1, \ldots, x^n),\ i = 1, \ldots, n+k$, such that

$$\operatorname{rk}\left(\frac{\partial y^i}{\partial x^q}\right) \;=\; n$$

Let

$$\det\left(\frac{\partial y^i}{\partial x^q}\right)\bigg|_{x^0} \;\neq\; 0\ , \qquad i = 1, \ldots, n$$

According to the Inversion Theorem, the transformation

$$\left(x^1, \ldots, x^n,\, y^{n+1}, \ldots, y^{n+k}\right) \;\to\; \left(y^1, \ldots, y^n,\, y^{n+1}, \ldots, y^{n+k}\right)$$

is locally invertible, so we can introduce the new coordinate system

$$y' \;=\; \left(x^1, \ldots, x^n,\, y^{n+1}, \ldots, y^{n+k}\right)$$

near the point $y_0 = F(x_0)$.

The imbedding $M^n \to M^{n+k}$ can be given now near the point y_0 by the set of equations $z^1 = 0,\ \ldots,\ z^k = 0$, where

$$z^1 \;=\; y^{n+1} - \varphi_{n+1}\,(x^1, \ldots, x^n)\ ,\ \ \ldots\ \ ,\ z^k \;=\; y^{n+k} - \varphi_{n+k}\,(x^1, \ldots, x^n)$$

We have also

$$\det\left(\frac{\partial z^p}{\partial y^{n+q}}\right) \;\neq\; 0\ ,$$

so the values $(x^1, \ldots, x^n,\, z^1, \ldots, z^k)$ give also a local coordinate system in $\mathbb{R}^{n+k}$.

Corollary is proved.

Lecture 4. Manifolds and Submanifolds: Implicit functions and Inversion. Vectors and Covectors.

According to the previous lecture, we can formulate here the following

Statement.

Let us have an imbedding

$$F: \; M^n \to N^{n+k} \; , \qquad (\mathrm{rk}_x F = n \, , \forall x \, , \; F(x) \neq F(y) \, , \forall x \neq y) \; ,$$

such that either M^n is compact or for every compact set $X \subset N$ the intersection $X \cap F(M)$ is compact.

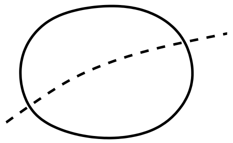
Then:

For every point $x \in M^n \subset N^{n+k}$ there exists local coordinate system $(x^1, \ldots, x^n, \, z^1, \ldots, z^k)$ in some $U \ni x$ such that the submanifold M^n is given in U by the system

$$z^1 = 0 \; , \; \ldots \; , \; z^k = 0$$

Conclusion.

Every C^∞ - manifold can be given by a set of **local** equations in $\mathbb{R}^N$: $M^n \to \mathbb{R}^N$ (proved for compact M^n).



Remark. Not all manifolds can be given by **global** nondegenerate set of equations in $\mathbb{R}^N$

$$\varphi_1 (y^1, \ldots, y^N) = 0 \; , \; \ldots \; , \; \varphi_{N-n} (y^1, \ldots, y^N) = 0$$

(Proof later).

$\mathbb{RP}^2$ **can not** be given by **nondegenerate global set of equations**.

"Tangent Manifold" $T^*(M^n)$.

Let us denote again by τ a tangent vector $(\tau^1, \ldots, \tau^n)$ in M^n attached to a point $x \in M^n$.

'Tangent Manifold" $T^*(M^n)$:

$$M^n = \cup_\alpha \left(U_\alpha , \, x_\alpha^1, \ldots, x_\alpha^n \right) \qquad (\text{Atlas}, \; C^\infty)$$

$$T^*(M^n) \;=\; \cup_\alpha \left(U_\alpha \times \mathbb{R}^n \,,\, x^1_\alpha, \dots, x^n_\alpha \,,\, \tau^1, \dots, \tau^n\right)$$

Change of coordinates:

Let us put in the intersection of Charts U_α and U_β: $x = x_\alpha,\ y = x_\beta$. We have then

$$x^i \;=\; x^i(y) \;=\; x^i(y^1, \dots, y^n) \qquad (\text{in } U_\alpha \cap U_\beta)$$

The components of the same vector in the coordinates (x) and (y):

$$(x,\, \tau) \;\leftrightarrow\; (y,\, \tau')$$

are connected by the relations:

$$\tau^i \;=\; \tau'^j \,\frac{\partial x^i}{\partial y^j} \qquad \left(\sum_j \text{ is assumed}\right)$$

Covectors $(\eta_1, \dots, \eta_n)$ are attached to the points $x \in M^n$.

Change of coordinates

$$(x,\, \eta) \;\leftrightarrow\; (y,\, \eta')$$

$$\eta'_j \;=\; \eta_i \,\frac{\partial x^i}{\partial y^j} \;, \qquad \eta_i \;=\; \eta'_j \,\frac{\partial y^j}{\partial x^i}$$

$$T_*(M^n) \;=\; \cup_\alpha \left(U_\alpha \times \mathbb{R}^n \,,\, x^1_\alpha, \dots, x^n_\alpha \,,\, \eta_1, \dots, \eta_n\right)$$

Scalar product (invariant):

$$\langle \tau \,,\, \eta \rangle \;=\; \tau^i \, \eta_i$$

Conclusion: Spaces of vectors and covectors are dual.

Basis:

$$\text{vectors}: \; e_i \;\leftrightarrow\; \frac{\partial}{\partial x^i}$$

$$\text{covectors}: \; e^i \;\leftrightarrow\; d\,x^i$$

$$\text{Vector field}: \quad f^i \,\frac{\partial}{\partial x^i} \qquad (\text{Vector fields})$$

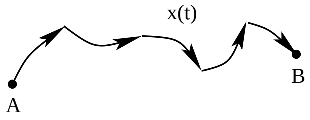
$$\text{Covector field}: \quad g_i \, dx^i \qquad (\text{Differential } 1-\text{forms})$$

Vector Fields = Dynamical Systems (Autonomous)

Integration of 1-forms along the path: let

$$\omega = g_i(x)\, dx^i$$

and the path is given by a **piecewise smooth** one-parametric curve



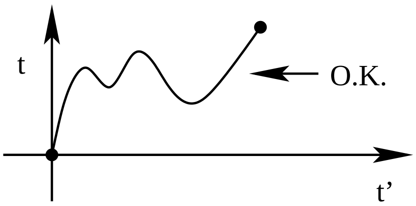
$$\vec{x}(t) = \big(x^1(t), \ldots, x^n(t)\big)$$

Definition.

$$\int_{x(t)} \omega = \int_A^B g_i(x)\, dx^i = \int_{t_0}^{t_1} \left(g_i\,(x(t))\, \frac{dx^i}{dt} \right) dt$$

Properties.

Integral **does not** depend on the choice of coordinates in M^n and "time" t.



Nonmonotonic changes of time $t = t\,(t')$ are also admissible.

Inner Product of tangent vectors

$$g_{ij}(x)\ \tau^i\, \tilde{\tau}^j = \langle \tau\,,\, \tilde{\tau} \rangle$$

Change of coordinates $x = x(y)$:

$$g_{ij}(x(y))\ \frac{\partial x^i}{\partial y^k}\, \frac{\partial x^j}{\partial y^s} = g'_{ks}(y)$$

Let Tensor field $g_{ij}(x)$ be symmetric:

$$g_{ij} = g_{ji}\ :\ \langle \tau\,,\, \tilde{\tau} \rangle = \langle \tilde{\tau}\,,\, \tau \rangle$$

We will require also that tensor $g_{ij}(x)$ is nondegenerate.

Under the above conditions we say that $g_{ij}(x)$ defines a **Pseudoriemannian Metric** on M^n.

Types of inner products ("Types of Geometry"):

$g_{ij}\, dx^i\, dx^j \;>\; 0$ - **Riemannian Metric**.

Special Types (p, q) of Pseudoriemannian metric:

$p = 0:$ - Riemannian

$p = 1:$ - Lorentzian

$p = q:$ - Ultrahyperbolic

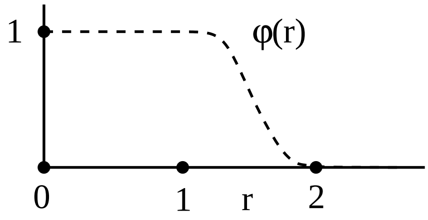
Theorem.

In every C^∞ - manifold there exists a C^∞ Riemannian Metric.

Proof. Take GDA (Good Double Atlas)

$$M^n \;=\; U_1 \cup \ldots \cup U_Q \;=\; V_1 \cup \ldots \cup V_Q$$

($|\vec{x}_\alpha| < 1$ in U_α and $|\vec{x}_\alpha| < 2$ in V_α) and the function $\varphi(r)$ introduced in Lecture 2.



Consider quadratic forms

$$G_\alpha \;=\; \sum_{j=1}^{n} \left(d\tilde{x}_\alpha^j\right)^2 \quad , \qquad \tilde{x}_\alpha^j \;=\; x_\alpha^j \cdot \varphi(r_\alpha)$$

Take Riemannian metric

$$G \;=\; \sum_\alpha G_\alpha$$

(We assume that every point belongs to **finite** number of domains V_α).

We claim that it is positive.

Theorem is proved.

Remark.

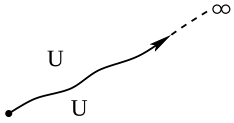
Lorentzian Nondegenerate Metric does NOT exist in $\mathbb{S}^2$ (or $\mathbb{RP}^2$). [Topology claims that Euler characteristics should be 0.]

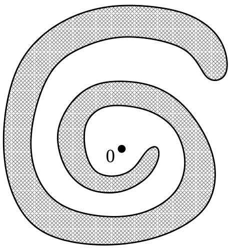
Homework 1.

1. Local coordinates: ρ and φ in $\mathbb{R}^2$.

a) Prove that ρ can not be chosen as a Cartesian coordinate in **the whole domain** $\mathbb{R}^2 \backslash 0$ (**whole**).

b) Prove that φ can be defined as a one-valued function in any domain not containing **closed paths** surrounding 0 .





Define φ in these domains.

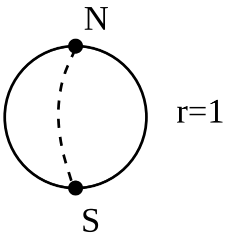
2. Spherical coordinates in $\mathbb{R}^3 \backslash 0$

$$x \;=\; r\,\sin\theta\,\cos\varphi \quad , \qquad y \;=\; r\,\sin\theta\,\sin\varphi \quad , \qquad z \;=\; r\,\cos\theta$$

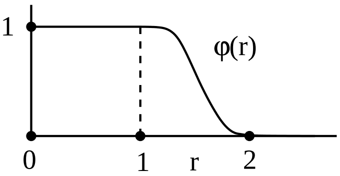
$$r^2 \;=\; x^2 + y^2 + z^2$$

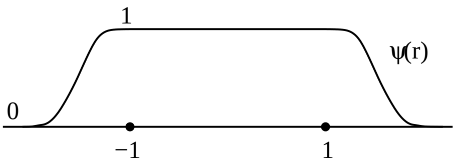
Let $r = 1$.

a) In which domains θ is a good Cartesian coordinate?
b) In which domains (φ, θ) are good local coordinates?



3. Construct C^∞ functions





using

$$f(x) \;=\; \begin{cases} 0 & , \quad x \le 0 \\ e^{-1/|x|^2} & , \quad x \ge 0 \end{cases}$$

Lecture 5. Manifolds and vectors fields. Important Examples.

Classes of manifolds:

a) All C^∞ - manifolds $M^n \subset \mathbb{R}^N$ for given N.

b) Manifolds defined by the Global Nondegenerate systems of Equations in $\mathbb{R}^N$

$$f_1 = 0, \ \ldots \ , \ f_{N-n} = 0$$

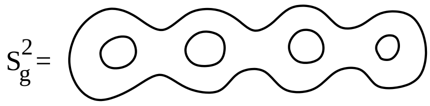
(Important cases: $n = 2, \ N = n + 1 = 3$).

Classification of manifolds:

$$\underline{n=1}: \qquad \underset{\substack{\uparrow \\ \text{compact}}}{\mathbb{S}^1} \quad , \quad \underset{\substack{\uparrow \\ \text{noncompact}}}{\mathbb{R}}$$

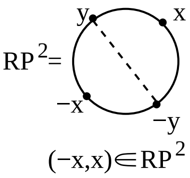
$\underline{n=2}$: compact **orientable** manifolds are

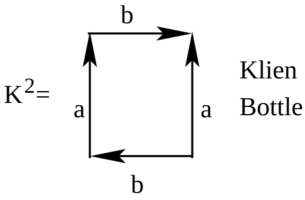
$\mathbb{S}^2, \ \mathbb{T}^2, \ \ldots, \ \mathbb{S}^2_g.$



nonorientable:

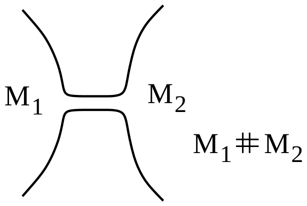
$\mathbb{RP}^2, \ \mathbb{K}^2$ (Klein Bottle), **other**.





"Connected sum" of previous manifolds:

$$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \ \ldots \ \# \mathbb{RP}^2$$
$$\mathbb{K}^2 \ = \ \mathbb{RP}^2 \# \mathbb{RP}^2$$



Noncompact 2 - manifolds.

Important manifolds:

$\mathbb{RP}^n$ - real projective spaces

$\mathbb{CP}^n$ - complex projective spaces

$\mathbb{QP}^n$ - “quaternionic” projective spaces

Groups: $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$, $O_n \supset SO_n$, $O_{p,q} \supset SO_{p,q}$ (case $p = 1$ - Lorentz Groups), $U_n \supset SU_n$, $U_{p,q} \supset SU_{p,q}$

Stiefel Manifolds

Grassmann Manifolds

Isometry groups of $\mathbb{R}^n$

Isometry groups of $\mathbb{R}^{p,q}$

where

Euclidean Metric: $g_{ij} \ = \ \delta_{ij}$

Lorentzian Metric: $g_{ij} \ = \ \operatorname{diag}(1, -1, \dots, -1)$, i.e.

$$g_{ij} \ = \ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}$$

Vector fields and Dynamical Systems.

Vectors on Manifolds M^n

$$M^n \ = \ \cup_\alpha \left(U_\alpha,\, x^1_\alpha, \dots, x^n_\alpha\right)$$

$\tau \ = \ (\tau^1, \dots, \tau^n)$.

Covectors on Manifold M^n: $\eta \ = \ (\eta_1, \dots, \eta_n)$.

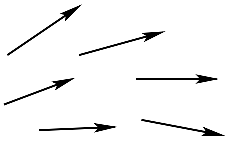
$T^*(M^n)$ - space of all vectors on M^n.

$T_*(M^n)$ - space of all covectors on M^n.

Vector field

$$\tau(x) \ = \ \underset{\text{locally}}{(\tau^1(x), \dots, \tau^n(x))}$$

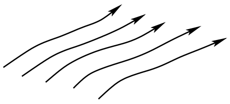
generates dynamical system:



$$\dot{x}^i \ = \ \tau^i(x) \ , \qquad i = 1, \dots, n$$

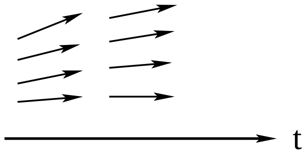
$$\text{Solution} \ = \ \text{curve } \vec{x}(t): \quad \dot{x}^i \ = \ \tau^i(x)$$

Cauchy Theorem: Let $\tau(x_0) \neq 0$. There exists system of local coordinates $(y^1, \dots, y^n)$ such that $\tau \ = \ (1, 0, \dots, 0)$. In this coordinate system we have



$$\underset{\text{solution}}{x(t)} \quad \to \quad y(t) \;=\; (t,\, \underset{\text{const}}{y_0^2, \dots, y_0^n})$$

Let $\tau(x_0) = 0$. Consider $M^n \times \mathbb{R} = M'$ and vector field $\tau' = (\tau, 1)$ in M' near $x = x_0$.



$\tau'(x_0) \neq 0$.
Apply Theorem of Cauchy.

One-parametric group generated by vector field.

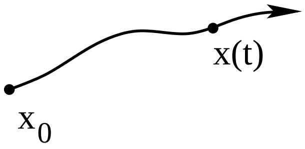
The equation

$$\dot{x} \;=\; \tau(x)$$

generates a (local) one parametric group of invertible transformations of (local domains) in M^n:

$$S_t : M^n \to M^n$$

$$x(0) \;=\; x_0 \;, \quad x(t) \;=\; S_t(x_0)$$



$$S_0 \;=\; I \quad , \quad S_{-t} \;=\; S_t^{-1} \quad , \quad S_{t+t'} \;=\; S_t \circ S_{t'} \;=\; S_{t'} \circ S_t$$

Commuting Vector Fields: $\tau_1(x)$, $\tau_2(x)$.

Definition:

$$[\tau_1\,,\,\tau_2] \;=\; \tau_1 \circ \tau_2 \;-\; \tau_2 \circ \tau_1$$

$$\tau \;=\; \tau^i(x)\,\frac{\partial}{\partial x^i}$$

$$\tau\,\eta \;=\; \tau^i\,\frac{\partial}{\partial x^i}\,\eta^j\,\frac{\partial}{\partial x^j} \;=\; \tau^i\,\eta^j\,\frac{\partial^2}{\partial x^i \partial x^j} \;-\; \tau^i\,\frac{\partial \eta^j}{\partial x^i}\,\frac{\partial}{\partial x^j}$$

Corollary

$$[\tau\,,\,\eta] \;=\; \tau^i\,\frac{\partial \eta^j}{\partial x^i}\,\frac{\partial}{\partial x^j} \;-\; \eta^i\,\frac{\partial \tau^j}{\partial x^i}\,\frac{\partial}{\partial x^j} \;=\; \left(\tau^i\,\frac{\partial \eta^j}{\partial x^i} \;-\; \eta^i\,\frac{\partial \tau^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$$

Remark. Sometimes we denote operator $\tau(f)$ acting on scalar functions by $\nabla_\tau f$ because it coincides in this case with covariant derivative of the scalar field f along vector field τ.

Statement: Let $\tau(x_0) \neq 0$, $\eta(x_0) \neq 0$. Then $[\tau, \eta] \equiv 0 \;\Rightarrow\;$ there exists local coordinate system $(y^1, \ldots, y^n)$ such that

$$\tau = (1, 0, \ldots, 0) \;, \qquad \eta = (0, 1, 0, \ldots, 0)$$

They generate a (local) commutative group $\mathbb{R}^2$:
S_t – shifts by (τ)
$S'_{t'}$ – shifts by (η)

$$S'_{t'} \circ S_t \;=\; S_t \circ S'_{t'}$$

Examples.
1. $\tau = \text{const}$ in $\mathbb{R}^n$: group of shifts $x \to x + t \cdot \text{const}$.
2. $\tau^i = a^i_j x^j$ (linear)

$$\dot{x}^j \;=\; \tau^j(x) \;, \qquad x_0 \to x(t) \;, \quad x(0) = x_0$$

- $x(t)$ generate linear maps S_t.
3. $\operatorname{Tr} a^i_j = 0 \;\Rightarrow\; S_t \in SL_n(\mathbb{R})$.
4. $a^i_j = -a^j_i \;\Rightarrow\; S_t \in SO_n$.

Lecture 6. Group Manifolds. Lie Algebra. Important Examples.

Group Manifolds.

1) $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $GL_n(\mathbb{Q})$.

The notation $\mathbb{Q}$ represents here the "noncommutative field" of quaternions:

$$\{a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d\}$$

where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \;, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k} \;, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \;, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

$\dim GL_n(\mathbb{R}) \;=\; n^2$
$\dim GL_n(\mathbb{C}) \;=\; 2n^2$
$\dim GL_n(\mathbb{Q}) \;=\; 4n^2$

Question: prove that $GL_n(\mathbb{R})$ has 2 components. $GL_n(\mathbb{C})$ is connected.

2) $SL_n(\mathbb{R})\,,\; SL_n(\mathbb{C}) \;:\; \det \;=\; 1$
Equation : $\underline{\det A \;=\; 1}$ in $\mathbb{R}^{n^2}$ or in $\mathbb{C}^{n^2}$
Is this equation nondegenerate ?

3) $O_n\,,\; A\,A^t \;=\; I\,,$
a) Is this set of equations NONDEGENERATE in $GL_n(\mathbb{R}) \;\in\; \mathbb{R}^{n^2}$?
b) Is SO_n - connected manifold?
c) $\dim O_n \;=\; n(n-1)/2$

4) $O_{p,q} \;:\; \langle A\eta\,,\, A\zeta\rangle \;=\; \langle \eta\,,\, \zeta\rangle$
a) $p=1\,,\; q=n\;, \quad \langle \eta\,,\, \zeta\rangle \;=\; \eta^0\zeta^0 \;-\; \sum_{\alpha=1}^{n} \eta^\alpha\zeta^\alpha\;, \quad (1,n)$
b) $\langle \eta\,,\, \zeta\rangle \;=\; \sum_{\alpha=1}^{p} \eta^\alpha\zeta^\alpha \;-\; \sum_{\alpha=p+1}^{p+q} \eta^\alpha\zeta^\alpha\;, \quad (p,q)$

5) **Unitary group:** $\zeta,\, \eta \;\in\; \mathbb{C}^n$

$$\langle \zeta\,,\, \eta\rangle \;\;=\;\; \overline{\langle \eta\,,\, \zeta\rangle} \;\;=\;\; \sum_{i\le p} \zeta^i\,\bar\eta^i \;-\; \sum_{i>p} \zeta^i\,\bar\eta^i$$

$$U_{p,q} \;: \quad \langle A\zeta\,,\, A\eta\rangle \;=\; \langle \zeta\,,\, \eta\rangle \quad , \qquad (q=0\;:\;U_n)$$

$$SU_n\,,\, SU_{p,q} \;:\; \det A \;=\; 1$$

How to introduce local coordinates in the group manifold $M^n = G$?
Let $A(t) \;\in\; G\,,\; A(0) \;=\; I\,.$

$$\left.\frac{dA}{dt}\right|_{t=0} \;=\; B \;\in\; \text{Lie Algebra}$$

Another form

$$\text{Lie Algebra} \quad\to\quad \frac{dA}{dt}\,A^{-1} \;\text{ or }\; A^{-1}\,\frac{dA}{dt}$$

Groups SO_n , O_n :

$$\langle A(t)\zeta\,,\, A(t)\eta\rangle \;=\; \langle \zeta\,,\, \eta\rangle \quad , \qquad \frac{d}{dt}\,\langle A(t)\zeta\,,\, A(t)\eta\rangle \;=\; 0\;,$$

$$A(t) \;=\; A(0) \;+\; \underset{\substack{\uparrow \\ B}}{\dot{A}(0)\,t} \;+\; O(t^2)\;, \qquad A(0) \,=\, 1$$

Lemma 1: $B^t \,=\, -B$ (i.e. $\dot{A}^t(0) \,=\, -\dot{A}(0)$).

Proof.

$$\langle A(t)\zeta\,,\,A(t)\eta\rangle \;=\; \langle \zeta\,,\,\eta\rangle \;+\; t\,[\langle B\zeta\,,\,\eta\rangle \,+\, \langle \zeta\,,\,B\eta\rangle] \;+\; O(t^2)$$

$$\frac{d}{dt}\,\langle A(t)\zeta\,,\,A(t)\eta\rangle\bigg|_{t=0} \;=\; 0 \quad\Rightarrow\quad \langle B\zeta\,,\,\eta\rangle \,+\, \langle \zeta\,,\,B\eta\rangle \;=\; 0$$

i.e.

$$\langle B\zeta\,,\,\eta\rangle \;\equiv\; -\,\langle \zeta\,,\,B\eta\rangle$$

Lemma is proved.

Lemma 2: $\exists\,\epsilon \,>\, 0$ such that for any $A \,\in\, SO_n\,,\; ||A - I|| \,<\, \epsilon$ we have $A \,=\, e^B\;,\; B^t \,=\, -B\,.$

Proof.

For small enough ϵ consider the convergent series

$$B \;=\; \log{(I + A - I)} \;=\; A - I \;-\; (A - I)^2/2 \;+\; (A - I)^3/3 \;-\; \ldots$$

$$B^t \;=\; \log{(I + A^t - I)} \;=\; A^t - I \;-\; (A^t - I)^2/2 \;+\; (A^t - I)^3/3 \;-\; \ldots$$

We can write then: $A \,=\, e^B\;,\; A^t \,=\, e^{B^t}$. Besides that, from the commutativity of all the terms of the series we can write also:

$$e^{B + B^t} \;=\; A\,A^t \;=\; I$$

which implies $B^t \,=\, -B$ for small enough B.

Lemma is proved.

Local coordinates in $GL_n(\mathbb{R})$ near I can be also taken from a small ball in the space $\mathbb{R}^{n^2} \,=$ space of matrices.

Lemma 3: For small enough B we have: $\operatorname{Tr} B \,=\, 0 \;\Leftrightarrow\; \det e^B \,=\, 1$.

Proof.

For the diagonal matrices $B \,=\, \operatorname{diag}{(b_1, \ldots, b_n)}$ we obviously have the relation $\det e^B \;=\; \prod e^{b_i} \;=\; \exp{(\operatorname{Tr} B)}$. The same property is then also evident for the diagonalizable matrices $B = S^{-1} \circ \operatorname{diag}(b_1, \dots, b_n) \circ S$ from the series representation of the matrix e^B. Since the set of diagonalizable matrices is dense in the matrix space and the functions $\exp(\operatorname{Tr} B)$ and $\det e^B$ are analytic functions of the matrix entries we actually have

$$\det e^B = e^{\operatorname{Tr} B}$$

for any matrix B. The statement of the Lemma for small enough B follows then from the properties of the function e^x.

Lemma is proved.

We can see then that in the local coordinate system in $GL_n(\mathbb{R})$, given by the entries of the matrix B near $I \in GL_n(\mathbb{R})$, the equation $\det A = 1$ has the form $\operatorname{Tr} B = 0$.

Conclusion. This equation is linear and NONDEGENERATE.

Group SO_n: Equations for SO_n in the coordinate system, given by the entries of B near $I \in GL_n(\mathbb{R})$, have the form $\underline{B^t = -B}$.

Unitary group $A\bar{A}^t = I$.

$U_n \subset GL_n(\mathbb{C})$, we can put again $A = e^B$ near $I = GL_n(\mathbb{C})$ and introduce a local coordinate system, given by the entries of B in $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. In this coordinate system:

a) Group $SL_n(\mathbb{C})$ is given by equation $\operatorname{Tr} B = 0$ (2 real equations).

b) Group U_n is given by equations $B^t = -\bar{B}$ $(\bar{B}_t = -B)$, i.e. $\underline{n^2 \text{ linear equations over } \mathbb{R}}$.

$\dim U_n = n^2$ (over $\mathbb{R}$). $U_1 = \mathbb{S}^1 = SO_2$ $\dim U_2 = 4$, is $SU_2 = \mathbb{S}^3$?

$$A \in SU_2 \ : \Rightarrow \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a\bar{c} + b\bar{d} = 0$, $|a|^2 + |b|^2 = 1$, $|c|^2 + |d|^2 = 1$, $|ad - bc| = 1$.

So, we have

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1,$$

i.e. $SU_2 = \mathbb{S}^3$.

Quaternion Group: $GL_n(\mathbb{Q})$.

$$\mathbb{Q} : \ \{q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \ , \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k} \ , \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \ , \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

$$\bar{q} \ \equiv \ a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$$

$$GL_2(\mathbb{Q}) \ = \ \{q \, : \, q\,\bar{q} \, = \, 1\} \ = \ \{a^2 + b^2 + c^2 + d^2 \, = \, 1\} \ = \ \mathbb{S}^3 \ = \ SU_2$$

Pauli matrices:

$$i\,\sigma_x \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad i\,\sigma_y \ = \ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ , \quad i\,\sigma_z \ = \ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Correspondence to the **basic quaternions** ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$):

$$i\,\sigma_x \ \leftrightarrow \ \mathbf{i} \ , \quad i\,\sigma_y \ \leftrightarrow \ \mathbf{j} \ , \quad i\,\sigma_z \ \leftrightarrow \ \mathbf{k}$$

$$SO_3 \ = \ SU_2/\mathbb{Z}_2 \ = \ \mathbb{S}^3/\mathbb{Z}_2 \ = \ \mathbb{RP}^3$$

Indeed, consider the norm-conserving transformations

$$q \ \to \ q_1\,q\,\bar{q}_1 \ , \quad q_1\,\bar{q}_1 \ = \ 1$$

Space $\mathrm{Span}\,\{1\}$ and the orthogonal space $\mathrm{Span}\,\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are invariant. Two transformations coincide iff : $q_1' \ = \ \pm q_1$.

$$SO_4 \ = \ \mathbb{S}^3 \times \mathbb{S}^3/\mathbb{Z}_2$$

Consider the transformations

$$q \ \to \ q_1\,q\,q_2 \ , \quad q_1\,\bar{q}_1 \ = \ 1 \ , \ q_2\,\bar{q}_2 \ = \ 1$$

Two transformations coincide iff : $(q_1' \, , \, q_2') \ = \ \pm(q_1 \, , \, q_2)$.

Lecture 7. Group Manifolds: Compact Lie groups. Most important Examples. Noncompact Lie groups. Most important Examples. Lie Algebras. Gradient-like Systems.

Group manifolds.

Compact groups: O_n , SO_n , U_n , SU_n , $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $Sp_n = GL_n(\mathbb{Q}) \cap SO_{4n}$, $SO_2 = U_1 = \mathbb{S}^1$, $SO_3 = SU_2/\mathbb{Z}_2$, $SU_2 = Sp_1 = \mathbb{S}^3$.

Local coordinates (near I):
1. SO_n : $B^t = -B$, $\dim SO_n = n(n-1)/2$
2. U_n : $B^t = -\bar{B}$, $\dim U_n = n^2$

Noncompact groups: $\mathbb{R}^n$, $\mathrm{Iso}\,(\mathbb{R}^n) \underset{O_n}{\longrightarrow} \mathbb{R}^n$ (shifts), $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$, $O_{p,q}$, $U_{p,q}$, $Sympl_n(?)$.

$$\mathbb{R}^{p,q} : \quad (e_1, \ldots, e_n) \ , \quad \langle e_i \, , \, e_j \rangle \ = \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & -1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & -1 \end{pmatrix}$$

$$A \in O_{p,q} : \quad \langle A\zeta \, , \, A\eta \rangle \ = \ \langle \zeta \, , \, \eta \rangle \ , \quad \zeta, \eta \in \mathbb{R}^{p,q}$$

$$A \in U_{p,q} : \quad \langle A\zeta \, , \, A\eta \rangle \ = \ \langle \zeta \, , \, \eta \rangle \ , \quad \zeta, \eta \in \mathbb{C}^{p,q}$$

Symplectic (skew-symmetric) product: $\langle \zeta \, , \, \eta \rangle \ = \ -\langle \eta \, , \, \zeta \rangle$:

$$\mathbb{R}^{2n} : \quad (e_1', e_1'', \ldots, e_n', e_n'') \ , \quad \langle e_i' \, , \, e_j' \rangle \ = \ 0 \ , \ \langle e_i'' \, , \, e_j'' \rangle \ = \ 0 \ , \ \langle e_i' \, , \, e_j'' \rangle \ = \ \delta_{ij}$$

$$g_{ij} \ = \ \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 0 \end{pmatrix}$$

$A \in Symp_n$: $\langle A\zeta \, , \, A\eta \rangle \ = \ \langle \zeta \, , \, \eta \rangle$, $\zeta, \eta \in \mathbb{R}^{2n}$ (Symplectic Space)

Lie Algebras:

1. SO_n : $B^t = -B$, (real entries).
2. U_n : $B^t = -\bar{B}$, (complex entries).

$$SO_{p,q} : \quad \langle B\zeta \, , \, \eta\rangle = -\langle \zeta \, , \, B\eta\rangle \; , \quad B - ?$$

Spaces $R^{p,q}$ with pseudoriemannian inner product ("metric"):

$$g_{ij} = \langle e_i \, , \, e_j\rangle = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \end{pmatrix}$$

Consider matrix $B = (b^i_j)$. We have

$$(B\zeta)^i = b^i_j \, \zeta^j \; , \quad \langle B\zeta \, , \, \eta\rangle = g_{ki} \left(b^i_j \, \zeta^j \, \eta^k\right) = \tilde{b}_{kj} \, \zeta^j \, \eta^k$$

Claim. For Lie Algebra we have $\tilde{b}_{kj} = -\tilde{b}_{jk}$ (Not for B !) .

Proof.

$$\langle B\zeta \, , \, \eta\rangle = g_{ki} \, b^i_j \, \zeta^j \, \eta^k = \tilde{b}_{kj} \, \zeta^j \, \eta^k$$

$$\langle \zeta \, , \, B\eta\rangle = g_{ki} \, b^i_j \, \zeta^k \, \eta^j = \tilde{b}_{kj} \, \zeta^k \, \eta^j = \tilde{b}_{jk} \, \zeta^j \, \eta^k$$

Statement is proved.

Conclusion.

1) Let $A(t) \in G$, $A(t) = I + B\,t + O(t^2)$ and $\langle A\zeta \, , \, A\eta\rangle = \langle \zeta \, , \, \eta\rangle$. Then we have $\tilde{B}^t = -\tilde{B}$, $\tilde{B}_{kj} = g_{ki} \, b^i_j$. Here G is any group ($G\,O_{p,q}$, $Sympl_n$, ...).

2. Lie Algebra of every group $O_{p,q}$ can be identified with the set of skew-symmetric matrices

$$\tilde{B}_{kj} = -\tilde{B}_{jk} = g_{ki} \, b^i_j$$

Matrix B is equal to

$$b^i_j = g^{ik} \, g_{ks} \, b^s_j = g^{ik} \, \tilde{b}_{kj}$$

with definition

$$\left(g^{ik}\right) \;=\; \left(g_{ki}\right)^{-1}$$

General definition

Let Riemannian (pseudoriemannian) manifold M^n be given with Atlas of Charts U_α, $(x^1_\alpha, \ldots, x^n_\alpha)$ and "metric" $g^{(\alpha)}_{ij}(x)$ in the domain U_α. For 2 tangent vectors attached to x we have

$$\langle \tau\,,\, \eta \rangle \;=\; g_{ij}(x)\, \tau^i\, \eta^j$$

We define "metric" in the space of covectors $\xi \,=\, (\xi_i)$, $\kappa \,=\, (\kappa_j)$

$$\langle \xi\,,\, \kappa \rangle \;=\; g^{ij}(x)\, \xi_i\, \kappa_j$$

where $(g^{ij}) \,=\, (g_{kl})^{-1}$, i.e. $g^{ij}\, g_{jk} \,=\, \delta^i_k\,.$

Conclusion. $T^*(M^n) \,\cong\, T_*(M^n)\,.$

Remark. "Skew" metrics

$$g_{ij}(x) \;=\; -\, g_{ij}(x)$$

can be nondegenerate **only** for $n \,=\, 2k$ (n even). We have here $\langle \eta\,,\, \eta \rangle \,\equiv\, 0\,.$

Gradient vector field

Let function $f \,:\; M^n \to \mathbb{R}$ be given. Its gradient is:

$$\textbf{vector}:\;\; \nabla^g f(x) \;=\; \left(g^{ij}(x)\, \frac{\partial f}{\partial x^j} \right)$$

$$\textbf{covector}:\;\; d\,f \;=\; \left(\frac{\partial f}{\partial x^i} \right)$$

Gradient system is

$$\dot{x}^i \;=\; g^{ij}\, \frac{\partial f}{\partial x^j} \;=\; \eta^i(x)$$

Lemma. Let $h(x)$ be any function in M^n. We have

$$\frac{dh}{dt} \;=\; \eta^i(x)\, \frac{\partial h}{\partial x^i} \;=\; \langle \nabla^g h\,,\, \nabla^g f \rangle$$

Proof.

$$\frac{dh}{dt} \;=\; \eta^i(x)\,\frac{\partial h}{\partial x^i} \;=\; g^{ij}(x)\,\frac{\partial h}{\partial x^i}\,\frac{\partial f}{\partial x^j} \;=\; \langle dh\,,\, df\rangle \;=\; \langle \nabla^g h\,,\, \nabla^g f\rangle$$

Lemma is proved.

Corollary 1.

$$g^{ij} \;=\; -\,g^{ji} \quad\Rightarrow\quad \frac{df}{dt} \equiv 0$$

(Symplectic case).

Corollary 2. For Riemannian metric $g_{ij}\,\eta^i\,\eta^j \;>\; 0$ we have $df/dt \;>\; 0$ along the gradient system

$$\dot{x}^i \;=\; g^{ij}\,\frac{\partial f}{\partial x^j}$$

(because $(g^{ij}) \;=\; (g_{ij})^{-1})$.

Important example of Symplectic Manifold is $T_*(M^n)$

$$\dim T_*(M^n) \;=\; 2n \;, \qquad \text{Atlas}: \;\{U_\alpha\,,\; (x^1_\alpha, \ldots, x^n_\alpha, p^\alpha_1, \ldots, p^\alpha_n)\}$$

Fix α - number of Chart.

$$\text{Tangent basis}: \quad e'_i \;=\; \frac{\partial}{\partial x^i} \quad , \quad e''_i \;=\; \frac{\partial}{\partial p_i}$$

Inner Product is

$$\langle e'_i\,,\, e'_j\rangle \;=\; 0 \;\; , \quad \langle e''_i\,,\, e''_j\rangle \;=\; 0 \;\; , \quad \langle e'_i\,,\, e''_j\rangle \;=\; -\,\langle e''_j\,,\, e'_i\rangle \;=\; \delta_{ij}$$

$\tau \;=\; (e'_i)$ - tangent vector to M^n, $p \;=\; (e''_j)$ - tangent covector to M^n, $\langle \tau\,,\, p\rangle \;=\tau^i\,p_i$ - natural Invariant Inner Product.

Language of 2-forms

$$\Omega \;=\; \sum dx^i \wedge dp_i \;\; , \qquad g_{ij} \;=\; \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$((x^1, \ldots, x^n, p_1, \ldots, p_n)$ - local coordinates in $T_*(M^n)\,)$.

“Gradient systems” in the Symplectic Manifolds are called “Hamiltonian systems”.

Homework 2.

1. How many projection local coordinate systems for $\mathbb{S}^n \in \mathbb{R}^{n+1}$ are needed to cover all sphere? $n = 1, 2, 3, \dots$.

2. How many local coordinate systems are needed to cover $\mathbb{RP}^n$? Let $n = 1, 2, 3$.

$$x \in \mathbb{RP}^n = \left(\mathbb{R}^{n+1}\backslash 0\right)/x \sim \lambda x \ , \qquad \lambda \neq 0$$

$$(x^0, \dots, x^n) \sim (\lambda x^0, \dots, \lambda x^n) \ , \qquad \lambda \neq 0$$

3. Prove that $SO_2 = U_1 = \mathbb{S}^1$ and $SO_3 = \mathbb{RP}^3 = \mathbb{S}^3/\pm 1$.

4. Which matrices belong to the Lie Algebra of the group $SO_{2,1}$?

5. Find the group $O_{1,1}$. How many components does it have?

6. Prove that $SO_3 = SU_2/\pm 1$.

7. Prove that $SO_4 = \mathbb{S}^3 \times \mathbb{S}^3/\pm(1,1)$.

8. Prove that $GL_2(\mathbb{R})$ has 2 components. Same for O_n .

Lecture 8. Riemannian, Pseudo-Riemannian and Symplectic Geometries.Complex Geometry. Restriction of Metric to submanifolds. Length of curves and Fermát Principle.

I. Riemannian Geometry = Manifold + Riemannian Metric g_{ij} , $g_{ij}\,\eta^i\,\eta^j > 0$.

II. Pseudoriemannian Geometry = Manifold + Pseudoriemannian Metric g_{ij} , $\det g_{ij}(x) \neq 0$.

III. Symplectic Geometry = Manifold + Symplectic Inner Product $g_{ij}(x) = -\,g_{ji}(x)$, $\det g_{ij}(x) \neq 0$. Corollary: $\dim M = 2k$.
Main Example (Physics) = $T_*(M^n)$.

Flat Geometry:

$$M^n = \mathbb{R}^n \ , \qquad g_{ij} = \text{const}$$

1. Riemannian Case: $||\eta||_g^2 \,>\, 0$, $\langle e_i\,,\,e_j\rangle \,=\, \delta_{ij}$.

2. Pseudoriemannian Case: type p, q, $\langle e_i\,,\,e_j\rangle \,=\, \pm\delta_{ij}$.

3. Symplectic Case: M^{2k} ,

$$\langle e_i'\,,\,e_j'\rangle \;=\; 0 \;\;,\;\; \langle e_i''\,,\,e_j''\rangle \;=\; 0 \;\;,\;\; \langle e_i'\,,\,e_j''\rangle \;=\; -\,\langle e_j''\,,\,e_i'\rangle \;=\; \delta_{ij}$$

$$g_{ij} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

4. Complex Geometry: $(z^1, \dots, z^n)$, $z^l \,=\, x^l + iy^l$. **Complex Vectors:** $\eta \,=\, (\eta^1, \dots, \eta^n)$, $\eta^j \,\in\, \mathbb{C}$.

$$\langle \eta\,,\,\zeta\rangle \;=\; g_{ij}\,\eta^i\,\bar{\zeta}^j$$

$$\langle e_i\,,\,e_j\rangle \;=\; g_{ij} \;=\; \bar{g}_{ji}$$

$||\eta||^2 \,>\, 0$ − positive case, $||\eta||^2 \,=$ indefinite type → real form of type (p, q).

Volume element in M^n, $g_{ij}(x)$:

$$d^n\sigma \;=\; \sqrt{\det g_{ij}} \;\;,\;\; dx^1 \otimes \dots \otimes dx^n \;=\; \sqrt{g}\,d^nx$$

(Riemannian Case)

$$d^n\sigma \;=\; \sqrt{(-1)^q \det g_{ij}} \;\;,\;\; dx^1 \otimes \dots \otimes dx^n \;=\; \sqrt{\pm g}\,d^nx$$

(Pseudoriemannian Case)

Transformation Rule (**"Measure"**) :

$$x \;=\; x(y) \;\;, \quad d^nx \;=\; dx^1 \dots dx^n \;=\; \left|\det \frac{\partial x^i}{\partial y^j}\right| \, dy^1 \dots dy^n \;=\; |J|\,d^ny$$

Important Remark. For manifolds given by **Oriented Atlas** $(J > 0)$ we can write $d^n x = J\, d^n y$ (differential forms !), $J = \det ||\partial x^i / \partial y^j||$.

Lemma 1. Riemannian Metric in the manifold M^n defines Riemannian Metric in every submanifold $W^k \subset M^n$.

Proof.

Let $W^k \subset M^n$, locally we have for local coordinates y in M^n : $y^i = y^i(x^1, \ldots, x^k)$, $i = 1, \ldots, n$, where x represent some local coordinates in W^k.

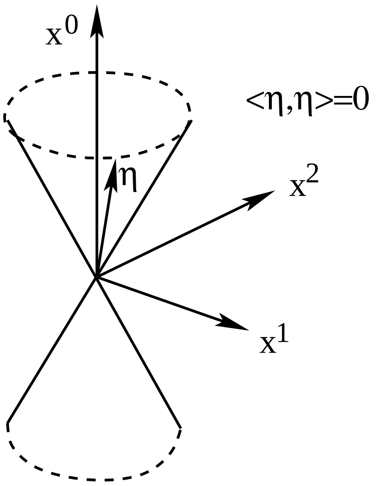
We define "**restriction** of metric"

$$g'_{ij}(x) \;=\; g_{sk}(y(x))\, \frac{\partial y^s}{\partial x^i} \frac{\partial y^k}{\partial x^j}$$

(restriction of inner product on every linear subspace of tangent space). It remains **positive**.

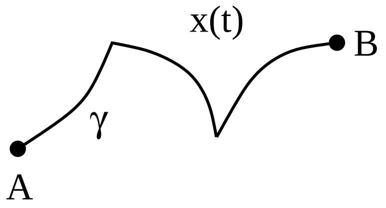
Lemma is proved.

Remark. The analogous lemma is wrong for Pseudoriemannian or Symplectic Geometry because after the restriction to linear subspace metric might become **degenerate**.



$\mathbb{R}^{1,2}$, light cone: $\langle \eta \,, \eta \rangle = 0$.

Riemannian Metric $\Leftrightarrow$ length of **piecewise smooth** curves.



$$l(\gamma) \;=\; \int_a^b \sqrt{g_{ij}(x(t))\,\dot{x}^i\,\dot{x}^j}\;dt$$

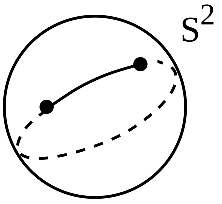
“Distance” $=$ $\min_\gamma l(\gamma)$

Statement. Riemannian metric transforms M^n into **metric** space

$$l\,(a,c) \;\leq\; l\,(a,b) \,+\, l(b,c)$$

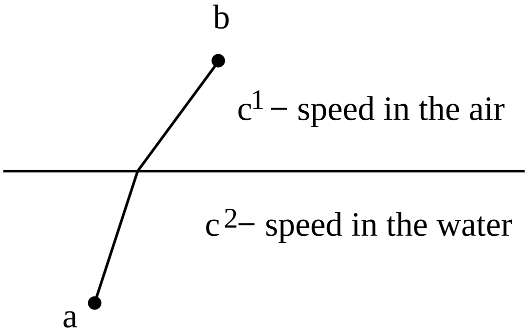
Another metric induced by imbedding $M^n \subset \mathbb{R}^N$ (Either M^n is compact or $M^n \cap D_\rho^N$ is compact for all $\rho \geq 0$) .

“Geodesics” $=$ **“Locally shortest”** paths



$\mathbb{R}^n$: geodesics $=$ **“straight lines”** .

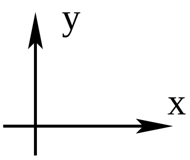
P. Fermat (XVII Century)



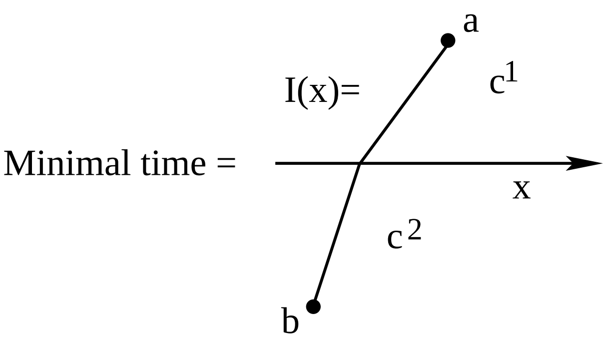
“Minimal Time Principle”: Light propagates from the point A to the point B along the path with minimal time among all piecewise smooth paths joining these 2 points

$$\text{Time} \;=\; \int_a^b \frac{|dl|}{c(x(t))} \;=\; \int_a^b \sqrt{\frac{dl^2}{c^2(x)}}$$

$$dl \;=\; \sqrt{dx^2 + dy^2}$$



$$\text{Let} \quad c(x) \;=\; \begin{cases} c_1\,, & y \,>\, 0 \\ c_2\,, & y \,<\, 0 \end{cases}$$



Minimize this integral:
$\min_x I(a, b, x) =?$

So we have a "Fermat Riemannian Metric"

$$g^F_{ij} \;=\; \frac{\delta_{ij}}{c^2(x)} \;\;, \qquad ||\eta||^2_F \;=\; ||\eta||^2_E\,\frac{1}{c^2} \;\;, \qquad ||\eta||_F \;=\; ||\eta||_E\,\frac{1}{c^2}$$

($F \;=\;$ **"Fermat"**, $E \;=\;$ **Euclid**) .

Speed of light in vacuum $c = c_{vac} \sim 3 \cdot 10^{10}\, cm/sec\,,$ $\underline{c_{media} \,<\, c_{vacuum}}$
.

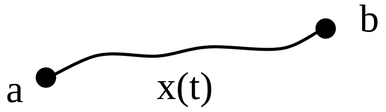
How to find geodesics?

Euler - Lagrange (XVIII Century)

Consider **more general** problem. Let $L(x, \eta)$ ("Lagrangian") be a smooth function in $T^*(M^n)$ (tangent manifold). Fix points $a, b \,\in\, M^n$. Find "extreme curves" for the action

$$S(\gamma) \;=\; \int_0^1 L\,(x(t),\, \dot{x}(t))\; dt$$

($\eta = \dot{x}$) on the piecewise smooth paths: $x(t)$, $x(0) = a$, $x(1) = b$.



Examples:

Geometry: action and length functionals

$$\text{a)} \quad L = \frac{1}{2} g_{ij}(x)\, \eta^i \eta^j = \frac{1}{2} ||\eta||^2$$

$$\text{b)} \quad L' = ||\eta|| = \sqrt{g_{ij}\, \eta^i \eta^j} \quad - \quad \textbf{length}$$

Physics: action functional

$$L = \frac{1}{2} ||\eta||^2 - U(x)$$

(gravity, electric fields)

$$L = \frac{1}{2} ||\eta||^2 - e\, U(x) + \frac{e}{c} A_i(x(t))\, \dot{x}^i$$

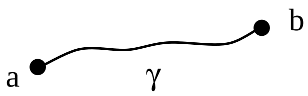
- electric field $E_i = -\,\partial U / \partial x^i$, magnetic field $B_{ij} = \partial A_i / \partial x^j - \partial A_j / \partial x^i$.

Lecture 9. Geodesics and Calculus of variations. Length and Action functionals. Examples.

Geodesics: M^n, $(x^1, \dots, x^n)$ (local coordinate system), $g_{ij}(x)$, $\underline{g_{ij}(x)\, \eta^i \eta^j > 0}$

.

Length of path $\{x(t)\} = \gamma$



$$l(\gamma) = \int_a^b \sqrt{g_{ij}(x(t))\, \dot{x}^i \dot{x}^j}\; dt$$

More general:

"Lagrangian" $L(x, \eta) : T^*(M^n) \to \mathbb{R}$ is given.
"Action functional" is given

$$S\{\gamma\} = \int_\gamma L\,(x(t),\, \dot{x}(t))\; dt$$

Examples:

1) $L = g_{ij}(x)\, \dot{x}^i\, \dot{x}^j/2 = ||\dot{x}||^2/2$
2) $L' = \sqrt{g_{ij}\, \dot{x}^i\, \dot{x}^j} = ||\dot{x}||$
3) $L'' = ||\dot{x}||^2/2 - U(x)$ (gravity or electricity)
4) $L''' = ||\dot{x}||^2/2 + (e/c)\, A_i(x)\, \dot{x}^i$ - magnetic field (its vector-potential).
5) "Relativistic Particle" (?)

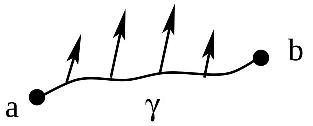
Geodesics: Either (1) or (2)

P. Fermat: $g_{ij}(x) = \delta_{ij}/c^2(x)$

"Variation" of path γ

$$\gamma + \epsilon\, \eta(x(t)) = \gamma_\epsilon$$

(locally it makes sense)

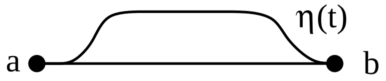


$\eta =$ vector field along (γ) (tangent to M^n).

Variation of Action:

$$S\,\{\gamma + \epsilon\, \eta\} = \int_a^b L\,(x(t) + \epsilon\, \eta(t),\ \dot{x} + \epsilon\, \dot{\eta})\; dt$$

Requirement: vector field should satisfy to some "boundary conditions". At the firs step we take vector fields $\eta(t)$ is C^∞ and **equal to zero** near the endpoints (a) and (b)



Extremal curve or critical point: (to find it necessary to solve following equation for all boundary conditions at the endpoints)

$$\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = 0$$

for **all** $\eta(t)$ (C^∞ and 0 near the endpoints).

Lemma (Euler - Lagrange).

Curve γ is **extremal** iff

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \;=\; \frac{\partial L}{\partial x^i}\ ,$$

$L \,=\, L(x,\,\dot{x})\,,\ (x,\,\dot{x}) \,\in\, T^*(M^n)\ .$

Proof. We have

$$\frac{dS}{d\epsilon}\,(\gamma + \epsilon\,\eta)\bigg|_{\epsilon=0} \;=\; \frac{d}{d\epsilon}\int_a^b L\,(x(t) \,+\, \epsilon\,\eta(t),\ \dot{x} \,+\, \epsilon\,\dot{\eta})\ dt \;=$$

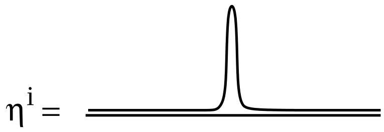
$$= \int_a^b \left(\frac{\partial L}{\partial x^i}\,\eta^i \;+\; \frac{\partial L}{\partial \dot{x}^i}\,\dot{\eta}^i\right)\,dt \;=\; \int_a^b \left(\frac{\partial L}{\partial x^i} \;-\; \frac{d}{dt}\,\frac{\partial L}{\partial \dot{x}^i}\right)\,\eta^i\ dt$$

which is true for all $\eta(t)$ which are C^∞ and equal to zero near a and b. Indeed, we have

$$\int_a^b \frac{\partial L}{\partial \dot{x}^i}\,\dot{\eta}^i\,dt \;=\; -\int_a^b \eta^i\,\frac{d}{dt}\,\frac{\partial L}{\partial \dot{x}^i}\,dt \;+\; \left(\eta^i\,\frac{\partial L}{\partial \dot{x}^i}\right)\bigg|_a^b \;=\; -\int_a^b \eta^i\,\frac{d}{dt}\,\frac{\partial L}{\partial \dot{x}^i}\,dt$$

because $\eta^i(a) \,=\, \eta^i(b) \,=\, 0$.

Take now $\eta \,=\, (0,\,\ldots,\,0,\,\eta^i,\,0,\,\ldots,\,0)$



Conclusion: We have the Euler - Lagrange System of ODE

$$\frac{\partial L}{\partial x^i} \;=\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

near $t = t_0$ for all $t_0 \in (a, b)$.

Lemma is proved.

Terminology:

$$\partial L/\partial \dot{x}^i \;=\; p_i \;=\; \text{“Momentum”}$$

$$\partial L/\partial x^i \;=\; f_i \;=\; \text{“force”}$$

$$\boxed{\dot{p}_i \;=\; f_i}$$

$$L \;=\; \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j \quad \Rightarrow \quad p_i \;=\; g_{ij}\, \dot{x}^j$$

$$\begin{array}{ccc} T^*(M^n) & \to & T_*(M^n) \\ \dot{x} & \to & p \\ \text{velocity} & & \text{momentum} \\ \text{vector} & & \text{(covector)} \end{array}$$

Equation of Geodesics:

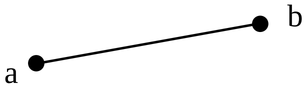
$$\dot{p}_k \;=\; \frac{\partial L}{\partial x^k} \;=\; \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k}\right) \dot{x}^i\, \dot{x}^j$$

Another form (for length) $L' \;=\; \sqrt{g_{ij}\, \dot{x}^i\, \dot{x}^j}$:

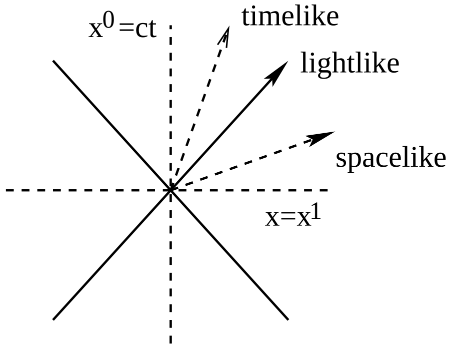
$$\frac{d}{dt}\left(\frac{p_k}{\sqrt{g_{ij}(x)\, \dot{x}^i\, \dot{x}^j}}\right) \;=\; \frac{\partial L'}{\partial x^k} \;=\; \frac{1}{2\sqrt{g_{ij}(x)\, \dot{x}^i\, \dot{x}^j}}\, \frac{\partial g_{ij}}{\partial x^k}\, \dot{x}^i\, \dot{x}^j$$

Conclusion. Let parameter t for the length functional L' is "natural", i.e. $t \;=\; l(\gamma)$ (length), $dt \;=\; \sqrt{g_{ij}(x(t))\, \dot{x}^i\, \dot{x}^j}$. Then we have same equations for both L and L' because $\sqrt{g_{ij}(x(t))\, \dot{x}^i\, \dot{x}^j} \;=\; 1$.

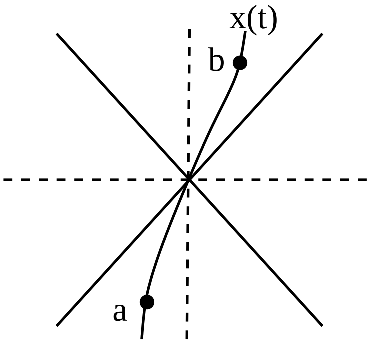
Corollaries.
1) Geodesics for $\mathbb{R}^n$, $g_{ij} \;=\; \delta_{ij}$, are the **straight lines**.



2) Relativistic Particles: (x^0, x^1, x^2, x^3),
$g_{ij} \;=\; \mathrm{diag}\,(1, -1, -1, -1)$.
Let $x^0 \;=\; ct$,
$\eta^2 \;<\; 0$ - "spacelike" vector,
$\eta^2 \;=\; 0$ - "lightlike" vector,
$\eta^2 \;>\; 0$ - "timelike" vector.



Requirement:
For every real material object we have:
$\langle \dot{x} \, , \, \dot{x} \rangle \; \geq \; 0$.
If the mass of the object $> \; 0$ then we have:
$\langle \dot{x} \, , \, \dot{x} \rangle \; > \; 0$.



"Time which you lived" :

$$\tau \quad = \quad \frac{1}{c} \int_a^b \sqrt{\langle \dot{x} \, , \, \dot{x} \rangle} \; dt \quad = \quad \frac{1}{c} \; \text{length} \; (\gamma)$$

Let

$$\vec{v} \;\; = \;\; (\dot{x}^1, \, \dot{x}^2, \, \dot{x}^3) \quad , \qquad \vec{w} \;\; = \;\; (\dot{x}^1/c, \;\, \dot{x}^2/c, \;\, \dot{x}^3/c)$$

Then

$$\sqrt{\langle \dot{x} \, , \, \dot{x} \rangle} \quad = \quad c \, \sqrt{1 \; - \; w^2}$$

For a particle of mass m we put :

$$L \quad = \quad - \, m \, c \, \sqrt{\langle \dot{x} \, , \, \dot{x} \rangle}$$

"Momentum":

$$p_i \quad = \quad \frac{\partial L}{\partial \dot{x}^i} \quad = \quad \frac{m \, \dot{x}^i}{\sqrt{1 - w^2}} \quad , \qquad i = 1, 2, 3$$

Energy :

$$\mathcal{E} \quad = \quad \dot{x}^i \, \frac{\partial L}{\partial \dot{x}^i} \; - \; L$$

Examples :

1) Geodesics

$$L \quad = \quad \frac{1}{2} \, g_{ij}(x) \, \dot{x}^i \, \dot{x}^j \quad \Rightarrow \quad \mathcal{E} \; = \; L$$

$$L' \quad = \quad \sqrt{g_{ij}(x) \, \dot{x}^i \, \dot{x}^j} \quad \Rightarrow \quad \mathcal{E} \; \equiv \; 0$$

2) Gravity or electric field

$$L'' = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j \ - \ U(x) \quad \Rightarrow \quad \mathcal{E} = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j \ + \ U(x)$$

3) Magnetic field

$$L''' = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j \ + \ \frac{e}{c}\, A_i(x)\, \dot{x}^i \quad \Rightarrow \quad \mathcal{E} = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j$$

Equations :

1) Geodesics

$$\dot{p}_k = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i\, \dot{x}^j \quad , \qquad p_k = g_{kj}\, \dot{x}^j$$

2) Gravity or electric field

$$\dot{p}_k \ - \ \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i\, \dot{x}^j = - \frac{\partial U}{\partial x^k} \quad , \qquad p_k = g_{kj}\, \dot{x}^j$$

3) Magnetic field

$$\dot{p}_k \ - \ \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i\, \dot{x}^j = \frac{e}{c}\, \frac{\partial A_i}{\partial x^k}\, \dot{x}^i \quad , \qquad p_k = g_{kj}\, \dot{x}^j \ + \ \frac{e}{c}\, A_k(x)$$

i.e.

$$\frac{d}{dt} \left(g_{kj}\, \dot{x}^j \right) \ - \ \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i\, \dot{x}^j = \frac{e}{c} \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right) \dot{x}^i$$

Magnetic field

$$B_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$$

Lecture 10. Variational problem and geodesics on Riemannian manifolds: Action Functional, Lagrangian, Energy, Momentum. Conservation of Energy and Momentum.

M^n, $(x^1, \ldots, x^n)$, $g_{ij}(x)$.

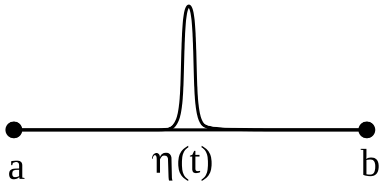
"Lagrangian": $L(x, \eta) \, : T^*(M^n) \, \to \, \mathbb{R}$.

$$S\{\gamma\} \;\; = \;\; \int_\gamma L\,(x(t),\, \dot{x}(t))\; dt \qquad\qquad \text{(Action)}$$

"Momentum": $p_i \;\; = \;\; \partial L/\partial \dot{x}^i \;\; = \;\; \partial L/\partial v^i$

$$\delta\, S\,(\gamma,\, \eta) \quad \to \quad \frac{dS}{d\epsilon}\; (\gamma + \epsilon\, \eta)\Big|_{\epsilon=0} \quad = \quad \int_a^b \left(\frac{\partial L}{\partial x^i} \;-\; \frac{d}{dt}\, \frac{\partial L}{\partial \dot{x}^i} \right) \eta^i \; dt$$

where $\eta(t)$ represents "variation" of the path γ .



Euler - Lagrange equation:

$$\delta\, S \;\; = \;\; 0 \qquad \leftrightarrow \qquad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \;\; = \;\; \frac{\partial L}{\partial x^i}$$

Examples

1) $L \;\; = \;\; g_{ij}(x)\, \dot{x}^i\, \dot{x}^j/2$

2) $L \;\; = \;\; \sqrt{g_{ij}\, \dot{x}^i\, \dot{x}^j}$

3) **Physics**

$$L \quad = \quad g_{ij}(x)\, \dot{x}^i\, \dot{x}^j/2 \quad - \quad \underset{\substack{\uparrow \\ \text{gravity,} \\ \text{electric} \\ \text{field}}}{U(x)} \quad + \quad \underset{\substack{\uparrow \\ \text{magnetic} \\ \text{field}}}{(e/c)\; A_i(x(t))\; \dot{x}^i}$$

"Energy"

$$\mathcal{E} \;\; = \;\; \dot{x}^i\, \frac{\partial L}{\partial \dot{x}^i} \;-\; L \;\; = \;\; v^i\, \frac{\partial L}{\partial v^i} \;-\; L$$

Energy Conservation Law

Theorem 10.1. For the Euler - Lagrange System we have the following conservation law:

$$\frac{d\mathcal{E}}{dt} = 0$$

(i.e. $\mathcal{E}$ is constant along the trajectories of the Euler - Lagrange System).

Proof.

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) = \ddot{x}^i \frac{\partial L}{\partial \dot{x}^i} + \dot{x}^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}\dot{x}^i - \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i =$$

$$= \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}\right]\dot{x}^i = 0$$

Theorem is proved.

Momentum Conservation Law

Theorem 10.2. Let $L(x, v)$ does not depend on x^1 :

$$L = L\left(x^2, \dots, x^n, v^1, \dots, v^n\right)$$

Then we have: $\dot{p}_1 = 0$ on the trajectories of the Euler - Lagrange System.

Proof.

$$\dot{p}_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = \frac{\partial L}{\partial x^1} = 0$$

Theorem is proved.

Definition. Vector field $\zeta = (\zeta^i(y))$ is called “Symmetry” of Lagrangian if $L(x, v)$ does not depend on x^1 in the local coordinate system (x) where $\zeta = (1, 0, \dots, 0)$.

Corollary. In the original system $(y^1, \dots, y^n)$ we have $\zeta = (\zeta^1, \dots, \zeta^n)$ The component $p_\zeta = p_i \zeta^i$ is a conservative quantity $\boxed{\dot{p}_\zeta = 0}$ because p_ζ is exactly the first component of p in the system $(x^1, \dots, x^n)$ where $\zeta = (1, 0, \dots, 0)$.

Examples.

1) $L \;=\; g_{ij}(x)\,\dot{x}^i\,\dot{x}^j/2$

Energy

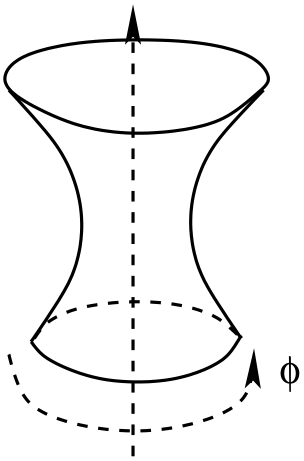
$$\mathcal{E} \;=\; \dot{x}^i\,\frac{\partial L}{\partial \dot{x}^i} \;-\; L \;=\; L$$

$\dot{\mathcal{E}} \;=\; 0 \;\;\Rightarrow\;$ - parameter along geodesics is **NATURAL** because

$$\mathcal{E} \;=\; \frac{1}{2}\,||\dot{x}||_g^2 \;=\; \text{const}$$

along trajectory.

2) Let a surface $M^2 \;\subset\; \mathbb{R}^3$ be invariant under **rotations** around z - axis:



$L \;=\; g_{ij}(x)\,\dot{x}^i\,\dot{x}^j/2$, $n \;=\; 2$, $x \;=\; (\rho,\,\varphi)$, surface $\Phi(\rho,\,z) \;=\; 0$, $\partial L/\partial\varphi \;=\; 0$.

"Angular Momentum"

$$p_\varphi \;=\; \frac{\partial L}{\partial \dot{\varphi}} \;=\; \text{const} \quad - \quad \textbf{along geodesics}$$

Geometrical meaning - later.

Consider now

$$L' \;=\; a_1 \cdot \frac{1}{2}\left[(\dot{x}^0)^2 \;-\; (\dot{x}^1)^2 \;-\; (\dot{x}^2)^2 \;-\; (\dot{x}^3)^2\right] \;=\; a_1 \cdot \frac{1}{2}\,\langle \dot{x}\,,\,\dot{x}\rangle$$

$$L'' \ = \ a_2 \cdot \sqrt{(\dot{x}^0)^2 \ - \ (\dot{x}^1)^2 \ - \ (\dot{x}^2)^2 \ - \ (\dot{x}^3)^2} \ = \ a_2 \cdot \sqrt{\langle \dot{x} \, , \, \dot{x} \rangle} \ , \quad t \ = \ x^0/c$$

Relativity

$$\text{Use } L' \ : \ \mathcal{E}' \ = \ L' \ , \quad p_i^{(4)} \ = \ \frac{\partial L'}{\partial \dot{x}^i}$$

time $\tau \ \sim$ length by theorem above so

$$p_i^{(4)} \ = \ \frac{\partial L'}{\partial \dot{x}^i} \ = \ a_1 \cdot \left(\frac{dx^0}{dt} \frac{dt}{d\tau}, - \frac{dx^1}{dt} \frac{dt}{d\tau}, \ldots, - \frac{dx^3}{dt} \frac{dt}{d\tau} \right)$$

$$\frac{dt}{d\tau} \ = \ 1 \ / \ \frac{d\tau}{dt} \ = \ \frac{1}{\sqrt{c^2 \ - \ \sum_{\alpha=1}^{3} (v^\alpha)^2}} \ = \ \frac{1}{\sqrt{c^2 - v^2}}$$

where

$$v \ = \ \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right)$$

4-D Formalism

So we have for the 4-momentum

$$p^{(4)} \ = \ a_1 \cdot \left(\frac{1}{\sqrt{1 - w^2}}, - \frac{dx^1/dt}{c\sqrt{1 - w^2}}, - \frac{dx^2/dt}{c\sqrt{1 - w^2}}, - \frac{dx^3/dt}{c\sqrt{1 - w^2}} \right)$$

where $w \ = \ v/c$.

Clearly we have to choose: $a_1 \ = \ mc$, where m is the mass of a particle.

3-D Formalism

Use L'' . **We use time** $t \ = \ x^0/c$, $x^0 \ = \ ct$.

$$L'' \ = \ a_2 \cdot \sqrt{(c^2 \ - \ (\dot{x}^1)^2 \ - \ (\dot{x}^2)^2 \ - \ (\dot{x}^3)^2}$$

$$p_\alpha^{(3)} \ = \ \frac{\partial L''}{\partial \dot{x}^\alpha} \ = \ - \, a_2 \cdot \frac{v^\alpha}{\sqrt{c^2 - v^2}} \ = \ - \, \frac{a_2}{c} \, \frac{v^\alpha}{\sqrt{1 - v^2/c^2}}$$

$v^\alpha \ = \ dx^\alpha/dt$, $\alpha \ = \ 1, \ldots, 3$. We choose $a_2 \ = \ - \, mc$.

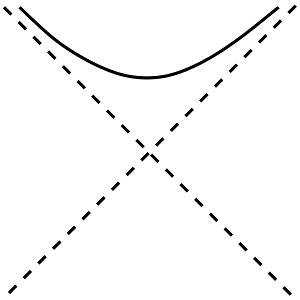
So we finally have

$$\mathcal{E}'' \ = \ \dot{x}^\alpha \, \frac{\partial L''}{\partial \dot{x}^\alpha} \ - \ L'' \ = \ \frac{mc^2}{\sqrt{1 - w^2}} \ , \qquad w \ = \ v/c$$

$\mathcal{E}''$ = "physical energy" $\mathcal{E}$. Let $p_0 = \mathcal{E}/c$. We have

$$p_0^2 - p_1^2 - p_2^2 - p_0^3 = m^2c^2$$

- "MASS SURFACE" in the space $\mathbb{R}^{1,3}$

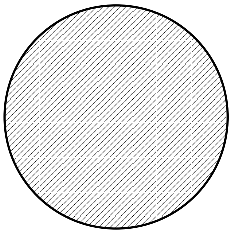


(Hyperbolic = Lobachevsky - Bolyai Space).

"Velocity" of particle

$$v = (v^1, v^2, v^3) \quad , \qquad \frac{v^i}{\sqrt{1 - v^2/c^2}} = p_i$$

We have "Poincare" Model for $3D$ Hyperbolic Space (ball)



Homework 3.

1. Prove that group $O(1,1)$ consists of transformations:

$$P \ , \quad P^2 = 1 \quad , \qquad T \ , \quad T^2 = 1 \quad , \qquad \begin{pmatrix} \operatorname{ch}\varphi & \operatorname{sh}\varphi \\ \operatorname{sh}\varphi & \operatorname{ch}\varphi \end{pmatrix}$$

$$U(1,1) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R} \ \text{(topologically)} \quad , \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$O(2) \cong \mathbb{Z}_2 \times \mathbb{S}^1 \ \text{(topologically)}$$

2. Prove equality $SU(1,1) = SL_2(\mathbb{R})$.

$$SU(1,1) = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad , \quad |a|^2 - |b|^2 = 1 \quad , \qquad SL_2(\mathbb{R}) : \ \{A, \ \det A = 1\}$$

Use change of basis $(e_1, e_2) \leftrightarrow (e = e_1 + ie_2, \ \bar{e} = e_1 - ie_2)$.

3. Prove that $SL_2(\mathbb{R}) \cong \mathbb{R}^2 \times \mathbb{S}^1$ (topologically).

4. Calculate Euclidean Riemannian metric of $\mathbb{R}^2$ in polar coordinates.

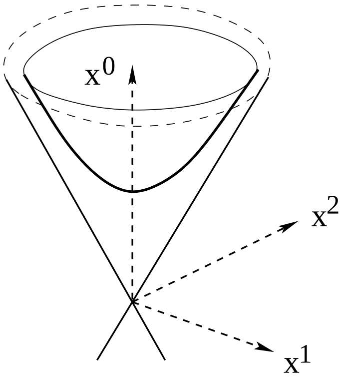
5. Calculate Riemannian Metric of sphere $\mathbb{S}^2$:

$$\mathbb{S}^2 : \; x^2 + y^2 + z^2 \; = \; 1 \quad , \qquad \mathbb{R}^3 : \; dl^2 \; = \; dx^2 + dy^2 + dz^2$$

in the spherical coordinates θ, φ :

$$z \; = \; r \, \cos\varphi \;\; , \quad x \; = \; r \, \sin\theta \, \cos\varphi \;\; , \quad y \; = \; r \, \sin\theta \, \sin\varphi \quad , \qquad (r \; = \; 1)$$

6. Calculate Riemannian Metric of “pseudosphere” (hyperbolic Lobachevsky plane)



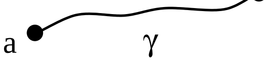
$\mathbb{H}^2$ in $\mathbb{R}^{1,2}$

$$\mathbb{H}^2 : \; \rho^2 \; = \; (x^0)^2 - (x^1)^2 - (x^2)^2 \; = \; 1 \quad , \qquad x^0 \; > \; 0$$

$$\mathbb{R}^{1.2} : \; dl^2 \; = \; (dx^0)^2 \; - \; d(x^1)^2 \; - \; d(x^2)^2$$

Lecture 11. Geodesics and Action Functional. Examples. Hamiltonian form of Euler - Lagrange equation. Conservation of Energy and Momentum. Euclidean, Spherical and Hyperbolic Geometries.

Geodesics:



$$S \, \{\gamma\} \;\; = \;\; \int_\gamma L(x, \, v) \, dt \quad , \qquad v \; = \; \dot{x}$$

$$L \;=\; \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j \quad , \qquad L(x,\eta) \;:\; T^*(M^n) \,\to\, \mathbb{R}$$

Euler - Lagrange Equations:

$$\dot{p}_i \;=\; \frac{\partial L}{\partial v^i} \quad , \qquad p_i \;=\; g_{ij}\, v^j$$

$$g_{ij}(x)\, \ddot{x}^j \;+\; \frac{\partial g_{ij}}{\partial x^k}\; \dot{x}^k\, \dot{x}^j \;\;=\;\; \frac{1}{2}\, \frac{\partial g_{jk}}{\partial x^i}\; \dot{x}^k\, \dot{x}^j$$

$$\ddot{x}^j\, g_{ij} \;\;=\;\; \left(\frac{1}{2}\, \frac{\partial g_{jk}}{\partial x^i} \;-\; \frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^k\, \dot{x}^j$$

$$\ddot{x}^p \;\;=\;\; g^{pi} \left(\frac{1}{2}\, \frac{\partial g_{jk}}{\partial x^i} \;-\; \frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^k\, \dot{x}^j$$

Let us introduce the "Christoffel Symbols"

$$\Gamma^p_{kj} \;\;=\;\; \Gamma^p_{jk} \;\;=\;\; \frac{1}{2}\; g^{pi} \left(\frac{\partial g_{ij}}{\partial x^k} \;+\; \frac{\partial g_{ik}}{\partial x^j} \;-\; \frac{\partial g_{jk}}{\partial x^i} \right)$$

We have then

$$\ddot{x}^p \;+\; \Gamma^p_{kj}\; \dot{x}^k\, \dot{x}^j \;\;=\;\; 0$$

Euclidean (Pseudoriemannian) Metric: $\Gamma^p_{kj} \;=\; 0$.

Examples:

1. $\mathbb{R}^2 \;:\; dx^2 \,+\, dy^2 \;\Rightarrow\; dz\, d\bar{z} \;\;=\;\; dr^2 \,+\, r^2 (d\varphi)^2$

2. $\mathbb{S}^2 \;:\; d\theta^2 \,+\, \sin^2\!\theta\, d\varphi^2 \;\Rightarrow\; \dfrac{dz\, d\bar{z}}{(1+|z|^2)^2}$

3. $\mathbb{H}^2 \;:\; d\chi^2 \,+\, \mathrm{sh}^2\chi\, d\varphi^2 \;\Rightarrow\; \dfrac{dz\, d\bar{z}}{(1-|z|^2)^2} \;\Rightarrow\; \dfrac{dz\, d\bar{z}}{y^2}$

Lemma 1. For the general metric of the form $dl^2 \;=\; d\rho^2 \;+\; \Phi^2(\rho)\, (d\varphi)^2$ every line $\varphi \,=\, \mathrm{const}$ is geodesics.

Proof. We have

$$L \;=\; \frac{1}{2}\, \left(\dot{\rho}^2 \,+\, \Phi^2(\rho)\, \dot{\varphi}^2 \right)$$

Euler - Lagrange equations:

$$\ddot{\rho} \;=\; \Phi\Phi'(\rho)\,\dot{\varphi}^2 \quad , \qquad \frac{d}{dt}\left(\Phi^2(\rho)\,\dot{\varphi}\right) \;=\; 0$$

So we have: $\dot{\varphi} \,=\, 0$ always satisfy to the Euler - Lagrange equations.
Lemma is proved.

Remark. Integrate Euler - Lagrange equations now:

$$\Phi^2\,\dot{\varphi} \;=\; a \;=\; \text{const} \;\Rightarrow\; \dot{\varphi} \;= a/\Phi^2(\rho)$$

and therefore: $\ddot{\rho} \;=\; \Phi\Phi'(\rho)\,a^2/\Phi^4(\rho) \;=\; a^2\,\Phi'(\rho)/\Phi^3(\rho)$,

$$\text{so :} \quad \ddot{\rho} \;=\; -\,a^2\,\frac{\partial V(\rho)}{\partial \rho} \quad , \quad V(\rho) \;=\; \frac{1}{2\Phi^2(\rho)} \quad ,$$

$$\text{so :} \quad \ddot{\rho}\,\dot{\rho} \;+\; a^2\,\frac{\partial V(\rho)}{\partial \rho}\,\dot{\rho} \quad \text{or} \quad \frac{\dot{\rho}^2}{2} \;+\; \frac{a^2}{\Phi^2(\rho)} \;=\; b \;=\; \text{const}$$

We have then

$$\dot{\rho} \;=\; \sqrt{2(a^2/\Phi^2(\rho) - b)}$$

and

$$t \;=\; \int \frac{d\rho}{\sqrt{2(a^2/\Phi^2(\rho) - b)}} \qquad \Rightarrow \qquad \rho \,=\, \rho(t) \;,\quad \dot{\varphi} \;=\; a/\Phi^2(\rho)$$

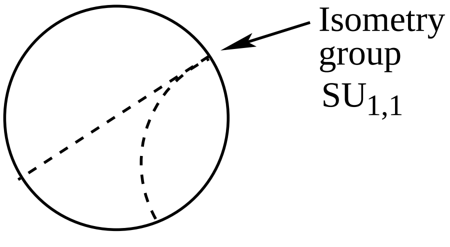
Conclusion. Every geodesics for the $M^2 \;=\; \{\mathbb{R}^2,\, \mathbb{S}^2,\, \mathbb{H}^2\}$ can be obtained from the straight line $\dot{\varphi} \,=\, 0$ (or $\varphi \,=\,$ const) by the action of the following groups:

1) $\mathbb{R}^2 :\; Iso\,(\mathbb{R}^2)$ $-$ straight lines (Euclidean space).

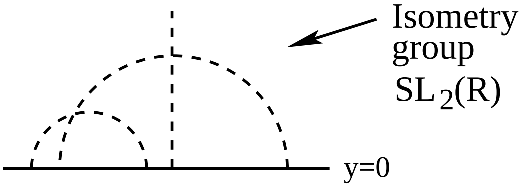
2) $\mathbb{S}^2 :\; SO_3$ $-$ big circles (round sphere).

3) $\mathbb{H}^2 :\; SO_{2,1}$ $(= SL_2(\mathbb{R}),\, SU(1,1)?)$ - circles orthogonal to the boundary in the Poincare model:

$$|z| \;<\; 1 \quad , \qquad dl^2 \;=\; \frac{dz\,d\bar{z}}{(1-|z|^2)^2}$$



Klein Model:
$y = \mathrm{Im}\, z > 0$, $dl^2 = dz\, d\bar{z}/y^2$.



How to prove that they represent the same metric?

Geodesic Flow = Dynamical system in $T^*(M^n) \cong T_*(M^n)$.

$$T^*(M^n): \quad \dot{x}^p = v^p \quad , \quad \dot{v}^p = \Gamma^p_{ij}\, x^i\, x^j$$

$$T^*(M^n): \quad \dot{p}_i = \frac{\partial L}{\partial x^i} \quad , \quad \dot{x}^i = g^{ik}(x)\, p_k \quad , \quad (p_i = g_{ij}(x)\, \dot{x}^j)$$

Theorem. Let

$$\mathcal{E}(x,p) = H(x,p) \quad , \quad \mathcal{E} = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L$$

Then we have

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} \quad , \quad \dot{x}^i = \frac{\partial H}{\partial p_i}$$

Proof. We change $(x, v) \to (x, p)$. By the Euler - Lagrange equations we have for $\mathcal{E} = v^i\, p_i - L(x, v(x,p))$:

$$1) \quad \left(\frac{\partial \mathcal{E}}{\partial p_j}\right)_x = \left(\frac{\partial v^i}{\partial p_j}\right)_x p_i + v^j - \left(\frac{\partial L}{\partial v^i}\right)_x \left(\frac{\partial v^i}{\partial p_j}\right)_x = v^j = \dot{x}^j$$

$$2) \quad \left(\frac{\partial \mathcal{E}}{\partial x^j}\right)_p = \left(\frac{\partial v^i}{\partial x^j}\right)_p p_i - \left(\frac{\partial L}{\partial x^j}\right)_v - \left(\frac{\partial L}{\partial v^i}\right)_x \left(\frac{\partial v^i}{\partial x^j}\right)_p = -\dot{p}_j$$

Theorem is proved.

Definition. $H(x, p)$ = **“Hamiltonian”** (function) (generator of time dynamics).

Remark. Hamiltonian system is a “skew-gradient” flow in the inner product of (x,p) - space

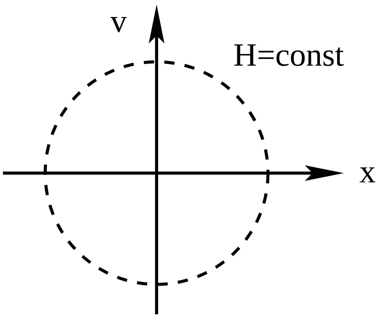
$$g_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -g_{ji}$$

2-form

$$\Omega = \sum_{i=1}^{n} dx^i \wedge dp_i$$

is invariant under the flow (later).

Example. $n = 1$, $H = p^2/2 + V(x)$, $L = \dot{x}^2/2 - V(x)$, $p = v$.



$$\frac{dH}{dt} \equiv 0 \quad , \quad \text{(energy conservation)}$$

For geodesics:

$$H = \frac{1}{2}\, g^{ij}(x)\, p_i\, p_j \quad , \quad L = \frac{1}{2}\, g_{ij}(x)\, \dot{x}_i\, \dot{x}_j \quad ,$$

$$(g^{ij}) = (g_{ij})^{-1} \quad , \quad g^{ik}\, g_{kj} = \delta^i_j \quad , \quad g_{ij} = g_{ji} \quad .$$

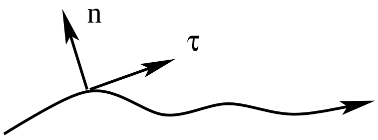
Lecture 12. Curvature of curves and surfaces. How to differentiate tangent vector fields?

Curvature.

1. Curvature of curves $\gamma(t) \subset \mathbb{R}^n$. Let $\dot{\gamma} \neq 0$. Choose normal parameter $t =$ length s , $\gamma \to \gamma(s)$, $|\gamma| = 1$, $\dot{\gamma} = \tau$.

Definition. $|\ddot{\gamma}| =$ curvature

Special case $n = 2$ (oriented plane) :



(τ, n) - orthonormal oriented frame.

Statement.

$$\frac{d\tau}{ds} = k \cdot n$$

$(k = \text{ curvature })$.

Proof.

$$\langle \tau , \tau \rangle = 1 \quad \Rightarrow \quad 2 \langle \tau , \frac{d\tau}{ds} \rangle = 0 \quad \Rightarrow \quad \dot{\tau} \perp n \quad \Rightarrow \quad \dot{\tau} = k \cdot n$$

Statement is proved.

Frené:

$$(\tau(s) , n(s)) = A(s) \ (\tau_0, n_0) \quad , \quad A(s) \in SO_2 \ ,$$

so

$$\frac{dA}{ds} A^{-1} = B(s) = - B^t(s)$$

So we have:

$$\frac{dA}{ds} = B(s) A(s) \quad \text{or} \quad \frac{dA}{ds} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} A(s)$$

or

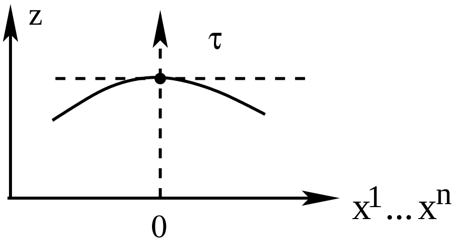
$$\dot{\tau} = k n \quad , \quad \dot{n} = - k \tau$$

2. Curvature of Hypersurfaces

$$M^n \subset \mathbb{R}^n \ (x^0, x^1, \ldots, x^n)$$

Let M^n is given by the equation

$$x^0 = z = F(x^1, \ldots, x^n)$$



Coordinate system is chosen in such a way that $z \perp M^n$ at the point P: $(x^1, \ldots, x^n) = (0, \ldots, 0)$, so we have near P:

$$\left.\frac{\partial z}{\partial x^j}\right|_P = \left.\frac{\partial F}{\partial x^j}\right|_{(0,\ldots,0)} = 0 \quad , \qquad j = 1, \ldots, n$$

Curvature form ("Second quadratic form") $k_{ij}\, dx^i\, dx^j$:

$$k_{ij}\, dx^i\, dx^j = \frac{\partial^2 F}{\partial x^i \partial x^j}\, dx^i\, dx^j$$

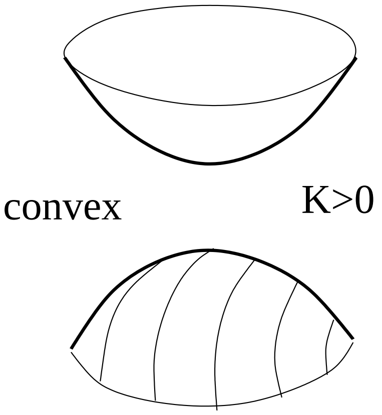
(at P only!)

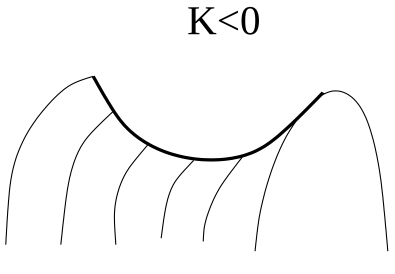
It describes curvature of **Normal Sections** (curves) in $\mathbb{R}^{n+1}$, $n \geq 2$.

$$''\text{Principal curvatures}'' = \text{eigenvalues } k_j$$

$$''\textbf{Gaussian Curvature}'' = \det \left(\frac{\partial^2 F}{\partial x^i \partial x^j} \right) = K$$

$$''\textbf{Mean Curvature}'' = \text{Tr} \left(\frac{\partial^2 F}{\partial x^i \partial x^j} \right)$$





Gauss Theorem. Gauss Curvature depends only on the Riemannian Metric in M^n. (Later).

Riemannian Metric for $M^2 \subset \mathbb{R}^3$. Let $z = F(x, y)$, where (z, x, y) is an orthonormal coordinate system in $\mathbb{R}^3$.

$$dl^2 = dz^2 + dx^2 + dy^2 = (F_x\, dx + F_y\, dy)^2 + dx^2 + dy^2 =$$

$$= \left(1 + F_x^2\right) dx^2 + \left(1 + F_y^2\right) dy^2 + 2\, F_x\, F_y\, dx\, dy$$

For the point $P \in M^n$ where $z \perp M^n$ we have $g_{ij}(x) = \delta_{ij}$.

Invariant form of **characteristic Polynomial**

$$\det\, (k_{ij} - \lambda\, g_{ij})$$

Here $(\lambda_1, \ldots, \lambda_n)$ - principal curvatures, $\sum \lambda_j$ - mean curvature, $\lambda_1 \ldots \lambda_n$ - Gaussian curvature, k_{ij} **transforms as a quadratic form in the tangent space.**

Our Goal : **Riemannian Curvature** and **Curvature in the vector bundles.**

Derivative of function

$$d \;:\quad f \;\to\; df \;=\; \sum f_{x^i}\, dx^i$$

covector field:

$$(\eta_i) \;=\; (f_{x^i})$$

Vector field:

$$\left(\zeta^i\right) \;\overset{d}{\to}\; \left(\partial_i \zeta^i\right)$$

Transformation rule $x = x(y)$

$$(x) \qquad \zeta^i \;=\; \zeta'^j\, \frac{\partial x^i}{\partial y^j} \qquad (y)$$

Consider now

$$\partial_k\, \zeta^i \;=\; \frac{\partial}{\partial x^k} \left(\zeta'^j\, \frac{\partial x^i}{\partial y^j} \right) \quad , \qquad \text{where} \quad \frac{\partial}{\partial x^k} \;=\; \frac{\partial y^p}{\partial x^k}\, \frac{\partial}{\partial y^p}$$

We have

$$\frac{\partial}{\partial x^k}\, \zeta^i \;=\; \frac{\partial y^p}{\partial x^k}\, \frac{\partial}{\partial y^p} \left(\zeta'^j\, \frac{\partial x^i}{\partial y^j} \right) \;=\; \frac{\partial y^p}{\partial x^k}\, \frac{\partial x^i}{\partial y^j}\, \frac{\partial \zeta'^j}{\partial y^p} \;+\; \frac{\partial y^p}{\partial x^k}\, \frac{\partial^2 x^i}{\partial y^p \partial y^j}\, \zeta'^j$$

Conclusion: Second derivative enters in the change of coordinates rule

$$\left(\frac{\partial}{\partial x^k}\,\zeta^i\right) \quad = \quad \underbrace{\left(\frac{\partial}{\partial y^p}\,\zeta'^j\right)\frac{\partial y^p}{\partial x^k}\,\frac{\partial x^i}{\partial y^j}}_{\text{TENSOR LAW}} \qquad + \qquad \underbrace{\frac{\partial^2 x^i}{\partial y^p \partial y^j}\,\frac{\partial y^p}{\partial x^k}\,\zeta'^j}_{\text{NONTENSOR LAW}}$$

For covectors:

$$\frac{\partial}{\partial x^k}\,(\eta_i) \;=\; \frac{\partial y^p}{\partial x^k}\,\frac{\partial}{\partial y^p}\left(\eta'_j\,\frac{\partial y^j}{\partial x^i}\right) \;=\; \left(\frac{\partial}{\partial y^p}\,\eta'_j\right)\frac{\partial y^p}{\partial x^k}\,\frac{\partial y^j}{\partial x^i} \;+\; \frac{\partial^2 y^j}{\partial x^k \partial x^i}\,\eta'_j$$

Conclusion 1. Skew symmetric expression $\partial_k\,\eta_i \;-\; \partial_i\,\eta_k$ transforms as a tensor (inner product)

$$(x) \qquad (\partial_k\,\eta_i \;-\; \partial_i\,\eta_k) \;=\; \left(\partial'_p\,\eta'_q \;-\; \partial'_q\,\eta'_p\right)\frac{\partial y^p}{\partial x^k}\,\frac{\partial y^q}{\partial x^i} \qquad (y)\ , \qquad x \;=\; x(y)$$

or

$$(x) \qquad \frac{\partial x^k}{\partial y^p}\,\frac{\partial x^i}{\partial y^q}\,(\partial_k\,\eta_i \;-\; \partial_i\,\eta_k) \;=\; \partial'_p\,\eta'_q \;-\; \partial'_q\,\eta'_p$$

which makes sense even if number of $(x)'$s (n) is not equal to number of $(y)'$s (m) !

$$f\ :\quad \begin{array}{ccc} (y) & \to & (x) \\ N^m & \to & M^n \end{array}$$

We call the corresponding tensor “differential 2-form”

$$h_{ij}(x) \;=\; -\,h_{ji}(x)$$

$$h'_{pq} \;=\; h_{ij}\,\frac{\partial x^i}{\partial y^p}\,\frac{\partial x^q}{\partial y^q} \qquad , \qquad h \;\to\; f^*(h)$$

$$f^*\ :\quad \begin{array}{ccc} h' & \leftarrow & h \\ 2-\text{form in } N^m & & 2-\text{form in } M^n \end{array}$$

Conclusion 2. Quantities $\partial_i\,\eta_j$ and $\partial_k\,\zeta^p$ are **not** tensors. How to make tensors?

"Covariant derivatives"

$$\nabla_i \eta^j = \partial_i \eta^j + \Gamma^j_{ik} \eta^k$$

$$\nabla_i \eta_j = \partial_i \eta_j - \Gamma^k_{ij} \eta_k$$

where Γ^i_{jk} are "Christoffel Symbols".

Requirements: $\nabla_i \eta^j$ and $\nabla_i \eta_j$ **are tensors**.

Difficulty: $\nabla_i \nabla_j \neq \nabla_j \nabla_i$, $\nabla_i \nabla_j - \nabla_j \nabla_i = \hat{R}_{ij}$ (**"Curvature"**).

More general (local picture): consider $\Psi(x) \in \mathbb{R}^L_x$, where $\mathbb{R}^L_x$ is a linear space with basis $(e_1(x), \ldots, e_L(x))$ depending on the point $(x^1, \ldots, x^n)$, $\Psi(x) = \sum \Psi^i e_i(x)$. Consider the set of linear equations

$$\partial \Psi^i / \partial x^j = A^i_{jk} \Psi^k (x^1, \ldots, x^n)$$

Can we solve this system?

Let us define Operators of "Covariant Derivatives"

$$\nabla_j = \frac{\partial}{\partial x^j} - A^i_{jk}(x)$$

Lemma. The set of linear equations (above) is solvable for all **"initial data"** $\Psi(x_0) = \Psi_0$ if and only if

$$\nabla_i \nabla_j - \nabla_j \nabla_i = R_{ij} = 0$$

for all $x \in \mathbb{R}^n$.

Proof. For every solution $\Psi(x)$ we have

$$\partial_i \Psi = A_i \Psi \qquad (A_i = \text{matrix})$$

and

$$\partial_i \partial_j \Psi = \partial_i (A_j \Psi) = \partial_j (A_i \Psi)$$

or

$$(\partial_i A_j - \partial_j A_i) \Psi + A_j (\partial_i \Psi) - A_i (\partial_j \Psi) = 0$$

or

$$(\partial_i A_j - \partial_j A_i - A_i A_j + A_j A_i) \Psi = 0$$

where

$$(\partial_i A_j - \partial_j A_i - A_i A_j + A_j A_i) = R_{ij} \qquad \text{(matrix)}$$

So we have $R_{ij}\,\Psi = 0$ for all $x \in \mathbb{R}^n$. Therefore $R_{ij} \equiv 0$.

Lemma is proved.

$$R_{ij} = [\nabla_i , \nabla_j]$$

Examples.

a) $L = n$, Ψ - vector field, $A^i_{jk} = -\Gamma^i_{jk}$.

b) $L = n$, Ψ - covector field, $A^i_{jk} = \Gamma^i_{jk}$.

c) $L = 1$, A_i - scalar values, $A_i A_j - A_j A_i = 0$,

$$\partial_i A_j - \partial_j A_i = R_{ij}$$

Lecture 13. Vector bundles. Connection and Curvature. Parallel transport.

Vector bundles and Curvature.

Vector bundle = Family of Linear Spaces $\mathbb{R}^N_x$ depending on parameter $x \in X$ and "locally trivial":

For every point $x \in X$ there exists an open set $U \ni x$ such that we can choose a basis $[e^U_1(x), \ldots, e^U_N(x)] \in \mathbb{R}^N_x$ continuously depending on $x \in X$. For $\underline{x \in U \cap V}$ we have

$$e^U_i(x) = a^{j\,UV}_i(x)\, e^V_j(x)$$

Real case: $a^i_j \in GL_n(\mathbb{R})$

Complex case: $a^i_j \in GL_n(\mathbb{C})$

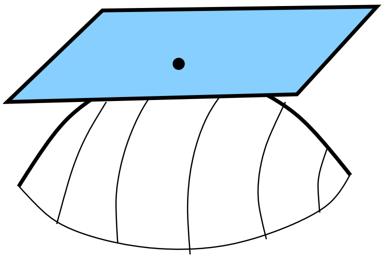
Orthogonal case: $a^i_j \in O_n$

Unitary case: $a^i_j \in U_n$

- For all pairs U, V (!).

Other groups are also possible and define the “structural group” of the bundle.

Our case: X is a C^∞-**manifold** and $a^i_j(x)$ are C^∞-**functions**.



Example. Let $X = M^k \subset \mathbb{R}^L$. Consider family of tangent k-spaces to M^k.

Map $G : M^k \to G_{k,L}$ (Grassmann Manifold).

Curvature is local quantity.

Locally vector bundle is given by the product $U \times \mathbb{R}^n$ according to the choice of the basis $e_1(x), \ldots, e_n(x)$ in $\mathbb{R}^n$.

Consider set of linear ODE's

$$\frac{\partial \Psi^j}{\partial x^i} = A^j_{ip}(x)\, \Psi^p(x) \quad , \quad \Psi = \sum \Psi^j(x)\, e_j$$

$i = 1, \ldots, n$, $j, p = 1, \ldots, N$.
or:

$$\nabla_i \Psi^j = \partial_i \Psi^j - A^j_{ip} \Psi^p = 0$$

Is this system solvable?

For $n = 1$ it is true.

For $n > 1$ **it may be wrong! “Curvature”** R_{ij} of this **“Differential Geometric Connection”** $A^j_{ip}(x)$ **should be equal to zero:**

$$[\nabla_i , \nabla_q] = \nabla_i \nabla_q - \nabla_q \nabla_i = \left[\frac{\partial}{\partial x^i} - A^j_{ip}(x) \, , \, \frac{\partial}{\partial x^q} - A^j_{ql}(x) \right] = 0$$

As we saw in the previous lecture, we can formulate the following Lemma:

Lemma. The set of linear equations (above) is solvable for all **“initial data”** $\Psi(x_0) = \Psi_0$ if and only if

$$\nabla_i \nabla_j - \nabla_j \nabla_i = R_{ij} = 0$$

for all $x \in \mathbb{R}^n$.

$$R_{ij} = [\nabla_i , \nabla_j]$$

Examples.

a) $L = n$, Ψ - vector field, $A^i_{jk} = -\Gamma^i_{jk}$.

b) $L = n$, Ψ - covector field, $A^i_{jk} = \Gamma^i_{jk}$.

c) $L = 1$, A_i - scalar values, $A_i A_j - A_j A_i = 0$,

$$\partial_i A_j - \partial_j A_i = R_{ij}$$

“Gauge Transformations”, Curvature

We have $\partial_i \Psi = A_i \Psi$ (Matrix Form), $A_i = A^k_{ij}(x)$, so

$$\nabla_i = \partial_i - A_i \quad , \qquad \nabla_q = \partial_q - A_q \ ,$$

$$[\nabla_i , \nabla_q] = R_{iq} = -\frac{\partial A_q}{\partial x^i} + \frac{\partial A_i}{\partial x^q} + A_i A_q - A_q A_i$$

(Matrices in $\mathbb{R}^N$) .

1) For tangent (cotangent) case $N = n$.

2) For scalar case $N = 1$, n is any.

3) For euclidean case $\langle e_i , e_j \rangle = \delta_{ij}$ we should have $(A_i)^k_j = -(A_i)^j_k$ (skew symmetry).

Change of Basis:

$$e_i = b^i_j(x)\, e'_j \quad , \qquad [e = B\, e']$$

or

$$\Psi^i\, e_i = \Psi^i\, b^j_i\, e'_j \quad , \qquad \Psi'^j = \Psi^i\, b^j_i$$

$$\vec{\Psi}' = B(x)\, \vec{\Psi}$$

Lemma 2. Let

$$\frac{\partial \Psi^j}{\partial x^i} = A^j_{ik}\, \Psi^k(x)$$

Then for the new basis (e'_j) we have

$$\frac{\partial \Psi'^j}{\partial x^i} = A'^j_{ik}\ \Psi'^k(x)$$

where

$$A'_i = B\, A_i\, B^{-1} + \frac{\partial B}{\partial x^i}\, B^{-1}$$

Proof. We have $\Psi' = B(x)\,\Psi$, so

$$\frac{\partial \Psi'^j}{\partial x^i} = B\, A_i\, \Psi + \frac{\partial B}{\partial x^i}\, \Psi = B\, A_i\, B^{-1}\, \Psi' + \frac{\partial B}{\partial x^i}\, B^{-1}\, \Psi'$$

Lemma is proved.

Conclusion.

$$A'_i = B\, A_i\, B^{-1} + \frac{\partial B}{\partial x^i}\, B^{-1}$$

Let $B^{-1} = G(x)$, we have then

$$A'_i = G^{-1}\, A_i\, G - G^{-1}\, \frac{\partial G}{\partial x^i}$$

- **Gauge Transformation**.

Homework 4.

1. Prove that the group $O(1,1)$ (connected component) can be written in the form

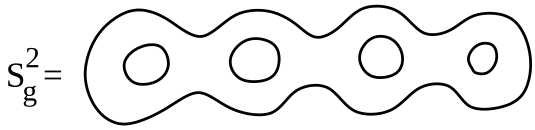
$$\begin{pmatrix} \mathrm{ch}\,\psi & \mathrm{sh}\,\psi \\ \mathrm{sh}\,\psi & \mathrm{ch}\,\psi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-w^2}} & \frac{w}{\sqrt{1-w^2}} \\ \frac{w}{\sqrt{1-w^2}} & \frac{1}{\sqrt{1-w^2}} \end{pmatrix}$$

2. a) Prove that every isometry of $\mathbb{R}^2$, preserving orientation, is either shift or rotation around some point.

b) Prove that every isometry of $\mathbb{R}^2$, inverting orientation, is **product of reflection and shift** along this line (line of reflection).

3. How many local coordinate systems are needed to cover $\mathbb{RP}^2$? Find cover by 3 domains.

4. Find cover of sphere $\mathbb{S}^2_g$ with g handles by 2 systems of local coordinates.



5. Introduce "pseudospherical" coordinates

$$x^0 \;=\; \rho\,\mathrm{ch}\,\theta \;\;,\;\; x^1 \;=\; \rho\,\mathrm{sh}\,\theta\,\cos\varphi \;\;,\;\; x^2 \;=\; \rho\,\mathrm{sh}\,\theta\,\sin\varphi$$

Restrict metric

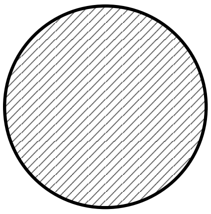
$$(dx^0)^2 \;-\; (dx^1)^2 \;-\; (dx^2)^2$$

on the "pseudoshere" $\rho \,=\, 1$. Calculate it.

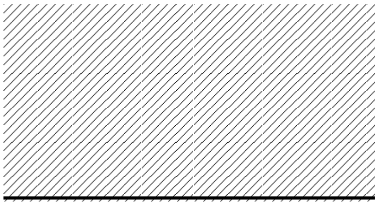
6. Prove that the following metrics are equivalent:

$$a)\;\; |z| \,<\, 1 \quad , \quad dl^2_{(1)} \;=\; \frac{dz\,d\bar{z}}{(1-|z|^2)}$$

$$b)\;\; \mathrm{Im}\,w \,>\, 0 \quad , \quad dl^2_{(2)} \;=\; \frac{dw\,d\bar{w}}{(\mathrm{Im}\,w)^2}$$



(P)



(K)

Use transformation

$$w \;=\; \frac{a\,z \,+\, b}{c\,z \,+\, d}$$

Find a, b, c, d such that **ball** maps into **upper plane**.

7. Find isometry groups for P and K in the form

$$z \;\to\; \frac{a\,z \,+\, b}{c\,z \,+\, d}$$

Prove that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1) \;\text{ for } P \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \;\text{ for } K$$

Lecture 14. Vector bundles. Connection and Curvature. Parallel transport.

Vector bundle=“locally trivial” family of vector spaces $\mathbb{R}^N_x$, $x \in X$ with “local bases” $[e^U_1(x), \dots, e^U_N(x)]$, $x \in U \subset X$, $\forall x \ \ \exists U \ni x$.

For $x \in U \cap V$

$$e^U_i(x) = a^{j,UV}_i(x) e^N_j(x).$$

“Group” $a^{j,UV}_i(x) \in G \subset \mathrm{GL}_N(\mathbb{R})$

Differential geometrical connection (on vector bundle).

Locally matrix-valued functions $A_i(x) = (A_i)^j_k(x)$ are given for all $x \in U \subset X$, and operators $\nabla^U_i = \partial_i - A^U_i$ acting on functions $\Psi^U : U \to \mathbb{R}^N$, $\Psi^U = (\Psi^{j,U})$,

$$\nabla^U_i \Psi^U = \partial_i \Psi^U - A^U_i \Psi^U.$$

In intersection $x \in U \cap V$ we have

$$\Psi^U = g^{UV} \Psi^V, \ \ G(x) = g^{UV}(x),$$

such that:

$$A^V_i = G^{-1}(x) A^U_i G(x) - G^{-1} \partial_i G$$

(gauge equivalence).

“Curvature”

$$R^U_{ij} = \nabla^U_i \nabla^U_j - \nabla^U_j \nabla^U_i \ \ \text{(matrix functions)}.$$

Theorem (Later):

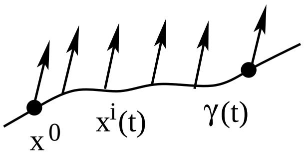
$$R^V_{ij} = G^{-1} R^U_{ij} G \ \ (!)$$

ξ-tangent vector to $X = M^n$ in the point $x \in M^n$, $x^1, \dots, x^n$.

$$\nabla^U_\xi = \left(\frac{\partial \Psi^U}{\partial x^i} - A_i \Psi^U \right) \xi^i$$

– covariant derivative along $\xi \in T^*_X$.

Curve $\gamma = \{x^i(t)\}$. Covariant derivative along γ:

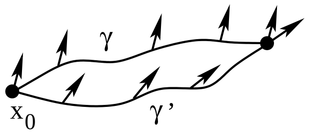


$$\nabla^U_\gamma = \nabla^U_{\dot{x}(t)} = \dot{x}^i \nabla^U_i \Psi^U.$$

Definition. "Parallel" vector field along the line $\gamma(t)$:

$$\nabla^U_{\dot{x}(t)} \Psi^U(x(t)) \equiv 0 \quad \text{for all} \quad t.$$

"Parallel transport" •+ vector $\Psi^U(x_0) = \Psi_0$ along the line $\gamma(t)$, $\gamma(0) = x_0$.



Results for γ_1 and γ_2 may be different!

Geodesics: $N = n$, vector bundle is **tangent** vector bundle to $X = M^n$, $\mathbb{R}^n_x = T^*_x$, $e_1, \ldots, e_n$ – standard basis in tangent spaces in local coordinates $e_i \equiv \partial/\partial x^i$ for $U \subset X$, $X = M^n$.

Definition. $\gamma(t) = \{x^j(t)\}$ is **"geodesic curve"** if $\nabla^U_{\dot{x}(t)} \dot{x}(t) \equiv 0$.

Lemma 1. Curve $\gamma(t) = \{x^j(t)\}$ is geodesic iff the following equation is true:

$$\ddot{x}^j + \tilde{\Gamma}^j_{ik} \dot{x}^i \dot{x}^k = 0,$$

where $\tilde{\Gamma}^j_{ik} = -A^j_{ik}(x)$ (Connection).

Proof. By definition, we have

$$\nabla^U_{\dot{x}(t)} \dot{x}(t) = \dot{x}^k \nabla^U_k \dot{x}(t) = \dot{x}^k \partial_k \dot{x}^j(t) - \dot{x}^k A^j_{ik}(x) \dot{x}^i(t) = \underbrace{\ddot{x}^j(t) + \tilde{\Gamma}^j_{ik} \dot{x}^i \dot{x}^k = 0.}_{\text{Geodesic equation}}$$

"Variational geodesics."

$$\ddot{x}^j + \Gamma^j_{ik} \dot{x}^i \dot{x}^k = 0$$

we need

$$\Gamma^k_{ij} = -\frac{1}{2}(A^k_{ij} + A^k_{ji}) \quad - \quad \textbf{symmetric part}$$

for these 2 equations coincide. Calculating variation, we obtain:

$$\Gamma^l_{ij} = \text{Symmetrization of} \left[g^{lq} \left[-\frac{1}{2} \frac{\partial g_{ij}}{\partial x^q} + \frac{\partial g_{iq}}{\partial x^j} \right] \right] = \frac{1}{2} g^{lq} \left[-\frac{\partial g_{ij}}{\partial x^q} + \frac{\partial g_{iq}}{\partial x^j} + \frac{\partial g_{qj}}{\partial x^i} \right].$$

Calculation of Connection and Curvature: (it is linear operation)

1. Covariant derivative of scalars is trivial:

$$\nabla_\xi f(x) = \xi^i \partial_i f.$$

2. Covariant derivatives of vectors, covectors and tensors like inner products are defined:

$$\nabla_i \xi^j(x) = \partial_i \xi^j + \Gamma^j_{ik}\xi^k \quad \text{vectors},$$

$$\nabla_i \eta_j(x) = \partial_i \eta_j + \tilde{\Gamma}^\rho_{ij}\eta_\rho \quad \text{covectors},$$

$$\nabla_i t_{jk}(x) = \partial_i t_{jk} + G^{pl}_{ijk} t_{pl} \quad \text{inner products of vectors},$$

such that

a) $\nabla_i(\xi^j \eta_j) = \nabla_i(\xi^j)\eta_j + \xi^j \nabla_i(\eta_j) = \partial_i(\xi^j \eta_j)$ (scalars).

b) $\nabla_i(\xi_k \eta_l) = \nabla_i(\xi_k)\eta_l + \xi_k \nabla_i(\eta_l)$ (product of two covectors).

c) $\nabla_i g_{kl} \equiv 0$, g_{kl} =**Riemannian Metric.**

Theorem. There exist unique symmetric connection such that

$$-\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} \tag{1}$$

$$\nabla_i t_{kl} = \partial_i t_{kl} - \Gamma^p_{ik} t_{pl} - \Gamma^p_{il} t_{kp} \tag{2}$$

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2} g^{is}\left[\frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right] \tag{3}$$

Proof of (1): we have (a):

$$\nabla_i(\eta^k \xi_k) = \partial_i i(\eta^k \xi_k) = (\partial_i \eta^k + \Gamma^k_{is}\eta^s)\xi_k + \eta^k(\partial_i \xi_k + \tilde{\Gamma}^s_{ik}\xi_s), \quad \Rightarrow \quad \Gamma = -\tilde{\Gamma}.$$

Proof of (2): we have "Leibnitz"(b):

$$\nabla_i(\eta_l \xi_k) = (\partial_i \eta_l + \tilde{\Gamma}^s_{ik}\eta_s)\xi_k + \eta_l(\partial_i \xi_k + \tilde{\Gamma}^s_{ik}\xi_s),$$

so we see that (2) is true for the "products" $(\eta_l \xi_l)$, so it is true for linear combinations of products and so it is true fro all tensors of the type (t_{kl}). Proof of the formula for Γ^k_{ij} – **next lecture**.

Lecture 15. Connections in tangent bundle. Curvature. Ricci curvature. Einstein equation. Spaces of constant curvature.

Differential-geometrical connections in tangent bundle $T^*(M^n)$, $x^1, \dots, x^n$.

$$\begin{aligned}
&A^i_{jk}(x) = \Gamma^i_{jk}(x) \\
&\nabla_i \eta^j(x) = \partial_i \eta^j + \Gamma^j_{ik}\eta^k, && \text{(vectors)} \\
&\nabla_i \eta_k(x) = \partial_i \eta_k + \tilde{\Gamma}^j_{ik}\eta_j, && \text{(covectors)} \\
&\nabla_i t_{jk}(x) = \partial_i t_{jk} + G^{pl}_{ijk} t_{pl}, && \text{(inner products - tensors)}
\end{aligned}$$

Axioms:

1. $\nabla_i f(x) = \partial_i f(x)$, scalars.
2. $\nabla_i(\xi^j \eta_j) = \nabla_i(\xi^j)\eta_j + \xi^j \nabla_i(\eta_j)$.
3. $\nabla_i(\xi_k \eta_l) = \nabla_i(\xi_k)\eta_l + \xi_k \nabla_i(\eta_l)$.
4. $\nabla_i g_{kl} \equiv 0$, (g_{kl} =Riemannian Metric).

Items 2-3 are Leibnitz rule.

Definition. Connection in $T^*(M)$ is "symmetric" if $\Gamma^k_{ij}(x) = \Gamma^k_{ji}(x)$ (**"torsion"=0**).

"Torsion tensor" $= T^k_{ij}(x) = \Gamma^k_{ij}(x) - \Gamma^k_{ji}(x)$.

Theorem. There exists a unique differential-geometrical connection on $T^*(M)$ extended to covectors and tensors as above, **symmetric** and **compatible with Riemannian metric** $\nabla_i g_{kj} = 0$.

Proof.

Step 1: Prove that $\tilde{\Gamma}^k_{ij} = -\Gamma^k_{ij}$ (follows from the Axioms 1 and 2).

Step 2: Prove that $\nabla_i t_{jk} = \partial_i t_{jk} - \Gamma^s_{ij} t_{sk} - \Gamma^s_{ik} t_{js}$.

Proof. From the Step 1 we have $\tilde{\Gamma} = -\Gamma$. From the Axiom 3 we have our result for $\nabla_i(\sum_P \xi^{(p)}_k \eta^{(p)}_j)$, $t_{kj} = \sum_P \xi^{(p)}_k \eta^{(p)}_j$. Every tensor t_{kj} can be presented in that form (may be as a series).

Step 3: From the condition $\nabla_i g_{kj} = 0$ we have

$$\partial_i g_{kj} = \Gamma^s_{ij} g_{ks} + \Gamma^s_{ik} g_{sj}$$

Let us solve the system of linear equations for the triple (ijk) using condition $\Gamma^k_{ij} = \Gamma^k_{ji}$ (3 equations for 3 unknown quantities $\Gamma^k_{ij} = \Gamma^k_{ji}$) or $\Gamma_{ij,k} = g_{ks}\Gamma^s_{ij}$.

Solution:

$$\Gamma_{ij,k} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_i g_{kj} - \partial_k g_{ij}\right).$$

Our equations are:

$$\partial_i g_{kj} = \Gamma_{ij,k} + \Gamma_{ik,j}$$

$$u = \partial_i g_{kj}, \quad v = \partial_k g_{ij}, \quad w = \partial_j g_{ki},$$

$$a = \Gamma_{ij,k}, \quad \Rightarrow \quad a = \frac{u - v + w}{2},$$

$$\Gamma_{ij;k} = \frac{1}{2}\left(\partial_i g_{kj} - \partial_k g_{ij} + \partial_j g_{ki}\right).$$

$$\Gamma^k_{ij} = g^{ks}\Gamma_{ij;s} = \frac{1}{2} g^{ks}\left(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij}\right).$$

For the case $\Gamma^k_{ij} = \Gamma^k_{ji}$, $g_{ij} = g_{ji}$ we obtain the same formula as for geodesics (for Calculation of Variations)

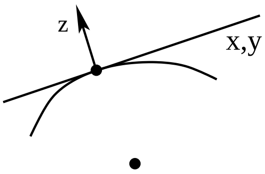
$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0.$$

Curvature.

$$(R^p_k)_{ij} = \nabla_i\nabla_j - \nabla_j\nabla_i = \hat{R}_{ij}, \quad (where) \;\; \hat{R}_{ij} \;\; \text{are matrices.}$$

$$\nabla_i\nabla_j - \nabla_j\nabla_i = [\partial_i + \Gamma_i, \partial_j + \Gamma_j] = \frac{\partial \Gamma_j}{\partial x_i} - \frac{\partial \Gamma_i}{\partial x_j} + \underbrace{\Gamma_i\Gamma_j - \Gamma_j\Gamma_i}_{\text{product of matrices}}.$$

Example. Consider special coordinates for $M^2 \subset \mathbb{R}^3$



$(z \perp M^2$, x, y – local coordinates near $x_0 = (0,0))$,

$$\delta_{ij} = g_{ij}(0), \quad z = F(x,y), \quad g_{xx} = 1 + F_x^2,$$

$$g_{xy} = F_x F_y, \quad g_{yy} = 1 + F_y^2$$

We have finally:

$$g_{ij}(0) = \delta_{ij}, \quad \partial_i g_{kl}(0) = 0, \;\Rightarrow\; \Gamma^i_{jk}(0) = 0,$$

$$\frac{\partial \Gamma^k_{ij}}{\partial x^l} = ? \ \textbf{Calculation is required.}$$

Conclusions from calculations (later):

1. $R_{ij;kl} = g_{is} R^s_{j;kl} = R_{kl;ij}$.

2. $R_{ji;kl} = -R_{ij;kl} = R_{ij;lk}$.

For $n = 2$ we have: $i, j, k, l = 1, 2$. The whole curvature tensor $R_{ij,kl}$ can be defined by 1 scalar function R (why?)

$$R_{ij} = R^k_{i,kj} \quad \text{Ricci Curvature}$$

$$R = R^i_i, \quad R^i_j = g^{is} R_{sj}, \quad \text{Scalar Curvature}$$

Gauss Theorem. $R/2$ = Gaussian Curvature of the surface $M^2 \subset \mathbb{R}^3$ (Calculation later).

For $n = 2$ and $M^2 = S^2, \mathbb{R}^2, H^2$ we have $R = \text{const}$.

For $n = 3$ we have: Ricci Curvature R_{ij} completely determines the whole tensor $R_{ij,kl}$. Why? They have the same number of components.

Curvature of $\mathbb{R}^n$, S^n, H^n.

$$R = \begin{cases} 0, \\ > 0, \\ < 0. \end{cases} \quad ?$$

Curvature of conformally Euclidean metric $g_{ij} = \phi^2(x)\delta_{ij}$.
What is “Curvature of 2-directions”?
Einstein equation: $n = 4$, g_{ij}-indefinite.

$$R_{ij} - \frac{1}{2} R g_{ij} = \lambda g_{ij}, \quad \text{no matter, only gravity.}$$

“cosmological constant” $\lambda \neq 0$.
Curvature of metrics in the compact groups like SO_n, U_n, …?
What does it mean – “Constant curvature”?

$$\nabla_i R_{ij,kl} \equiv 0 \quad \text{“locally symmetric spaces”}.$$

Compact groups, in particular.

Appendix to Lecture 15.

$$\Gamma^k_{ij} = g^{ks}\Gamma_{ij;s} = \frac{1}{2}g^{ks}\left(g_{sj,i} + \partial_j g_{is,j} - g_{ij,s}\right).$$

Our notation:

$$\partial f/\partial x^k \equiv f, k$$

$$\hat{R}_{iq} = \frac{\partial\hat{\Gamma}_q}{\partial x_i} - \frac{\partial\hat{\Gamma}_i}{\partial x_q} + \hat{\Gamma}_i\hat{\Gamma}_q - \hat{\Gamma}_q\hat{\Gamma}_i, \quad \text{where } \hat{\Gamma}_i = \Gamma^k_{ij}.$$

Our system:

$$g_{ij}(0) = \delta_{ij},\ g_{ij,p}(0) = 0,\ g^{ij}(0) = \delta_{ij} = g_{ij}(0),\ g^{ij}{}_{,p}(0) = 0$$

$$\hat{R}_{iq} = R_{jk;iq} = R^j{}_{k;iq}, \qquad (x, y = 0, 0).$$

$$\frac{\partial\Gamma^j_{qk}}{\partial x^i} = \frac{1}{2}(g_{jk,iq} + g_{qj,ik} - g_{qk,ij})$$

$$\frac{\partial\Gamma^j_{ik}}{\partial x^q} = \frac{1}{2}(g_{jk,iq} + g_{ij,qk} - g_{ik,qj})$$

$$2R_{jk;iq} = g_{jk,iq} + g_{qj,ik} - g_{qk,ij} - g_{jk,iq} - g_{ij,qk} + g_{ik,qj} =$$

$$= g_{qj,ik} - g_{qk,ij} - g_{ij,qk} + g_{ik,qj}$$

(All formulas are valid at the point 0,0)

Lecture 16. Tensor fields. Curvature as a tensor field. Gaussian curvature for surfaces in $\mathbb{R}^3$.

- What is a tensor field?
- **Calculations:** Curvature is a tensor field.
- Calculation of Curvature in the special coordinates.
- Algebraic properties of Curvature Tensor.
- Gauss Theorem for $M^2 \subset \mathbb{R}^3$.

- Curvature Tensor at the most symmetric spaces $\mathbb{R}^n$, S^n, H^n.

What is a Tensor? (Scalars, vectors, covectors, inner products, Riemannian Curvature.)

M^n, $x^1, \dots, x^n$ -local coordinates. Tensor field of the type (k, l) is defined by components:

$$T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x), \quad \dim = n^{k+l}, \quad \text{Vector bundle } \Psi : M^n \to \text{Tensors},$$

such that for $x = x(y)$ we have change

$$\acute{T}^{p_1,\dots,p_k}_{q_1,\dots,q_l}(y) = T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x(y)) \left(\frac{\partial x^{j_1}}{\partial y^{q_1}} \cdots \right) \left(\frac{\partial y^{p_1}}{\partial x^{i_1}} \cdots \right)$$

Examples: $T^i(x)$ – vector field.

$$\acute{T}^p(y) = T^i(x(y)) \frac{\partial y^p}{\partial x^i},$$

$$\acute{T}_q(y) = T_j(x(y)) \frac{\partial x^j}{\partial y^q},$$

Inner products:

$$\acute{g}_{pq} = g_{ij} \frac{\partial x^i}{\partial y^p} \frac{\partial x^j}{\partial y^q}.$$

Linear operators:

$$\acute{a}^p_q = a^i_j \frac{\partial y^p}{\partial x^i} \frac{\partial x^j}{\partial y^q}.$$

Inner product of covectors:

$$\acute{g}^{pq} = g^{ij} \frac{\partial y^p}{\partial x^i} \frac{\partial y^q}{\partial x^j}.$$

Operations: linear (sum), product of tensors $\overset{(1)}{T}{}^I_J \cdot \overset{(2)}{T}{}^P_Q = T^{IP}_{JQ}$, permutation of indexes: $g_{ij} \to g_{ji}$.

"Trace": $T^{\dots i \dots}_{\dots i \dots} = \sum_i T^{\dots i \dots}_{\dots i \dots}$.

Calculations.

$$\Psi = G\Psi', \ \nabla_i = \partial_i + \Gamma_i, \ \nabla'_i = \partial_i + \Gamma'_i, \ \Gamma'_i = G^{-1}\Gamma_i G + G^{-1}\partial_i G.$$

$$R_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i = \partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i.$$

Theorem: $R'_{ij} = \nabla'_i \nabla'_j - \nabla'_j \nabla'_i = G^{-1} R_{ij} G$.

Proof: We have $\partial_i G^{-1} = -G^{-1}\, \partial_i G\, G^{-1}$, because $\partial_i (G^{-1}G) = 0 = \partial_i G^{-1} G + G^{-1}\, \partial_i G = 0$. O.K.

$$\partial_i \Gamma'_j = \partial_i (G^{-1}\Gamma_j G + G^{-1}\partial_j G) = -(G^{-1}(\partial_i G) G^{-1}\Gamma_j G) +$$

$$+ G^{-1}\partial_i \Gamma_j G + G^{-1}\Gamma_j \partial_i G - G^{-1}\partial_i G\; G^{-1}\partial_j G + G^{-1}\partial_i \partial_j G,$$

$$\Gamma'_i \Gamma'_j = (G^{-1}\Gamma_i G + G^{-1}\partial_i G)(G^{-1}\Gamma_j G + G^{-1}\partial_j G) =$$

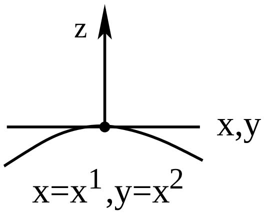
$$= G^{-1}\Gamma_i \Gamma_j G + G^{-1}(\partial_i G) G^{-1}\Gamma_j G + G^{-1}\Gamma_i \partial_j G + (G^{-1}\partial_i G)(G^{-1}\partial_j G)$$

Conclusion: $\hat{R}'_{ij} = G^{-1} \hat{R}_{ij} G$.

Corollary. $R^i_{j;kl}$ is a tensor for **tangent bundle** $T^*(M^n)$, $\hat{R}_{kl} = R^s_{p;kl}$.

Gauss Curvature:

$z = F(x, y)$,
$g_{11} = 1 + F_x^2$,
$g_{12} = 1 + F_x F_y$,
$g_{22} = 1 + F_y^2$,



$z \perp x, y$,
$F_x = F_y = 0$ at $0, 0$.

"Curvature form" $(x = 0, y = 0)$, matrix $g_{ij}(0, 0) = \delta_{ij} = g^{ij}(0, 0)$.

$$K = \det \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix} = F_{xx}F_{yy} - F_{xy}^2, \quad x = 0,\ y = 0.$$

Calculate Riemannian Curvature: $R_{12;12}$ – only this component is nontrivial.

$$g_{12;12} = F_{xx}F_{yy} + F_{xy}^2, \quad g_{11;22} = 2F_{xy}^2, \quad g_{22;11} = 2F_{xy}^2, \quad x, y = 0, 0.$$

Using formulas for $R_{ij;kl}(0, 0)$ in these coordinates (See Appendix to the previous Lecture), we obtain the following results:

$$R = 2R_{12;12} = 2K, \quad R_{ij} = R^p_{i;pj}, \quad R = R^p_p.$$

Results are valid in all coordinates.

Consider a pair of vector fields ξ, η. Define $\nabla_\eta = \eta^i \nabla_i$.

Curvature:

$$\hat{R}_{\eta\xi} = \nabla_\eta \nabla_\xi - \nabla_\xi \nabla_\eta - \nabla_{[\eta,\xi]},$$

$$[\eta, \xi] = [\eta^i \partial_i, \xi^j \partial_j] = (\eta^i \ \partial \xi^j / \partial x^i - \xi^i \ \partial \eta^j / \partial x^i) \partial_j .$$

We have

$$\hat{R}_{\eta,\xi} = \hat{R}_{ij} \eta^i \xi^j \quad \text{(Matrix)}$$

for any vector bundle.

Quadratic Form (tangent bundle, symmetric connection)

$$R_{ij;kl} = R_{kl;ij} = -R_{ji;kl} - R_{ij;lk},$$

$< ij >= - < ji >$ – basic vectors in $\wedge^2 \mathbb{R}^n$.

Curvature along 2-direction: let η, ξ be unit orthogonal vectors (in the Riemannian Metric)

$$|\eta| = 1, \quad |\xi| = 1, \quad < \eta, \xi >= 0.$$

Then the sectional curvature is:

$$\underbrace{R_{ij;kl} \ \eta^k \xi^l \eta^i \xi^j}_{\text{Sum by } k,l} = R < \eta \wedge \xi, \eta \wedge \xi > .$$

Metrics: S^n, $\mathbb{R}^n$, H^n.

Quadratic form $R_{ij;kl}$ is determined by one constant R. All curvatures if all 2-directions are the same at all points:

- Positive for S^n.
- Negative for H^n.
- 0 for $\mathbb{R}^n$.

Symmetry Groups are O_{n+1} for S^n, Iso($\mathbb{R}^n$) and $O_{1,n}$ for H^n. Dimension of these groups is $n(n+1)/1$. Any points can be mapped to any point (homogeneous space). Any unit tangent vector cam be mapped to any unit tangent vector.

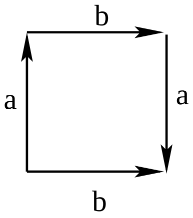
Every pair (x, v) can be mapped to every pair (y, w), where x, y are points, v, w are tangent vectors at the points x, y respectively, $|v| = |w| = 1$.

Remark. Group O_n acts homogeneously on the space $V_{n,k}$ of orthonormal k-frames $(\tau_1, \dots, \tau_k)$ in $\mathbb{R}^n$, $\tau_i \perp \tau_j$, $|\tau_i| = 1$.

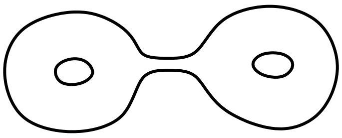
$V_{n,k}$="Stiefel Manifold", $V_{n,1} = S^{n-1}$, $V_{n,n} = O_n$.

Homework 5.

1. Prove that $\mathbb{RP}^2 \backslash \text{point}$ is diffeomorphic to MOBIUS BAND.



2. Prove that KLEIN BOTTLE is the same as $\mathbb{RP}^2 \# \mathbb{RP}^2$, where $\#$ is "connected sum"



3. Prove that metric of $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$ can be written in the form

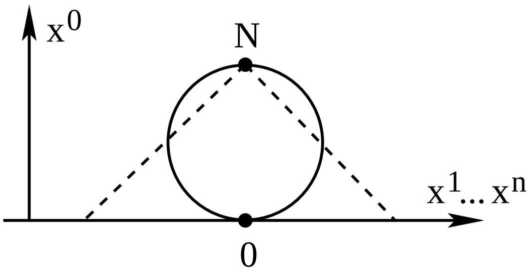
$$\text{a)} \quad dl^2 = d\rho^2 + \sin^2\!\rho \, (d\Omega)^2 \quad , \quad \mathbb{S}^n \, ,$$

where $(d\Omega)^2$ is the metric of $\mathbb{S}^{n-1}$,

$$\text{b)} \quad dl^2 = dr^2 + r^2 \, (d\Omega)^2 \quad , \quad \mathbb{R}^n \, ,$$

$$\text{c)} \quad dl^2 = d\chi^2 + \text{sh}^2\rho \, (d\Omega)^2 \quad , \quad \mathbb{H}^n \, .$$

4. Introduce "conformal coordinates" in $\mathbb{S}^n$

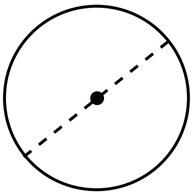


Prove that

$$dl^2 \;=\; \frac{\sum (dx^i)^2}{(1+r^2)^2} \quad , \qquad r^2 \;=\; \sum_{i=1}^{n} (x^i)^2$$

5. Prove that straight lines passing through center are geodesics for the metric of $\mathbb{S}^2$:

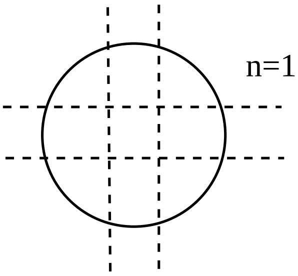
$$dl^2 \;=\; 4\,\frac{dz\,d\bar{z}}{(1+|z|^2)^2}$$



Homeworks 2, 3, 4. Solutions.

Homework 2. Solutions.

1. Projection Coordinates for $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. We need $2(n+1)$ coordinate domains

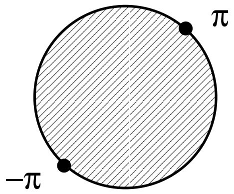


2. $\mathbb{RP}^n$: $(x^0, \ldots, x^n) \;\sim\; (\lambda x^0, \ldots, \lambda x^n)$, $\quad \lambda \neq 0$,

$$U_j \;=\; \left\{x^j \neq 0\right\} \;\Rightarrow\; (x^0, \ldots, 1, \ldots, x^n)$$

- local coordinates, $U_j \;=\; \mathbb{R}^n$.

3. $SO_3 \;=\; \mathbb{RP}^3$: axis + angle $\leq \pi$.



$$(\varphi = \pi) \cong (\varphi = -\pi) \ .$$

4. $SO_{1,2}$: $\tilde{B}_{ij} = -\tilde{B}_{ji}$, $B_i^k = g^{ks}\,\tilde{B}_{si}$,

$$g^{ks} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So we have B_i^k = product of (g^{ks}) by skew symmetric matrix $\tilde{B}_{si}$.

5. $O_{1,1} \ni T, P, PT, 1$ (in different connected components). We have $T^2 = P^2 = 1$, $PT = TP \ - \mathbb{Z}_2 \times \mathbb{Z}_2$. Component of 1 is

$$\begin{pmatrix} \operatorname{ch}\psi & \operatorname{sh}\psi \\ \operatorname{sh}\psi & \operatorname{ch}\psi \end{pmatrix}$$

6 - 7. Quaternions $\mathbb{R}^4 \ni q$

$$SO_3 : \ q \to q_1\,q\,q_1^{-1} \quad , \quad |q_j| = 1 \ , \quad j = 1, 2$$

$$SO_3 : \ q_1 \text{ and } q_2 \ - \ \text{any } |q_j| = 1$$

8.

a) $GL_n(\mathbb{R})$ has 2 components only.
b) $GL_n(\mathbb{C})$ is connected.

(b) Proof.

Set of matrices with distinct eigen-values is dense. Write linear operators in basis of eigenvectors for $GL_n(\mathbb{C})$:

$$\begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix} \quad , \quad \lambda_j \neq 0 \ , \quad \lambda_j \in \mathbb{C}\backslash 0$$

it is connected space.

(b) is proved.

(a) Proof.

Let our field is $\mathbb{R}$. Choose the same type basis consisting of 1-dim blocks if λ_j is real or 2-dim blocks $(\lambda, \bar{\lambda})$ for complex λ. We have for the forms of blocks

$$\lambda_j \in \mathbb{R} \; - \; (\lambda_j)$$

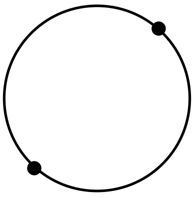
$$\lambda_j \in \mathbb{C} \; - \; a_j \begin{pmatrix} \cos b_j & \sin b_j \\ \sin b_j & \sin b_j \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Deform $\lambda_j \to \pm 1$, $a_j \to \pm 1$, $b_j \to 0$.

Now remember that

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is connected with 1



$$\xi \xrightarrow{A} -\xi$$

$$\xi \in \mathbb{R}^2$$

Multiplying our matrices by

$$A \; = \; \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

many times we see that there are 2 components ($\det > 0$, $\det < 0$) .

(a) is proved.

Homework 3. Solutions.

1. Already was solved in HW2.

2.

$$SU(1,1) \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad \langle A\,\psi \, , \, A\,\varphi \rangle \; = \; \langle \psi \, , \, \varphi \rangle$$

vectors $(a,b) = \zeta$, $(c,d) = \eta$

$$\langle \zeta \, , \, \zeta \rangle \; = \; 1 \quad , \quad \langle \eta \, , \, \eta \rangle \; = \; -1 \quad , \quad \langle \zeta \, , \, \eta \rangle \; = \; 0 \qquad \Rightarrow$$

$$|a|^2 - |b|^2 = 1 \ , \quad |c|^2 - |d|^2 = -1 \ , \quad a\,\bar c - b\,\bar d = 0$$

solution:

$$\begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \ , \quad a\,\bar a - b\,\bar b = 1$$

$$SL_2(\mathbb{R}): \quad e_1\ ,\ e_2 \quad \to \quad a\,e_1 + b\,e_2\ ,\ c\,e_1 + d\,e_2$$

$$a\,d - b\,c = 1\ , \quad a, b, c, d \in \mathbb{R}$$

Take basis

$$e = e_1 + i\,e_2\ , \quad \bar e = e_1 - i\,e_2$$

$$e_1 = (e + \bar e)/2\ , \quad e_2 = (e - \bar e)/2i$$

In the new basis we have

$$e \ \to \ \frac{1}{2}\Big[\Big(a + d - i\,(b - c)\Big)e + \Big(a - d + i\,(b + c)\Big)\bar e\Big] = q\,e + p\,\bar e$$

$$\bar e \ \to \ \frac{1}{2}\Big[\Big(a - d - i\,(b + c)\Big)e + \Big(a + d + i\,(b - c)\Big)\bar e\Big] = \bar p\,e + \bar q\,\bar e$$

where

$$q\,\bar q - p\,\bar p = \frac{1}{4}\Big((a{+}d)^2 + (b{-}c)^2 - (a{-}d)^2 - (b{+}c)^2\Big) = a\,d - b\,c = 1$$

4. $SL_2(\mathbb{R}) \cong \mathbb{S}^1 \times \mathbb{R}^2$ (?)

$$A \in SL_2(\mathbb{R}) \quad \Rightarrow \quad A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \times \begin{pmatrix} \lambda & \mu \\ 0 & 1/\lambda \end{pmatrix}$$

where $\lambda \in \mathbb{R}^+ \backslash 0 \cong \mathbb{R}$, $\mu \in \mathbb{R}$.

Remark. Every "semisimple" Lie Group is topologically a product: $G \cong K \times \mathbb{R}^N$, where K is a "compact group" and $\mathbb{R}^N$ is a "Borel subgroup" (upper triangle).

4.

$$dl^2 = d\theta^2 + \sin^2\theta\,(d\varphi)^2 \quad , \quad \mathbb{S}^2 \subset \mathbb{R}^3$$

For polar coordinates in $\mathbb{R}^2$: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ we have

$$dl^2 = d\rho^2 + \rho^2\,(d\varphi)^2$$

5.

$$\mathbb{H}^2 : \quad z^2 - x^2 - y^2 = 1$$

$$z = \operatorname{ch}\theta \ , \quad x = \operatorname{sh}\theta \cos\varphi \ , \quad y = \operatorname{sh}\theta \sin\varphi$$

$$- \, dl^2 = d\theta^2 + \operatorname{sh}^2\theta \, (d\varphi)^2$$

Homework 4. Solutions.

1. We have parametrization (physics)

$$\operatorname{ch}\psi = \frac{1}{\sqrt{1-w^2}} \ , \quad \operatorname{sh}\psi = \frac{w}{\sqrt{1-w^2}}$$

because

$$(\operatorname{ch}\psi)^2 - (\operatorname{sh}\psi)^2 = 1$$

Remark. Physics: let $w = v/c$.
Lorentz transformation:

$$x^0 = c\,t = \frac{1}{\sqrt{1-v^2/c^2}} \, x'^0 + \frac{v/c}{\sqrt{1-v^2/c^2}} \, x'^1$$

$$x^1 = \frac{v/c}{\sqrt{1-v^2/c^2}} \, x'^0 + \frac{1}{\sqrt{1-v^2/c^2}} \, x'^1$$

So we have

$$t = \frac{1}{\sqrt{1-v^2/c^2}} \, t' + \frac{v/c^2}{\sqrt{1-v^2/c^2}} \, x'$$

$$x = \frac{v}{\sqrt{1-v^2/c^2}} \, t' + \frac{1}{\sqrt{1-v^2/c^2}} \, x'$$

Note, that we get the Galilean Transformation

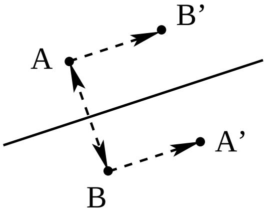
$$t \simeq t' \ , \quad x \simeq x' + v\,t'$$

in the case $v/c \ll 1$.

2. a) $Iso\,(\mathbb{R}^2)_+ \cong$ shifts + rotations SO_2 .

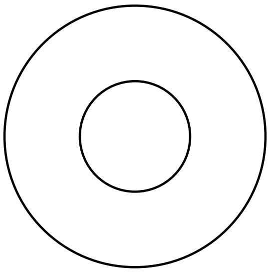
$$1 \to SO_2 \to Iso\,(\mathbb{R}^2)_+ \to \text{shifts} = \mathbb{R}^2$$

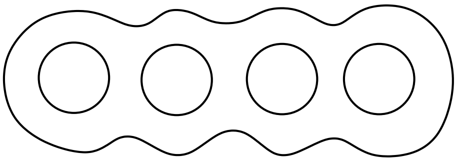
b) $Iso\,(\mathbb{R}^2)_- \;\cong\;$ reflection $+$ shift along the reflexion line .



3. Already was solved for $\mathbb{RP}^n$: $\vec{x} \neq 0$, $(x^0, \ldots, x^n) \sim \lambda\,(x^0, \ldots, x^n)$, $(n+1)$ systems :

$$U_i \; : \; (x^0,\, \ldots,\, 1,\, \ldots,\, x^n)$$

4. $\mathbb{S}^2_g \subset \mathbb{R}^3$  Two such domains for $\mathbb{T}^2$, $g = 1$.



Two such domains for any g (glue them along the boundary strips).

Lecture 17. Differential forms.

Last chapter.

1. Geodesics, Calculus of Variations, Fermat Principle, Lagrangian, Action Functional (Length and "Kinetic Energy"=**Natural Parameter**), Euler-Lagrange equations, Momentum, Energy, Conservation Laws, Examples.

2. Curvature of curves, Curvature of Hypersurfaces in $\mathbb{R}^{n+1}$, Quadratic (2nd) Form, Principal Curvatures and Gaussian Curvature: definition via the special coordinates.

3. Vector Bundles and Differential-Geometrical (Linear) Connection, Curvature, Gauge Transformation for Connection and Curvature. $\boxed{\text{Parallel Transport.}}$

4. Tangent Bundle and “Christoffel Symbols”, Cotangent Bundle, Tensor Bundles (Inner Products), Geodesics (new definition), compatibility with Riemannian Metric. Symmetric Connections. Formulas for Christoffel Symbols and Riemannian Curvature. Special Coordinates.

5. Symmetries of the Riemannian Curvature Tensor, Ricci Tensor and Scalar Curvature. Examples. Einstein Equations ($n = 4$). Curvature for $n = 2$. Gauss Theorem. Curvature for $n = 3$. Curvature of $\mathbb{R}^n$, S^n, H^n.

Next chapter. Differential forms.

- $m = 0$: Differential 0-form is a scalar function
$$f(x) : M^n \to \mathbb{R}.$$
- $m = 1$: Differential 1-form is a covector field ω written in the form
$$\omega = \sum \omega_i(x) dx^i.$$
- $m = n$ Differential n-form = object of integration:
$$\Omega = f(x) dx^1 \wedge \ldots \wedge dx^n, \quad \text{(locally)}$$
such that for $x = x(y)$ we have
$$\Omega = f(x(y)) dx^1 \wedge \ldots \wedge dx^n, \quad dx^i = \frac{\partial x^i}{\partial y^j} dy^j, \quad \text{(summation in } j)$$

Definition. Differential k-from in M^n is a smooth quantity Ω which locally in every Chart of Atlas $(x^1, \ldots, x^n)$ can be written in the form
$$\Omega = \sum_I f_I(x) \underbrace{dx^{i_1} \wedge \ldots \wedge dx^{i_k}}_{dx^I} = f_I dx^I,$$

where $I = (i_1 < i_2 \ldots < i_k)$, and for $x = x(y)$ we have

$$\Omega = f_I(x(y))dx^i = g_J(y)dy^J, \quad dy^J = dy^{j_1} \wedge \ldots \wedge dy^{j_k}, \quad j_1 < j_2 < \ldots < j_k.$$

The symbols dx^i form an Associative Algebra under multiplication $\wedge$ (exterior product), such that

$$dx^i \wedge dx^j = -dx^j \wedge dx^i,$$

(bilinear, associative.)

Examples:

1. $$(\sum_i a_i dx^i) \wedge (\sum_j b_j dx^j) = \sum_{ij} a_i b_j dx^i \wedge dx^j = \sum_{i<j} (a_i b_j - a_j b_i) dx^i \wedge dx^j.$$

2. Change of coordinates:
$$x = x(y), \quad dx^i = \frac{\partial x^i}{\partial y^j} dy^j, \quad \text{(summation in } j\text{)}$$

3. **Corollary:**
   (a) $k \leq n$.
   (b) $\Omega_n = f(x) dx^1 \wedge \ldots \wedge dx^n$.

Lemma. C^∞ differential forms is a ring $\wedge^*(M^n, \mathbb{R})$ such that every C^∞-map $f : M^n \to N^m$ there is a natural "induced map" of rings:

$$f^* : \wedge^*(N^m) \to \wedge^*(M^n),$$

commuting with all algebraic operations (addition, multiplication $\wedge$).

Proof. "Change of coordinates"

$$x = x(y), \quad y \xrightarrow{f} x, \quad \Rightarrow \left[dx^i \overset{f^*}{\Rightarrow} \frac{\partial x^i}{\partial y^j} dy^j \ \text{ (sum) } \right]$$

commutes with change of coordinates in N and M and with all algebraic operations like $\wedge$.

For scalar functions we have:

$$\phi : N \to \mathbb{R}, \quad f^*\phi(y) = \phi(x(y)) \ \text{ by definition.}$$

For forms

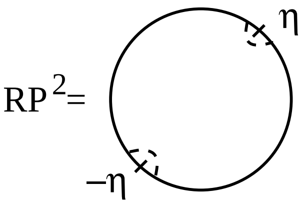
$$f^*dx^i = \frac{\partial x^i}{\partial y^j}(x(y))dy^j \quad \text{by definition,}$$

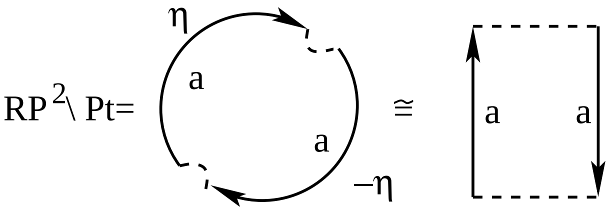
products are mapped into products.

The proof is finished.

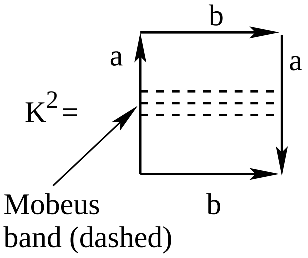
Homework 5. Solutions.

1.





2.

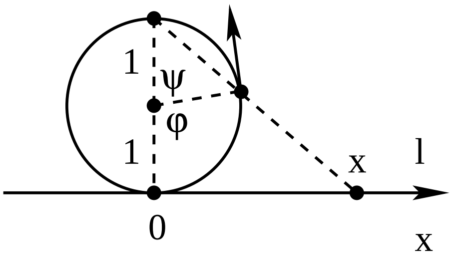


$\mathbb{R}P^2 \sharp \mathbb{R}P^2 = ?.$

$K^2 \backslash$ MOBEUS $\cong$ MOBEUS

3. Already was done

4. "Conformal coordinates" in $\mathbb{S}^n$

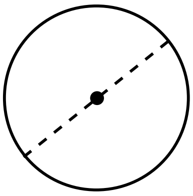


$$S^n \backslash N \cong \underset{x^1,\dots,x^n}{\mathbb{R}^n}$$

$$dl^2 \;=\; \text{const} \cdot \frac{\sum (dx^i)^2}{(1+|x|^2)^2} \;?$$

a) $n \;=\; 1$ (line): $x \;=\; 2\,\mathrm{tg}\,\psi$
b) $n \;=\; 2$ **preservation of angles**

5.



$$|z| \;<\; 1 \quad , \quad dl^2 \;=\; 4\,\frac{dz\,d\bar{z}}{(1+|z|^2)^2}$$

Central line $=$ geodesics? Metric does not depend on the angle φ !

Lecture 18. Differential forms. De Rham operator.

Differential forms.

Consider a C^∞-manifold M^n with Atlas of Charts. Locally k-forms are:

$$\Omega = \sum_{i_1<i_2<\ldots<i_k} f_{i_1\ldots i_k}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_k}.$$

Multiplication: Associative and

$$dx^i\wedge dx^j = -dx^j\wedge dx^i.$$

$$\begin{array}{ll} \Omega = 0 & \text{for } \; k>n \\ \Omega = f(x)dx^1\wedge\ldots\wedge dx^n & \text{for } \; k=n \\ \Omega = f(x) & \text{for } \; k=0 \\ f(x)\cdot\Omega = \Omega\cdot f(x) & \end{array}$$

Change of coordinates: $x = x(y)$

$$dx^i \overset{f^*}{\Rightarrow} \frac{\partial x^i}{\partial y^j}dy^j \;\; \text{(sum)}.$$

Functorial properties: $g : \underset{y}{M} \to \underset{x}{N}$, $x = x(y)$.

$$G^*\Omega = \sum f_I(x(y))dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad dx^i \ \Rightarrow \ \frac{\partial x^i}{\partial y^j}(x(y))dy^j.$$

Algebra of forms.

$$\wedge^*(M) = \sum_{k=0}^{n} \wedge^k(M),$$

$$g^*(\Omega_1 \wedge \Omega_2) = g^*\Omega_1 \wedge g^*\Omega_2.$$

Integration of forms.

1. $$\int_{\pm P} f(x) = \pm f(P), \ k = 0, \ \ \pm P = \text{“point with orientation”}.$$

2. $$\int_D f(x)dx^1 \wedge \ldots \wedge dx^n = \text{ordinary integral}$$

   D-domain (local). To integrate over large domains, one can write

   $$\int_D \Omega = \sum_q \phi_q(x)\Omega,$$

   where

   $$\sum_q \phi_q(x) \equiv 1, \quad \phi_q \ \text{ is a } C^\infty \text{ function,}$$

   $\phi_q \equiv 0$ outside of small domain $D_q^n \subset M^n$. The family ϕ_q is called “Partition of Unity”.

3. Let Ω be a k-form and $g : D^k \to M^n$, g is C^∞. By definition:

   $$\int_{g(D^k)} \Omega = \int_{D^k} g^*\Omega \quad - \quad \text{integral of } k\text{-form in } k\text{-ball } D^k.$$

Properties: Integration of forms does **not** depend on local coordinates both in M^n and D^k (restrict the k-form to the k-dimensional body and integrate).

Differentiation of forms:

$$d : \wedge^k(M^n) \to \wedge^{k+1}(M^n)$$

1. df is usual differential.
2. $d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0$.
3. $d(f dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$.

Example:

$$d(f_i dx^i) = df_i \wedge dx^i = \sum_{i,j} \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i<j} \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) dx^i \wedge dx^j.$$

Examples of forms:

Momentum (p_i) in **Calculus of Variations.** Differential of function $df = \sum \frac{\partial f}{\partial x^i} dx^i$.

Electric field (E_i)-covector field.

2-forms: Magnetic field in $\mathbb{R}^3$. $B = B_{\alpha\beta} dx^\alpha dx^\beta$.

Electromagnetic field in $\mathbb{R}^4 = (x^0, x^1, x^2, x^3)$.

$$F = F_{ij} dx^i \wedge dx^j = cF_{oj} dt \wedge dx^\alpha + B_{\alpha\beta} dx^\alpha \wedge dx^\beta,$$

$i, j = 0, 1, 2, 3$, $\alpha, \beta = 1, 2, 3$, $x^0 = cdt$, $F_{0\alpha} = E_\alpha$ – electric field, $B_{\alpha\beta} = F_{\alpha\beta}$ – magnetic field.

Symplectic 2-form in the cotangent space $T_* M^n$ with local coordinates (x^i, p_i)

$$\Omega = \sum_{i=1}^{n} dx^i \wedge dp_i$$

$$\underbrace{\Omega \wedge \ldots \wedge \Omega}_{n \text{ times}} \; - \text{ volume forms in } T_*(M^n).$$

It defines nondegenerate inner product given by matrix

$$g_{ij} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \quad \text{skew symmetric.}$$

Kähler Riemannian Metric in Complex Space (Manifolds) with complex linear coordinates $(z^1, \dots, z^n)$.

$$dl^2 = \sum_{\alpha,\beta} g_{\alpha,\bar{\beta}} dz^\alpha d\bar{z}^\beta > 0.$$

Coordinates: $(z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n)$. Associated 2-form is

$$\Omega = \frac{i}{2} \sum_{\alpha,\beta} g_{\alpha,\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \quad z = x + iy.$$

Examples: n=1.

$$dl^2 = g_{\alpha\beta} dz^\alpha d\bar{z}^\beta = \begin{cases} dz d\bar{z} : & \mathbb{R}^2 \\ \dfrac{dz d\bar{z}}{(1 \pm |z|^2)^2} : & \begin{matrix} S^2 \\ H^2 \end{matrix} \end{cases}$$

$$\Omega = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(1 \pm |z|^2)^2} = \frac{dx \wedge dy}{(1 \pm (x^2 + y^2))^2} \quad - \text{Area Form.}$$

Theorem: The operator d is well-defined in $\wedge^k(M^n)$ by the definition above. It commutes with C^∞-maps $g : M \to N$ and has the following properties:

1. $d(\Omega_k \wedge \Omega_l) = (d\Omega_k) \wedge \Omega_l + (-1)^k \Omega_k \wedge (d\Omega_l)$.
2. $d \circ d = 0$.

Proof:

a)

$$d\Big(g^* f(x(y))\Big) = g^* df : \quad d(g^* f) = d(f(x(y))) = \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j} dy^j = g^* df.$$

b)

$$d(dx^i) = d(0) \wedge dx^i = 0,$$

$$d(g^* dx^i) = d\left(\frac{dx^i}{dy^j} dy^j\right) = \frac{\partial^2 x^i}{\partial y^j \partial y^q} dy^q \wedge dy^j \equiv 0.$$

So d commutes with the map g^* for $k = 1$.

For $k > 1$ our result is obvious by the following reason: $d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0$. Let us prove now that the Statement 1 of the Theorem. Let $\Omega_k = f \cdot dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, $\Omega_l = g \cdot dx^{j_1} \wedge \ldots \wedge dx^{j_l}$.

$$d(\Omega_k \wedge \Omega_l) = d\left[(f \cdot dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \wedge (g \cdot dx^{j_1} \wedge \ldots \wedge dx^{j_l})\right] =$$
$$= (df \cdot g + f \cdot dg) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} =$$
$$= df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge g dx^{j_1} \wedge \ldots \wedge dx^{j_l} +$$
$$+ (-1)^k f dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dg \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} =$$
$$= (d\Omega_k) \wedge \Omega_l + (-1)^k \Omega_k \wedge (d\Omega_l).$$

Let us prove now that $d \circ d = 0$.

$$d(f \cdot dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

$$d \circ d(f \cdot dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = d(df) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} - df \wedge d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \equiv 0.$$

Theorem is proved.

We have:

$$\underbrace{d\,(A_j dx^j)}_{\text{“covector field”}} = \underbrace{\sum_{i<j} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) dx^i \wedge dx^j.}_{\text{“curl”=2-form}}$$

Let $n = 3$, $B_{12} \to \tilde{B}^3$, $B_{13} \to -\tilde{B}^2$, $B_{23} \to \tilde{B}^1$.

$$d(B_{12} dx^1 \wedge dx^2 + B_{13} dx^1 \wedge dx^3 + B_{23} dx^2 \wedge dx^3) = \underbrace{\left(\frac{\partial \tilde{B}^1}{\partial x^1} + \frac{\partial \tilde{B}^2}{\partial x^2} + \frac{\partial \tilde{B}^3}{\partial x^3} \right)}_{\mathrm{div}\tilde{B}} dx^1 \wedge dx^2 \wedge dx^3$$

Faraday Laws:

$n = 3:$ a) $dB = 0,$ Magnetic field

b) $dE = \frac{1}{c}\frac{\partial B}{\partial t}, \quad E = E_\alpha dx^\alpha,$

$n = 4:$ $d(F_{ij} dx^i \wedge dx^j) = 0.$ Electromagnetic field

Lecture 19. Differential forms. Cohomology.

Consider the algebra of differential forms on a manifold N^m. Let $\Omega \in \wedge^k(N^m)$. Then $d\Omega \in \wedge^{k+1}(N^m)$. Consider a map $f : M \to N$. Let us check, that $f^*d\Omega = df^*\Omega$.

Proof. Let $\phi : N \to \mathbb{R}$, $f^*\phi(y) = \phi(x(y))$.

$$f^*d\phi(y) = f^*\left(\frac{\partial\phi}{\partial x^i}dx^i\right) = \frac{\partial\phi}{\partial x^i}(x(y))f^*(dx^i) = \frac{\partial\phi}{\partial x^i}(x(y))\frac{\partial x^i}{\partial y^j}dy^j =$$

$$\frac{\partial(f^*\phi)}{\partial y^j}dy^j. \text{ O.K. } f^*d = df^*,$$

for scalars and dx^i. But the product $\wedge$ commutes with f^*. Every form is a combination of $\phi(x) \cdot dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. So we have

$$f^*d = df^* \text{ for all } k \geq 0.$$

Homology (Cohomology).

$$H^k(M^n, \mathbb{R}) = \operatorname{Ker} d / \operatorname{Im} d \text{ in } \wedge^k (M^n).$$

$\operatorname{Ker} d \subset \wedge^k : \{\Omega \mid d\Omega = 0\}$ closed forms

$\operatorname{Im} d \subset \wedge^k : \{\Omega \mid \Omega = d\Omega'\}$ exact forms

Examples: a) $k = 0 \Rightarrow \Omega = \phi$ is locally constant.
Conclusion: $H^0(M^n, \mathbb{R}) = \mathbb{R}^p$, where p is the number of components.

Closed Forms.

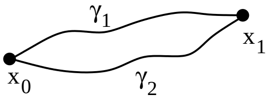
1. Forms with constant coefficients in $\mathbb{R}^n$ or in $T^n = \mathbb{R}^n/\mathbb{Z}^n$: $\Omega = \sum a_I dx^I$, $a_I = \text{const}$.

2. Closed 1-form

$$\omega = \sum \psi_i(x)dx^i, \quad d\omega = \sum_{i<j}\left(\frac{\partial\psi_j}{\partial x^i} - \frac{\partial\psi_i}{\partial x^j}\right)dx^i \wedge dx^j.$$

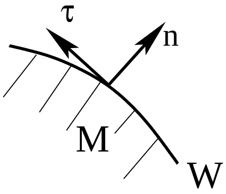
$$d\omega = 0 \;\Leftrightarrow\; \frac{\partial \psi_j}{\partial x^i} - \frac{\partial \psi_i}{\partial x^j} \equiv 0,$$

$$\phi(x) = \int\limits_{x_0}^{x} \omega \quad \Rightarrow \quad d\phi = \omega \text{ if } \phi \text{ is well-defined.}$$



Necessary condition: $d\omega = 0$.

Stokes Formula. Consider an oriented manifold M^n with boundary $\partial M^n = W^{n-1}$ with "induced" orientation. What is it?



Consider "external" normal vector n to $W \subset M^n$ and tangent frame τ to W. Let (n, τ) form an oriented frame in M^n at the point P. We say that τ is an oriented $(n-1)$-frame to W in the induced orientation.

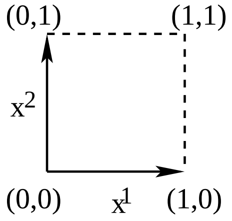
Theorem. For every $(n-1)$-form Ω in M^n we have

1. $$\int\limits_{W^{n-1}}\!\!\cdots\!\int \Omega = \int\limits_{M^n}\!\!\cdots\!\int d\Omega,$$

   W^{n-1} is a closed C^∞ manifold, M^n is compact "manifold with boundary W".

2. For every manifold N, $n-1$-form Ω in $\wedge^{n-1}(N)$ and mapping $f : M^N \to N$ we have
   $$\int\limits_{W^{n-1}}\!\!\cdots\!\int f^*\Omega = \int\limits_{M^n}\!\!\cdots\!\int f^* d\Omega,$$

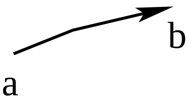
Proof for the case $M^n = I^n$ (cube) with coordinates $(x^1, \ldots, x^n)$, $0 \le x^i \le 1$,



$W = \partial I^n$ (I^n="ball").

a) Case $n = 1$. We have

$$\int\limits_a^b \Psi' dx = \Psi(b) - \Psi(a), \quad \Omega = \Psi(x).$$

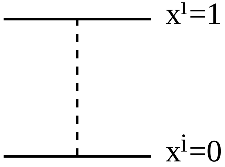


b) Case $n > 1$. We have

$$\Omega_{n-1} = \sum_{i=1}^{n} \Psi_i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n = \sum_{i=1}^{n} \Omega_n^{(i)},$$

where $\widehat{dx^i}$ means that this multiplier is omitted.

Consider every summand $\Omega_n^{(i)}$ separately.



$$\int\limits_{\partial I^n} \cdots \int \Omega_n^{(i)} = \int\limits_{[x^i=0,1]} \cdots \int \Omega_n^{(i)} =$$

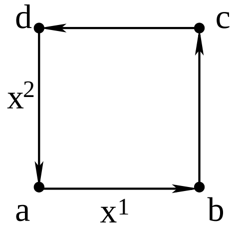
$$= (-1)^{i-1} \left[\int \cdots \int \Psi_i(x^1, \ldots, x^i = 1, \ldots, x^n) dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n - \right.$$

$$\left. - \int \cdots \int \Psi_i(x^1, \ldots, x^i = 0, \ldots, x^n) dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \right] =$$

$$= (-1)^{i-1} \int\limits_{I^n} \cdots \int \partial_i \Psi_i(x^1, \ldots, x^i, \ldots, x^n) dx^1 \wedge \ldots \wedge dx^i \wedge \ldots \wedge dx^n =$$

$$= \int\limits_{I^n} \cdots \int d\Omega_n^{(i)}.$$

Let $n = 2$. Orientation for $n = 2$:



Here $a = (0,0)$, $b = (1,0)$, $c = (1,1)$, $d = (0,1)$, $(n,\tau) = (x^1, x^2)$.

$$\Omega = \Psi_1(x^1,x^2)dx^1 + \Psi_2(x^1,x^2)dx^2,$$

$$d\Omega = \left(\frac{\partial\Psi_2}{\partial x^1}(x^1,x^2) - \frac{\partial\Psi_1}{\partial x^2}(x^1,x^2)\right) dx^1 \wedge dx^2,$$

$$\int\limits_{\partial I^2} \Omega = \int_0^1 \Psi_1(x^1,0)dx^1 + \int_0^1 \Psi_1(1,x^2)dx^2 + \int_1^0 \Psi_1(x^1,1)dx^1 + \int_1^0 \Psi_1(0,x^2)dx^2$$

Let $\Omega = \Psi_1 dx^1$ for $n = 2$. We have

$$d\Omega = \frac{\partial\Psi_1}{\partial x^2} dx^2 \wedge dx^1 = -\frac{\partial\Psi_1}{\partial x^2} dx^1 \wedge dx^2$$

$$\iint\limits_{I^2} d\Omega = -\int\limits_0^1 \left(\int\limits_0^1 \frac{\partial\Psi_1}{\partial x^2} dx^2\right) dx^1 = -\int\limits_0^1 dx^1 \left(\Psi_1(x^1,1) - (\Psi_1(x^1,0)\right) =$$

$$= \int\limits_0^1 dx^1 \left(\Psi(x^1,0) - (\Psi(x^1,1)\right).$$

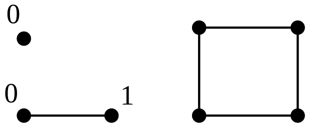
We have

$$\int\limits_{\partial I^2} \Omega = \int_0^1 \Psi_1(x^1,0)dx^1 - \int_0^1 \Psi_1(x^1,1)dx^1.$$

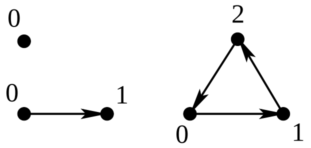
These two expressions coincide.

"Ordinary homology" for the C^∞ - manifolds:

- $\phi : I^k \to M^n$ – "singular cube".



- $\psi : \sigma^k \to M^n$ – "singular simplex".
  $\sigma^k =< 0, 1, \dots, k >$.



ϕ, ψ are assumed to be C^∞ maps.

"Boundary":

$\partial I^k = \partial(I^{k-1} \times I^1) = \partial(I^{k-1}) \times I^1 + (-1)^{k-1} I^{k-1} \partial I^1$.

$\partial \sigma^k = \sum\limits_{i=0}^{n} (-1)^i < 0, 1, \dots, \widehat{i}, \dots, k >= \sum\limits_{i=0}^{n} (-1)^i \sigma_i^{k-1}$.

"Degenerate cube" $I^k \xrightarrow[\text{projection}]{} I^{k-1} \xrightarrow{\phi} M^n$.

Every form Ω_k defines a **linear form**

$$\begin{array}{ll} C^\infty - \text{singular cubes} & \Rightarrow \mathbb{R} \\ C^\infty - \text{singular simplices} & \Rightarrow \mathbb{R}, \end{array}$$

(integration of Ω along singular cubes, singular simplices, degenerate cubes $\to 0$).

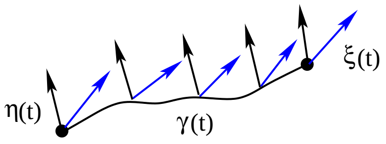
Homework 6.

1. Find all geodesics for $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$ (for $\mathbb{H}^n$ in P - model and in K - model.

2. Find all closed geodesics in torus $\mathbb{T}^2$ with euclidean metric.

3. Find all closed geodesics in $\mathbb{RP}^2$ (metric of constant positive curvature).

4. Prove that **parallel transport** along any path preserves inner product



$\gamma(t) : [x^i(t)]$,

$\langle \eta(t), \zeta(t) \rangle =$ const if $\nabla_{\dot{x}}\, \eta = 0$, $\nabla_{\dot{x}}\, \zeta = 0$

$(\nabla_s\, g_{ij} \equiv 0)$.

Lecture 20. Differential forms and tensors. Categorial properties. Cohomology and Stocks formula. Homotopy invariance of Cohomology.

Category of C^∞-manifolds and C^∞-maps. Functors like spaces of tensors with low indices. Differential k-forms $\wedge^k(M^n)$: exterior multiplication $\wedge$, operator d and integration, their properties. Stokes formula fro integration. Singular cubes and simplices, boundary operator ∂. Cohomology, forms and singular complexes.

a) $H^k_\wedge(M^n, \mathbb{R})$ (forms).

b) $H^k(M^n, \mathbb{R})$ (simplices and cubes).

c) $H_k(M^n, \mathbb{R})$ (simplices and cubes).

a) $\underbrace{Ker\, d}_{\text{closed forms}} \;/\; \underbrace{Im\, d}_{\text{exact forms}} = H^k_\wedge(M^n, \mathbb{R})$.

b) $\underbrace{Ker\, \partial^*}_{\text{cocycles}} / \underbrace{Im\, \partial^*}_{\text{coboundaries}} = H^k(M^n, \mathbb{R})$ (cubes, simplices).

c) $\underbrace{Ker\, \partial}_{\text{cycles}} / \underbrace{Im\, \partial}_{\text{boundaries}} = H_k(M^n, \mathbb{R})$ (cubes, simplices).

$\partial I^k = \partial(I^{k-1} \times I^1) = \partial(I^{k-1}) \times I^1 + (-1)^{k-1} I^{k-1} \partial I^1$.

$\partial \sigma^k = \sum\limits_{i=0}^{n} (-1)^i \sigma_i^{k-1}, \quad \sigma_i^{k-1} =< 0, 1, \ldots, \widehat{i}, \ldots, k >$

"simplicial chains":

$$\sum_s \lambda_s(\sigma^k, \phi_s), \quad \phi_s : \sigma^k \to M^n \quad (C^\infty).$$

"Simplicial cochains" = functional ψ on simplicial chains:

$$\psi(\text{chain}) = \psi \left[\sum_s \lambda_s(\sigma^k, \phi_s) \right] = \sum_s \psi(\sigma^k, \phi_s).$$

“Boundary operator:” ∂ : k-chains$\rightarrow$ $k-1$-chains (linear).

$$\partial(\sigma^k,\phi)=\sum_{i=0}^{n}(-1)^i(<0,1,\dots,\widehat{i},\dots,k>,\phi),$$

where ϕ is naturally defined at the boundary simplices.

“Coboundary operator:” ∂ : k-cochains$\rightarrow$ $k+1$-cochains.

$$<\partial^* a,b>=<a,\partial b>,\quad a \text{ is a cochain},\quad b \text{ is a chain}.$$

“Special cochains”$=$ C^∞-form Ω

$$<\Omega,b>=\int\limits_b \Omega=\sum_s \lambda_s \int\limits_{(\sigma^k,\phi_s)} \Omega,\qquad b \text{ is a chain}.$$

“Stokes formula”

$$\int\limits_b d\Omega=<d\Omega,b>=<\Omega,\partial b>=\int\limits_{\partial b}\Omega,$$

Conclusion: Stokes formula defines a correct homomorphism

$$H^k_\wedge(M^n,\mathbb{R})\to H^k_{\text{simplices}}(M^n,\mathbb{R}).$$

In Algebraic topology:

$$H^k_{\text{simplices}}(M^n,\mathbb{R})\to \operatorname{Hom}(H_k(M^n),\mathbb{R}).$$

Isomorphism

$$H^k_\wedge(M^n,\mathbb{R})\cong \operatorname{Hom}(H_k(M^n),\mathbb{R})$$

was claimed by Poincaré in 1895 and proved by De-Rham in 1930s.

Lemma 1. Cohomology $H^k_\wedge(M^n,\mathbb{R})$ form a ring (operation $\wedge$) such that

$$a\wedge b=(-1)^{kl} b\wedge a,\quad a\in H^k_\wedge,\quad a\in H^l_\wedge.$$

Proof. Let Ω, Ω' represent a, b, i.e. $\Omega+\operatorname{Im} d)\cong a$, $\Omega'+\operatorname{Im} d)\cong b$. Define

$$a\wedge b\cong(\Omega\wedge\Omega'+\operatorname{Im} d)\ \text{ in }\ H^{k+l}(M^n,\mathbb{R}).$$

We have

$$(\Omega + du) \wedge (\Omega' + dv) = \Omega \wedge \Omega' + du \wedge \Omega' + \Omega \wedge dv + du \wedge dv,$$

but

$$du \wedge \Omega' = d(u \wedge \Omega'), \quad \Omega \wedge dv = \pm d(\Omega \wedge v), \quad du \wedge dv = d(u \wedge dv).$$

The Lemma is true.

Instead of $\mathbb{R}$ maybe any ring, for **noncommutative ring** may take place $a \wedge b \neq \pm b \wedge a$. For **associative** rings multiplication of forms and homology is **associative.**

Definition. C^∞ maps $g, h : N \to M$ are homotopic if there exists a C^∞ map $F : N \times I \to M$ such that $F\big|_{t=1} = g$, $F\big|_{t=0} = h$.

Theorem. For homotopic maps g, h induced maps of cohomology coincide.

$$g^* = h^* : H^k_\wedge(\underset{(x)}{M}, \mathbb{R}) \to H^k_\wedge(\underset{(y)}{N}, \mathbb{R}).$$

Proof. Let $\Omega \in \wedge^k(M)$ be a k-form and $F^*\Omega \in \wedge^k(N \times \mathbb{R})$ be a k-form. Coordinates in $N \times \mathbb{R}$ are $y^1, \ldots, y^n, t$. Every form can be written $u = a + dt \wedge b$, where a, b do **not** contain dt.

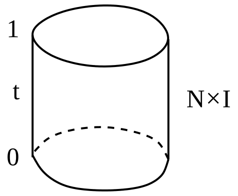
Define operator:

$$D : \wedge^k(N \times \mathbb{R}) \to \wedge^{k-1}(N)$$

by formula

$$Du = \int\limits_0^1 b(t)dt, \quad u = a + dt \wedge b, \quad a \to 0.$$

We have:



Lemma. $Ddu + dDu = u\big|_{t=1} - u\big|_{t=0}$.

Proof.

$$du = d_y a + dt \wedge \dot{a} - dt \wedge d_y b,$$

$$Ddu = \int_0^1 \dot{a} dt - \int_0^1 d_y b dt = a\Big|_{t=1} - a\Big|_{t=0} - d_y \int_0^1 b dt,$$

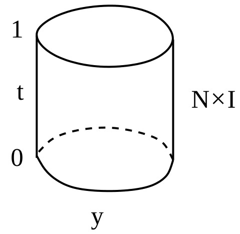
but

$$a\Big|_{t=1} = u\Big|_{t=1}, \quad a\Big|_{t=0} = u\Big|_{t=0}, \quad \int_0^1 b dt = Du, \quad d_y \int_0^1 b dt = dDu.$$

O.K.

Finally we have for every closed k-form Ω in M:

$$\Omega \to F^*\Omega \to DF^*\Omega \text{ in } \wedge^{k-1}(N).$$



$$d\Omega = 0, \quad dF^*\Omega = 0,$$

so we have

$$dD(F^*\Omega) + \cancel{Dd(F^*\Omega)} = F^*\Omega\Big|_{t=1} - F^*\Omega\Big|_{t=0} = g^*\Omega - h^*\Omega.$$

So $g^*\Omega = h^*\Omega + \operatorname{Im} d$,

$$g^* \equiv h^* : H^k(M) \to H^k(N).$$

Theorem is proved.

Corollary. For every contractible manifold M like point, Ball, space $\mathbb{R}^n$ and so on cohomology are the same:

$$H^*_\wedge(\text{point}) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Poincaré Lemma. For every manifold M^n every closed k-form Ω is “locally exact” $(k > 0)$:

$$\Omega = d\omega \text{ in contractible open domain } U \subset M^n.$$

Lecture 21. Homotopy invariance of Cohomology. Examples of differential forms. Symplectic and Kähler manifolds.

Homotopy invariance of cohomology.

$$C^\infty - \text{maps:} \qquad \begin{array}{l} g : N \to M \\ h : N \to M \end{array}$$

are homotopic if exists a map F such that:

$$F :\to N \times I \to M, \quad F\Big|_{t=1} = g, \quad F\Big|_{t=0} = h.$$

Theorem. If g and h are homotopic, then

$$g^* = h^* : H^k(M, \mathbb{R}) \to H^k(N, \mathbb{R}).$$

Proof. Let $u \in \wedge^*(N \times I)$. Then u can be uniquely written as:

$$u = a + dt \wedge b = a_I(y,t)dy^I + b_I(y,t)dt \wedge dy^I.$$

Define the following operator D:

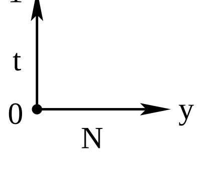
$$Du = \left[\int\limits_0^1 b_I(y,t)dt \right] dy^I.$$

Lemma.

$$Ddu + dDu = u\Big|_{t=1} - u\Big|_{t=0}.$$

Proof.

$$du = d_y a + dt \wedge \dot{a} - dt \wedge d_y b,$$



$$d_y Du = d_y \int_0^1 b dt = \int_0^1 (d_y b) dt$$

$$Ddu = D(d(a + dt \wedge b)) = D(d_y a + dt \wedge \dot{a} - dt \wedge d_y b) =$$

$$= a\Big|_{t=1} - a\Big|_{t=0} - d_y \int_0^1 bdt = a\Big|_{t=1} - a\Big|_{t=0} - dDu.$$

O.K.

Proof of the Theorem.

$$F :\to N \times I \to \underset{\Omega}{M},$$

$$F^* d\Omega = df^*\Omega = 0,$$

$$dD(F^*\Omega) + \cancel{Dd(F^*\Omega)} = F^*\Omega\Big|_{t=1} - F^*\Omega\Big|_{t=0} = h^*\Omega - g^*\Omega.$$

$$g^* \equiv h^* : H^k(M) \to H^k(N).$$

O.K.

Homotopy equivalent manifolds N, M:

$$\exists \quad N \underset{\phi}{\to} M \underset{\psi}{\to} N$$

such that

$$\begin{aligned} \psi \cdot \phi \quad &\text{is homotopic to } \mathbb{1}_N : \ N \to N \\ \phi \cdot \psi \quad &\text{is homotopic to } \mathbb{1}_M : \ M \to M. \end{aligned}$$

Corollary: M and N are homotopy equivalent implies:

$$\begin{aligned} \phi^* &: \ H^*(M) \to H^*(N) \\ \psi^* &: \ H^*(N) \to H^*(M). \end{aligned}$$

are isomorphisms of rings.

In particular, $H^*(\mathbb{R}^n) = H^*(D^n) = H^*(M) = H^*(\text{point})$, where M **is any contractible manifold** (map $\mathbb{1} : \ M \to M$ is homotopic to const: $M \to$ point).

Poincaré Lemma. Every closed k-form Ω is "locally exact" ($k > 0$).

Examples of forms

1. $k = 0$, $df = 0 \Rightarrow f = \text{const}$ (in every component).

$$H^0(M) = \mathbb{R}^p, \quad \text{where} \;\; p = \text{number of components.}$$

2. M^n is an oriented manifold. Every n-form Ω is **closed**: $d\Omega = 0$ (obvious). Riemannian metric in M^n generates a "volume form"

$$\Omega = d^n\sigma = \sqrt{\det g_{ij}}dx^1 \wedge \ldots \wedge dx^n \quad \text{locally in oriented atlas.}$$

Form $\Omega = d^n\sigma$ is closed and **not exact** for closed (compact) manifolds because

$$\int\limits_{M^n} \Omega = \text{volume}\, M > 0.$$

So $H^n(M^n) \neq 0$ for closed oriented manifolds. For connected nonoriented manifolds and and manifolds with boundary we have $H^n(M^n) = 0$ (not proved yet).

3. **"Nondegenerate 2-forms":**

$$\Omega = \sum_{i<j} \Omega ij dx^i \wedge dx^j \quad \text{(locally)}$$

$$\det \Omega_{ij} \neq 0, \quad \Omega_{ij} = -\Omega_{ji} \;\; \text{we have} \;\; n = 2k.$$

Theorem. For every 2-form in $2k$-dimensional manifold we have:

$$\frac{1}{k!}\underbrace{\Omega \wedge \ldots \wedge \Omega}_{k \text{ times}} = \sqrt{\det \Omega_{ij}}dx^1 \wedge \ldots \wedge dx^{2k}.$$

Proof. Let $M^k = \mathbb{R}^n$ and $\Omega_{ij} = \text{const}$.

Step 1. Choose such basis that

$$\Omega_{ij} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \Omega = \sum_{i=1}^{k} dx^i \wedge dp_i.$$

Step 2. Proof Theorem in this basis:

$$\left(\sum_{i=1}^{k} dx^i \wedge dp_i\right)^k = k!\; dx^1 \wedge dp_1 \wedge \ldots \wedge dx^k \wedge dp_k.$$

We have:

$$(dx^i \wedge dp_i) \wedge (dx^j \wedge dp_j) = (dx^j \wedge dp_j) \wedge (dx^i \wedge dp_i).$$

Step 3. Return to the original basis. Both sides are n-forms and transform in the same way.

So Theorem is proved.

Lemma. Every skew-symmetric inner product can be reduced to the form

$$\Omega_{ij} = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ -\mathbb{1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Proof (for nondegenerate forms). For every vector e there exists e' such that $< e, e' >= 1$. Find orthogonal compliment to the subspace (e, e'). Iterate this process. Our Lemma follows.

Remark. $\sqrt{\det \Omega_{ij}}$ is a polynomial of the matrix entries if $\Omega_{ij} = -\Omega_{ji}$. It is called **"Pfaffian".**

Definition. Manifold M^{2k} is called "simplistic" if there exists a 2-form $\Omega_{ij} dx^i \wedge dx^j$ such that $\det \Omega_{ij} \neq 0$ and $d\Omega = 0$.

Examples.

a) $T_*(M^k)$. We have

$$\Omega = \sum_{i=1}^{k} dx^i \wedge dp_i.$$

b) Kähler (complex manifolds). Complex coordinates $(z^1, \ldots, z^n)$ (locally).

$$o < g_{\alpha\beta} dz^\alpha d\bar{z}^\beta - \text{ Riemannian Metric.}$$

$$\Omega = \frac{i}{2} \sum_{\alpha,\beta} g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta, \quad d\Omega = 0, \quad \det g_{\alpha\beta} \neq 0.$$

Corollary. Every simplectic manifold is oriented.

Proof. $\Omega^k = \sqrt{\det \Omega_{ij}} dx^1 \wedge \ldots \wedge dx^{2k}$ is a **"volume form".**

Lecture 22. Symplectic Manifolds and Hamiltonian Systems. Poisson brackets. Preservation of Symplectic form by Hamiltonian System.

Symplectic form. M^n, Ω, $d\Omega = 0$, $n = 2k$, $\Omega^k = k! d^n\sigma$ – volume.

Property (without proof): locally there exists a system of coordinates $x^1, \dots, x^k$, $p_1, \dots, p_k$ such that

$$\Omega = \sum_{i=1}^{k} dx^i \wedge dp_i.$$

Example. Consider cotangent bundle of a C^∞ manifold $M^n = T_* N^k$, with coordinates (x, p). Change of coordinates

$$x = x(y), \quad \acute{p}_j = p_i \frac{\partial x^i}{\partial y^j}$$

Lemma 1. 2-form Ω is well-defined (independent of the choice of coordinates in N).

Proof. $\Omega = \sum\limits_i dx^i \wedge dp_i$, $\acute{\Omega} = \sum\limits_{i=1}^{k} dy^i \wedge d\acute{p}_i$. Why $\Omega = \acute{\Omega}$.

$$d\acute{p}_j = dp_i \frac{\partial x^i}{\partial y^j} + p_i \frac{\partial^2 x^i}{\partial y^j \partial y^k} dy^k$$

$$\sum_j dy^j \wedge d\acute{p}_j = \sum_{ij} dy^j \wedge dp_i \frac{\partial x^i}{\partial y^j} + \xcancel{dp_i \frac{\partial^2 x^i}{\partial y^j \partial y^k} dy^j \wedge dy^k} =$$

$$= \sum_{ij} \frac{\partial x^i}{\partial y^j} dy^j \wedge dp_i = \sum_i dx^i \wedge dp_i.$$

O.K.

More generic simplectic form in $T_* N$ is

$$\sum dx^i \wedge dp_i + \frac{e}{c} B = \Omega_B, \quad B = \sum_{i<j} b_{ij}(x) dx^i \wedge dx^j, \quad dB = 0.$$

For $k = 3$ it is **"Magnetic Field"** (correction of simplectic form).

Hamiltonian system in: $T_*N = M$ is defined by "Hamiltonian" $H(x,p)$,

$$\dot{x}^i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial x_i}$$

Another form: Take covector field (dH). Take inner product given by simplistic form $\Omega = \sum dx^i \wedge dp_i$.

$$\Omega_{ij} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

Construct vector field $\eta_H = \Omega^{-1}(dH)$.

Dynamical System.

$$\dot{z} = \eta(z), \quad z = (x,p),$$
$$\dot{x} = H_p, \quad \dot{p} = -H_x.$$

"Poisson Bracket".

$$\{f,g\} = < df, dg >_\Omega = - < dg, df >_\Omega .$$

It provides the structure of Lie algebra.

For any functions $f(x,p)$ we have:

$$\dot{f} = \{H, f\}.$$

We have $\{H,H\} = 0$ – conservation of energy.

Geodesics. $M = T_*N$, $H = \frac{1}{2} g^{ij} p_i p_j$. Lagrangian $L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$,

$$H = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L, \quad p_i = \frac{\partial L}{\partial \dot{x}^i}.$$

Euler-Lagrange equations:

$$\dot{p}_i = \frac{\partial L}{\partial x^i} \quad \Leftrightarrow \quad \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0.$$

Theorem 1. In every **compact** closed simplectic manifold M^n $(n = 2k)$ with form Ω (nondegenerate, we have

$$H^{2j}(M) \neq 0, \quad j = 1, \dots, k.$$

Proof. We know that $\Omega^k = k!$ volume, so $\int \Omega^k \neq 0$. Therefore $H^{2j} \neq 0$, because Ω^{2j} can not be exact (if Ω^{2j} is exact then Ω^{2k} is exact.)

Theorem 2. For any simplectic manifold M^n and Hamiltonian system with Hamiltonian $H : M^n \to \mathbb{R}$ simplectic form is preserved by mapping $S_t : M^n \to M^n$ (time shift by our system).

Proof. Locally we choose such coordinates $(x^1, \dots, x^k, p_1, \dots, p_k)$ that $\Omega = \sum\limits_{i=1}^{k} dx^i \wedge dp_i$ (Darboux).

Our system is:

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{x}^i = \frac{\partial H}{\partial p_i}$$

Small time shift is:

$$\begin{aligned} S_t :\ & p_i \to p_i + \dot{p}_i t + O(t^2) \\ & x^i \to x^i + \dot{x}^i t + O(t^2) \end{aligned}$$

$$\begin{aligned} S_t^* p_i &= p_i + \dot{p}_i t + O(t^2) \\ S_t^* x^i &= x^i + \dot{x}^i t + O(t^2) \end{aligned}$$

We have

$$S_t^*(dx^i \wedge dp_i) = \sum_i (dx^i + \dot{x}^i t) \wedge (dp_i + \dot{p}_i t) = \sum_i \left[dx^i + td\frac{\partial H}{\partial p_i} \right] \wedge \left[dp_i - td\frac{\partial H}{\partial x^i} \right] =$$

$$= \sum_i dx^i \wedge dp_i + t \left[d\frac{\partial H}{\partial p_i} \wedge dp_i - dx^i \wedge d\frac{\partial H}{\partial x^i} \right] =$$

$$= \Omega + \sum_{ij} t \left[\cancel{\frac{\partial^2 H}{\partial p_i \partial p_j} dp_j} \wedge dp_i + \frac{\partial^2 H}{\partial p_i \partial x^j} dx^j \wedge dp_i - \cancel{dx^i \wedge \frac{\partial^2 H}{\partial x^i \partial x^j}} dx^j - dx^i \wedge \frac{\partial^2 H}{\partial x^i \partial p_k} dp_k = \right]$$

$$= \Omega$$

Theorem is proved.

Remark 1. “Multivalued Hamiltonian”= closed 1-form dH.

Remark 2. “Poisson structure”: Ω may be degenerate, Ω^* is well-defined but degenerate ($\Omega^* = \Omega^{ij}$).

Remark 3. **The Poisson Structure** is a symplectic inner product of covectors which defines a Lie Algebra of functions (the Poissoon Bracket):

$$\{f,g\} =< df, dg >_\Omega$$

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

Here we have $\Omega^{ij} = (\Omega)^{-1}$ if it is nondegenerate. This definition implies in that case $d\Omega = 0, \Omega = \Omega_{ij} dx^i \wedge dx^j$.

Remark 4. According to the Darboux Theorem, every nondegenerate Symplecttic Geometry with $d\Omega = 0$ is always "flat", i.e. there exists system of local coordinates such that Ω is constant (and reduced to standard canonical form.)

Homework 7.

1. Let action functional is

$$S\{\gamma\} \;=\; \int_\gamma g_{ij}\, \dot{x}^i\, \dot{x}^j\, dt \;+\; \int_\gamma A_i(x)\, dt$$

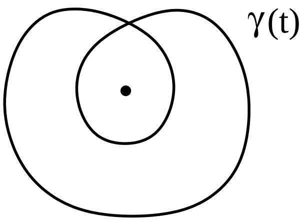
where $\gamma = \{x^i(t)\}$.

Prove that the Euler - Lagrange equation depends on the form

$$d\left(A_i\, dx^i\right) \;=\; \Omega$$

(only).

2. Let $\gamma(t)$ be a plain curve, closed.



Prove that:

$$\oint k(s)\, ds \;=\; 2\pi \times n \quad , \qquad n = \text{integer}$$

where s is natural parameter and $k(s)$ is curvature.

3. Calculate **area form** in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$ in the spherical (pseudospherical) coordinates (polar for $\mathbb{R}^2$).

4. Prove that

$$d\varphi \ = \ c \cdot \frac{xdy - ydx}{x^2 + y^2}$$

in the plane $\mathbb{R}^2$.

5. Let a complex line bundle C^1_x be given over M^n, $(x^1, \ldots, x^n)$ with an imaginary Differential - Geometric Connection $\{A^U_j \, dx^j \ = \ iA^U_{j\mathrm{Real}} \, dx^j\}$, $U \subset M^n$, $A^U_{j\mathrm{Real}} \in \mathbb{R}$. Let the bundle is "unitary" (i.e. in the domains $U \cap V$ change of basis $e_U(x) \ = \ g^{UV}(x) \, e_V(x)$, $g^{UV} \ \in \ U_1 \ = \ \mathbb{S}^1 \quad (e^{i\psi}))$.

Prove that 2-form

$$H \ = \ \left(H^U\right) \ = \ \left(dA^U_{\mathrm{Real}}\right) \ = \ \left(dA^V_{\mathrm{Real}}\right)$$

in $U \cap V$ is well defined **as a closed 2-form** in $\Lambda^2(M^n)$.

$$d\,H \ = \ 0$$

(first Chern class). Its integrals along the closed 2-submanifolds are always $2\pi n, n \in Z$.

Remark. This fact (well-known in Topology of characteristic classes) was also discovered by Dirac in 1930-50s in process of Heisenberg-Schrodinger quantization of "Magnetic Monopole": the Hilbert space of state is in fact space of sections of complex line bundle where vector-potential of magnetic field is a covariant derivative of sections. (The Feinman quantization through the path integral leads to different topology: the Action functional for magnetic monopole is in fact a closed 1-form on the space of paths whose periods along 1-cycles in this space should be quantized, i.e. they should be equal to $2\pi n, n \in Z$ assuming that the Plank constant is equal to one in our units.)

Lecture 23. Volume element in Compact Lie groups. Averaging of differential forms and Riemannian Metric. Cohomology of Homogeneous Spaces.

Homogeneous spaces. $M^n = G/H$, G – Lie group (compact).

Theorem. Let group G be compact. There exists Riemannian Metric in $M = G/H$ s.t. $g^* g_{ij}(x) = g_{ij}(x)$, $g \in G$, $g : M \to M$, $g(x)$- action of the group G.

Remark. We assume that there exists a G**-invariant volume form** Ω in the group G. **We call it** $d\sigma(g)$**.**

Proof of the Theorem. Take any Riemannian Metric $g_{ij}(x)$ in M. Consider family of metrics $g^* g_{ij}(x) = g^h_{ij}(x)$ in M, depending on $g \in G$ as parameter. Integrate:

$$\int\limits_G g^g_{ij}(x) d\sigma(g) = G_{ij}(x).$$

1. This is a Riemannian (positive!) metric.
2. This metric is G-invariant because for $\psi \in G$ we have

$$\psi^* G_{ij} = \int\limits_G \psi^* g^g_{ij}(x) d\sigma(g) = \int\limits_G g^{\psi g}_{ij}(x) d\sigma(g) = \int\limits_G g^{\psi g}_{ij}(x) d\sigma(\psi g)$$

The last identity is based on the following:

Lemma 1. In every compact Lie Group there exists a double-invariant volume form.

Proof. Consider left-invariant metric in $G : g_{ij}$ and its volume form Ω : $h^* g_{ij} = g_{ij}$, $h^* \Omega = \Omega$. But this n-form is also right-invariant because for $g \to hgh^{-1}$ this form maps into itself (space of right-invariant n-forms is 1-dimensional and group is compact).

So the Theorem is proved.

Theorem 2. For homogeneous manifolds with **compact** group G: $M = G/H$ every closed differential form is homologous to an invariant closed form.

Proof. Let $d\omega = 0$ in $\wedge^k(M)$. Consider the integral

$$\overline{\omega} = \frac{1}{|G|} \int\limits_G h^* \omega \, d\sigma(h),$$

$$|G| = \int\limits_G d\sigma(h), \quad h \in G, \quad h : M \to M.$$

1. This is an invariant form.
2. This is a closed form:

$$d\overline{\omega} = \frac{1}{|G|}\int\limits_G h^*\omega \, d\sigma(h) = \frac{1}{|G|}\int\limits_G h^*(d\omega)\, d\sigma(h) = 0.$$

3. This from represents the same cohomology class as ω: all forms $h^*\omega$ belongs to the **same** cohomology class if the group is **connected**. So all "integral sums" belong to the same cohomology class.

So our Theorem follows.

Examples.

1. Spheres S^n, $G = SO_{n+1}$. Only invariant forms are $k = 0$ (scalars) and $k = n$. So we have:

$$H^0(S^n) = H^n(S^n) = \mathbb{R}.$$

2. Tori $T^n = \mathbb{R}^n/\mathbb{Z}^n$, $G = T^n$ (abelian group). Spaces of invariant forms are $\wedge^k\mathbb{R}^n$. All of them are closed: $d\omega = 0$ for invariant forms, so $H^*(T^n) = \wedge(v_1, \dots, v_n)$.
3. $S^{n_1} \times \dots \times S^{n_k}$. Forms $v_1, \dots, v_k$, $\dim v_k = k$. We have $v_j^2 = 0$ **only** relations.

$$H^*(S^{n_1} \times \dots \times S^{n_k}) = \{v_1, \dots, v_k, v_j^2 = 0\}.$$

4. Cohomology of $\mathbb{R}P^{2n+1}$ are the same as for S^{2n+1} ($\boxed{H^*(\ ,\mathbb{R})}$).

   Cohomology of $\mathbb{R}P^{2n}$ are 0 ($k > 0$).

   Cohomology of $\mathbb{C}P^n$ are generated by 2-form Ω and its powers:

$$1, \Omega, \Omega^2, \dots, \Omega^n.$$

Action of SO_n and U_n in $\mathbb{R}^n$ and $\mathbb{C}^n$. Invariant differential forms:

1. $\mathbb{R}^n$, $dx^1, \ldots, dx^n$, $A \in SO_n$.

$$A(dx^i) = \sum_j a^i_j dx^j, \quad AA^t = \mathbb{1}.$$

Invariant differential forms are:

$$k = 0, \quad \text{scalar}, \quad k = n, \; dx^1 \wedge \ldots \wedge dx^n.$$

No other invariant forms!

2. $\mathbb{C}^n$, $dz^1, \ldots, dz^n, d\bar{z}^1, \ldots, d\bar{z}^n$, $A \in U_n$.

$$\left.\begin{array}{l} A(dz^i) = \sum\limits_j a^i_j dz^j, \\ A(d\bar{z}^i) = \sum\limits_j \bar{a}^i_j dz^j \end{array}\right\}$$

Invariant forms are:

$k = 0$ – scalars.

$k = 2$ (2-forms) $\frac{i}{2} \sum\limits_j dz^j \wedge d\bar{z}^j = \Omega$.

$k = 2l$, Ω^l.

For $l = n$ we have the "volume element"

$$\Omega = \pm \left(\frac{i}{2}\right)^n dz^1 \wedge \ldots dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n.$$

For $n = 1$.

$\mathbb{R}^2 = \mathbb{C}^1$: dz, $d\bar{z}$, $A = (e^{i\phi}) = U_1$.

$$\left.\begin{array}{l} dz \to e^{i\phi} dz \\ d\bar{z} \to e^{-i\phi} d\bar{z} \end{array}\right| \quad dz \wedge d\bar{z} \to dz \wedge d\bar{z}.$$

No other invariant differential forms for $\mathbb{C}^n$ and $G = U_n$ **(with constant coefficients).**

Important homogeneous spaces G/H with compact group H.

1. $S^n = SO_{n+1}/SO_n$.
   $S^{2n-1} = U_n/U_{n-1}$.
   $S^{4n-1} = Sp_n/Sp_{n-1}$.
2. $\mathbb{C}P^n = U_{n+1}/U_1 \times U_n = SU_{n+1}/U_n$.
3. Stiefel Manifolds.
   $V_{n,k} = SO_n/SO_{n-k}$ (orthogonal k-frames in $\mathbb{R}^n$).
   $V_{n,k}^{\mathbb{C}} = U_n/U_{n-k}$ (complex version).
4. Grassmann Manifolds.
   $G_{n,k} = SO_n/SO_k \times SO_{n-k}$.
   $G_{n,k}^{\mathbb{C}} = U_n/U_k \times U_{n-k}$
5. Lie Groups (compact, simple) G
   $G = \{G \times G/G\}$, $g \to h_1 g h_2^{-1}$, $h_1 \in G$, $h_2 \in G$.
6. "Principal" homogeneous spaces
   $G = G/(1)$, $H = 1$.

Lecture 24. Volume element in Compact Lie groups. Averaging of differential forms and Riemannian Metric. Cohomology of Homogeneous Spaces.

Theorem. Let G be a compact Lie group with invariant volume element $d\sigma(h)$:

For $h_1,\ h_2 : G \to G,\ \ \phi : h \to h_1 h h_2^{-1}$ we have
$\phi^* d\sigma(h) = d\sigma(h)$.

Let G act (from the left) in the manifold $M\ \ h : M \to M$, $h \in G$. Group G is connected. Then

1. Every closed differential form is homologous to an invariant form.

2. Operator of “Averaging” maps $H^k(M)$ as a unit homomorphism:

$$\overline{\omega} = \frac{1}{|G|} \int\limits_G h^*\omega \; d\sigma(h), \quad h \in G,$$

$$0 = d\omega \;\Rightarrow\; d\bar{\omega} = 0, \;\; \omega - \bar{\omega} = dv, \;\; \bar{\omega} = dv \;\Rightarrow\; \bar{\omega} = d\bar{v},$$

i.e.

$$H^*(M) \cong H^*_{\mathrm{inv}}(M) \;\leftarrow\; \text{calculated with } G - \text{invariant forms only.}$$

Proof was given at the previous lecture.

Examples.

1. S^n, $G = SO_{n+1}$, $H = SO_n$. Invariant forms $k = 0, n$ **only.** $H^0 = \mathbb{R}$, $H^n = \mathbb{R}$.

2. $G = S^3 = SU_2$, $H = (1)$. Invariant forms $k = 0, 1, 2, 3$

$$\wedge^0 = \mathbb{R}, \;\; \wedge^1 = \mathbb{R}^3, \;\; \wedge^2 = \mathbb{R}^3, \;\; \wedge^0 = \mathbb{R},$$

$$d : \wedge^1 \to \wedge^2 \;\text{ is an isomorphism.}$$

3. $U_{n+1}/U_1 \times U_n = \mathbb{C}P^n = SU_{n+1}/U_n$. $G = SU_{n+1}$, $H = U_n$. H acts on $\mathbb{C}^n$ (tangent to 1). Constant forms in $\mathbb{C}^n$ (all closed).

Basis: $dz^{i_1} \wedge \ldots \wedge dz^{i_k} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_l}$, type (k, l).

Action: $a \in U_n$,

$$A(dz^i) = \sum_j a^i_j dz^j,$$
$$A(d\bar{z}^i) = \sum_j \bar{a}^i_j dz^j$$

For $n = 1$.

$$\left.\begin{array}{l} dz \to e^{i\phi} dz \\ d\bar{z} \to e^{-i\phi} d\bar{z} \end{array}\right| \;\Rightarrow\; \text{invariant form } \; dz \wedge d\bar{z}.$$

$k \geq 1$: invariant form are $\sum\limits_{i=1}^{n} dz^i \wedge d\bar{z}^i$ and all Ω^k,$k = 1, \ldots, n$.

No other invariant forms (!).

Cohomology rings.

1. Tori $T^n = S^1 \times \ldots \times S^1 = \mathbb{R}^n/\mathbb{Z}^n$

$$H^*(T^n) = \wedge(v_1, \ldots, v_n), \quad \dim V_j = 1.$$

2. Spheres S^n, $G = SO_{n+1}$. Only invariant forms are $k = 0$ (scalars) and $k = n$. So we have:

$$H^0(S^n) = H^n(S^n) = \mathbb{R}.$$

3. $S^{n_1} \times \ldots \times S^{n_k}$, $G = SO_{n_1+1} \times \ldots \times SO_{n_k+1}$.

$$H^*(S^n) = \{1, v_n | v_n^2 = 0\}.$$

$H^*(S^{n_1} \times \ldots \times S^{n_k}) = \wedge^*(1, v_{n_1}, \ldots, v_{n_k} | v_j^2 = 0)$, **no other relations.**

4. $\mathbb{C}P^n$.

$H^*(\mathbb{C}P^n) = \{1, u, u^2, \ldots, u^n\}$, $\dim u = 2$, $u^{n+1} = 0$, $G = SU_{n+1}$, $H = U_n$.

5. Homology of $V_{n,k}$, $G_{n,k}$ can be computed using forms

$$H^*(\ , \mathbb{R}) =?$$

$$\begin{aligned} V_{n,k} &= SO_n/SO_{n-k}; & G_{n,k} &= SO_n/SO_k \times SO_{n-k}, \\ V^{\mathbb{C}}_{n,k} &= U_n/U_{n-k}; & G^{\mathbb{C}}_{n,k} &= U_n/U_k \times U_{n-k}. \end{aligned}$$

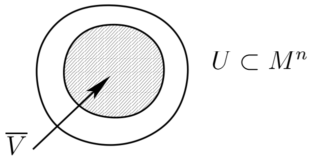
6. Lie Groups G (compact): $G = G \times G/G$.

Lecture 25. Approximation and Transversality.

Differential topology. C^∞-manifolds, Atlases, Charts, Orientations, C^k-maps, rank of map, C^∞-submanifolds: (existence), $M^n \subset \mathbb{R}^N$ (M^n compact).

Approximation. Every continuous map of compact manifolds $f : \underset{x}{M^n} \to \underset{y}{N^m}$, $y(x)$ can be **approximated** by **homotopic** C^∞-map $g : M^n \to N^m$: $\max\limits_{x} |f(x), g(x)| < \epsilon$.

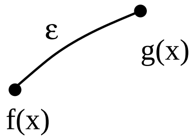
More exactly. Let $f : M \to N$ be C^0-map which is C^∞ in open domain $U \subset M^n$. For every domain V, $\overline{V} \subset U \subset M^n$ approximation $g : M^n \to N^m$ can be chosen such that $f \equiv g$ in V.



(**"Locality Property".**)

Lemma. For $\epsilon > 0$ small enough maps f and g are homotopic.

Proof. Take Riemannian Metric $g_{ij}(y)$ in N^m.



For $\epsilon > 0$ small enough geodesics joining $f(x)$, $g(x)$ is unique in the ball of radius 2ϵ.

Our homotopy it such that every point moves from $f(x)$ to $g(x)$ along this short geodesics homogeneously and reaches its end at $t = 1$. O.K.

Corollary. Homotopy class (C^∞) for c^∞ maps $M \to N$ are the same as C^0 homotopy classes.

Transversality. Let $W^{m-k} \subset N^m$ be a C^∞-submanifold. A map $f : M^n \to N^m$ is called **"transversal along** W**"** iff for every $x \in f^{-1}(W)$ map of tangent spaces

$$\begin{tikzcd}(\pi \circ df) :\ T^n_x \arrow[r, "df"] \arrow[rr, bend right] & T^m_{f(x)} \arrow[r, "\pi", "\text{normal projection}"'] & T^m_N / T^{m-k}_W \ \cong \underset{\text{normal plane}}{\mathbb{R}^k}\end{tikzcd}$$

has rank k.

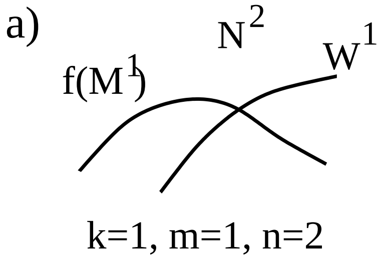
Examples.

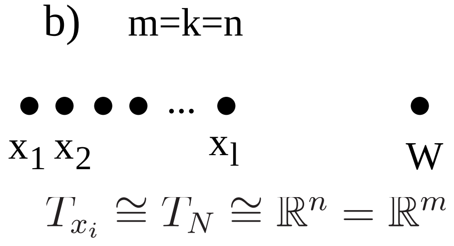
1. $n < k \quad \Rightarrow \quad f^1(\Omega) = \emptyset$ empty.

2. $n = k$:

(a) $f^1(W)$ are isolated points $x_1, \dots, x_l$, $(df)_{x_k}$ has rank k.

(b) **Special case:** $m = k$, $\dim W = 0$ (point).



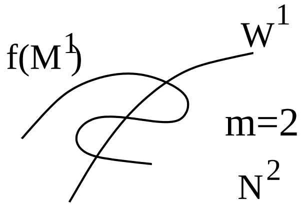


Theorem (without proof).

1. Every map can be approximated by a map transversal along $W \subset N$ (approximation is "local").
2. For every map f submanifold $W \subset N$ can be approximated ("locally") by $\tilde{W} \in N$ such that f is transversal along $\tilde{W}$.

"Locality" = "remains unchanged in the almost entire area where it already was transversal".

Theorem 1. Let map $f : M^n \to N^n \supset W^{n-k}$ be transversal along $W \in N$. Then $f^{-1}(W)$ is a C^∞-submanifold in M of codimension (and map $df\big|_{\text{normal}}$ is **isomorphic**).



Proof. Let $y^1, \dots, y^m$ be local coordinates in N^m and W^{m-k} is given locally by equations: $W : y^1 = \ldots = y^k = 0$.

For the transversal map $f : M \to N$ along $W \subset N$ we have functions

$$y^1(f(x)) = f^*y^1, \dots, f^*y^k = y^k(f(x)),$$

and the inverse image $f^{-1}(W)$ is given by equations:

$$\begin{array}{ccccc} y^1(f(x)) = 0, & \ldots, & y^k(f(x)) = 0 & & \\ \| & & \| & & \\ z^1(x) & \ldots & z^k(x) & = V^{n-k} \subset M \end{array}$$

These equations are **NONDEGENERATE** because of **TRANSVERSALITY** long W (i.e. at the points $f(x) \cap W$).

So we have nondegenerate submanifold

$$V^{n-k} \subset M$$

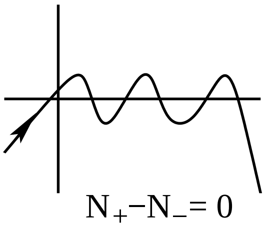
and map

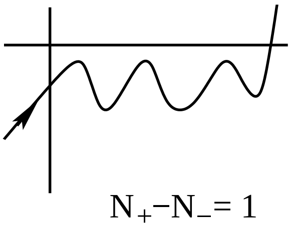
$$V^{n-k} \to W^{n-k}$$

is isomorphic along normal k-planes by definition of **TRANSVERSALITY**.

Examples. Function $\mathbb{R}^n \xrightarrow{f} \mathbb{R} \supset W =$ point. Level $f(x) = y_0$ is transversal iff $df \neq 0$ for all $f^{1}(y)$. **Topological invariants.**

1. Let $n = 1$.





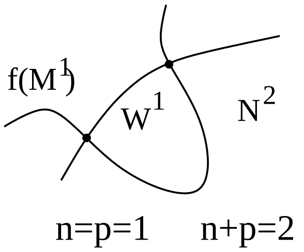
   (a) **local** = numbers N_+ and N_- of "positive" and "negative" points.
   (b) **global** = $N_+ - N_-$.

2. **Degree of map.** $f : M^n \to N^n$ – closed (oriented).

$$f^{-1}(\text{point}) = N_+ \cup N_-,$$

   - degree (mod 2) is $N_+ - N_-$ (mod 2).
   - degree (over $\mathbb{Z}$) is $N_+ - N_-$ (both M^n, N^n are oriented).

3. **Intersection index.**



$$M^n \xrightarrow{f} N^{n+p} \supset W^p.$$

$$(M^n, f) \circ W^p \in \mathbb{Z}$$

if oriented, otherwise modulo 2.

IMPORTANT 1ST CASE: "Empty set transversality"

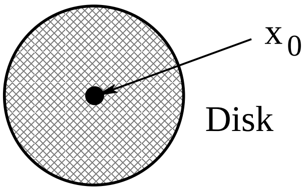
$$f : M^n \to N^m \supset W^{m-k}, \quad n < k, \quad f^1(\Omega) = \emptyset \quad \text{(transversal map)}.$$

Homework 8.

1. Prove that the expression $\sum_i p_i dx^i$ (locally in $T_*(N^k)$) defines correctly 1-form in $T_*(N^k)$.

2. Prove that **geodesic flow** (Hamiltonian $H(x,p) = g^{ij}(x)\, p_i\, p_j\, /\, 2$) preserves the form $\sum_i p_i\, dx^i$ at the level $H = const$ in the manifold $T_*(M^n)$.

3. Prove that every closed k-form Ω in M^n is homologous to the form $\Omega' - \Omega = d\omega$, such that $\Omega' \equiv 0$ in the disc near the point x_0 :



4. Prove that the Euler - Lagrange equation in $T_*(N^k)$ for the functional

$$S\,\{\gamma\} \;=\; \int_\gamma \left(\sum_i p_i\, dx^i \;-\; H\, dt \right) \quad ,$$

$\gamma = \{x(t),\, p(t)\}$, $H = H(x,p)$, is exactly a hamiltonian system

$$\dot{p}_i \;=\; -\frac{\partial H}{\partial x^i} \quad , \quad \dot{x}^i \;=\; \frac{\partial H}{\partial p_i}$$

5. Prove that the flow $\dot{x}^i = \xi^i(x)$ preserves the volume form $\Omega = dx^1 \wedge \cdots \wedge dx^n$ if

$$\sum_{i=1}^{n} \frac{\partial \xi^i(x)}{\partial x^i} \;=\; 0 \qquad (\; \Leftrightarrow \quad S_t^*\, \Omega \;=\; \Omega\,)$$

Homework 6 and Homework 7. Solutions.

Homework 6. Solutions.

1. Geodesics for $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$. Metric

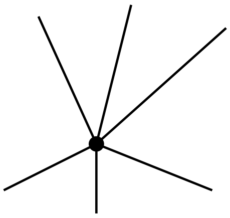
$$\text{a)} \quad dl^2 \;=\; d\rho^2 \;+\; \sin^2\rho\,(d\Omega)^2 \quad , \quad \mathbb{S}^n \;,$$

where $(d\Omega)^2$ is the metric of $\mathbb{S}^{n-1}$,

$$\text{b)} \quad dl^2 \;=\; dr^2 \;+\; r^2\,(d\Omega)^2 \quad , \quad \mathbb{R}^n \;,$$

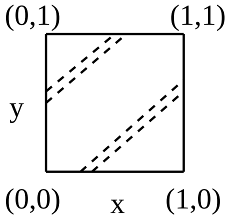
$$\text{c)} \quad dl^2 \;=\; d\chi^2 \;+\; \text{sh}^2\rho\,(d\Omega)^2 \quad , \quad \mathbb{H}^n \;.$$

Straight lines through zero $\eta_0 \in \mathbb{S}^{n-1}$ is geodesics (equation does not depend on angles on $\mathbb{S}^{n-1}$).



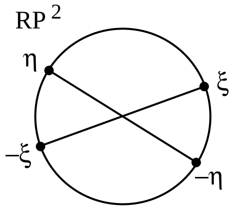
Other geodesics will be found by the action of groups SO_{n+1} (for $\mathbb{S}^n$), $SO_n * \mathbb{R}^n$ (for $\mathbb{R}^n$), and $SO_{n,1}$ (for $\mathbb{H}^n$).

2. Closed geodesic in $\mathbb{T}^2$: straight lines with rational parameters

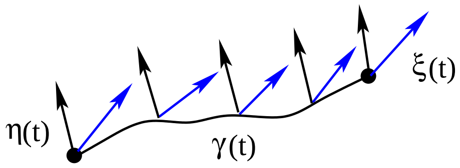


$$y \;=\; \frac{m}{n}\,x \;+\; \text{const}$$

3. Closed geodesic in $\mathbb{RP}^2$ (all nongomotopic to zero)



4. Parallel transport along the path $\gamma(t)$, $x^1(t), \dots, x^n(t)$, connection $\nabla g_{ij} \equiv 0$.



$$\frac{d}{dt}\langle \eta(t)\,,\,\zeta(t)\rangle \;=\; \frac{d}{dt}\left(g_{ij}\,\eta^i(t)\,\zeta^j(t)\right) \;=$$

$$=\;\left(\nabla_{\dot{x}}\,g_{ij}\;\eta^i(t)\;\zeta^j(t)\right) + \left(g_{ij}\;\nabla_{\dot{x}}\,\eta^i(t)\;\zeta^j(t)\right) + \left(g_{ij}\;\eta^i(t)\;\nabla_{\dot{x}}\,\zeta^j(t)\right) \;=\; 0$$

(Leibnitz Identity).

Homework 7. Solutions.

1.

$$S\,\{\gamma\} \;=\; \underset{\displaystyle S_0(\gamma)}{\underset{\uparrow}{\int_\gamma g_{ij}\,\dot{x}^i\,\dot{x}^j\,dt}} \;+\; \underset{\displaystyle S_1(\gamma)}{\underset{\uparrow}{\int_\gamma A_i(x)\,dt}}$$

$A \,=\, A_i(x)\,dx^i$ is one-form in M^n .

$$\mathrm{EL}_{(i)} \;=\; \mathrm{EL}_{0(i)} \;+\; \frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{x}^i}\right) \;-\; \frac{\partial L_1}{\partial x^i} \;=\; 0$$

$$\frac{\partial L_1}{\partial \dot{x}^i} \;=\; A_i(x)\;, \qquad \frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{x}^i}\right) \;=\; \frac{\partial A_i}{\partial x^k}\,\dot{x}^k\;, \qquad \frac{\partial L_1}{\partial x^i} \;=\; \frac{\partial A_j}{\partial x^i}\,\dot{x}^j$$

$$\mathrm{EL}_{(i)} \;=\; \mathrm{EL}_{0(i)} \;+\; \sum_k \frac{\partial A_i}{\partial x^k}\,\dot{x}^k \;-\; \sum_j \frac{\partial A_j}{\partial x^i}\,\dot{x}^j \;=$$

$$=\; \mathrm{EL}_{0(i)} \;+\; \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)\dot{x}^k \;=\; \mathrm{EL}_{0(i)} \;+\; B_{ik}\,\dot{x}^k$$

$(dA \,=\, B)$.

2.

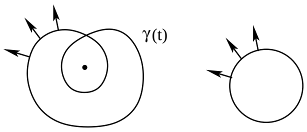
$$\int_\gamma \overbrace{k(s)\,ds}^{d\,\varphi} \;=\; 2\pi\, n \quad ?$$

γ - closed

φ - **rotation of normal (and tangent) vector**.

$$k(s) \;=\; \frac{d\dot{x}(s)}{ds}$$

s - natural parameter.



Another form:

$$\gamma \;\xrightarrow{G}\; \mathbb{S}^1 : \;\; \textbf{Gauss Map} \; : \; G^*(d\varphi) \;=\; k(s)\,ds$$

(check!)

3. Area form

$$\mathbb{R}^2 \; : \quad dx \wedge dy \;=\; r\,dr \wedge d\varphi \quad , \qquad r \;=\; \sqrt{\det g_{ij}}$$

$$dl^2 \;=\; dr^2 \,+\, r^2\,d\varphi^2 \quad , \qquad g_{ij} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad , \quad \sqrt{g} \;=\; r$$

$$\mathbb{S}^2 \; : \quad \frac{4\,dz \wedge d\bar{z}}{(1+|z|^2)^2} \;=\; \sin\theta\,d\theta \wedge d\varphi$$

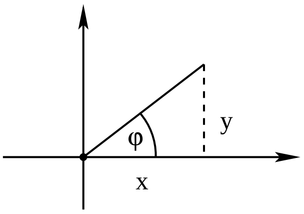
$$dl^2 \;=\; \frac{4\,dz\,d\bar{z}}{(1+|z|^2)^2} \;=\; d\theta^2 \,+\, \sin^2\theta\,d\varphi^2 \quad , \qquad g_{ij} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$\mathbb{H}^2 \; : \quad \frac{4\,dz \wedge d\bar{z}}{(1-|z|^2)^2} \;=\; \operatorname{sh}\chi\,d\chi \wedge d\varphi$$

$$dl^2 \;=\; \frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2} \;=\; d\chi^2 \,+\, \operatorname{sh}^2\chi\,d\varphi^2 \quad , \qquad g_{ij} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sh}^2\chi \end{pmatrix}$$

4.

$$d\varphi \;=\; \text{const} \cdot \frac{xdy - ydx}{x^2+y^2} \quad , \qquad \varphi \;=\; \operatorname{arctg}\frac{y}{x}$$



5. Complex Line bundle C^1_x over M^n , $(x^1, \ldots, x^n)$:

$$M^n = \cup_\alpha U_\alpha \quad , \quad x^1_\alpha, \ldots, x^n_\alpha$$

Let $U_\alpha = U$, $U_\beta = V$, we have in $U \cap V$ for the bases $e_U(x)$, $e_V(x)$:

$$e_U(x) = g^{UV}(x)\, e_V(x) \quad , \quad g^{UV}(x) = e^{i\varphi^{UV}(x)}$$

- U_1 - **connection**, group $G = U(1)$. We have for the connection $A^U_i(x)\, dx$ in $U \cap V$:

$$A^U = \left(g^{UV}\right)^{-1} A^V g^{UV} - \left(g^{UV}\right)^{-1} dg$$

where

$$\left(g^{UV}\right)^{-1} A^V g^{UV} = A^V \quad , \quad \left(g^{UV}\right)^{-1} dg = i\, d\varphi(x)$$

Finally, we have for A^U_{Real} :

$$A^U_{\text{Real}} = A^V_{\text{Real}} - d\varphi(x)$$

Conclusion:

$$d\, A^U_{\text{Real}} = A^V_{\text{Real}} = H$$

(same for all U^α).

Obviously $d\, H = 0$.

Remark 1.

a) $H \in H^2(M^n, \mathbb{R})$ is the first Chern class of a line bundle.

b) **It does NOT depend on the choice of connection.**

Proof. Let us have $\tilde{A}^U = A^U + i\Psi^U$, $\tilde{A}^V = A^V + i\Psi^V$. We have then in $U \cap V$: $\Psi^U = \Psi^V = \Psi$. So we get, that Ψ is a globally defined 1-form in M^n and

$$\tilde{H} = H + d\Psi$$

where $d\Psi$ is an exact form in M^n .

c)

$$\frac{1}{2\pi}\int_{2-cycle} H$$

is **integer** $(\in \mathbb{Z})$.

Lecture 26. Transversality. Imbeddings and Immersions of Manifolds in Euclidean Spaces.

Transversality.

$$f : M^n \to N^m \supset W^{m-k}$$

(along W)

$$T^n_x \xrightarrow{df} T^m_{f(x)} \xrightarrow{\pi} T^m_N / T^{m-k}_W \cong \mathbb{R}^k,$$

$$\mathrm{rk}(\pi \circ df) = k \ \text{ for all } \ x \in f^{-1}(W).$$

Example. $n < k \ \Rightarrow \ f^{-1}(W) = \emptyset$.

Applications (imbeddings and immersions).

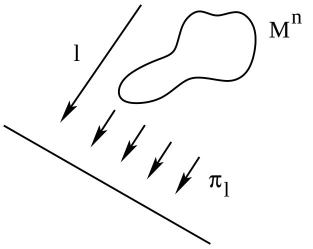
Theorem 1. Every map $f : M^n \to \mathbb{R}^N$ can be approximated by **NONDEGENERATE** imbedding if $N \geq 2n+1$ (M^n is compact C^∞ manifold).

Proof.

Step 1. Consider any C^∞-map $M^n \xrightarrow{\phi} \mathbb{R}^N$ and C^∞ imbedding $M^n \xrightarrow{\psi} \mathbb{R}^Q$.

Their product $\phi \times \psi : M^n \to \mathbb{R}^{N+Q} = \mathbb{R}^P$ gives us C^∞ imbedding $\Phi : M^n \subset \mathbb{R}^{N+Q} = \mathbb{R}^P$, $N + Q > 2n + 1$.

Step 2. Project the imbedding $M^n \in \mathbb{R}^P$ into the space $\mathbb{R}^{P-1}$ along the vector (direction) l (unit vector $\pm l$ in S^{N+Q-1}).



Consider the projection $\pi_l \circ \Phi : M^n \to \mathbb{R}^{P-1}$.

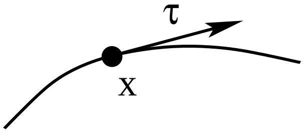
Lemma 1. For all vectors $\pm l$ outside of the image $R : M^n \times M^n \backslash \Delta \to S^{P-1}$

$$R(x,y) = \frac{x-y}{|x-y|} \in S^{P-1}, \;\; \pm l \notin R(M^n \times M^n \backslash \Delta).$$

the projection map $\pi_l \circ \Phi$ is imbedding.

Proof. $\pi_l(x) = \pi_l(y) \;\Rightarrow \pm l \in T(M \times M)$.

Lemma 2.



Consider the map $T_1^*(M^n) \xrightarrow{R_1} S^{P-1}$, where $(x,\tau) \in T_1^*(M^n)$, $|\tau| = 1$, τ is tangent to M^n at x. $R_1(x,\tau) = \tau$.

Let $\pm l \notin \operatorname{Im} R_1$. Then the projection $\pi_l \circ \Phi : M^n \to \mathbb{R}^{P-1}$ is an nondegenerate immersion.

Proof. No one tangent vector to $M^n \in \mathbb{R}^P$ belongs to the kernel of the projection if l not parallel to τ for all (x,τ).

Lemma is true. O.K.

Proof of the Theorem 1. Let Φ be any imbedding $M^n \subset \mathbb{R}^N$ where $N > 2n+1$. We project it along the direction l, $\pm l \notin \operatorname{Image}(M \times M \backslash \Delta)$. **Almost all vectors** have this property because $\dim \operatorname{Image}(M \times M \backslash \Delta) = 2n$, less then $\dim S^{N-1} \subset \mathbb{R}^N$, $N-1 > 2n$.

Here Transversality works. We apply transversality to the pair

$$M \times M \backslash \Delta \to S^{N-1} \supset (\text{point}).$$

Out Theorem follows (nondegeneracy see below).

Theorem 2. Every map $M^n \to \mathbb{R}^N$ can be approximated by immersion, if $N \geq 2n$.

Proof is similar. Image of the map $T^*(M) \to S^{N-1}$, $(x,\tau) \to \tau$, $\tau \in \mathbb{R}^n_x$, $|\tau| = 1$ does not touch "typical" point $(\pm l) \in S^{N-1}$, because $\dim T^*(M)$ is equal to $2n-1 < N-1$, if $N > 2n$.

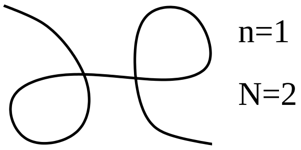
So for $N > 2n$ we can project imbeddings along π_l and image in $\mathbb{R}^{N-1}$ remains an immersion $M^n \to \mathbb{R}^{N-1}$.

So for $N-1 \geq 2n$ we can project preserving immersion.

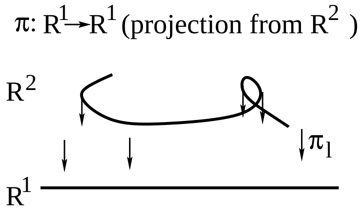
Theorem follows.

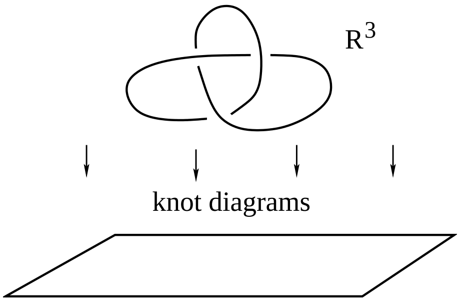
Example.

- $N = 2n$. Projection to $\mathbb{R}^{N-1}$ may be singular (not immersion).
- $N = 2n + 1$. Projection to $\mathbb{R}^{N-1}$ may be not imbedding.



Singularities of projection:





Remark 1. Our results are true for all manifolds (C^∞):

a) $M^n \to N^N$, $N > 2n$, **imbeddings are dense.**

b) $M^n \to N^N$, $N \geq 2n$, **immersions are dense.**

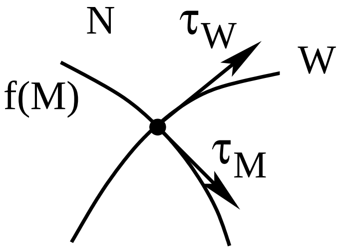
Remark 2. Our results are true for noncompact manifolds.

Lecture 27. Intersection Index and Degree of Map.

Intersection Index:

$$f : M^n \to N^m \supset W^{m-k}, \quad f \text{ transversal along } W, \ \mathbf{k} = \mathbf{n}.$$

Intersection $f(M^n) \cap W^{m-k}$ is "transversal" in the point $x \in M$ if linear spaces $(df)_x \cdot \mathbb{R}^n_x \in \mathbb{R}^m_{f(x)}$ and $\mathbb{R}^{m-k}_{f(x)}$ (tangent to W) jointly generate $\mathbb{R}^m$ (tangent to N^m in $f(x)$).



- τ_N - oriented frame to N.
- τ_M - oriented frame to M.
- τ_W - oriented frame to W.

"Sign" of intersection point $x_i = \operatorname*{sign}_{x_j} [\tau_N / \tau_M \tau_W]$.

1. **Intersection number:**

$$M \circ W \ \text{ in } \ N \quad = \sum_{x_j \in f^{-1}(W)} (-1)^{\operatorname{sign}(x_j)}$$

2. **Nonoriented Case:** intersection number is from $\mathbb{Z}_2$.

Degree of map.

$$f : M^n \to N^n \supset W = (\text{point}), \quad k = n = m.$$

f-transversal along W:

$$f^{-1}(W) = x_1 \cup \ldots \cup x_p \in M^n,$$

M^n – oriented (closed). Degree of f [$\deg f$] is equal to

$$\deg f = \sum_{x_j \in f^{-1}(W)} (-1)^{\operatorname{sign}(x_j)}$$

$$\operatorname{sign}(x_j): \quad \tau_N / \tau_M = \operatorname{sign} \det \left(df \Big|_{x_j} \right),$$

$$\underbrace{df \Big|_{x_j}}_{\text{linear map}} : \mathbb{R}^n_{\text{tangent to } M} \to \mathbb{R}^n_{\text{tangent to } N}.$$

$\deg f$ is a particular case of intersection number for the case $\dim W = 0$ (W=point). If M and/or N are nonoriented,

$$\deg_W f \in \mathbb{Z}_2.$$

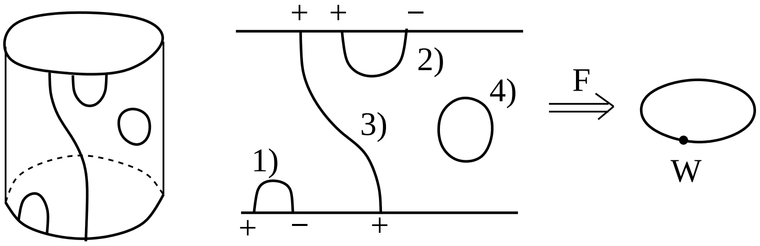
Theorem 1. Degree of map and Intersection Index are the same for homotopic maps.

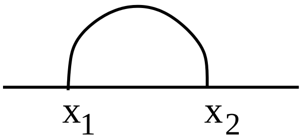
Proof. Consider homotopy F such that

$$F : M \times I \to N \supset W, \quad F\big|_{t=0} = f, \quad F\big|_{t=1} = g, \quad F \in C^{\infty},$$

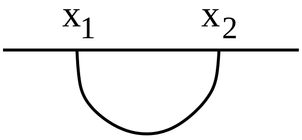
and F is transversal along W.

Consider $F^{-1}W$. We have picture with 4 possibilities:

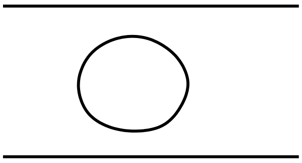


1. 

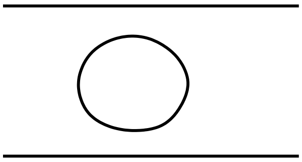
$\operatorname{sign}(x_1) = -\operatorname{sign}(x_2)$.

2. 

$\operatorname{sign}(x_1) = -\operatorname{sign}(x_2)$.

3. 

no point for $t = 0, 1$.

4. 

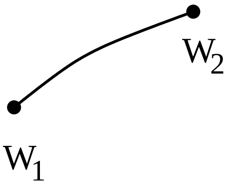
sign remains unchanged.

Conclusion. Total sum remains unchanged.

O.K.

Corollary ($N = S^n$): degree of map does **not** depend on the point $W \subset N^n$.

Proof. Let $N = S^n$. Rotate sphere S^n ϕ_t, such that $\phi_0 = \mathbb{1}$, $\phi_1(W_1) = W_2$,



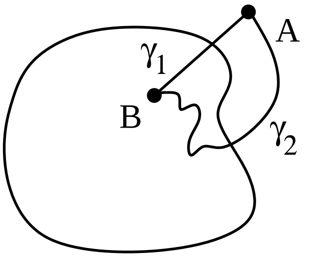
Consider homotopy process

$$\phi_t \circ f = F(x,t), \quad F\big|_{t=o}(x) = W_1, \quad F\big|_{t=1}(x) = W_2.$$

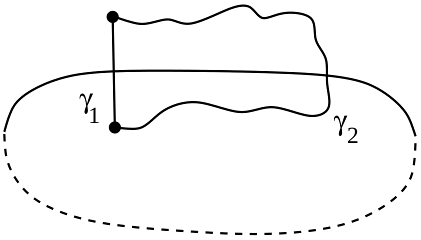
O.K.

Examples.

1. **Jordan Theorem.**



$S^1 \subset \mathbb{R}^2$ (C^∞-imbeddings). Two paths γ_1, γ_2 connecting a pair of points A, B.



$(\gamma_1 \cup \gamma_2) \circ S^1 = 0$ (Homework 9).
If $\gamma_1 \circ S^1 = 1 \;\Rightarrow\; \gamma_2 \circ S^1 = \pm 1$.

Conclusion. If $\gamma_1 \circ S^1 = 1$, then every path γ_2 connecting A and B crosses S^1.

Remark. Intersection of 2 closed submanifolds $M^n \circ W^k \in \mathbb{R}^m$, $m = n + k$ **is equal to 0.**

2. **Gauss Theorem:** Let $w = f_n(z)$ be a polynomial in $\mathbb{R}^2 = \mathbb{C}$. It defines a map $f : S^2 \to S^2$, $\deg f = n$, **plus** every point $f^{-1}(w)$ is positive. So we have exactly n points in $f^{-1}(w) : z^1, \ldots, z^n$ in **transversal** case.

3. $F : M^n \to N^n$ – map of closed oriented manifolds, $m = n = k$, W-point.

   Theorem.

   $$\int\limits_M f^*(\Omega) = \underbrace{(\deg f)}_{\text{integer}} \int\limits_N \Omega, \qquad \text{(Why?)}$$

   Application in Geometry will be presented.

Lecture 28. Intersection Index and Degree of Map.

Intersection index and degree of map are homotopy invariant.

Theorem. Let M^n, N^n be oriented (closed) manifolds and $f : M^n \to N^n$ be a C^∞-map. Then for every n-form Ω in N we have

$$\int\limits_M f^*(\Omega) = \underbrace{(\deg f)}_{\text{integer}} \cdot \int\limits_N \Omega.$$

Proof. By transversality theorem, almost all points $W \in N$ are "transversal" along $W \in N$. The set of transversal points in N is open and has measure=1. So the integral $\int\limits_N \Omega$ depends only on the set of transversal points in $w \in N$. Let $U \ni w$ be a small open set containing w such that all points in U are transversal. Let U be connected set.

We have $f^{-1}(U) = U_1 \cup \ldots \cup U_l$ where $U_j \xrightarrow{f} U$ is a diffeomorphism preserving $(+)$ or reversing $(-)$ orientation with sign $= (-1)^{s_j}$, $j = 1, \ldots, l$.

We have

$$\int\limits_{U_j} f^*\Omega = (-1)^{s_j} \int\limits_U \Omega,$$

and

$$\int\limits_{f^{-1}(U)} f^*\Omega = \left(\int\limits_U \Omega\right) \cdot \left(\sum_j (-1)^{s_j}\right).$$

We have

$$\deg f = \sum_j (-1)^{s_j}, \quad (\deg f \text{ calculated at the point } w \in U),$$

by definition.

We know that $\deg f$ does not depend on w: it is same for all transversal points $w \in N$, and their measure is 1.

So we calculated

$$\int_M f^*(\Omega) = (\deg f) \cdot \int_N \Omega.$$

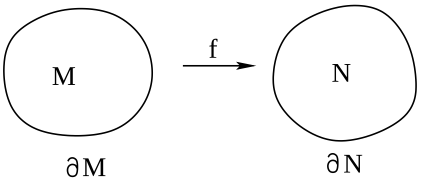
Theorem is proved.

Corollaries.

1. This Theorem is true also for manifolds with boundary

   $f : (M, \partial M) \to (N, \partial N)$

   assuming that $f(\partial M) \subset \partial N$.

   In this case $\deg f_M = \deg f_{\partial M}$ (Why?) (Homework 10).

2. Degree of map $f : M \to N$ is well-defined for "proper" maps f such that f^{-1}(compact) is compact.

   If Ω is such that $\int_N \Omega < \infty$, we have

   $$\int_M f^*(\Omega) = (\deg f) \cdot \int_N \Omega.$$

   (See examples of polynomial mappings)

   $$f = P_n(x) : \underset{(x)}{\mathbb{R}} \to \underset{(x)}{\mathbb{R}}, \quad y = a_0 x^n + \ldots + a_n.$$

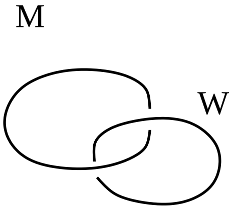
   (See Homework 10)

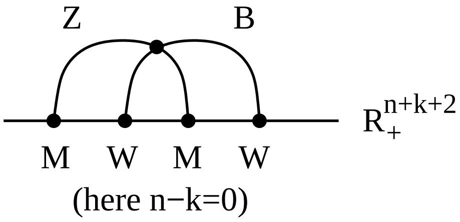
What is "Linking number" for 2 submanifolds $\{M^n, W^k\}$ in $\mathbb{R}^{n+k+1}$?

a) $\{M^n, W^k\} = M^n \circ B^{k+1}$, $\partial B^{k+1} = W^k$ in R^{n+k+1}.

b) $\{M^n, W^k\} = Z^{n+1} \circ W^k$, $\partial Z^{k+} = M^n$ in R^{n+k+1}.

c) $\{M^n, W^k\} = Z^{n+1} \circ B^{k+1}$ in R_+^{n+k+2}.



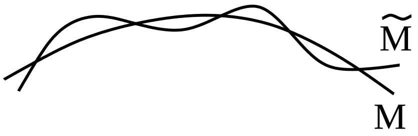


Intersection Number (in manifolds).

1.
$$M^n \circ W^k = (-1)^{kn}\ W^k \circ M^n \quad \text{(oriented case)}.$$

2. Self-intersection ($M^n = W^n$).
$$M^n \circ M^n = ? \ \text{ in } \ N^{2n}.$$

Perturb $M = (\tilde{M})$.

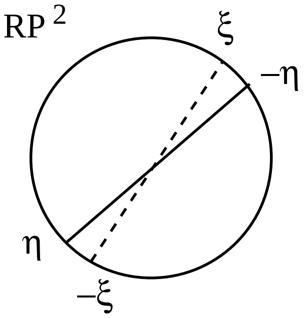


$M \circ M = \tilde{M} \circ M$.

Corollary. $M^n \circ M^n = 0 = -M^n \circ M^n$, $n = 2k + 1$.

Modulo 2 (nonorientable case).

Example.



$M^1 \circ M^1 = 1$.

Fixpoint (Lefschetz) problem:

$$f : M^n \to M^n \ \ \text{(orientable)}.$$

Solutions: $f(x) = x$.

Algebraic number of solutions: Let $\Delta \subset M \times M$, $(x, x) \in \Delta$. "Lefschetz Number" $(x, f(x)) \in \Delta_f$.

$$L(f) = \Delta_f \circ \Delta.$$

Homework 9.

1. Prove that for all open 2-manifolds we have $H^2(M^2, \mathbb{R}) = 0$.

2. Prove that degree of map $f : \mathbb{S}^1 \to \mathbb{S}^1$ can be calculated by

$$f(x + 2\pi) = f(x) + 2\pi m \ , \quad m \in \mathbb{Z} \ , \quad m = \deg f$$

(2π - circles).

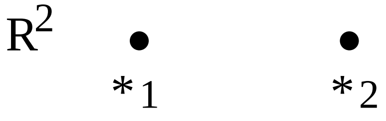
3. Prove that intersection index of 2 closed curves in any domain $U \subset \mathbb{R}^2$ is equal to 0.

4. Prove that every map $M^n \to \mathbb{S}^m \backslash W^{m-k}$ is homotopic to zero for $n < k - 1$.

5. Prove that every map $\mathbb{S}^n \to \mathbb{S}^m$ is homotopic to zero for $n < m$.

6. Calculate cohomology ring $H^*(SO_4, \mathbb{R}) = ?$

7. Calculate $H^1(M^2, \mathbb{R}) = ?$ for $M^2 = \mathbb{R}^2 \backslash (*_1 \cup *_2)$:



Lecture 29. Intersection Index and Degree of Map.

Intersection Index, Degree of Map.

For $M^n \circ W^k \subset \mathbb{R}^{n+k}$ we have Intersection Index.

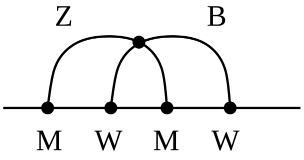
$$\text{For } f : M^n \to N^n \text{ we have } \deg f.$$

They have the following properties:

1. They are homotopy invariant.
2. $M^n \circ W^k = (-1)^{kn}\, W^k \circ M^n$.
3. $M \circ M = 0$ for $n = k = 2l + 1$ (oriented case, $M \circ M \in \mathbb{Z}$).
4. In nonoriented case $M^n \circ M^n$ may be $\neq 0$.
5. Linking number $\{M^n, W^k\}$ in $\mathbb{R}^{n+k+1}$.

$$M = \partial Z, \quad W = \partial B, \quad M \cap W = \emptyset, \quad \{M^n, W^k\} = M \circ B = Z \circ W.$$

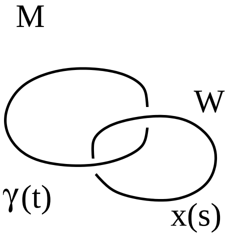
Let $M, W \in \mathbb{R}^{n+k+1} = \partial\mathbb{R}_+^{n+k+2}$



We have $B \circ Z = \{M, W\}$.

Gauss formula:

Let $M = S^1 \in \mathbb{R}^3$, $W = S^1 \in \mathbb{R}^3$.



$$\det\begin{vmatrix} d\gamma^1 & d\gamma^2 & d\gamma^3 \\ dx^1 & dx^2 & dx^3 \\ \gamma^1 - x^1 & \gamma^2 - x^2 & \gamma^3 - x^3 \end{vmatrix}$$

$$\|$$

$$\{M, W\} = \frac{1}{4\pi} \oint\limits_M \oint\limits_W \frac{(d\vec{\gamma}(t) \times d\vec{x}(s), \vec{\gamma}(t) - \vec{x}(s))}{|\vec{\gamma}(t) - \vec{x}(s)|^3}$$

$M = \gamma(t)$, $W = x(s)$,
$\gamma(t) \neq x(s)$,
$\phi = \{\gamma(t), x(s)\}$,
$\phi : T^2 \to \mathbb{R}^3 \times \mathbb{R}^3 \backslash \Delta$.

$$\boxed{\text{Is it closed 2-form in } \mathbb{R}^3 \times \mathbb{R}^3 \backslash \Delta?}$$

Fixed Point Problem.

$$f : M^n \to M^n, \quad f(x) = x \quad (?)$$

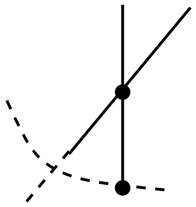
Algebraic (Lefschetz) number of fixpoints:

$$L(f) = \Delta \circ \Delta_f \text{ in } M^n \times M^n, \quad (x,x) \in \Delta, \quad (x, f(x)) \in \Delta_f,$$

$$\Delta = \Delta_f \;\Rightarrow\; f(x) = x.$$

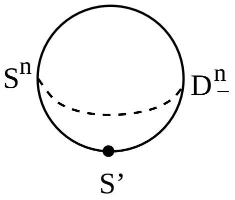
1. **Special case** $f \sim 0 :\; M^n \to \text{const}.$

$$\Delta \circ \delta_f = 1.$$



Example. $M^n = S^n$, $\phi : S^n \to D^n$ (south hemisphere).

Map: $S^n \xrightarrow{\phi} D^n_- \xrightarrow{f} D^n_- \xrightarrow{\psi} S^n$



$$g = \psi \circ f \circ \phi \sim 0 \text{ (obvious).}$$

Conclusion: $\Delta \circ \Delta_f = 1$. **There exists fix point** (at least one).

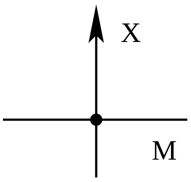
2. **Special case** $f \sim 1 : M \to M$ (identical). We have

$$\Delta \circ \Delta_f = \Delta \circ \Delta = \begin{cases} 0, \; n = 2k+1, \\ \chi(M^n), \; n = 2k. \end{cases}$$

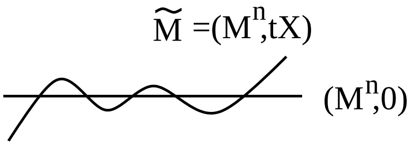
Example: "Euler Characteristics" $\chi(M^n)$.

$$S_t(X) : M^n \to M^n, \quad X - \text{ vector field,}$$

$$S_t(X) \sim 1, \quad M^n \subset T^*(M^n)$$



$\chi(M^n) = \tilde{M} \circ M = ?$



Consider "gradient" vector (covector) field df for $f : M \to \mathbb{R}$.

$$T_*(M) \cong T^*(M), \text{ we use } T_*(M).$$

$$\tilde{M} \circ M = ?$$

Critical point: $(df)\big|_{x_k} = 0$. Critical point x_k is **nondegenerate** iff the quadratic form

$$d^2 f = \left(\frac{\partial^2 f}{\partial x^j \partial x^j} \right)_{x_k} dx^i dx^j$$

is non-degenerate.

"Sing" of x_k is equal to the number of negative squares in the form $(d^2 f)_{x_k}$, which is the **"Morse index"** $n(x_k)$.

$$\operatorname{sign}_f x_k = (-1)^{\text{Morse Index}(x_k)} = (-1)^{m(x_k)}.$$

Lemma. Intersection Index $M \circ M$ in $T_*(M)$ is equal to:

$$M \circ M = \sum_{j=0}^{n} (-1)^j m_j(f) = \chi(M^n) \text{ for orientable manifolds,}$$

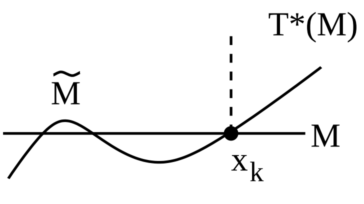
$$m_j(f) = \sharp\text{points } x_k \text{ with Morse Index } = j.$$

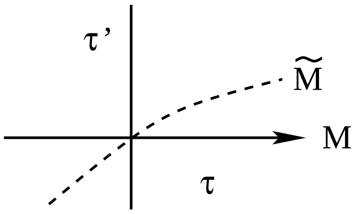
Example, $n = 2$. $\chi(M^2) = m_0(f) - m_1(f) + m_2(f)$, m_0 – minima, m_1 – saddles, m_2 – maxima.

Proof of Lemma. Variation of manifold M given by vector field X is $S_X(t) : x \to x + tX(x) + O(t^2)$ where $t \searrow 0$. For the fix point $X(x) = 0$ we have:

$$S_X(t) : \vec{x} \to \vec{x} + tA\vec{x} + O(t^2), \text{ where } x_k = (0, \ldots, 0) \text{ and } A = (A^i_j) = \left(\frac{\partial X^i}{\partial x^j} \right)_{x_k}.$$

Now calculate intersection index $\tilde{M} \circ M$ in $T_*(M^n) = T^*(M^n)$.





Local coordinates in $T^*(M)$ are $(x^1, \dots, x^k, \eta^1, \dots, \eta^k)$, $\vec{\eta}$ – tangent vector.

Basis is: $\tau_1 = \frac{\partial}{\partial x^1}$, $\dots$, $\tau_n = \frac{\partial}{\partial x^n}$, $\tau_1' = \frac{\partial}{\partial \eta^1}$,$\dots$, $\tau_n' = \frac{\partial}{\partial \eta^n}$.

Tangent space to $M \subset T^*(M)$ is span$(\tau_1, \dots, \tau_n)$ at the point x_k (fixpoint for X).

Tangent space to $\tilde{M} = S_X(t)M$ for $t \searrow 0$ is span$(\tau_1 + tA\tau_1', \dots, \tau_n + tA\tau_n')$ (neglecting term $O(t^2)$).

$$(\tau_1 + tA\tau_1', \dots, \tau_n + tA\tau_n', \tau_1, \dots, \tau_n) - \quad \text{basis} \quad \tilde{M} \circ M.$$

sing is equal to $\operatorname{sign} \det A$.

O.K.

Let $g_{ij}X^j = df$ and $g_{ij}(x_k) = \delta_{ij}$. We have

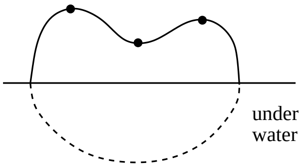
$$\operatorname{sign} \det A\big|_{x_k} = \operatorname{sign} \det \left(\frac{\partial X^i}{\partial x^j} \right)_{x_k} = \operatorname{sign} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)_{x_k} = (-1)^{\sharp \text{ negative squares}}.$$

Lemma is proved.

So Euler Characteristics is:

1. Total algebraic singularity of vector field.
2.
$$\chi(M^n) = \sum_{j=0}^{n} (-1)^j m_j(f), \quad \text{for} \quad f : M \to \mathbb{R}.$$
$$m_j(f) = \sharp \text{points } x_k \ \text{ with Morse Index } = j.$$

Example. Maxwell (XIX century).



Floating island, no beaches.
$m_0 - m_1 + m_2 = 1$.

Lecture 30. Two applications of differential forms: Degree of Map and Hopf Invariant.

Two applications of differential forms.

I) Homotopy theory: Degree of map $S^n \xrightarrow{f} S^n$ (spheres may be replaced by any closed orientable manifolds:)

A) Take n-form $\Omega \in \wedge^n(S^n)$, $\int\limits_{S^n} \Omega = 1$.

Definition.

$$\deg f = \int\limits_{S^n} f^*(\Omega)$$

Proof.

Let F be a homotopy from f to g: $F : S^n \times I \to S^n$, $(0 \leq t \leq 1)$, $F(x, 0) = f$, $F(x, 1) = g$. Consider $F^*(\Omega))$. We have $dF^*(\Omega) = F^*(d\Omega) = 0$.

$$\int\limits_{(S^n,1)} \underset{\underset{g^*\Omega}{\uparrow}}{F^*(\Omega)} - \int\limits_{(S^n,0)} \underset{\underset{f^*\Omega}{\uparrow}}{F^*(\Omega)} = \int\limits_{(S^n \times I)} dF^*(\Omega) = 0.$$

$$\Omega \to \Omega + d\omega \ \Rightarrow \int\limits_{(S^n,t)} \Omega + d\omega = \int\limits_{(S^n,t)} \Omega.$$

Theorem is proved.

We calculated integral $\int\limits_{S^n} f^*\Omega$ geometrically and proved geometrically that $\int\limits_{S^n} f^*\Omega = \left(\int\limits_{S^n} \Omega\right) \cdot \deg f$.

B) Consider $S^3 \to S^2$ and fix 2-form Ω in $\wedge^2(S^2)$ such that $\int\limits_{S^2} \Omega = 1$.

Let $f^*\Omega = d\omega$, $\omega \in \wedge^1(S^3)$.

Define **"Hopf Invariant"**

$$H(f) = \int\limits_{S^3} \omega \wedge f^*\Omega.$$

Theorem. $H(f)$ **is homotopy invariant.**

Proof. Consider $F : S^3 \times I \to S^2$, $\int\limits_{S^2} \Omega = 1$. Take $F^*(\Omega) \in \wedge^2(S^3 \times I)$. Take

$$F^*(\Omega) = d\omega, \quad \omega \in \wedge^2(S^3 \times I), \quad \omega' = \omega\big|_{t=0}, \quad \omega'' = \omega\big|_{t=1}.$$

We have

$$d(\omega \wedge F^*(\Omega)) = d\omega \wedge F^*(\Omega) = F^*(\Omega) \wedge F^*(\Omega) = F^*(\Omega \wedge \Omega) = 0.$$

So

$$\begin{tikzcd}
\int\limits_{t=1} \omega \wedge F^*(\Omega) & - & \int\limits_{t=0} \omega \wedge F^*(\Omega) & = \int\limits_{S^3 \times I} (F^*\Omega)^2 = 0. \\
\int\limits_{S^3} \omega'' \wedge g^*\Omega \arrow[u] & & \int\limits_{S^3} \omega' \wedge f^*\Omega \arrow[u] &
\end{tikzcd}$$

Here

$$F\big|_{t=1} = g, \quad F\big|_{t=0} = f.$$

$\boxed{\text{How to calculate } H(f) \text{ geometrically?}}$

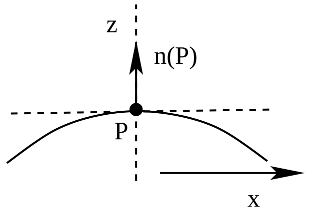
Let $\Omega \to \Omega' = \Omega + dv$: $H(t)$ is the same.

Let $\omega \to \omega' = \omega + u$, $du = 0$: $H(t)$ is the same.

Theorem is proved.

II) Riemannian Geometry: **Gaussian Curvature.**

Let $M^n \subset \mathbb{R}^{n+1}(x^0, x^1, \ldots, x^n)$ and M^n is locally (near $P \in M^n$) defined by equation:



$$x^0 = z(x^1, \ldots, x^n), \quad P = (0, 0, \ldots, 0), \quad (dz)\big|_P = 0.$$

Consider map (Gauss).

$$Q \xrightarrow{G} n(Q) \in S^n \subset \mathbb{R}^{n-1}, \quad Q \in M^n.$$

Definition $d\sigma$ – volume element at S^n.

$$K^* = G^* d\sigma.$$

n-form K^* is the "Gauss form" or "Curvature Form". Let $d\sigma_M$ is a volume element of M^n induces from $M^n \subset \mathbb{R}^n$ Riemannian Metric. We have by definition

$$K^* = K d\sigma_M.$$

In our system of coordinates $z = x^0 = z(x^1, \ldots, x^n)$, $z \perp M^n$ (at P), we have $g_{ij}(P) = g_{ij}(0) = \delta_{ij}$ and $(\partial g_{ij}/\partial x^q) = 0$. So we have

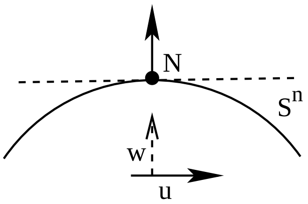
$$(d\sigma_M)_P = dx^1 \wedge \ldots \wedge dx^n, \quad K^* = K dx^1 \wedge \ldots \wedge dx^n,$$

and K is ordinary Gauss Curvature:

$$K_P = \det \left(\frac{\partial^2 z}{\partial x^i \partial x^J} \right)_P, \quad P = (0, 0, \ldots, 0).$$

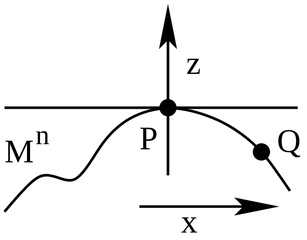
Theorem 1. $K^* = K d\sigma_M = G^*(d\sigma)$, where $d\sigma$ is invariant volume element in $S^n \subset \mathbb{R}^{n+1}$.

Proof. Let us describe volume element $d\sigma_N$ in S^n. Let S^n be given by equations (N="north pole")



$w^2 + u_1^2 + \ldots + u_n^2 = 1,$
$w = \sqrt{1 - (u^1)^2 - \ldots - (u^n)^2},$
$N = (0, 0, \ldots, 0),$
$u^1, \ldots, u^n$ – local coordinates.

Construct **Gauss map:**



$G : Q \to n_Q \in S^n, \ \ Q = (x^1, \ldots, x^n)$
$M^n : \ -z(x^1, \ldots, x^n) + x^0 = 0,$
n_Q – normal vector.

$$n_Q = \frac{(1, -z_{x^1}, \ldots, -z_{x^n})}{\sqrt{1 + (z_{x^1})^2 + \ldots + (z_{x^n})^2}}$$

We see that $P \to N$, where $P : \vec{x} = 0$.

The volume element of sphere at N is:

$$d\sigma = du^1 \wedge \ldots \wedge du^n, \quad u - \text{ local coordinates in } S^n, \quad u^0 = w.$$

Map:

$$\vec{u} = n_Q = \frac{(1, -\nabla_z)}{\sqrt{1 + (\nabla_z)^2}},$$

$$w = \frac{1}{\sqrt{1 + (\nabla_z)^2}}, \quad u^j = \frac{-\partial z/\partial x^j}{\sqrt{1 + (\nabla_z)^2}}, \quad j = 1, \ldots, n.$$

Calculate Jacobian:

$$J_0 = \det\left(\frac{\partial u^j}{\partial x^k}\right)\Bigg|_T, \qquad \boxed{x = 0, 0, \ldots, 0}.$$

$J_0 =?$. Remember that $(\nabla Z)_P \equiv 0$. So we have:

$$J_0 = \det\left(\frac{\partial \vec{u}}{\partial \vec{x}}\right)\Bigg|_{(0,\ldots,0)} = (-1)^n \cdot \det\left(\frac{\partial^2 z}{\partial x^i \partial x^i}\right)\Bigg|_{(0,\ldots,0)},$$

So we have

$$K d\sigma_M = G^*(d\sigma) = (-1)^n \cdot \det\left(\frac{\partial^2 z}{\partial x^i \partial x^i}\right)\Bigg|_{(0,\ldots,0)} dx^1 \wedge \ldots \wedge dx^n.$$

For $n = 2k$ we have $(-1)^n = 1$.

Let $n = 2$**.** By definition

$$K = \det\left(\frac{\partial^2 z}{\partial x^i \partial x^i}\right)\Bigg|_T, \quad \text{where} \quad z = x^0 = z(x, y), \quad \nabla Z\big|_T = 0.$$

Theorem is proved.

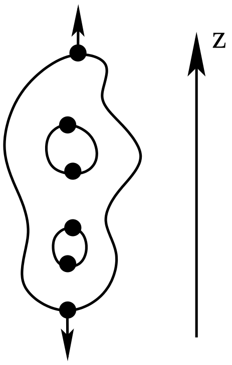
Conclusion:

$$\int\limits_{M^2} K d\sigma = (\deg G) \cdot \iint\limits_{S^2} d\sigma.$$

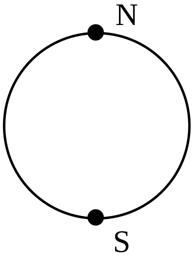
How to calculate $\deg G$**?** Let $n = 1, 2$. For $n = 2$ we expressed Gauss Curvature through the Curvature Tensor (Riemannian).

How to calculate $\deg G$ in $\mathbb{R}^3$ (the same consideration can be done for any compact $M^n \subset \mathbb{R}^{n+1}$).

$M^2 \subset \mathbb{R}^3$, $G : M^2 \to S^2$, $h : M^2 \to \mathbb{R}$.



Lemma 1. Let $S, N \in S^2$ (the South and the North poles) be transversal points for G. Then $\nabla h = 0$ iff $P_j \in G^{-1}(N)$ or $P_j \in G^{-1}(S)$.



Lemma 2. For the function z on M^n we have[1]:

$$\mathrm{sign}_{P_j} = (-1)^{\text{Morse Index in } G^{-1}(N)}$$

$$\mathrm{sign}_{S_j} = (-1)^{n+\text{Morse Index in } G^{-1}(N)}$$

Calculation:

$$\chi(M^n) = \sum_{j=0}^{n} (-1)^j m_j(z) = \deg_N G + (-1)^n \deg_S G = \begin{cases} 0, & n = 2k+1, \\ 2\deg G, & n = 2k. \end{cases}$$

Lecture 31. Comparison of notations of our Lectures with book of Do Carmo “Riemannian Geometry”.

We recommend book Do Carmo (D.C.) “Riemannian Geometry”. Let us make some comparison of notations and prove some theorems (f.i. “local minimality” of geodesics).

Manifolds:

---

[1] Morse Index of $z(\vec{x})$ at P, $(\nabla z)_P = 0$ in equal to the $\sharp$ of negative squares in the form $(d^z)_P$.

D.C. “Differential Structure”, **Hausdorff**.

Our Course: “Atlas of Charts”, C^∞ manifolds, **metric spaces**, **“Double good Atlas”**.

Local coordinates - same.

Imbedding and Immersions $M^n = \mathbb{R}^N$ (proof of existence is missing in D.C.).

Partition of unity (proof is missing in D.C.) - we proved for compact manifolds.

Tangent vectors, basis of T^n_τ as $\partial_i = \partial/\partial x^i$ in local coordinates $(x^1, \ldots, x^n)$.

Vector field $X = \sum a^i(x)\,\partial_i$ (locally).

Our notation $X = (a^i)$ (index is “upper”). Action of vector field on functions $f(x)$:

$$\text{D.C. :} \quad X(f) = \sum a^i(x)\,\frac{\partial f}{\partial x^i} = a^i\,\frac{\partial f}{\partial x^i}$$

Commutator

$$[X\,,\,Y] = X\,(Y(t)) - Y\,(X(t))$$

$$X = (a^i)\ , \quad Y = (b^i)\ , \quad [X\,,\,Y]^i = \left(a^j\,\frac{\partial b^i}{\partial x^j} - b^j\,\frac{\partial a^i}{\partial x^j}\right)$$

C^∞-maps : $M^n \xrightarrow{f} N^m$, differential of map, rank :

$$d\,f_\tau \,:\, T^n_\tau \to T^m_{f(\tau)} \qquad \text{(linear)}$$

Implicit function theorem / local inversion theorem of the map (no proof in D.C. and in our course).

Approximation and Transversality are missing in D.C. or presented in the highly reduced form - see C.P.

Examples of manifolds - similar.

Orientation.

Covectors (basis (dx^i) in local coordinates) are missing in D.C. Covector fields $\omega = \sum_i u_i(x)\,dx^i$ (1-forms).

Manifolds $T^*(M^n)$ (vectors), Atlas, Charts.

Manifolds $T_*(M^n)$ (covectors, missing in D.C.)

Riemannian and Pseudoriemannian Metric = inner product (symmetric, nondegenerate) is $\langle X\,,\, Y\rangle$, $g_{ij} = \langle \partial_i\,,\, \partial_j\rangle$.

Symplectic inner products and differential forms are missing in D.C.

Volume element $d^n\,\sigma = \sqrt{\det g_{ij}}\; dx^1 \wedge \cdots \wedge dx^n$ (locally).

Induced Metric for submanifolds $M^n \subset \mathbb{R}^N$ or $M^n \subset N^k$ (good only for positive Riemannian Metrics).

Lie Groups, right and left invariant vector fields, right and left invariant Riemannian Metric on Compact Lie Groups, **bi-invariant metric** and inner product of right-invariant vector fields (D.C.).

Riemannian Metric and length of curves. Existence of Riemannian metric in every manifold (proved by partition of unity in D.C.; another proof - imbedding in $\mathbb{R}^N$). Volume provided by Riemannian metric.

Existence of biinvariant volume on a compact group G. Existence of invariant metric in the spaces with action of compact group G ("averaging procedure" - both in D.C. and our course).

Connections ("affine") - considered in D.C. for tangent bundles only: vector field X, $X = a^i(x)\,\partial_i$,

$$\nabla_X Y = \sum_i a^i\,\nabla_i Y \quad , \quad Y = b^j(x)\,\partial_j$$

$$\nabla_i Y = b^j\,(\nabla_i\,\partial_j) + \frac{\partial b^j}{\partial x^i}\,\partial_j = \left(\frac{\partial b^j}{\partial x^i} + b^j\,\Gamma^k_{ij}\right)\partial_j$$

for components, $\nabla_i(\partial_j) = \Gamma^k_{ij}\,\partial_k$.

Symmetric Connection" $\Gamma^k_{ij} = \Gamma^k_{ji}$ $\{\nabla_i\,\partial_j = \nabla_j\,\partial_i\}$.

Curve

$$c(t) = \{x^j(t)\} \ , \quad X = (\dot{x})$$

Let V be a vector fields, $V = (v^j)$, $X(t)$ - tangent to c.

$$\frac{d}{dt}(V) = \nabla_X V = \left(\frac{dv^j}{dt} + v^q\,\dot{x}^k\,\Gamma^j_{qk}\right)\partial_k$$

D.C. : $X_j \leftrightarrow \partial_j$, $\nabla_{X_i} X_j = \Gamma^k_{ij}\,\partial_k$, $X = (\dot{x}^j(t))$ along $c(t)$.

Tensors. Examples:

1. Scalars (type $(0,0)$) - $f(x)$.
2. Vectors (type $(1,0)$) - (u^i) $\sum u^i \partial_i$.
3. Covectors (type $(0,1)$) - (v_i) $\sum v_i \, dx^i$.
4. Inner products of vectors g_{ij} (type $(0,2)$) .
5. Inner products of covectors g^{ij} (type $(2,0)$) .
6. Operators (a^i_j) (type $(1,1)$) .

General Tensors (type (k,l)) .

Components $T^{i_1 \dots i_k}_{j_1 \dots j_l}(x)$

$e_1, \dots, e_n$ - basis of tangent vectors

$e^1, \dots, e^n$ - basis of covectors

$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k} \otimes e^{j_1} \otimes e^{j_2} \otimes \dots \otimes e^{j_l}$ - basis in tensor space (k,l).

Operations : **Linear**, **Tensor product**

$$(k,l) \otimes (p,q) \subset (k+p, l+q)$$

$$T'^{i'}_{j'} \otimes T'^{i''}_{j''} = T^{i' i''}_{j' j''}(x)$$

Permutation of indices (lower or upper)

Trace

$$T^{i' \, i \, i''}_{j' j j''} \xrightarrow{\text{Trace}} T^{i' \, i''}_{j' \, j''} = \sum_{i=1}^{n} T^{i' \, i \, i''}_{j' \, i \, j''}$$

Examples: $A = (a^i_j)$, $a^i_j \to a^i_j = \operatorname{Tr} A$ (trace), $g_{ij} \to g_{ji}$ - permutation.

Extension of connection to all tensor fields. **Axioms**.

a) Trivial for scalar fields $\nabla_i f = \partial_i f$.

b) **Satisfies to Leibnitz formula** for **tensor products**.

c) Annihilates Riemannian metric $\nabla_i g_{kl} \equiv 0$.

d) Commutes with trace

$$\nabla_X \langle V , W \rangle = u^i \frac{\partial}{\partial x^i} \langle V , W \rangle = \langle \nabla_X V , W \rangle + \langle V , \nabla_X W \rangle$$

where $X = (u^i)$,

$$\nabla_k \langle V , W \rangle = \nabla_k \left(g_{ij} \, v^i \, w^j \right) = \langle \nabla_k V , W \rangle + \langle V , \nabla_k W \rangle$$

$\nabla_k = \nabla_{X_k}$, $X_k = \partial_k$.

For parallel Transform

$$\frac{d}{dt} \langle V , W \rangle = \langle \frac{dV}{dt} , W \rangle + \langle V , \frac{dW}{dt} \rangle$$

$$\frac{dV}{dt} = 0 \, , \ \frac{dW}{dt} = 0 \quad \Rightarrow \quad \frac{d \langle V , W \rangle}{dt} = 0$$

Levi - Civita Theorem. Symmetric connection compatible with metric g_{ij} is given by the formula

$$\Gamma^p_{jk} = \frac{1}{2} \, g^{pi} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

“symmetric”: $\nabla_k \, \partial_j = \nabla_j \, \partial_k \ \leftrightarrow \ \nabla_X Y - \nabla_Y X = [X , Y]$.

Homework 10.

1. Prove that for complex polynomial

$$w = P_n(z) : \ \mathbb{S}^2 \to \mathbb{S}^2$$

we have $\deg f = n$.

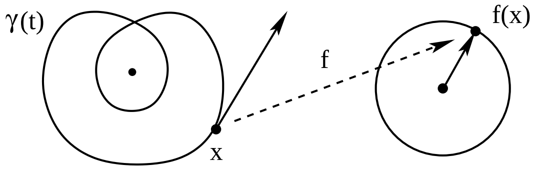
2. Calculate degree of rational map

$$w = \frac{P_n(z)}{Q_k(z)} \ , \quad \text{(irreducible fraction)}$$

3. Calculate degree of **real** polynomial

$$y = a_0 \, x^n + a_1 \, x^{n-1} + \ldots + a_n \ : \ \mathbb{R} \to \mathbb{R} \ , \quad a_j \in \mathbb{R}$$

4. Let $\gamma(t) = \gamma(t + 2\pi) : \ \mathbb{S}^1 \to \mathbb{R}^2$ and $\dot{\gamma} \neq 0$.



Consider “Gauss Map” :

$$\mathbb{S}^1 \xrightarrow{f} \mathbb{S}^1 \quad , \qquad f\,(\dot\gamma) \;=\; \dot\gamma/|\dot\gamma| \in \mathbb{S}^1$$

Prove that $f^*(d\varphi) \;=\; k(s)\,ds$, where s - natural parameter, k - curvature. Calculate $\deg f \;=\; ?$

5. Consider 2-form in $\mathbb{R}^3\backslash 0$:

$$\Omega \;=\; x\,dy\wedge dz \;-\; y\,dx\wedge dz \;+\; z\,dx\wedge dy$$

Prove that

$$d\left(\frac{\Omega}{(x^2+y^2+z^2)^{3/2}}\right) \;=\; 0$$

Prove that

$$\Omega|_{\text{unit sphere}} \;=\; \text{area element in } \mathbb{S}^2$$

which is SO_3 - invariant.

Homeworks 8, 9. Solutions.

Homework 8. Solutions.

1. Prove that $\sum_i p_i dx^i$ is 1-form in $T_*(N^k)$ (local coordinates $(x^i,\, p_j)$).

Change of coordinates $x \;=\; x(y)$,

$$(\text{system } x) \qquad p_i\,\frac{\partial x^i}{\partial y^j} \;=\; p'_j \qquad (\text{system } y)$$

So we have

$$\sum_j p'_j\,dy^j \;=\; \sum_{i,j} p_i\,\frac{\partial x^i}{\partial y^j}\,dy^j \;=\; \sum_i p_i\,dx^i$$

Remark. $\sum dp_i\wedge dx^i$ is a well-defined 2-form also (done before).

2. Prove that geodesic flow with Hamiltonian $H(x,p) = g^{ij}(x)\, p_i\, p_j \,/\, 2$ preserves the form $\sum_i p_i\, dx^i$ [correction due to student: it preserves the form $\sum_i p_i\, dx^i$ restricted to the $(2n-1)$ - dimensional submanifold $T_1(M^n)$, $|\dot{x}| = 1$ (unit tangent vectors)].

Proof. Calculation shows that for the shift S_t along the flow we have for geodesic flow

$$\frac{d}{dt}\, S_t^* \left(\sum_i p_i\, dx^i \right) \Bigg|_{t=0} = dH$$

For geodesic flow have

$$H(x,p) = \frac{1}{2}\, g^{ij}(x)\, p_i\, p_j = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j$$

so $T_1(M^n)$ is given by the condition $H = 1$ and $dH = 0$ on $T_1(M^n)$.

Calculation: We have

$$S_t^*\,(p_i) = p_i + t\,\dot{p}_i + O(t^2) \quad , \quad S_t^*\,(x^i) = x^i + t\,\dot{x}^i + O(t^2)$$

For $\dot{x}^i = H_{p_i}$, $\dot{p}^i = -\, H_{x^i}$ we have then

$$\frac{d}{dt}\, S_t^*(p_i) \Bigg|_{t=0} = -\, H_{x^i} \quad , \quad \frac{d}{dt}\, S_t^*(x^i) \Bigg|_{t=0} = H_{p_i}$$

$$\frac{d}{dt}\, S_t^*(dx^i) \Bigg|_{t=0} = dH_{p_i} = H_{p_i p_j}\, dp_j + H_{p_i x^j}\, dx^j$$

Finally:

$$\frac{d}{dt}\, S_t^* \left(\sum_i p_i\, dx^i \right) \Bigg|_{t=0} = -\, H_{x^i}\, dx^i + p_i \left[H_{p_i p_j}\, dp_j + H_{p_i x^j}\, dx^j \right] =$$

$$= d \left(p_i \frac{\partial H}{\partial p_i} - H \right) = dL$$

where L is a “Lagrangian”.

For

$$H(x,p) = \frac{1}{2}\, g^{ij}(x)\, p_i\, p_j = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i\, \dot{x}^j$$

we have $H = L$ and also $dH = dL$.

2 is proved.

3. Locally every closed form is exact: $\omega = d\nu$ **in the domain** $U \subset M^n$. Let U is the ball $|x|^2 < \epsilon$. Multiply ν by function φ (C^∞, $\varphi \equiv 1$ in U and $\varphi \equiv 0$ outside of the ball $|x|^2 > 2\epsilon$).

Our form is $\omega - d(\varphi\nu)$.

4. Let

$$S\{\gamma\} = \int_\gamma \left(\sum_i p_i(t)\,\dot{x}^i(t)\,dt - H(p,x)\,dt \right) ,$$

$\gamma = \{x(t), p(t)\}$, $L = \sum p_i\,\dot{x}^i - H(x,p)$. We have from the Euler Lagrange equations

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} = -\frac{\partial H}{\partial x^i}$$

$$\dot{x}^i - \frac{\partial H}{\partial p_i} = \frac{\partial L}{\partial p_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) = 0$$

Remark. We can write

$$S\{\gamma\} = \int_\gamma \left(d^{-1}(\Omega) - H\,dt \right)$$

where Ω is the **simplectic form**.

In our case

$$\Omega = \sum dp_i \wedge dx^i = d\left(\sum p_i\,dx^i\right)$$

If cohomology class $[\Omega] \in H^2(M^{2n})$ is nonzero (Dirac Monopole, Compact simplectic Manifolds) the expression $S\{\gamma\}$ defines correctly a "Multivaled Functional" or closed 1-form on the space of paths γ (f.i. closed paths).

5. Consider volume element $dx^1 \wedge \cdots \wedge dx^n = \Omega$ and the flow $\dot{x}^i = \xi^i(x)$. We have

$$\left.\frac{d}{dt}\,(S_t^*\Omega)\right|_{t=0} =$$

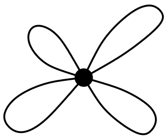
$$= \lim_{t\to 0} \frac{1}{t}\left[d(x^1 + t\xi^1) \wedge \cdots \wedge d(x^n + t\xi^n) - dx^1 \wedge \cdots \wedge dx^n \right] =$$

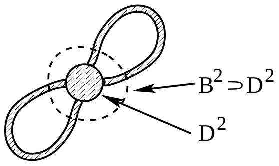
$$= \left(\frac{\partial \xi^1}{\partial x^1} + \ldots + \frac{\partial \xi^n}{\partial x^n} \right) dx^1 \wedge \cdots \wedge dx^n = \operatorname{div}\vec{\xi}(x) \cdot \Omega$$

Homework 9. Solutions.

1. Let $M^2 = \mathbb{R}^2 \backslash (\text{point})$. It is homotopy equivalent to $\mathbb{S}^1$, so $H^2(M^2) = H^2(\mathbb{S}^1) = 0$.

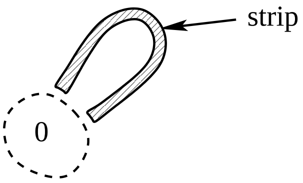
Every open 2-manifold is homotopy equivalent to “graph” like





For every 2-form in D^2 we have $\omega = d\nu$ (in D^2).

Construct function $\varphi \equiv 1$ in D^2 and $\varphi \equiv 0$ outside of domain $B^2 \supset D^2$. So $\omega \sim \omega - d(\varphi\nu) = \omega'$. And ω' is nonzero only in strips

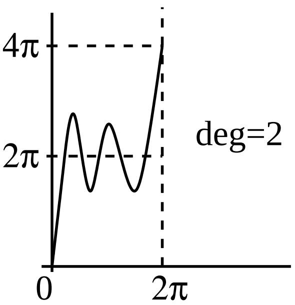


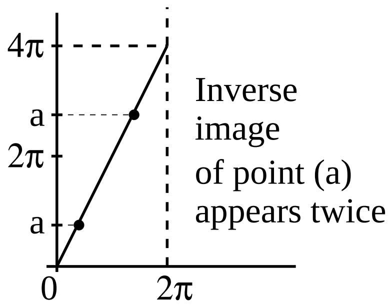
Every such strip is homotopy equivalent to circle. Our result follows.

2. Degree of map $\mathbb{S}^1 \to \mathbb{S}^1$, $m = \deg f$?

$$f(x + 2\pi) = f(x) + 2\pi m \quad , \quad m \in \mathbb{Z}$$

Proof.

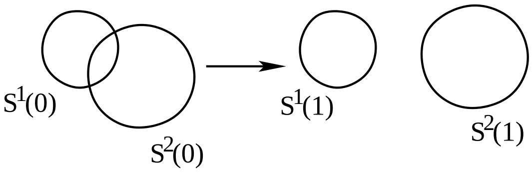




2 is proved.

3. Deform one circle far: $S_1(t)$, $0 \le t \le 1$, $S_2(0) = S_2$. We have

$$S_1 \circ S_2(0) = S_1 \circ S_2(1) = \emptyset$$



4. Every map $M^n \to \mathbb{S}^{n+k}$ is homotopic to zero for $k > 0$ because it is homotopic to C^∞ (approximation), and image of C^∞-map does not cover $\mathbb{S}^{n+k} = \mathbb{S}^m$. Homotopy process

$$M^n \times I \xrightarrow{F} \mathbb{S}^m$$

can be approximated by smooth (C^∞) map F transversal along W^{m-k}.

Here $m = n + k$. For $n < k - 1$ we have

$$F\,(M^n \times I) \cap W^{n-k} = \emptyset$$

So our statement follows.

5. Same argument for $f : \mathbb{S}^n \to \mathbb{S}^m$, $n < m$: C^∞ - map can not cover $\mathbb{S}^m$ if $m > n$.

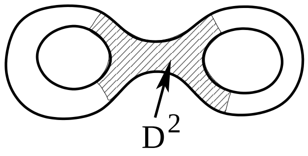
6. $H^*(SO_4, \mathbb{R}) = ?$ We have $SO_4 = \mathbb{S}^3 \times \mathbb{S}^3 / \pm (1,1)$. Group $G = SO_4 \times SO_4$. G - **invariant torus** in SO_4 appear (as well as for $\mathbb{S}^3 \times \mathbb{S}^3$) only in dimensions 0, 3, 6 . So we have for $H^*(SO_4, \mathbb{R})$ in different dimensions:

$$H^*(SO_4, \mathbb{R}) : \left\{ \begin{array}{cccc} 1 \quad , & \omega_1 \quad , & \omega_2 \quad , & \omega_1 \wedge \omega_2 \\ & & & \\ D = 1 & D = 3 & D = 3 & D = 6 \end{array} \right\}$$

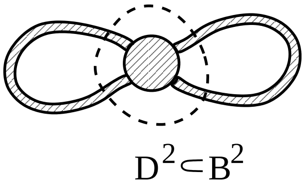
$H^0 = \mathbb{R}$, $H^3 = \mathbb{R} \oplus \mathbb{R}$, $H^6 = \mathbb{R}$.

7. We have

$\mathbb{R}^2 \backslash (*_1 \cup *_2) \cong$



$\cong$



Use arguments same as in Problem 3 above for calculation of H^2

$$H^1\left(\mathbb{R}^2 \backslash (*_1 \cup *_2)\right) = \mathbb{R} \oplus \mathbb{R}$$

Lecture 32. Geodesics. Gauss Lemma.

1. **Geodesics:** smooth curves $\gamma(t) \;=\; \{x^i(t)\}$ such that

$$\frac{D}{dt}\,\frac{d\vec{x}}{dt} \;\equiv\; 0 \qquad (\,\text{or } \nabla_{\dot{x}}\,\dot{x} \;=\; 0\,)$$

Equation

$$\ddot{x}^i \;+\; \Gamma^i_{kj}\,\dot{x}^k\,\dot{x}^j \;=\; 0$$

Calculus of variation:

a) **Length**

$$l(\gamma) \;=\; \int_a^b |\dot{x}(t)|\;dt \;=\; \int_a^b L_0(x,\,\dot{x})\;dt$$

where

$$L_0(x,\,\dot{x}) \;=\; \sqrt{g_{ij}(x)\,\dot{x}^i\,\dot{x}^j}$$

is a “Lagrangian”.

b) **“Action”**

$$S(\gamma) \;=\; \int_a^b \frac{1}{2}\;g_{ij}(x)\,\dot{x}^i\,\dot{x}^j\;dt \;=\; \int_a^b L_1(x,\,\dot{x})\;dt$$

Euler - Lagrange equation (EL)

$$\frac{d}{dt}\,\frac{\partial L}{\partial \dot{x}^i} \;=\; \frac{\partial L}{\partial x^i}$$

Momentum:

$$p_i \;=\; \frac{\partial L}{\partial \dot{x}^i}$$

Energy:

$$E \;=\; p_i\,x^i \;-\; L$$

Conservation of energy:

$$\frac{dE}{dt} \;\equiv\; 0$$

For $L \,=\, L_1$ we have

$$E \;=\; L_1 \;=\; \frac{1}{2}\;g_{ij}(x)\,\dot{x}^i\,\dot{x}^j \;=\; \frac{1}{2}\;\|\dot{x}\|^2$$

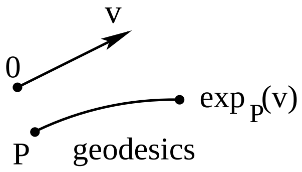
The condition $\underline{E \,=\, \text{const}}$ means that the parameter t along geodesics is **NATURAL** ($\sim$ length).

For $L = L_0$ **with natural parameter** and L_1 Euler - Lagrange equations coincide.

Exponential Map:

V - domain in $T_P(M^n) = \mathbb{R}^n$, $P = M^n$.

$\exp_P \,:\, V \to M^n$, $v \in T_P$, $\exp_P(V)$:



"Geodesic ball": $B_\epsilon(0)$ - ball in T_P ,

$$\exp_P\,(B_\epsilon(0)) \quad - \quad \textbf{geodesic ball in } M^n$$

It exists for ϵ , - small enough (depends on $P \in M^n$).

For compact M^n it exists for any **radius**.

For symmetric connection we have $\nabla_i\,\partial_j = \nabla_j\,\partial_i$ (or $\nabla_{X_i} X_j = \nabla_{X_j} X_i$ for $X_i = \partial_i$). Let $s\,(u,v)$ be a parametrized surface

$$s \;=\; \{x^1(u,v),\, \ldots,\, x^n(u,v)\}$$

Lemma 1.

$$\frac{D}{\partial v}\frac{\partial s}{\partial u} \;=\; \frac{D}{\partial u}\frac{\partial s}{\partial v}$$

where

$$\frac{\partial s}{\partial u} \;=\; \frac{\partial x^i}{\partial u}\,\partial_i \;=\; X \quad , \quad \frac{\partial s}{\partial v} \;=\; \frac{\partial x^j}{\partial v}\,\partial_j \;=\; Y$$

or $\nabla_X Y = \nabla_Y X$ at s.

Proof. Vector fields X, Y commute at the surface s, so $[X, Y] = 0$ at s. But we have

$$\nabla_X Y \;-\; \nabla_Y X \;-\; [X\,,\,Y] \;=\; 0$$

at s.

Lemma is proved.

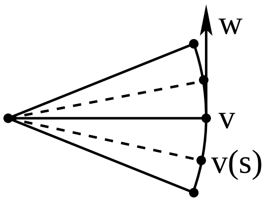
Gauss Lemma. Let $\exp_P(v)$ be defined for $v \in T_R = \mathbb{R}^n$, and $w \in T_P$. Then we have

$$\langle (d\exp_P)_v\,(v)\,,\,(d\exp_P)_v\,(w)\rangle \;=\; \langle v\,,\,w\rangle$$

Proof. For $v \;\|\; w$ lemma is obvious because $\gamma(t) = \exp_P(tv)$ is geodesics for $0 \le t \le 1$.

Let now $w \perp v$, so $\langle v, w\rangle = 0$. Put

$$w \;=\; \left.\frac{dv}{ds}\right|_{s=0}, \quad v(0) = v, \quad |v(s)| \;=\; \text{const}$$



For small $\epsilon > 0$ **all** $u = t\,v(s)$, $|s| < \epsilon$, have well-defined $\exp_P(u)$, $0 \le t \le 1$.

Consider surface $f: A \to M^n$, parametrized by (t,s):

$$A: \quad f(t,s) \;=\; \exp_P\,(tv(s)) \quad, \quad 0 \le t \le 1\;, \quad |s| < \epsilon$$

where $f(t, s_0)$ - geodesics through P.

By definition

$$\left.\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle\right|_{t=1,s=0} \;=\; \langle (d\exp_P)_v\,(w)\,,\,(d\exp_P)_v\,(v)\rangle$$

We have also

$$\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \;=\; \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \;+\; \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle \;=\; \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \;=$$

$$=\; \langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \;=\; \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \;=\; 0\;,$$

so $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle$ is $t-$independent.

Calculate it for $t \to 0$:

We have

$$\frac{\partial f}{\partial t}(t,0) \;=\; (d\exp_P)_{tv}\,(tw)$$

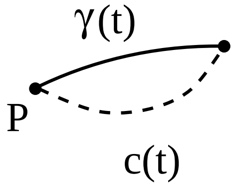
and

$$\frac{\partial f}{\partial s}(t,0) \ \to \ 0 \quad , \qquad t \to 0 \ .$$

So $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \ \equiv \ 0$ for all t , and $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \Big|_{t=1,s=0} \ = \ 0$

Lemma is proved.

Theorem. Let $\epsilon > 0$ is small enough and $\exp_P(B_\epsilon(0)) \subset M^n$ is a geodesic ball. Let $c(t)$ be a smooth curve joining $\gamma(0)$ and $\gamma(1)$ where γ is a geodesics started in P .



Then length of $c(t)$ is: $l(c) \geq l(\gamma)$, and $l(c) = l(\gamma)$ implies $c = \gamma$.

Proof. Let $c \subset B_\epsilon$ (round geodesic ball). Curve $c(t)$ can be written as

$$c \ = \ \exp_P(r(t) \cdot v(t)) \quad , \qquad |v(t)| = 1 \ ,$$

where $v(t)$ is a curve in tangent space T_P , and $r(t)$ is such that $0 < r \leq 1$.

Let $c(t_1) \neq P$ for all $t_1 \in (0,1]$ and $r(t) > 0$ for all $t_1 \in (0,1]$.

We have for

$$f(r(t),t) \ = \ \exp_P(r(t) \cdot v(t)) \quad , \qquad |v(t)| = 1 \quad :$$

$$\frac{dc}{dt} \ = \ \frac{\partial f}{\partial r} r'(t) \ + \ \frac{\partial f}{\partial t}$$

By Gauss Lemma (above) we have

$$\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle \ = \ 0$$

We have also $|\partial f / \partial r| = 1$, so

$$\left|\frac{dc}{dt}\right|^2 \ = \ |r'(t)|^2 \ + \ \left|\frac{\partial f}{\partial t}\right|^2$$

and

$$\int_{\epsilon}^{1} \left|\frac{dc}{dt}\right| \, dt \quad \geq \quad \int_{\epsilon}^{1} |r'(t)| \, dt \quad \geq \quad r(1) \, - \, r(0)$$

For $\epsilon \to 0$ we have $r(0) = 0$. So we get

$$l\,(c) \;\geq\; r(1) \;=\; l\,(\gamma)$$

At the same time the condition $\partial f/\partial t \equiv 0$ means $v(t) = v(0) = \text{const}$, i.e. $c = \gamma$ (geometrically for $r'(t) \geq 0$, otherwise: $l\,(c) > l\,(\gamma)$) .

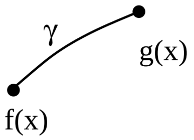
Theorem is proved.

Lecture 33. Local minimality of geodesics.

Last lecture: For every point $P \in M^n$ there exists an ϵ-ball (geodesic) B containing P as a center such that geodesics is a shortest line between P and any point $Q \in B$.

Corollary 1. Let X be compact space and $f,\, g \,:\, X \to M^n$ such 2 maps that distance $\rho\,(f(x),\, g(x))$ is smaller than the minimum of all $\epsilon\,(P)$ for all $P \in f(X)$. Then f and g are homotopic.

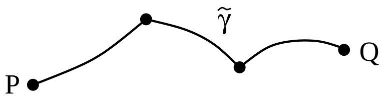
Proof. Move $f(x)$ to $g(x)$ monotonically along unique minimal geodesics joining these points



Corollary is proved.

Corollary 2. Let M^n be compact (or **metrically complete**) Riemannian manifold. For every 2 points $P,\, Q$ with distance $\rho\,(P,\, Q) = d$ there exists geodesics γ, $l(\gamma) = d$, joining P and Q.

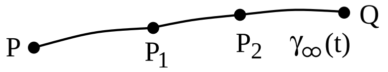
Proof. Let us remind that $d(P,\, Q) = \min_{\tilde{\gamma}} l\,(\tilde{\gamma})$, $\tilde{\gamma}$ joins P and Q and $\tilde{\gamma}$ is smooth or piecewise smooth curve:



Let us remind you the "Arcellát Principle" claiming that the set of piecewise smooth curves of length $\leq \Gamma$ is **"precompact"**. It means in particular that every "Cauchy Sequence" of curves for which length is well-defined.

Consider sequence of piecewise smooth curves γ_n such that $l(\gamma_n) \to d$, $d = \rho(P, Q)$. Limit curve γ_∞ is such that $l(\gamma_\infty) = d$, but it can be not smooth (even not piecewise smooth).

Take 2 points P_1, P_2 at γ_∞ such that $\rho(P_1, P_2) < \epsilon$ (small enough).



Replace γ_∞ from P_1 to P_2 by small geodesics joining them. By construction new curve is shorter than γ_∞ or γ_∞ is geodesics between P_1 and P_2. So, our statement follows.

Corollary is proved.

Remark. In the book D.C. the "Hopf - Rinow" theorem is proved for this result. In particular, it claims that $\exp_\tau$ is defined for all vectors in $T_\tau(M^n)$ in complete Riemannian manifold. For compact manifold it simply follows from the fact that manifold $T_1(M^n)$ is compact and geodesic flow is well defined for all time.

However, we presented here the "Arcellát" argument which was used in many other variational problems since the great work of Hilbert who was first to prove existence of minimizing geodesics in XIX Century.

See Hopf - Rinow theorem in D.C.

Now we return to Curvature

$$R(X, Y)\, Z \quad - \quad \text{curvature tensor}$$

$$X = (u^i),\ Y = (v^j),\ Z = (z^k),\ W = (w^l).$$

$$R(X, Y) = \big(\nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[X,Y]}\big)$$

\- curvature operator.

Index notations

$$\left(R^l_k\right)_{ji} = (\nabla_j \nabla_i - \nabla_i \nabla_j) = \hat{R}_{ji} = -\hat{R}_{ij} = (R_{ji})^j_k$$

(Here $[\partial_i, \partial_j] = 0$, $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$, $W = \partial_l$).

$R(X, Y) -$ linear 0-order operator on tangent space (matrix).

operator

$$R\,(X,\,Y) \;=\; R^k_{l,ij}\,u^i\,v^j \;=\; \hat{R}_{X,Y}$$

vector

$$R\,(X,\,Y)\;Z \;=\; R^k_{l,ij}\,u^i\,v^j\,z^l$$

number

$$\langle R\,(X,\,Y)\;Z\,,\,W\rangle \;=\; R^k_{l,ij}\,u^i\,v^j\,z^l\,w^p\,g_{kp}$$

Formulas for symmetric connection were given, and for $R^l_{k,ij}$ through Γ^k_{ij}. (Especially for coordinate system such that $g_{ij}(P) \,=\, \delta_{ij}\,,\ \partial_k\,\Gamma^p_{ij}|_P \,=\, 0$).

Homework 11.

1. Compact Lie groups, bi-invariant Riemannian metric $g \;\to\; h_1\,g\,h_2^{-1}$. Prove that e^{At} are geodesics (for $SO_3 \;\cong\; \mathbb{RP}^3$ and $SU_2 \;\cong\; \mathbb{S}^3$). A is from Lie Algebra: $A^t \;=\; -\,A\;\;(G \;=\; SO_n)$, $A^t \;=\; -\,\bar{A}$, $\mathrm{tr}\,A \;=\; 0$ for $G \;=\; SU_n$.

2. Prove that the form

$$\langle A^R\,,\,A^R\rangle \;=\; L(g,\,\dot{g})$$

gives right-invariant Riemannian metric where $A^R(t) \;=\; \dot{g}(t)\,g^{-1}(t)$, $\langle A\,,\,A\rangle$ - any inner product in $\mathbb{R}^n \;=\;$ Lie Algebra ($n \;=\; k(k-1)/2$ for SO_k , $n \;=\; k^2-1$ for SU_k). It is bi-invariant if $\langle A\,,\,A\rangle \;=\; \mathrm{Tr}\,A^2$, $A^t \;=\; -\,A$ and $\langle A\,,\,A\rangle \;=\; \mathrm{Tr}\,A\,\bar{A}$, $\mathrm{tr}\,A \;=\; 0$, $A^t \;=\; -\,\bar{A}$ (groups $G \;=\; SO_k$ and SU_k). For left-invariant metric take $A^L \;=\; g^{-1}(t)\,\dot{g}(t)$.

3. Find all geodesics in SU_{2n} joining I and $-\,I$.

4. Calculate intersection matrices for all compact 2-manifolds (closed, modulo 2).

5. Prove that for any open domain in compact 2-manifold $U \,\subset\, M^2$ intersection matrix of 1-cycles modulo 2 has finite rank (i.e. factor-space by the annihilator is finite-dimensional).

Lecture 34. Riemannian Curvature. Examples.

Consider M^n , $\psi \in \mathbb{R}^m_x$.

Curvature tensor.

Connection: ∇_X , $X = \partial_i$, $\nabla_X \psi = \partial_i \psi + \hat{\Gamma}_i \psi$

Curvature:

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = \hat{R}$$

$$X = \partial_i \ , \quad Y = \partial_j \ , \quad \hat{R}_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i \ : \quad \mathbb{R}^m_x \to \mathbb{R}^m_x$$

1-form (locally) - **connection**

$$A = \hat{\Gamma}_i \, dx^i$$

2-form (locally) - **curvature**

$$R = \hat{R}_{ij} \, dx^i \wedge dx^j = dA + A \wedge A$$

(Matrix Multiplication).

$$\hat{R} = dA + \frac{1}{2} [A \wedge A]$$

- **same**; **written through the commutator - Lie Algebra multiplication**.

Gauge Transformation: $\psi = G(x) \, \varphi(x)$

$$\hat{\Gamma}_i \quad \to \quad G^{-1} \, \hat{\Gamma}_i \, G - G^{-1} \, \partial_i \, G \ :$$

$$\nabla_i \, (\psi) = \nabla_i \, (G\varphi) = \partial_i \, (G\varphi) + \hat{\Gamma}_i \, (G\varphi)$$

$$\text{or}: \quad G \left(\partial_i \, \varphi + G^{-1} \, \partial_i G \, \varphi + G^{-1} \, \hat{\Gamma}_i \, G \, \varphi \right)$$

$$\hat{R}_{ij} \quad \to \quad G^{-1} \, \hat{R}_{ij} \, G$$

(calculation).

Conclusion: for tangent bundle $\hat{R}_{ij} = (R^l_{k,ij})$ is **tensor**.

Symmetric connection compatible with Riemannian metric (**tangent bundle**)

$$\nabla_i \langle X \, , \, Y \rangle \;=\; \langle \nabla_i \, X \, , \, Y \rangle \;+\; \langle X \, , \, \nabla_i \, Y \rangle$$

Corollary:

$$\Gamma^k_{ij} \;=\; \frac{1}{2} \, g^{ks} \left(- \frac{\partial g_{ij}}{\partial x^s} \;+\; \frac{\partial g_{is}}{\partial x^j} \;+\; \frac{\partial g_{sj}}{\partial x^i} \right)$$

Geodesic Equation

$$\ddot{x}^i \;+\; \Gamma^i_{jk}(x) \; \dot{x}^i \, \dot{x}^k \;=\; 0$$

1) Euler - Lagrange equation for action

$$S \;=\; \int_\gamma \frac{1}{2} \; ||\dot{x}||^2 \; dt \qquad \text{or}$$

2) $\nabla_{\dot{x}} \, \dot{x} \;=\; 0$.

Calculation of Curvature: Choose coordinate system such that

$$g_{ij}(P) \;=\; \delta_{ij} \quad , \qquad \partial g_{ij} / \partial x^k \, |_P \;=\; 0$$

in such point

$$R^s_{ijk} \;=\; \frac{\partial}{\partial x^j} \, \Gamma^s_{ik} \;-\; \frac{\partial}{\partial x^i} \, \Gamma^s_{jk}$$

Symmetries of Curvature: Let $R_{ij,kl} \;=\; g_{is} \, R^s_{j,kl}$ then

$$R_{ij,kl} \;=\; -\,R_{ij,lk} \;=\; -\,R_{ji,kl} \quad , \qquad R_{kl,ij} \;=\; R_{ij,kl}$$

Bianchi:

$$R_{ij,ks} \;+\; R_{jk,is} \;+\; R_{ki,js} \;=\; 0$$

So $\hat{R}$ is quadratic form on the space $\Lambda^2 \, T_P \;=\; \Lambda^2 \, \mathbb{R}^n$ satisfying to Bianchi identity (above) (check D.C. pages 91 - 93).

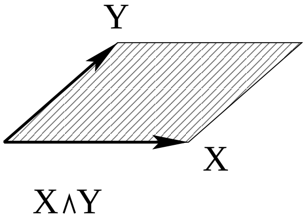
Sectional Curvature

$$K\,(X, Y) \;=\; \frac{R\,(X, Y, X, Y)}{\text{Area } (X, Y)}$$

X, Y - vectors,

$$\text{Area}\,(X,Y) \;=\; |X \wedge Y| \;=\; \sqrt{|X|^2\,|Y|^2 \,-\, \langle X,Y\rangle^2} \;=$$

$$= \;\sqrt{\det \begin{pmatrix} \langle X,X\rangle & \langle X,Y\rangle \\ \langle X,Y\rangle & \langle Y,Y\rangle \end{pmatrix}}$$



$$X \wedge Y \;\in\; \Lambda^2\, T_P \;=\; \Lambda^2\, \mathbb{R}^n$$

$R\,(X \wedge Y\,,\, X \wedge Y)$ - quadratic form on $\Lambda^2\,\mathbb{R}^n$.

$K \,>\, 0$ - “Positive Curvature” (like $\mathbb{S}^n$)

$K \,<\, 0$ - “Negative Curvature” (like $\mathbb{H}^n$)

Ricci Curvature: (see D.C.)

$$R_{ji} \;=\; R_{ij} \;=\; R^s_{i,sj}$$

Scalar Curvature $(R \;\to\; K \;=\; \text{const}\,\cdot\, R\,)$.

$$R^i_j \;=\; g^{is}\, R_{sj} \quad , \qquad R \;=\; R^i_i$$

$$\text{Ric} \;=\; \frac{1}{n-1}\; R_{ij}$$

Notation (D.C.)

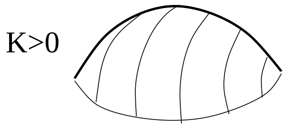
$$\text{Ric}_P\,(X) \;=\; \frac{1}{n-1} \sum_i \langle R\,(X, Z_i)\,X\,,\, Z_i\rangle$$

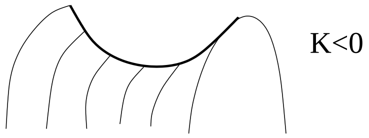
Here $|X| = 1$ and $Z_1, \ldots, Z_{n-1}$ are orthonormal basis in $R^n \,=\, T_P$, $Z_i \perp X$. We have

$$K \;=\; \frac{1}{n} \sum_j \text{Ric}_P\,(Z_j)$$

2 - dimensional Manifolds

$R \; = \; R_{12,12}$ - the only nonzero component.





$R \; = \;$ Gaussian Curvature $M^2 \subset \mathbb{R}^3$.

3 - dimensional Manifolds

R_{ij} determines all Curvature Tensor.

4 - dimensional Manifolds

Einstein Equation:

$$R_{ij} \; - \; \frac{1}{2} R \, g_{ij} \; = \; \lambda \, T_{ij} \; + \; \mu \, g_{ij}$$

where the term $\lambda \, T_{ij}$ represents "Matter" and $\mu \, g_{ij}$ is the "cosmological term" (or "dark energy").

Lie Groups and "symmetric" spaces (**"constant curvature tensor"**)

$$\nabla_s \left(R^i_{j,kl} \right) \; \equiv \; 0$$

Homeworks 10, 11. Solutions.

Homework 10. Solutions.

1. $f \, : \, \mathbb{S}^2 \to \mathbb{S}^2$, $f(z) \, = \, z^n \, + \, a_1 \, z^{n-1} \, + \, \ldots \, + \, a_0$.

We have homotopy $f_t(z) \, = \, z^n \, + \, t \, (a_1 \, z^{n-1} \, + \, \ldots \, + \, a_0)$. For $t = 0$ we have $f_0 \, = \, z^n$. For $t = 1$ we have $f_1 \, = \, f(z)$.

For $f_0 \, = \, z^n$ we have exactly n roots of equation

$$z^n \; = \; 1$$

The point $1 = W \in \mathbb{S}^2$ is transversal. So we have $\deg f_0 \, = \, \deg f_1 \, = \, n$.

2. Rational function $f \, = \, P/Q \, = \, w$.

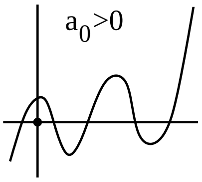
Solve equation

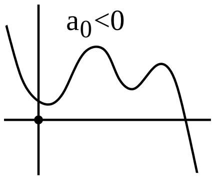
$$P(z) \; = \; w \, Q(z)$$

(Fraction is irreducible).

Take point w_0 such that roots are distinct. We have $\deg f = \max(\deg P, \deg Q)$.

3. Real polynomial $f = a_0 x^n + \ldots + a_n$, $a_0 \neq 0$. Let $n = 2k+1$. Then:





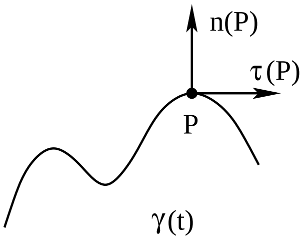
$a_0 > 0 : \deg f = 1$,

$a_0 - 0 : \deg f = -1$.

Consider the case $n = 2k$.

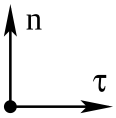
4. **Gauss Map:**

$\gamma(t) = \gamma(t + 2\pi)$,
$\dot\gamma \neq 0$.
$t = s \cdot \text{const}$,



s - **natural parameter**.
$P \to n(P) \in \mathbb{S}^1$.

$d\tau = k(s)\,ds = d\varphi$ - obvious.



5. 2-form

$$\Omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$$

1. Rotate in (y,z) - plane:

$$x \to x \quad , \qquad dy \wedge dz \to dy \wedge dz$$

2. Rotate in (x,y) - plane:

$$z \to z \quad , \qquad dx \wedge dy \to dx \wedge dy$$

3. Rotate in (x,z) - plane:

$$y \to y \quad , \qquad dx \wedge dz \to dx \wedge dz$$

For (y,z) - plane we have: rotation of Ω is equal to rotation of $x\,dy \wedge dz$ plus rotation of $dx \wedge (z\,dy - y\,dz)$ where

$$x \to x \quad , \qquad z\,dy - y\,dz \quad \to \quad z\,dy - y\,dz$$

Similar arguments we have for other relations.

So our form is invariant under all 3 rotations. Therefore it is invariant under the whole SO_3.

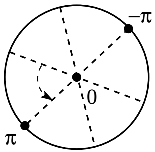
Homework 11. Solutions.

1. Compact Lie group, Bi-Invariant Metric ($SO_3 = \mathbb{RP}^3$ and $SU_2 = \mathbb{S}^3$) .

Let $A^t = -A$ be real 3×3 matrix. $e^{At} \in SO_3$. In some basis we have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and e^{At} are rotations around Z - axis. It is a **straight line** in $\mathbb{RP}^3 = SO_3$ presenting it as a ball B^3 (radius π). So all these geodesics are straight lines from 0 to π. So we proved it for SO_3. For SU_2 it is the same geodesics because $SU_2 / \pm 1 = SO_3$.



2. We have for $g \to g g_0$:

$$\dot{g}(t)\, g^{-1}(t) \;\to\; \frac{d}{dt}(g(t)g_0)\;(g(t)g_0)^{-1} \;\to\; \dot{g}(t)\, g^{-1}(t)$$

Same proof for $g \to g_0 g$ and $g^{-1}\dot{g}$ for left shifts. So $\langle A(t), A(t)\rangle$ is right (left) invariant inner product for $A = \dot{g}\, g^{-1}$ $(A = g^{-1}\, \dot{g})$.

Right and Left shifts of the group commute with each other

$$R_{h_1} L_{h_2} = L_{h_2} R_{h_1} \;:\qquad g \;\to\; h_2\, g\, h_1$$

Right shift maps right invariant metric into itself, and left shift maps right invariant metric into another right invariant metric $\tilde{g}_{ij}$.

$$R^*_{h_1}(g_{ij}) = g_{ij} \quad , \qquad L^*_{h_2}(g_{ij}) = \tilde{g}_{ij}$$

For $h_2 = h_1^{-1}$ metric $\tilde{g}_{ij}$ can be obtained from g_{ij} at the Lie Algebra by conjugation. So $\tilde{g}_{ij} = g_{ij}$ for inner product $\langle A, A\rangle$ invariant under conjugation. For SO_n it is $\langle A, A\rangle = -\operatorname{Tr} A^2$, $A^t = -A$.

3. Geodesics in SU_n are e^{At} (if it starts at I for $t = 0$). So we need: $e^{AT} = -I$ for some T. We have the following basis for su_2:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Let us put

$$A = \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} , \quad a \in \mathbb{R}$$

Fix time moment T such that $e^{iaT} = -1$. We have then $e^{AT} = -I$. All other solutions can be obtained by the change of basis in $\mathbb{C}^2$ (same T) :

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \rightarrow g \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g^{-1} ,$$

$g \in SU_2$. We have $g \in SU_2/U_1$.

4. All compact 2-manifolds M^2 :

$$H_1\left(M^2, \mathbb{Z}_2\right) = \begin{cases} \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \ \ (2g \text{ times}, \ M^2 \text{ is orientable}), \\ \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \ \ (k \text{ times}, \text{ for } M^2 = \mathbb{RP}^2\# \ldots \#\mathbb{RP}^2), \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \ , \quad M^2 = K^2 \ . \end{cases}$$

$$M^2 = \mathbb{S}^2_g : \quad \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \quad \text{over} \quad \mathbb{Z}_2$$

$$M^2 = \mathbb{RP}^2\# \ldots \#\mathbb{RP}^2 : \quad \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \quad k \text{ times}$$

$$M^2 = K^2 : \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$K^2 \,\#\, \mathbb{RP}^2 \;\;=\;\; \mathbb{T}^2 \,\#\, \mathbb{RP}^2 \;\;=\;\; \mathbb{RP}^2 \,\#\, \mathbb{RP}^2 \,\#\, \mathbb{RP}^2 \qquad (?)$$

$$K^2 \;\;=\;\; \mathbb{RP}^2 \,\#\, \mathbb{RP}^2 \qquad (?)$$

5. Let U be an open domain in any compact 2-manifold M^2 . Then intersection matrix

$$H_1\,(U,\,\mathbb{Z}_2) \;\;\to\;\; H_1\left(M^2,\,\mathbb{Z}_2\right)$$

depends only on image of cycles in M^2 where it has finite rank. So inner product in $H_1\,(U,\,\mathbb{Z}_2)$ also has finite rank.