SOME PROBLEMS IN THE TOPOLOGY OF MANIFOLDS CONNECTED WITH THE THEORY OF THOM SPACES

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In this paper we consider smooth manifolds W^i which are contained in Euclidean space R^{n+i} and whose normal bundle has as its structure group a subgroup G of the group $O(n)$. We will introduce an equivalence relation into the collection of all such manifolds. The set $V_n^i(G)$ of equivalence classes of manifolds will generate an abelian group. The equivalence relation and group operation will be introduced into our collection of manifolds in exactly the same way as in the classical interpretation by Pontryagin of the homotopy groups of spheres by means of equipped manifolds and as in the well-known construction of inner homology groups by Thom-Rohlin. Following L. S. Pontryagin, a manifold W^i which lies in the Euclidean space R^{n+i} will be said to be *G-equipped* if a fixed G-bundle structure is given in the normal bundle of the manifold W^i. A G-equipped manifold $W^i \subset R^{n+i}$ will be said to be *equivalent to zero* if there exists a smooth manifold $N^{i+1} \subset R^{n+i+1}$ with boundary $W^i \subset R^{n+i}$ such that the normal G-bundle can be extended from the manifold W^i onto the manifold N^{i+1}. It is evident that, for arbitrary groups $G_1 \subset O(n_1)$ and $G_2 \subset O(n_2)$, the operation of direct product of manifolds induces the pairing

$$V_{n_1}^i(G_1) \otimes V_{n_2}^j(G_2) \to V_{n_1+n_2}^{i+j}(G_1 \times G_2) \tag{1}$$

(the group $O(n_1) \times O(n_2)$ being considered as imbedded in the group $O(n_1 + n_2)$ by the usual map). We will confine ourselves here to the study of the groups $V_n^i(SO(n))$ for $i < n-1$, the groups $V_{2n}^i(u(n))$ for $i < 2n-2$, and the groups $V_{4n}^i(\mathrm{Sp}(n))$ for $i < 4n-4$. It is easy to see that these groups are independent of n. Accordingly, we will denote these by the symbols V_{SO}^i, V_u^i, and V_{Sp}^i. It is evident that the above pairing permits us to consider the direct sums $V_{SO} = \Sigma V_{SO}^i$, $V_u = \Sigma V_u^i$, and $V_{\mathrm{Sp}} = \Sigma V_{\mathrm{Sp}}^i$ as graded rings.

Theorem 1. *The quotient ring of the ring V_{SO} modulo 2-torsion is isomorphic to the polynomial ring on generators V_{4i} of dimension $4i$, $i \geq 0$. The ring V_u is isomorphic to the polynomial ring on generators u_{2i} of dimension $2i$, $i \geq 0$. The algebras $V_{\mathrm{Sp}} \otimes Z_p$, $p > 2$, and $V_{\mathrm{Sp}} \otimes Q$ (where Q is the field of rational numbers) are isomorphic to the algebras of polynomials on generators t_{4i} of dimension $4i$, $i \geq 0$. The rings V_{SO}, V_u, and V_{Sp} do not have p-torsion for $p > 2$, and the ring V_u has no 2-torsion.*

For the ring V_{SO}, this theorem has been announced by Milnor [5]. The structure of the groups V_{SO}^i is studied in [1, 4] by other methods. For a proof of Theorem 1 we remark first that, utilizing the well-known constructions of Thom [3] it is not difficult to prove the isomorphisms $V_n^i(G) \approx \pi_{n+i}(M_n(G))$ where $M_n(G)$ is the space constructed by Thom in his investigation of the G-realization of cycles. Further, it can be shown that the pairing

$$\pi_{n_1+i}(M_{n_1}(G)) \otimes \pi_{n_2+j}(M_{n_2}(G)) \to \pi_{n_1+n_2+i+j}(M_{n_1+n_2}(G_1 \times G_2)),$$

which corresponds to the pairing (1) by virtue of these isomorphisms, is induced by the natural homeomorphism

$$\rho: \quad M_{n_1}(G_1) \times M_{n_2}(G) / M_{n_1}(G_1) \vee M_{n_2}(G) \rightarrow M_{n_1+n_2}(G_1 \times G_2).$$

Thus, the study of the groups $V_n^i(G)$ and of the pairing (1) reduces to the study of the homotopy properties of the spaces $M_n(G)$. Corresponding to the groups V_{SO}, V_u, and V_{Sp}, we introduce the graded groups $H_{SO}(p)$, $H_u(p)$, and $H_{Sp}(p)$ whose homogeneous components are the fixed cohomology groups over the field Z_p of the corresponding spaces of Thom. Evidently, these groups can be considered as modules over the algebra $A = A_p$ of Steenrod.

Lemma 1. *The module $H_{SO}(p)$, for $p > 2$, admits a system of generators u_ω which are in one-to-one correspondence with the partitions ω of multiples of four whose terms are multiples of four but are not of the form $2p^t - 2$. The degree of the generator u_ω is equal to the sum of the terms of the partition ω. For any element $x \in H_{SO}(p)$ we have the relation $\beta x = 0$, where β is the homomorphism of Bochstein. All non-trivial relations among the elements of the module $H_{SO}(p)$ are expressible in terms of these relations. The module $H_{Sp}(p)$ is, for $p > 2$, isomorphic to the module $H_{SO}(p)$. The module $H_u(p)$, for any $p \geq 2$, is described in exactly the same way as the module $H_{SO}(p)$ was for $p > 2$, except that partitions of arbitrary even numbers into even summands which are not of the form $2p^t-2$ are permitted. The module $H_{Sp}(2)$, as in the case $p > 2$, admits a system of generators u_ω, but the condition that the summands are not of the form $2p^t - 2$ is replaced by the condition that the summands are not of the form $4(2^t-1)$. Also, for any element $x \in H_{Sp}(2)$, instead of the relation $\beta x = Sq^1 x = 0$, we have the relation $Sq^2 x = 0$. All non-trivial relations among the elements of the module $H_{Sp}(2)$ follow from these relations. The module $H_{SO}(2)$ is the direct sum of a free module and a module M_ω with one generator u_ω, where ω is an arbitrary partition of a multiple of four into summands which are multiples of four. The degree of the generator u_ω is the sum of the terms of the partition ω. The only non-trivial relation in the module M_ω is the relation $Sq^1 u_\omega = 0$.*

It can be shown further that the mappings

$$H_{SO}(p) \rightarrow H_{SO}(p) \otimes H_{SO}(p),$$
$$H_u(p) \rightarrow H_u(p) \otimes H_u(p),$$
$$H_{Sp}(p) \rightarrow H_{Sp}(p) \otimes H_{Sp}(p),$$

which are induced by the homeomorphism ρ, are given by the formula

$$\rho^*(u_\omega) = \sum_{\substack{(\omega_1, \omega_2) = \omega \\ \omega_1 \neq \omega_2}}' [u_{\omega_1} \otimes u_{\omega_2} + u_{\omega_2} \otimes u_{\omega_1}] + \sum_{(\omega_1, \omega_1) = \omega} u_{\omega_1} \otimes u_{\omega_1}.$$

Because of Lemma 1, the study of the modules $H_{SO}(p)$ and $H_{Sp}(p)$ for $p > 2$ and of the modules $H_u(p)$ for $p \geq 2$ is reduced to the study of the module M_β which has a single generator (whose degree we will consider to be zero) and which is defined by the relation $\beta x = 0$ for all $x \in M_\beta$. It is easy to see that if a diagonal map $M_\beta \rightarrow M_\beta \otimes M_\beta$ is defined, the group

$$\operatorname{Ext}_A(M_\beta, Z_p) = \Sigma \operatorname{Ext}_A^{s,t}(M_\beta, Z_p)$$

becomes an algebra. A proof, based on the construction by Adams [2] of the basis of the algebra of Steenrod, can be given of the following

Lemma 2. *The algebra $\operatorname{Ext}_A(M_\beta, Z_p)$ is isomorphic to the algebra of polynomials generated by the elements $h_t \in \operatorname{Ext}_A^{1, 2p^t-1}(M_\beta, Z_p)$, $(p \geq 2,\ t \geq 0)$.*

Theorem 1 now follows easily from Lemmas 1 and 2 of the properties of the spectral sequence of

Adams [2]. As a supplement to Theorem 1 we have

Theorem 2. *The quotient ring of the ring* V_{Sp} *modulo 2-torsion is not isomorphic to a ring of polynomials. Indeed, it has generators* $x \in V_{\mathrm{Sp}}^4$ *and* $y \in V_{\mathrm{Sp}}^8$ *of infinite order which satisfy the relation* $2^k(x^2-4y) = 0$.

The proof of Theorem 2 is analogous to the proof of Theorem 1. However, in view of Lemma 1, instead of the module M_β we must consider the module M_1 with one generator which is defined by the relations $\mathrm{Sq}^1 z = \mathrm{Sq}^2 z = 0$ for all $z \in M_1$. In this case, the group $\mathrm{Ext}_A(M_1, Z_2)$ also becomes an algebra. However, Lemma 2 is replaced by the following lemma:

Lemma 2'. *The algebra* $\mathrm{Ext}_A(M_1, Z_2) = \Sigma\, \mathrm{Ext}_A^{s,t}(M_1, Z_2)$ *is isomorphic to the cohomology algebra of a certain algebra B which possesses the following properties:* 1) *the algebra B contains a central sub-algebra C which admits a system of generators* $a_{r,0} \in C^{(2^r-1)}$ $(r \geq 2)$ *which satisfy the relations* $\alpha_{r\,0}^2 = 0$ *(and their consequences) and no others;* 2) *the algebra* $B//C$ *is commutative and admits the system of generators* $\alpha_0 \in B//C^{(1)}$, $\alpha_{r,1} \in B//C^{(2^{r+1}-2)}$ $(r \geq 1)$ *which satisfy the relations* $\alpha_0^2 = 0$ *and* $\alpha_{r,1}^2 = 0$ *and no others;* 3) *in the spectral sequence of Serre-Hochschild of the pair* (B, C) *we have the following relations:*

$$d_2(1 \otimes h_{r,0}) = h_0 h_{r-1,1} \otimes 1,$$
$$d_3(1 \otimes h_{r,0}^2) = h_{1,1} h_{r-1,1}^2 \otimes 1,$$
$$d_i(1 \otimes h_{r,0}^4) = 0, \quad i \geqslant 2,$$

where $h_{r,0}$ *is a generator of the algebra* $H^*(C)$ *determined by the equation* $(h_{r,0}, \alpha_{r,0}) = 1$, *while* h_0 *and* $h_{r,1}$ *are generators of the algebra* $H^*(B//C)$ *determined by the equations* $(h_0, \alpha_0) = 1$ *and* $(h_{r,1}, a_{r,1}) = 1$.

Since Theorems 1 and 2 are in essence theorems about the homotopy groups of Thom spaces, they can be applied to the problem of representing the integral cycles on manifolds in the form of submanifolds.

Theorem 3. *The integral homology class of* Z_{n-i} *of a compact, closed, orientable, smooth manifold* M^n *is represented by a submanifold* $W^{n-i} \subset M^n$ *if* $i > [n/2]+1$, *while for* $k < n-i-2(p-1)$ *the groups* $H_k(M^n)$ *have no p-torsion for any* $p > 2$. *The integral homology class of* Z_{n-2i} *of the manifold* M^n *has a* $U(i)$*-realization if* $2i > [n/2]+1$, *while for* $k < n-2i-2(p-1)$ *the groups* $H_k(M^n)$ *have no p-torsion for any* $p \geq 2$.

Letting K denote an arbitrary finite polyhedron, we have the following corollary of Theorem 3:

Corollary. *If, for* $q > i-2(p-1)$, *the groups* $H_q(K)$ *have no p-torsion for any* $p > 2$, *then every cycle* $Z_i \in H_i(K)$ *may be represented as the continuous image of the fundamental cycle of a certain orientable manifold* W^i.

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