TOPOLOGY OF FOLIATIONS

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Introduction

This paper contains the detailed presentation of the proofs of results announced in [5]–[7]. The basic object under study is a foliation of codimension 1 on a compact manifold M^n, given by a one-dimensional nonsingular "Pfaffian form" ω. The equation $\omega = 0$ then determines a field of $(n-1)$-planes on M^n, which we shall always suppose smooth. When the Frobenius integrability condition $\omega \wedge d\omega = 0$ is satisfied, this field of $(n-1)$-planes determines a smooth family of connected $(n-1)$-hypersurfaces fibering the manifold, called "leaves" ($d\omega$ is the exterior differential of ω). If the manifold has a boundary, we require that every component of the latter be a leaf. Other reasonable boundary conditions are of course possible, but we shall not consider them here. The basic questions to be dealt with concern the relation between the topological properties of the manifold and those of the leaves. The basic results, perhaps, are for the case $n = 3$. While for $n = 2$ only the torus and the Klein bottle have nonsingular foliations, the latter exist in the three-dimensional case on an arbitrary manifold and may have various highly complicated properties (in this regard see § 4). The hardest result of the paper is that of §§ 7–8, concerning a closed leaf, e.g., on S^3 or on $S^2 \times S^1$. Note also the separate result of § 5.

Among the authors who have had an essential influence on the outcome of this paper, I must mention Haefliger [3, 4] and Reeb [8, 9],

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Translated by J. A. Zilber.

I wish to express my appreciation to D. V. Anosov, V. I. Arnol'd and Ja. G. Sinaĭ, whose outstanding work on dynamical systems closely related to a special type of foliation called my attention to this subject. Some results of § 4 were communicated to me by H. Zieschang, to whom also I express my appreciation.

§ 1. General concepts. Connected components

For convenience, we give here and in §§ 2, 3 the general concepts of foliation theory which we shall be using. Without further mention, we shall always be dealing with smooth (or sometimes analytic) foliations on compact manifolds with or without boundary, the boundary consisting of finitely many component leaves.

Definition 1.1. A foliation is called *orientable* if its normal in some Riemannian metric can be so oriented at every point that the orientation depends continuously on the point.

Definition 1.2. By an *orientation* of the foliation is meant a choice of direction for the normals; this direction will be called the *positive* one, the other the *negative*.

Definition 1.3. By a *transverse segment* is meant a curve with two end points (or else void) which is nowhere tangent to the foliation. For an oriented foliation, a transverse segment has a *positive direction*. The notion of a *closed transversal*, and its orientation, is defined similarly.

Definition 1.4. We shall say that leaf A is *greater than or equal to* leaf B, in a foliation on a manifold M^n, if there exists a transverse segment leading from a point of A to a point of B in the positive direction. We write $A \geq B$ when this is the case, and also write $A \geq A$. We have the elementary

Lemma 1.1. *The relation $A \geq B$ is independent of the choice of points on A and B; furthermore, if $A \geq B$ and $B \geq C$, then $A \geq C$.*

The proof follows immediately from the fact that the foliation is smooth and the leaves A, B and C are connected. Let x_0, x_1 be points on A, and y_0, y_1 points on B with a transverse segment l going from x_0 to y_0 (Figure 1). Join x_1 to x_0 by a curve l', and y_0 to y_1 by a curve l''. Then the segment $l'l''$ goes from x_1 to y_1 and can be approximated arbitrarily closely by a transverse segment from x_1 to y_1. This being quite obvious, we leave the remainder of the proof to the reader. As for the second half of the lemma, we only mention that its proof is entirely similar: the segment $l_1(\overrightarrow{y_0y_1})l_2$ is approximately a transversal (Figure 1b).

We may now define the notion of "connected component" of a foliation: two leaves A and B belong to the same connected component if and only if $A \geq B$ and $B \geq A$.

By Lemma 1.1, the whole manifold M^n is partitioned into a disjoint union of connected components, each of which is itself a union of entire leaves. This definition is of course meaningful only for oriented foliations, and we shall assume henceforth chat we are dealing with the latter.

We now have

Theorem 1.1. *Only the following three types of connected components are possible for a foliation.*

a) *The entire manifold is the only connected component.*

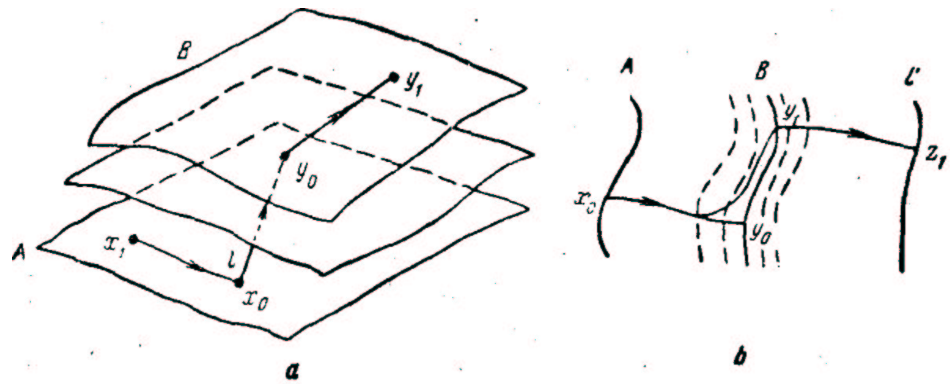

FIGURE 1

b) *The connected component consists of a single leaf; in this case it (the leaf) is closed, and there is no closed transversal passing through it. Conversely, if a leaf has no closed transversal passing through it, it is closed and is a connected component.*

c) *The closure of the connected component is a manifold $V^n \subset M^n$ with boundary dV^n, where the latter consists of finitely many component leaves of type* b)*, and the connected component itself is $V^n \setminus dV^n$.*

Proof. Assuming we do not have case a), then two cases are a priori possible: 1) the component consists of a single leaf; 2) the component contains at least two leaves.

In the first case it is obvious, by the definition of a component, that no closed transversal passes through the leaf. We shall now prove that the leaf is compact.

Lemma 1.2. *Any noncompact leaf A has a closed transversal passing through it.*[1]

Proof. Since M^n is compact and A is not, there exist two points $x, y \in A$ which are close in the metric of M^n but far from each other on A. Join them by a curve l on A and by a small transverse segment l' in M^n (Figure 2). It is obvious, as in Lemma 1.1, that the closed curve ll' can be approximated arbitrarily closely by a closed transversal which intersects A in, e.g., the point $x \in A$. This proves the lemma. □

We have thus proved all the assertions in case b). Now suppose a component S contains more than one leaf but is not all of M^n. Let A be a limit leaf for the leaves of S. This means that arbitrarily close to some point $x \in A$ there exist leaves belonging to S, while the leaf A itself does not belong to S; such a leaf always exists, since a component, provided it consists of more than one leaf, contains with any leaf all nearby leaves. The foliation being orientable, the leaf A has, near the point $x \in A$, "right" and "left" sides in the positive and negative directions respectively of the normal to the leaf.

We have then the easy

Lemma 1.3. *For sufficiently small $\epsilon > 0$, the leaves belonging to the component S occur in the ϵ-neighborhood of x on only one side of x (right hand or left hand),*

[1]It follows easily from Lemma 1.2 that the limit of compact leaves is a compact leaf (Reeb).

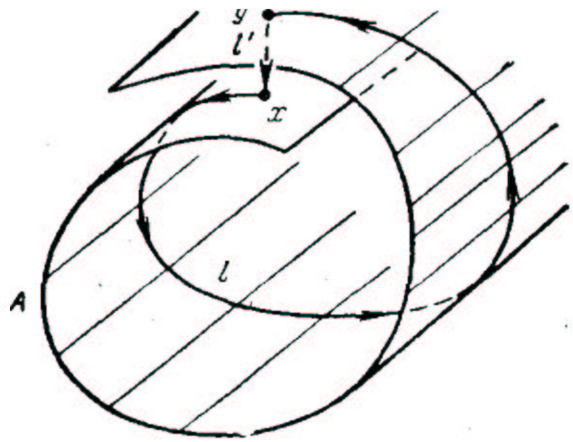

FIGURE 2

and the number ϵ may be chosen independent of the point x on the limit leaf A of the leaves of S.

Proof. Suppose that, on the contrary, the leaves of S occur on both sides of the point $x \in A$; let B and C be two such leaves, B on the left and C on the right of x, and so close to x that the normal to the foliation at x intersects B and C at distances no greater than 2ϵ from x; since M^n is compact, ϵ may be taken independent of $x \in M^n$. By definition of right and left sides, we have: $B \geq A \geq C$. But $C \geq B$, since $B, C \in S$; hence $A \geq C$ and $C \geq A$, i.e., $A \in S$. This contradiction proves the lemma. □

Now observe that it follows from Lemma 1.3 that on the side on which the component S abuts on the leaf A, it entirely fills out the ϵ-neighborhood of A on that side, where ϵ is the number chosen in Lemma 1.3. We see now that the leaf has no closed transversal passing through it, so that, by Lemma 1.2, it is compact. We have now proved all the assertions of the theorem. □

The relation $\geq$, defined above for leaves, carries over to connected components, for which it constitutes a partial order relation. We have thus associated a simple invariant with the foliation; namely, the partially ordered set of its connected components.

§ 2. INVARIANTS OF THE TYPE OF THE FUNDAMENTAL GROUP

Let x be a fixed point on the leaf A of a foliation on M^n, and consider the fundamental group $\pi_1(A)$ of A at x. An element $\alpha \in \pi_1(A)$ may be represented by a regular curve $f\colon S^1 \to A$, where $f(0) = f(2\pi) = x$. At each point $f(\lambda)$, $\lambda \in S^1$, erect the normal to A, on the same side for all points, say the right side. This family of normals determines a mapping $F\colon S^1 \times I \to M^n$ of an annulus, where $F|S^1 \times 0 = f$ and $F(\lambda, s)$ takes the point $(\lambda, s) \in S^1 \times I$ into the point of the normal erected at $f(\lambda) \in A$ which lies at a distance s along the normal from $f(\lambda)$. The mapping F will be called a *right normal fence* of the curve $\alpha \in A$. This "fence", of course, intersects the leaves of the foliation in general position (in fact, normally), i.e., along curves.

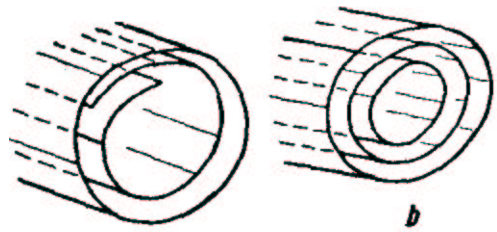

FIGURE 3

Definition 2.1. An element $\alpha \in \pi_1(A)$ is called a *right limit cycle* if the right normal "fence" $F\colon S^1 \times I \to A$, for some representative $f\colon S^1 \to A$ of $\alpha \in \pi_1(A)$, intersects the leaves sufficiently near A in curves which are not all closed (Figure 3).

In the opposite case, i.e., when all intersections of the "fence" with leaves near A at the point $x = f(0) = f(2\pi)$ are closed, we say that α is a *nonlimit cycle* on the right.

Definition 2.2. A limit cycle (on the right) is called *positive* if, in traveling along a curve $f\colon S^1 \to A$ representing $\alpha \in \pi_1(A)$, one finds that, after making the circuit from $x = f(0)$ to $x = f(2\pi)$, a nearby leaf on the positive side intersects the "fence" f in a point no further from x than before the circuit. A limit cycle α is called *negative* if α^{-1} is positive.

Similar definitions apply for left limit and nonlimit cycles.[2]

Lemma 2.1. *For an element $\alpha \in \pi_1(A)$, the property of being a limit cycle on the right or on the left is independent of the choice of the representative $f\colon S^1 \to A$ of α; while the property of being a nonlimit cycle remains unaltered under a free homotopy of the curve $f\colon S^1 \to A$, in which the initial point $f(0) = f(2\pi)$ travels along an arbitrary path.*

Proof. When a closed loop is deformed on A, its normal "fence" can be moved along with it. Hence the intersections of nearby leaves with the fence varying continuously, from which it follows that all the properties of being limit or nonlimit cycles, or of being positive or negative limit cycles on the right or left, are invariants or bound homotopy, i.e., are defined on the group $\pi_1(A)$; and that the property of being a nonlimit cycle is an invariant of free homotopy. This proves the lemma. □

As a corollary of Lemma 2.1, we observe that the set of nonlimit cycles on the right [resp. left] for a leaf A is a normal subgroup $N_1(A)$ [$N_2(A)$] of the group $\pi_1(A)$, and the sets of positive and negative right [left] cycles determine subsemigroups $P_1^+(A)$ [$P_2^+(A)$] and $P_1^-(A)$ [$P_2^-(A)$] of the group $P_1(A) = \pi_1(A)/N_1(A)$ [$P_2(A) = \pi_1(A)/N_2(A)$]. The groups $P_i(A)$ ($i = 1, 2$) will be called the limit cycle groups. The groups $P_w(A) = N_1(A)/N_1 \cap N_2$ and $P_l(A) = N_2(A)/N_1 \cap N_2$ will be called the *one-sided cycle groups*, on the right and left.

Remark 2.1. For an analytic foliation, it is obvious that $P_w(A) = P_l(A) = 0$, and also that $P_i(A) = P_i^+(A) \cup P_i^-(A) \cup 1$.

[2]In the western literature, e.g., [4, 8], limit cycles are discussed in the terminology of "holonomy groups".

Remark 2.2. Let us call a cycle $\alpha \in P_i(A)$ "rough" (or "coarse") if, in making a circuit of a representative $f\colon S^1 \to A$ of α, the point $F(0,s)$ of the normal "fence", for small s, goes into the point $F(0,\alpha(s))$, where $|\alpha'(0)| \neq 1$ ($\alpha(s)$ is called the "*sequential function*" and is independent of the choice of the representative in the homotopy class). It can be shown that if all nontrivial elements of the group $P_i(A)$ are "rough", then $P_i(A)$ is free abelian. To prove this, it suffices to observe that for a commutator $\alpha = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}$ the function $\alpha(s)$ is always such that $\alpha'(0) = 1$.

Remark 2.3. The following is a useful addendum to Theorem 1.1: if a compact leaf A lies on the boundary of a connected component of type c) in Theorem 1.1, and the component abuts on it on the right [left], then the group $P_1(A)$ [$P_2(A)$] is nontrivial, and sufficiently near A on the right [left] the component has no compact leaves in its interior.

Indeed, if $P_1(A)$ [$P_2(A)$] were trivial, the right [left] neighborhood of A would consist of compacts leaves, each by itself a connected component, leading to a contradiction. Furthermore, if near A there are compact leaves $A_1, \dots, A_k, \dots$, where $A_k \to A$ as $k \to \infty$, then the region between A_k and A is for large k diffeomorphic to $A \times I$, where $A \times 0$ is A and $A \times 1$ is diffeomorphic to A_k. It obviously follows that A cannot lie on the boundary of the component on this side.

Remark 2.4. We note in addition that if a connected component has a compact leaf strictly interior to it, then the leaf cannot be homologous to zero, since there exists a closed transversal intersecting it in general position (i.e., its intersection index with a closed curve is not zero). Hence the group $H_{n-1}(M^n)$ is in this case nontrivial.

We now define still another invariant of the foliation of a somewhat different type. Take any point x on a leaf A which is strictly interior to a connected component containing more than one leaf. Consider all positive closed transversals passing through the point $x \in A$. Partition them into regular homotopy classes, where the initial point is always at $x \in A$ (bound homotopy) and during the homotopy the transversal is never tangent to the leaves. (We note that for $n = 3$ one can separately consider classes of regular isotopy without selfintersections, but we shall not make a special study of this type of equivalence.)

Next, define the product ab of regular homotopy classes a, b of transversals. To do so, choose representatives $\bar{a}$ and $\bar{b}$ of the classes a and b passing normally through the chosen point $x \in A$. As in the usual definition of the product of paths, take the transversal $\overline{ab}$ and denote its class by ab. Then the following lemma is obvious.

Lemma 2.2. *The product ab of regular bound homotopy classes is independent of the choice of the representatives $\bar{a}, \bar{b}$ in the classes a and b. This product is associative* (*but not, in general, commutative*)*, and under it the set of regular homotopy classes of transversals bound at x becomes a semigroup $t(A,x)$. The semigroups $t(A,x)$ and $t(A,y)$ for different points $x, y \in A$ are isomorphic, the isomorphism depending only on the homotopy class on the leaf A of a path joining x and y on A. Thus, the group $\pi_1(A)$ acts on the semigroup $t(A,x) = t(A)$, and the orbits of this action are in a natural one-to-one correspondence with the free regular homotopy classes of transversals intersecting A.*

The proof is entirely parallel to that of the usual theorem concerning the fundamental group and its dependence on the point. We leave it to the reader.

The set of leaves intersecting a given closed transversal $\bar{a}$ will be denoted by Q_a; it is an invariant of the regular class a of $\bar{a}$. Simple examples show that within the same connected component of a foliation there may exist transversals intersecting different families of leaves (but not all leaves). We shall write $A \succ B$ if every closed transversal passing through A also passes through B, so that this gives rise to a certain new ordering of families of leaves. Note that if a leaf B is everywhere dense then for every A we have $A \succ B$ (obviously).

Theorem 2.1. *Within any connected component there exists a leaf A such that $A \succ B$ for every leaf B of the component.*

Proof. By definition of connected component, there always exists a closed transversal which lies within it. Let $\bar{a}$ be such a transversal, and a its class. Suppose the set of leaves Q_a fails to contain all leaves of the component. Note that Q_a is an open set. Let A be a leaf which is a limit leaf for Q_a and, indeed, lies strictly on the boundary of the open set Q_a and does not belong to the latter.

Now we need the simple

Lemma 2.3. *There exists a number $\epsilon > 0$, independent of the leaf A_ϵ, such that at least one point $x \in A$ is at a distance $> \epsilon$ from the boundary of the connected component.*[3]

Proof. Suppose that for every $\epsilon > 0$ there exists a leaf A which is entirely at a distance $< \epsilon$ from a boundary leaf A_0. For sufficiently small ϵ, there is a well-defined projection of A_ϵ onto A_0 along the normals to A_0, constituting a covering of A_0. The leaf A_ϵ lies entirely within the direct product $A_0 \times I_\epsilon$, where I_ϵ is the segment of the normal of length ϵ, and for each point $y \in A_0$ there exists on the segment $y \times I_\epsilon$ a limit point of the leaf A_ϵ furthest distance from A_0, the "upper point." For very small ϵ, this upper point is uniquely determined and depends continuously on y. Furthermore, locally (and therefore globally) the upper points all lie on the same leaf $\bar{A}_0$. Now the leaf A_0 is obviously compact, and we arrive at a contradiction; for the region between A_0 and $\bar{A}_0$ is diffeomorphic to $A_0 \times I$, and the leaves in this region are roughly parallel to A_0, so that by Remark 2.3 A_0 cannot lie on the boundary of the connected component, since $\bar{A}_0$ can be chosen arbitrarily close to A_0. This proves the lemma. $\square$

Returning now to the leaf A which was a limit leaf for Q_a, take on it a point x_1 at a distance $\geq \epsilon$ from the boundary of the component. Let $B \in Q_a$ be a leaf near x_1, and through B pass a transversal of class a (or a^{-1}) in the direction of A; this is meaningful. Call the transversal $\bar{\bar{a}}$. Proceeding along it, one passes through leaves closer and closer to A. This means that if we erect the normal to B on the same side and through the same point, then at first both of them, the normal and the transversal $\bar{\bar{a}}$, will intersect the same leaves, the normal monotonely approaching A from B and aiming at the point $x_1 \in A$. Let y be the first point on the normal lying on a leaf C which a fails to intersect. If there is no such point, this means that in a finite time the normal reaches the leaf B from which it began, while the transversal has made the circuit back to B.

Consider the point y (on C). By assumption, C lies in Q_a, since A is on the boundary of Q_a. We assert that $B = C$. The transversal $\bar{\bar{a}}$ has finite length and

[3]Translator's note: The A of the lemma is arbitrary, i.e., not necessarily the limit leaf A just mentioned.

an inclination to the leaves by an angle $> \alpha_0 > 0$. When we approach $y \in C$ along the normal, we also approach monotonically a point $z \in \bar{\bar{a}}$ at an angle to the leaves which is bounded below, passing through the same leaves. Hence z and y are on the same leaf, and furthermore, if $B \neq C$ we could keep on going.

Thus $B = C$, and the transversal $\bar{\bar{a}}$ completes its circuit at a point corresponding to the point y on the normal. Iterating the process (but now from the point y, and so on), we obtain the result that in any neighborhood of the point $x_1 \in A$ all the leaves of Q_a occur. Hence a transversal through x_1 necessarily intersects all the leaves in Q_a plus the leaf A. Through x_1 pass a closed transversal a_1, and consider Q_{a_1}. Iterate the process, if not all the leaves are in Q_{a_1}. Take a point x_2 on a leaf A_2 on the boundary of Q_{a_1}, at a distance $\geq \epsilon$ from the boundary of the component, etc. By transfinite induction, we find eventually a point x^- at a distance $\geq \epsilon$ from the boundary, through which there is a transversal intersecting all the leaves. This completes the proof of the theorem. □

Remark 2.5. Sacksteder proves in [11] that it no leaf in the (compact) limit set of a leaf A has a limit cycle, then either there exists a compact leaf in the limit set, or else A is everywhere dense. We observe, however, that if the limit set contains a compact leaf, then the latter always has limit cycles, so that the part of the statement that concerns a compact leaf should be dropped: either A is everywhere dense, or A is itself compact and all nearby leaves are compact, or the limit set of A contains a leaf with limit cycles (smoothness ≥ 2).

§ 3. Infinitesimal homomorphisms

We shall now define invariants of the foliation which are of a new type. Consider again a leaf $A \subset M^n$, and a map $f\colon X \to A$ of a complex X into the leaf. Suppose the image $f_*\pi_1(X) \subset \pi_1(A)$ lies entirely in the subgroup $N_1(A) \subset \pi_1(A)$. On the right hand side of A erect a "normal fence" of small height, and assume that X has a distinguished point $x_0 \in X$ which goes into the initial point $y \in A$, i.e., $f(x_0) = y$. Assume also that X is compact. Displace the point $y = f(x_0)$ to the right of A by a sufficiently small distance $\epsilon > 0$; the new point lies on a leaf which we call A_ϵ. We now try to "cover" the mapping $f\colon X \to A$ by a mapping $f_\epsilon\colon X \to A_\epsilon$ such that

a) $f_\epsilon(x_0) = y_\epsilon \in A_\epsilon$, where y_ϵ is the point distant ϵ from y along the normal to the right of y.

b) $\pi f_\epsilon(x) = f(x)$ for every point $x \epsilon X$, where π is the projection from A_ϵ onto A along the family of normals to A issuing from A on the right. The projection $\pi\colon A_\epsilon \to A$ is defined only for points of A_ϵ, at a distance $\leq \delta$ from A where δ is a constant independent of A or of the number ϵ (δ depends, in fact, on the rate of variation of the tangent planes to the foliation with respect to one another, and can be taken constant because of the smoothness of the foliation and the compactness of the containing manifold M^n). Since the complex X is compact, for ϵ sufficiently small relative to δ we can construct this "covering" of f, and so obtain a mapping $f_\epsilon\colon X \to A_\epsilon$. The number ϵ of course depends on f. Furthermore, f_ϵ is completely determined by the mapping f, the number ϵ and the choice of the initial points $x_0 \in X$, $y \in A$. One defines in a similar fashion left-displacement by a distance ϵ, except that for this it must be required that $f_*\pi_1(X)$ lie in $N_2(A)$.

We have the obvious

Lemma 3.1. *If the mappings f, g are (bound-) homotopic on the leaf A, then their displacements to the right [left] $f_\epsilon, g_\epsilon \colon X_\epsilon \to A_\epsilon$ are likewise (bound-) homotopic on the leaf A_ϵ, via a covering homotopy, for sufficiently small $\epsilon > 0$; and the mapping $f \colon X \to A$ is right-displaceable [left-displaceable] if and only if the same is true of a homotopic mapping $g \colon X \to A$.*

Proof. A homotopy is a mapping $F \colon X \times I \to A$; the conditions for F to be displaceable are satisfied simultaneously with those for $F|X \times 0 = f$ or $F|X \times 1 = g$. Note that $F|x_0 \times I = y \in A$. Hence, if we have constructed a covering of $F \colon X \times I \to A$ starting with the mapping $F_\epsilon|X \times 0 = f_\epsilon$ of the lower base, we obtain precisely the mapping g_ϵ on the upper base, and vice versa. This proves the lemma. □

Remark 3.1. The lemma does not apply to free homotopy, for if the image $F|x_0 \times I$ is a right limit cycle α, then the covering of the mapping $F|X \times 1$ does not give g_ϵ on the upper base when $F|X \times 0$ coincides with f_ϵ on the lower, but rather gives $g_{\alpha(\epsilon)}$ (for the notation $\alpha(\epsilon)$, see the definition of sequential function in Remarks 2.1, 2.2, 2.3).

Thus the bound homotopy classes of maps $f \colon (X, x_0) \to (A, y)$ are right-displaceable if and only if $f_*\pi_1(X) \subset N_1(A)$. We denote this set of displaceable classes by $\tilde{\tilde{\pi}}(X, A) \subset \pi(X, A)$. Displacement is of course always possible if $X = S^i$, $i > 1$, or if $X = S^1$ and the mapping f represents an element of $N_1(A)$. Similarly for the left side.

Definition 3.1. The elements $u, v \in \tilde{\tilde{\pi}}(X, A)$ are said to be *limitwise right-homotopic* if, for a pair of representatives $f \in u$, $g \in v$, where $f, g \colon (X, x_0) \to (A, y)$, the right displacements $f_\epsilon, g_\epsilon \colon X \to A$ are homotopic for all sufficiently small $\epsilon > 0$ (how small depends on the choice of map-representatives).

We have now

Lemma 3.2. *If $X = S^i$, $i \geq 1$, then the classes of limitwise right- [left-] homotopic elements of the groups $N_1(A)$ $[N_2(A)]$ for $i = 1$, or of the group $\pi_i(A)$ for $i > 1$, are the cosets of the subgroup of those elements which are limitwise right- [left-] homotopic to zero.*

Proof. By definition of right- [left-] ϵ-displacement of a mapping, the sum of two mappings of a sphere goes into the sum, and the inverse of an element goes into the inverse. Hence those elements which are limitwise homotopic to zero constitute a subgroup, for $i \geq 1$. Furthermore, this is a normal subgroup in $N_1(A)$ for $i = 1$, since under a free homotopy on A the disk which effects the homotopy to zero is drawn along with the mapping. It follows that we have in fact normality not only in $N_1(A)$, but in $\pi_1(A) \subset N_1(A)$.

This gives rise to an "infinitesimal homomorphism":

$$i > 1 \quad \begin{cases} 0 \to \Pi_i^1(A) \to \pi_i(A) \xrightarrow{q_i^1} G_i^1(A) \to 0, \\ 0 \to \Pi_i^2(A) \to \pi_i(A) \xrightarrow{q_i^2} G_i^2(A) \to 0, \end{cases}$$

$$i = 1 \quad \begin{cases} 0 \to \Pi_1^1(A) \to N_1(A) \xrightarrow{q_1^1} G_1^1(A) \to 0, \\ 0 \to \Pi_1^2(A) \to N_1(A) \xrightarrow{q_1^2} G_1^2(A) \to 0, \end{cases}$$

where $\Pi_i^j(A)$, $i \geq 1$, $j = 1, 2$, is the subgroup of those elements of $\pi_i(A)$ which are limitwise homotopic to zero, on the right for $j = 1$ and on the left for $j = 2$. □

Remark 3.2. If the leaf A is compact and $\pi_1(A)$ is finite, then by taking X to be the universal covering space of A we see that all nearby leaves are compact. This construction of the "infinitesimal homomorphisms" q_i^j is not specific for codimension 1, and thus we obtain the Reeb theorem on stability of a compact leaf with finite fundamental group.

Remark 3.3. But if the foliation is of codimension 1, we may take X to be the leaf A itself, and then all nearby leaves are diffeomorphic to A (for a non-orientable foliation we must take a two-sheeted covering of A). It has been shown by Reeb [8] that in this case all leaves are compact and, obviously, diffeomorphic to the original leaf A. [4] It follows that the foliation is a fiber bundle, whose base is either circle or a segment, and whose fibers are the leaves.

§ 4. Examples of foliations

1. The simplest example is given by a fiber bundle with a circle as base, the leaves being the fibers. In this case, the foliation is connected, and the semigroup $t(A)$ of closed transversals, for any leaf A, is monomorphically imbedded in the group $\pi_1(M^n)$, and consists of all elements of $\pi_1(M^n)$ which project into a positive multiple of the base-circle. There are no limit cycles, and the subgroups $\Pi_i^j(A)$ are trivial for all leaves. If the base is a segment, then each leaf is a connected component.

2. Consider a form ω on the torus T^n, where $\omega = \sum \omega_i \, d\phi_i$, the numbers ω_i are constants, and the ϕ_i are angular coordinates. The equation $\omega = 0$ determines a foliation on T^n which has no singularities, no limit cycles, and no "infinitesimal homomorphisms." If the set of numbers $(\omega_1, \dots, \omega_n)$ is linearly independent over the integers, then every leaf is diffeomorphic to R^{n-1}; while if the rank of the set $(\omega_1, \dots, \omega_n)$ over the integers is k, then a leaf is a quotient of R^{n-1} by the part of a lattice generated by $n - k$ vectors, so that $\pi_1(A)$ is isomorphic to $Z + \cdots + Z$ ($n - k$ terms). One may also consider more complicated forms on the torus.

An automorphism $h\colon T^n \to T^n$ of the torus which is given by a matrix H with integral coefficients such that one of the eigenvalues of H has absolute value > 1 and the others have absolute values < 1, determines a foliation of codimension 1 on T^n, two points x_i, x_j belonging to the same leaf A if $\lim_{k\to\infty} \rho(h^k x_i, h^k x_j) = 0$. Here, all leaves are R^{n-1}. They are everywhere dense in T^n. Similarly, one can consider an automorphism $h\colon M^n \to M^n$ with the same type of property (a U-cascade in the sense of Anosov [2]), namely, that for any point $x \in M^n$ the set of points $y \in M^n$ such that $\lim_{k\to\infty} \rho(h^k x, h^k y) = 0$ is locally a hypersurface in M^n which depends smoothly on the point $x \in M^n$. This determines a foliation without limit cycles. As regards the existence of U-cascades, see § 5.

A pair of transverse foliations is determined by a U-system in the sense of Anosov [2] on a manifold M^n, $n \geq 3$. Example: a geodesic flow on a surface of negative curvature, determining a dynamical system on the three-dimensional manifold of linear elements on the surface, where the trajectories fall into two-dimensional leaves of those "converging" and "diverging" relative to a given one. The leaves here are

[4] This follows from the fact that the limit of compact leaves is compact; see the footnote to Lemma 1.2.

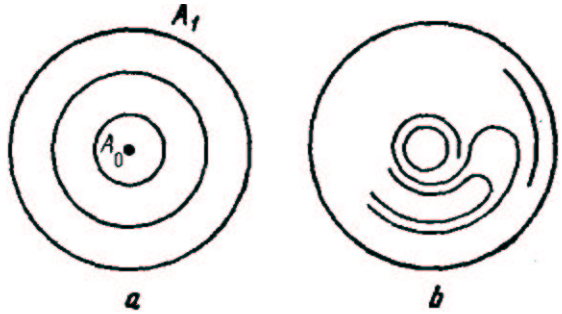

FIGURE 4

everywhere dense and are diffeomorphic to R^2 or, in countable number, to $S' \times R$, the circle $S' \times 0 \subset S^1 \times R$ being a two-sided limit cycle of foliation (and a periodic solution of the system itself).

3. Consider the solid torus $D^2 \times S^1$ with coordinates $x_1, x_2 \in D^2$, where $|x_1^2| + |x_2^2| \leq 1$ and $\phi \mod 2\pi \in S^1$. We define a foliation geometrically:

a) The cross-section $\phi = C$ of the foliation consists of the curves A_p: $A_p = \{x_1^2 + x_2^2 = p\}$, $1 \geq p \geq 0$, where $0 \leq \phi \leq 2\pi$ (Figure 4a); A_0 is evidently the point $O \in D^2$.

b) The cross-section $\lambda x_1 = \mu x_2$ of the foliation (Figure 4b) consists or the curves $t^2 = 2\phi^2/(1+\phi^2)$ where t is a parameter on the line $\lambda x_1 = \mu x_2$ proportional to distance and taking on values from -1 to $+1$.[5]

The boundary is a single closed leaf, the torus T^2.

This foliation, as well as any smooth foliation homeomorphic to it, will be called a *Reeb foliation* on $D^2 \times S^1$ (the homeomorphism of the family of planes being taken without regard for smoothness).

4. Consider a fiber bundle whose base is a circle and whose fiber is a surface P of genus g with boundary, the latter being a circle $S^1 = \partial P$. The space M^3 of this fiber bundle is a manifold whose boundary ∂M^3 is fibered over S^1 with fiber S^1. Let $\partial M^3 = S^1 \times S^1 = T^2$, and let the fibers $P \subset M^3$ meet the boundary ∂M^3 normally. Suppose given a decomposition $\partial M^3 = \partial P \times S^1$ as a direct product with coordinates ϕ_1, ϕ_2. Consider separately the product $T^2 \times I$, where I is the interval from 0 to 1, with coordinates ϕ_1, ϕ_2, s, with $0 \leq s \leq 1$. On $T^2 \times I$ define a "foliation" as follows:

a) The cross-section $\phi_1 = C \pmod{2\pi}$ consists of the curves (see Figure 5)

$$s^2 = \frac{2\phi_2^2}{1+\phi_2^2}, \quad 0 \leq s \leq 1.$$

b) The cross-section $\phi_2 = C$ consists of the curves $s = \text{const}$.

Now identify $\partial M^3 = (\phi_1, \phi_2)$ with $T^2 \times 0$ coordinate-wise. We obtain a diffeomorphic manifold $M_1^3 = M^3 \cup T^2 \times I$ with a foliation, whose boundary is a compact leaf, the torus T^2, while all leaves other than T^2 are open surfaces $P\partial P$ of genus g.

5. If we identify, in a natural fashion, $\partial(D^2 \times S^1) = T^2$ with $\partial M_1^3 = T^2$ via a diffeomorphism h, we obtain a smooth (non-analytic) foliation on the manifold

[5]Translator's note: The translator is unable to understand this equation and the corresponding one in Example 4. He would have expected $\phi = t^2/(1-t^2) + a$.

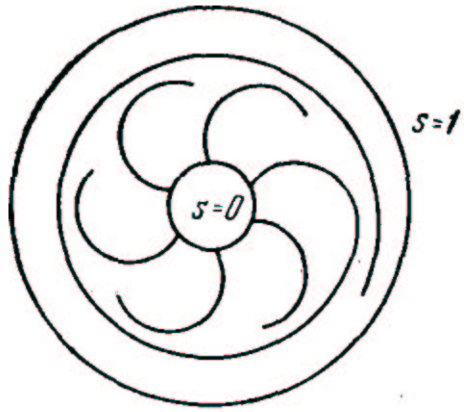

Figure 5

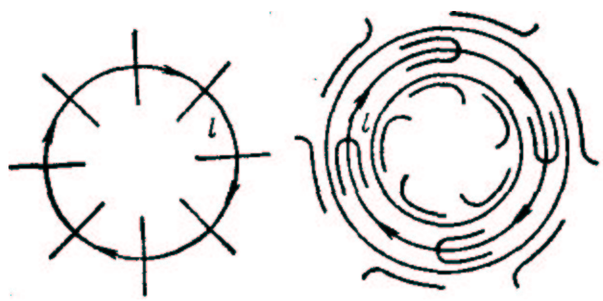

Figure 6

$M_1^3 \cup_h (D^2 \times S^1)$. For example, from the Reeb foliation on $D^2 \times S^1$ we obtain foliations on the sphere S^3, lens spaces L^3, and $S^2 \times S^1$, each having just one compact leaf T^2 and consisting of three connected components: two "Reeb" components and their common compact boundary leaf.

In addition, many of these fiber bundles of surfaces over S^1 also yield foliations on S^3 when we combine them with a Reeb foliation on $D^2 \times S^1$ via a suitable diffeomorphism of the torus $T^2 \to T^2$. We may in this fashion obtain a foliation on S^3 with a single "Reeb" component $D^2 \times S^1$, whose axis $0 \times S^1 \subset S^3$ is any knot in S^3 of which the complement has a finitely generated commutator-group (and is fibered with base S^1, according to the results of various authors). This construction is described by Zieschang [10].

6. The operation of "turbulizing" a foliation (due to Reeb) consists of the following. Suppose, for example, $n = 3$, and let l be a closed imbedded transversal of a foliation on M^3. Cut out a small tubular neighborhood of l from M^3. On the remaining manifold (with boundary) $\bar{M}^3$, construct a foliation as at the beginning of Example 4, with the torus as a compact boundary leaf. Having done so, paste back the tubular neighborhood of l (this tube is $D^2 \times S^1$), providing it with a Reeb foliation. The smooth foliation so obtained is called a *turbulization* of the original along the transversal l (Figure 6).

Thus from a given foliation we may obtain many others by turbulization. An example of this is

Lemma 4.1 (Zieschang). *Consider a finite number of circles in S^3, knotted or unknotted. Cut out tubular neighborhoods of these curves from S^3. The remaining manifold M^3 has as boundary a set of toruses $T_1^2 \cup \cdots \cup T_k^2$. Then on every such manifold there exists a smooth foliation, where the boundary consists of closed leaves.*

Proof. Consider a foliation on S^3 consisting of two solid toruses with a Reeb foliation on each. The axial transversal a of either one of the components is unknotted. Take k copies of a, close to each other and nonintersecting. Denote them by $a_1, \dots, a_k$. Although each is unknotted, a transversal of class $a_j^{l_j}$ may be knotted (we may approximate the l_j-fold multiple of the curve a_j by a curve which is not selfintersecting, but whether it is knotted depends on the particular approximation). We obtain in this fashion a closed braid in the sense of Artin; moreover, any knot (simple or multiple) may be reduced to this form, by classical results of Artin and others. The lemma can therefore be proved by cutting out a suitably chosen set of transversals of classes $a_j^{l_j}$ and applying the construction of example 4. □

From the lemma we easily obtain

Theorem 4.1 (Zieschang). *Every closed orientable manifold M^3 has a smooth foliation.*

Proof. As is well known, any three-dimensional manifold may be obtained by removing from S^3 finitely many tubes $(D^2 \times S^1)_i$ and pasting them back by suitably chosen automorphisms $h\colon T^2 \to T^2$ of the boundaries. The theorem now follows from Lemma 4.1 and example 4. □

§ 5. Foliations without limit cycles

We shall study here a class of foliations most nearly like fiber bundles; namely, those foliations without limit cycles. The principal result is contained in the following theorem.

Theorem 5.1. [6] *Let M^n be a compact closed manifold on which is given an orientable foliation, such that the groups $P_j(A)$ are trivial for all leaves A and for $j = 1, 2$. Then the universal covering space $\hat{M}$ of M^n is diffeomorphic to the direct product $\hat{A} \times R$ of the universal covering space $\hat{A}$ of any leaf A by a line R, the normal to the leaves on $\hat{M}$, the group $\pi_1(A)$ is monomorphically imbedded in $\pi_1(M^n)$ and is a normal subgroup of $\pi_1(M^n)$; and the quotient group $\pi_1(M^n)/\pi_1(A)$ is free abelian. If the quotient group is cyclic, then M^n is a fiber bundle with a circle as base and A as fiber, and the foliation is trivial.*

The proof will be carried out in several steps. To begin with, we show that the normal to the covering foliation in $\hat{M}$ intersects each leaf once and only once. Suppose the contrary, let l be a normal to the foliation on $\hat{M}$ which fails to intersect the leaf $\hat{A}$ but intersects all leaves to the left of $\hat{A}$ and close to it. We shall then

[6] The significance of the theorem consists in that such a foliation "has" many of properties of the "level surfaces" of a multivalued function, i.e., of the foliation determined by a closed nonsingular 1-form.

Figure 7

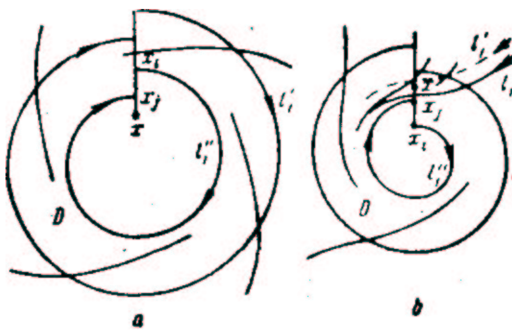

Figure 8

show that there must be a limit cycle near A, where A is the leaf covered by $\hat{A}$. We take the metric on $\hat{M}$ to be that induced by the metric on the compact manifold M^n. The leaves intersected by l will be denoted by $\hat{A}_t$, where t is the parameter of arc length along the normal. For t sufficiently large, the leaf $\hat{A}_t$ contains points which are distant from $\hat{A}$ no more than $\epsilon = \epsilon(t)$, where $\epsilon(t) \to 0$ monotonically as $t \to \infty$. On the other hand, the leaf $\hat{A}_t$, for any t, must contain points distant from $\hat{A}$ more than ϵ_0 (we fix this ϵ_0 for all $t > T$) (Figure 7). Otherwise, starting at a distance $\leq \epsilon_0$ from $\hat{A}$, the normal l would intersect $\hat{A}$, since the foliation and the metric in $\hat{M}$ were obtained via a covering from the compact manifold M^n.

Because of this, one can construct a transversal l_1 of the foliation, intersecting the same leaves $\hat{A}_t$ as l but situated at a distance $\leq \epsilon_0$ from $\hat{A}$ for the whole stretch $t > T$. Project l_1 into M^n. It lies near $A \subset M^n$. Hence l_1 projects normally into $\hat{A}$, and its projection l_1' into A is single-valued throughout.[7] Denote the projection of l_1' into A by l_1''; the latter is then a curve in A from which l_1' is distant always no more than ϵ_0. In M^n, let x be a limit point of l_1''. Construct the normal to the foliation at x. We now examine the behavior of the strip D between l_1' and l_1'', consisting of normals to the foliation of length $\leq \epsilon_0$, near x. We may suppose that D intersects the normal at x infinitely often along a segment of the normal, and that D intersects itself near x each time in a portion of a plane around the normal at x (Figure 8). For definiteness, suppose the intersections of the leaves with D travel toward l_1'' with decreasing t and intersect l_1' transversely with increasing t. Denote by $x_1, x_2, \ldots$ the points of return of l_1'' to the limit point x of l_1'', on the normal to x.

[7]Translator's note: There is some confusion here; l_1' is meant to be the projection of l_1 into M^n.

Case 1. The point x_j returns to x to the right of x_i for $j > i$ (Figure 8a).
Case 2. The point x_j returns to x to the left of x_i for $j > i$ (Figure 8b).
We may suppose that all the points x_k lie below x.

In Case 1, it is evident that on the part of D from x_i to x_j, closed up by a segment of the normal at x, there exists a limit cycle (Figure 8a).

In Case 2, this is not immediately clear, since the continuation of the intersection of l_1'' through x_j with this part of the strip leaves the strip transversely to l_1' (Figure 8b), while the continuation through x_i exits immediately from this part of the strip. Observe, however, that the curve l_1'' will continue to return to the point x. Now, it must needs return to the interval between x_i and x_j on the normal to x; since it is meaningless to speak of returning below, this is farther from x than x_i or x_j; and if it were to return to x above x_j but below x, this would contradict the fact that the "accompanying" transversal l_1' travels on the right-hand side of the leaves. Thus the foliation would have to have a limit cycle if a normal to $\hat{A}$ failed to intersect all the leaves on $\hat{M}$. This proves the first part of the theorem. If the normal intersected the same leaf twice, we should have on $\hat{M}$ a closed transversal, and on M^n a closed transversal homotopic to zero. By a theorem of Haefliger [4], we should then have on some leaf $A \subset M^n$ a limit cycle (perhaps one-sided). We conclude, therefore, that $\hat{M} = \hat{A} \times R$.

We now prove the second part of the theorem. The group $\pi_1(M^n)$ operates on $\hat{M}$ as a (discrete) group of motions, taking leaves into leaves. The leaves in $\hat{M}$ are indexed by the parameter t of the normal R; we may denote them by $\hat{A}_t$. The transformation $\alpha\colon \hat{M} \to \hat{M}$ induces a transformation $q(\alpha)\colon R \to R$, where $\hat{A}_{q(\alpha)t} = \alpha(\hat{A}_t)$. There is thus determined a representation $q\colon \pi_1(M^n) \to \operatorname{diff} R$.

Lemma 5.1. *The transformations $q(\alpha)\colon R \to R$ have no fixed points, provided they are nontrivial.*

Proof. If the transformation $q(\alpha)$ has a fixed point $t_0 \in R$, it has a fixed point $t_1 \subset R$ such that all sufficiently near points for, say $t > t_1$ (or less), are not fixed under $q(\alpha)$; i.e., $q(\alpha)t \neq t$ for $t_1 < t < t_1 + \epsilon$. Since the foliation is orientable, we must have $q(\alpha)t > t_1$ for $t > t_1$ and $t < t_1 + \epsilon$. Furthermore, since $q(\alpha)t_1 = t_1$, the path $a \in \pi_1(M^n)$ is deformable to the leaf A_{t_1}; (the leaf $\hat{A}_{t_1}$, being invariant under α, carries, under the projection onto A_{t_1}, a path of class α). Furthermore, the path of class a on the leaf A_{t_1} is a nontrivial limit cycle with sequential function $\alpha(s) = q(\alpha)(t_1 + s)$. This proves the lemma. □

It follows from Lemma 5.1 that the image $q\pi_1(M^n) \subset \operatorname{diff} R$ consists of transformations without fixed points. Thus, the group $q\pi_1(M^n)$ has an archimedian ordering, where we say that $q(\alpha) > q(\beta)$ if $q(\alpha)t > q(\beta)t$ for some one (or all) t. For it is obvious that if $q(\alpha) \neq q(\beta)$, there exists an integer n such that either $q(\alpha)^n > q(\beta)$ or $q(\alpha)^{-n} > q(\beta)$. Hence, by the Hölder theorem on archimedian-ordered groups, the group $q\pi_1(M^n)$ is free abelian.

It remains to prove that the kernel of the representation $q\colon \pi_1(M^n) \to \operatorname{diff} R$ is precisely $\pi_1(A)$. The kernel is the image of $\pi_1(A)$ in $\pi_1(M^n)$, and is independent, as is by now evident, of the leaf A. But since $\hat{M} = \hat{A} \times R$ and $\pi_1(\hat{M}) = 0$, we see that also $\pi_1(\hat{A}) = 0$, where $\hat{A}$ is the covering leaf in $\hat{M}$. All parts of the theorem are now proved, except for that concerning the special case that the group $q\pi_1(M^n) = \pi_1(M^n)/\pi_1(A)$ is free cyclic. Taking this case, let the generator be

$q(\alpha) \in q\pi_1(M^n)$. The quotient space $\hat{M}/\pi_1(A)$ is obviously the product $A \times R$, and the transformation operates on $\hat{M}/\pi_1(A)$. On the t-axis, choose a parameter $\lambda = \lambda(t)$ such that $q(\alpha)\lambda = \lambda + 1$ for all λ. For this choice of parameter, the transformation $a\colon \hat{M} \to \hat{M}$ induces a transformation $\hat{\alpha}\colon \hat{M}/\pi_1(A) = A \times R \to A \times R$, where the leaf (A, λ) goes into $(A, \lambda + 1)$; and we obtain a transformation $A \to A$, determining on M^n a fiber bundle structure with a circle as base. This completes the proof of the theorem.

Corollary 5.1. *$\hat{M}/\pi_1(A)$ is always $A \times R$, $M^n = A \times R/(Z + \cdots + Z)$.*

Corollary 5.2. *If every leaf is diffeomorphic to R^{n-1}, then the universal covering space $\hat{M}$ of M^n is R^n, and the fundamental group $\pi_1(M^n)$ is isomorphic to the free abelian group on n generators. In particular, M^n has the homotopy type of the torus T^n.*

Corollary 5.3. *A U-automorphism $h\colon M^n \to M^n$ with foliation of codimension 1, and so always for $n = 3$ (and sometimes for $n > 3$), can exist only on manifolds M^n obtained from R^n by factorization modulo a discrete group of transformations isomorphic to the free abelian group on n generators.*

(For the definition of U-automorphism, see § 4 and also [1].)

We remark that for $n = 2$ the corresponding assertion is a trivial consequence of the classification of surfaces, but for $n > 2$ such considerations cannot, of course, be made.

Remark 5.1. By the theorem of Sacksteder [11] mentioned in Remark 2.5, all leaves of the foliation are everywhere dense if the group $\pi_1(M^n)/\pi_1(A)$ is not cyclic and the smoothness class ≥ 2.

§ 6. Invariants of the manifold and infinitesimal homomorphisms. Some corollaries

The principal purpose of this section is to establish the relation between invariants of the manifold M^n (namely, the groups $\pi_1(M^n)$ and $\pi_2(M^n)$) and the nontriviality of the groups $\Pi_1^j \subset \pi_1(A)$ of paths "limitwise null-homotopic" on the right or on the left for the leaves of a given foliation on M^n. For the definition of the groups $\Pi_1^j(A)$, see § 3. The results obtained here will be essential later on.

Theorem 6.1. *Given a foliation on a manifold M^n, if one of the four conditions below is satisfied, then there exists a leaf $A \subset M^n$ within the manifold such that the group $\Pi_1^j(A)$ is nontrivial for $j = 1$ or 2.*

1) *The fundamental group $\pi_1(M^n)$ is finite.*

2) *The homotopy group $\pi_2(M^n)$ is nontrivial, but for every leaf B the group $\pi_2(B)$ is trivial.*

3) *There exists a leaf $B \subset M^n$ such that the inclusion homomorphism $\pi_1(B) \to \pi_1(M^n)$ has nontrivial kernel.*

4) *There exists a leaf $B \subset M^n$ such that the homomorphism $t(B) \to \pi_1(M^n)$ has nontrivial kernel, where $t(B)$ is the semigroup of closed transversals passing through the leaf B.*

Proof. Let us first examine conditions 1) and 2), which are the most important for our subsequent purposes. Consider condition 1). Since the group $\pi_1(M^n)$ is finite, there exists in M^n a closed transversal which lies strictly interior to M^n (if M^n has

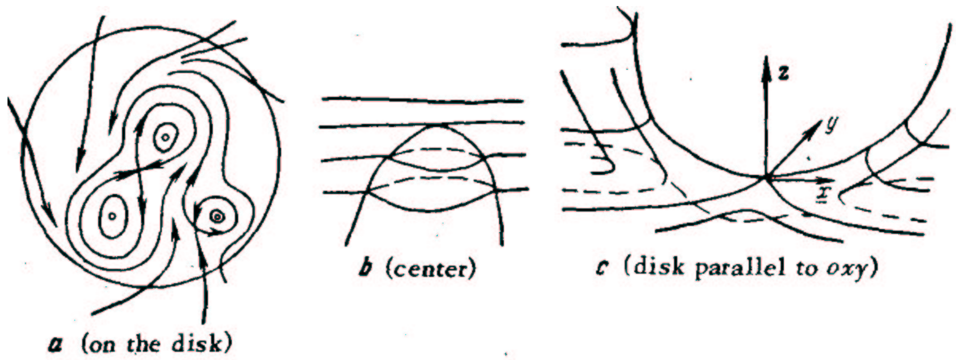

FIGURE 9

a boundary and the foliation is not a direct product $B \times l$, where B is a leaf) and is null-homotopic in M^n. Denote it by l. Span l by a regular disk $F\colon D^2 \to M^n$, where $F|_{\partial D^2} = l$. This is always possible, including the case $n = 3$, although the latter is not obvious (but known). We shall suppose that the disk is in general position with respect to the leaves of the foliation. There is then determined on the disk D^2, by intersection, a family of curves transverse to the boundary, with singular points of "central" and "saddle" type, the tangents at the singular points being of second order (see [4]); this is indicated in Figure 9.

Among the singular points on the disk there must be central ones. The closed curves surrounding a center are null-homotopic, on their respective leaves, provided they are sufficiently close to the center; this is evident from Figure 9b and the fact that the property of being null-homotopic is stable under displacement to a nearby leaf. If, nevertheless, some curve surrounding a center happens not to be null-homotopic on its leaf, then there is a closest one to the center with this property, i.e., such that those lying inside it on the disk are all null-homotopic on their respective leaves. This closest curve then determines a nontrivial element $\alpha \in \Pi_1^j(A)$, as required.

Suppose now that all such closed curves surrounding central singular points are null-homotopic on their respective leaves. Consider the closed curves on the disk which consist entirely of separatrices and inside of which, on the disk, lie only smooth curves surrounding a center. If any such is not null-homotopic on the corresponding leaf, it determines an element $\alpha \in \Pi_1^j(A)$ of the required sort.

Finally, if all such curves are still null-homotopic, we redefine the disk D^2 by replacing the regions surrounding centers by disks lying on the leaves and bounded by the above-described curves consisting of separatrices. The disk loses its regularity along these separatrix curves (it becomes "broken"). Consider the remaining closed curves on the disk which are unaffected by this process. Among them, by a theorem of Haefliger [4], must be one which is not null-homotopic, which in fact is a limit cycle, on the corresponding leaf. From among such closed curves on the disk, select an innermost; one such exists, although it may not be unique. Either it is regular or it consists of separatrices, and it is adjoined on (he inside only by regular curves, since the regions bounded by separatrices have already been filled on the corresponding leaves. It cannot be a limit cycle on its leaf from that side on which is situated the part of the disk interior to it (by Haefliger's theorem). Consequently, it becomes null-homotopic on the corresponding leaf upon being displaced toward the interior of the disk; the displacement is possible because it is not a limit cycle on that side. Thus the curve represents an element $\alpha \in \Pi_1^j(A)$, $\alpha \neq 1$.

We remark that part 3) is proved in an entirely similar fashion: a regular disk D^2 is constructed spanning a curve which represents a nonzero element of the kernel of the homomorphism $\pi_1(B) \to \pi_1(M^n)$; we need only observe that "saddle" points may now occur on the boundary of D^2, but this in no way changes matters, and the existence on the disk of the required curve not null-homotopic on the corresponding leaf is given here by the condition itself (and not by transversality on the boundary or existence of a limit cycle).

We proceed not to part 2). Consider a mapping of the sphere $S^2 \to M^n$ which is not deformable to any leaf. We may suppose the mapping is regular (which is always possible for $n \geq 3$) and is in general position with respect to the leaves. As before, there is determined on S^2, by intersection, a family of curves, with singular points of "central" and "saddle" type; furthermore, all these curves on S^2 may be taken to be either closed curves or else separatrices leading from one saddle point to another, since otherwise we should have a limit cycle on the corresponding leaf, by the Poincaré–Bendixson theorem for S^2. We may suppose, in addition, that among all these closed curves, both regular and "separatrix", all are null-homotopic on their respective leaves, since otherwise the situation reduces to the one already discussed. We shall show now that this mapping $S^2 \to M^n$ does in fact deform to a single leaf, thus arriving at a contradiction. We start the deformation at the centers, which necessarily exist on S^2. Take a center $x \in S^2$ and the family of closed curves surrounding it, which are by assumption null-homotopic on their respective leaves. At the center itself the curves degenerate into a point. Deform the image of S^2 in the manifold, starting at the center, so that the point is replaced by a disk close to it spanning a nearby curve on S^2 which surrounds the center.[8] This process can be carried onward continuously until we arrive at a closed curve consisting of separatrices. We do this for each center. The reason the process can be carried out is that if, in approaching a particular closed curve surrounding the center, the deformation fails to approach a limit, we need only recall that the curve is null-homotopic on the corresponding leaf, span it by a disk on the leaf, and displace the disk slightly "backward". On the neighboring leaf, the two disks, "fore" and "aft", meet in a common boundary and determine a mapping of the two-dimensional sphere into the leaf, which by assumption is null-homotopic, so that the one disk can be continuously deformed to the other, keeping the boundary fixed, and we may continue further with the deformation. We thus arrive at the separatrix curves surrounding the centers. Every region from a center out to the nearest separatrix is now on a single leaf.

We must now extend the deformation beyond the separatrices. But we have filled out each such central region by a single leaf. Hence the separatrix curves beyond which the deformation is to be extended are adjoined on the outside by closed regular curves, null-homotopic on their respective leaves, and we may continue the deformation, starting from these connected regions bounded by separatrices, in the same way as we started previously from the centers. By iterating the process, we

[8]Translator' note: The disk presumably lies entirely on one leaf at any moment of the deformation.

finally deform the mapping of the sphere into a single leaf and so arrive at the contradiction. We have thus proved part 2).

We shall dwell on part 4) more briefly. Consider two transversals l_1 and l_2 passing through a point $x \in B$, between which there exists no regular homotopy (bound at x) but there does exist a bound homotopy without the regularity condition. This homotopy can be thought of as a regular mapping of an annulus, $f\colon S^1 \times I \to M^n$, under which the image $f|0 \times I$ is concentrated on the leaf B near x, while the image $f(S^1 \times I)$ is in general position with respect to the leaves. We have again, now on the annulus, a family of curves with singular points of saddle and central type, with the segment $0 \times I$ going entirely into the one leaf B and containing no singular points. The curves are transverse to the boundary. This is essentially, in view of the properties of the curve $0 \times I$, a family of curves on a disk. If among the closed curves there exists one which is not null-homotopic, we arrive at the previous situation and find a nonzero element $\alpha \in \Pi_1^j(A)$; if they are all null-homotopic on their respective leaves, we redefine the homotopy so that there are no closed curves on the annulus, and the new homotopy is therefore in fact a regular homotopy between l_1 and l_2. We shall not carry out this argument in detail; it is similar to the proof of part 2), except for the requirement of regularity for the annulus, to fulfill which one need only observe how the saddle points and the central points "eat each other up" during the homotopy.

This completes the proof of the theorem. □

We can make some simple inferences from Theorem 6.1. Suppose all the groups $\Pi_1^j(M^n)$ are trivial. Then:

1°. If $n = 3$ and condition 2) fails to hold in that $\pi_2(B) \neq 0$ for some leaf B, then $M^3 = S^2 \times S^1$ or $P^2 \times S^1$ (with the trivial orientable foliation); this follows in an obvious way from the classification of surfaces. Note that if $\pi_1(M^3)$ is infinite and $\pi_2(M^3) = 0$, then the universal covering space $\hat{M}$ of M^3 is contractible, and the leaves on $\hat{M}$ are all planes, by part 3).

2°. For U-systems (see [1] or § 4) for which one of the transverse foliations is $(n-1)$-dimensional (and so always for $n = 3$), the groups $\Pi_1^j(B)$ are always trivial, since all leaves are either R^{n-1} or $S^1 \times R^{n-2}$, with the cycle $S^1 \times 0$ a limit cycle. Hence we obtain that $\pi_2(M^n) = 0$ and $\pi_1(M^n)$ is infinite. For the conclusions when $n = 3$, see the preceding paragraph (the triviality of $\pi_2(M^n)$ is new here).

3°. If all groups $\Pi_1^j(A)$ are trivial, then the fundamental group of any leaf lies in $\pi_1(M^n)$, as does the semigroup of transversals $t(A)$ for any leaf A. In this case, it is clear that the free regular homotopy classes of transversals through a leaf A are the conjugate classes of elements of $t(A) \subset \pi_1(M^n)$ by elements of $\pi_1(A) \subset \pi_1(M^n)$.

§ 7. The closed leaf theorem

The purpose of this section is to prove the following assertion.

Theorem 7.1. *Let $n = 3$, and suppose that for some leaf $A \subset M^3$ of a smooth orientable foliation on a compact manifold M^3 the group $\Pi_1^j(A)$ is nontrivial. Then A is compact and lies on the boundary of a connected component of the foliation.*

The proof is rather complicated and will be carried out in several steps. We shall suppose $j = 1$ and the leaf A lies strictly in the interior of M^3. We have

Lemma 7.1. *Suppose given, on a leaf $A \subset M^n$, a compact set $K \subset A$. Then there exists a number $\epsilon > 0$ such that no point $x \in M^n$ which lies at a positive distance $\leq \epsilon$ from any point $y \in K$ along the normal to A belongs to K. More strongly, we may take A to be a union of several leaves, $A = A_1 \cup \cdots \cup A_m$, and $K = K_1 \cup \cdots \cup K_m$, where $K_i \subset A_i$.*

The proof is simple; we leave it to the reader.

Now take a nontrivial element $\alpha \in \Pi_1^1(A)$, represent it by a regular curve $f = f_0 \colon S^1 \to A$, and erect on the curve $f(S^1) \subset A$, to the right of the leaf A, a "normal fence" $F \colon S^1 \times I \to M^3$, where $F|S^1 \times 0 = f$ and $F|\lambda \times I$ is, for any $\lambda \in S^1$, a segment of the curve beginning at the point $f(\lambda) \in A$ and traveling normally to the leaves, the segment being sufficiently small and depending on the point λ in such a way that the curve $F|S^1 \times 1$ lies entirely in a single leaf A_1; this is possible because $\alpha \in \Pi_1^1(A)$ is by definition a nonlimit cycle on the right. Furthermore, we may suppose that all the curves $F|S^1 \times t$ for $0 \leq t \leq 1$ have the same property; i.e., we may choose coordinates in the annulus $S^1 \times I$ so that this is so. Denote the curves $F(S^1 \times t)$ by f_t (where $f_0 = f$) and the corresponding leaves by A_t ($A_0 = A$), where for $t > 0$ all curves f_t are by hypothesis null-homotopic on the leaves A_t (for different t these may, in general, sometimes be the same leaf).

It follows from Lemma 7.1 that the curves f_t on the leaves A_t, $t > 0$, have no point in common with $f = f_0$ if the "normal fence" F is sufficiently short. Furthermore, the curves f_{t_1} have a single-valued projection to the left, along the normals of the foliation, onto the curves f_{t_2} for $t_2 < t_1$ (but not necessarily in the other direction); in particular, they all project onto f_0.

We have the important

Lemma 7.2. *For arbitrarily small $t_0 \geq 0$, there exists on the leaf A_{t_0} a closed curve $f_0' \colon S^1 \to A_{t_0}$ with the following properties.*

1°. f_0' is a closed piece of the curve f_{t_0}; it is regular and has only finitely many breaks at angles $\neq 0$.

2°. f_0' is not null-homotopic on the leaf A_{t_0} and belongs to the group $\Pi_1^1(A_{t_0})$.

3°. The displacements f_t' of the curve f_0' along the "fence" F to the right of A_{t_0} to the leaf A_{t_0+t} for sufficiently small $t > 0$, are closed pieces of the curves f_{t_0+t}; and each f_t', for small $t > 0$, bounds, on the leaf A_{t_0+t}, a regular mapping of a disk, $D_t \colon D^2 \to A_{t_0+t}$, which has only finitely many breaks, occurring at breaks of the tangent of the boundary $D_t|\partial D^2 = f_t'$ (regularity always means nondegeneracy of the tangent vectors). The leaf A_{t_0} will be denoted hereafter by $A_0' = A_{t_0}$.

Proof. Take a fixed $t_1 > 0$, and suppose the leaf A_0 fails to satisfy the conditions of Lemma 7.2. This implies that the regular curve (in general position) $f = f_0 \colon S^1 \to A = A_0$ is selfintersecting; that the curves f_t, for $t > 0$, are likewise selfintersecting; and that even on the universal covering space $\hat{A}_t$ of the leaf A_t, for $t > 0$, the closed covering curve $\hat{f}_t \colon S^1 \to \hat{A}_t$ (for $t > 0$, f_t is null-homotopic on $\hat{A}_t$) is selfintersecting on $\hat{A}_t$ and fails to bound a regular disk on $\hat{A}_t$. Now, the curve $\hat{f}_{t_1}$ on $\hat{A}_{t_1}$ has finitely many points of selfintersection, in general position. Proceed along $\hat{f}_{t_1}$ from the initial point to the first selfintersection, and then split up $\hat{f}_{t_1}$ into two parts, one of which is not selfintersecting (and has just one break), and the other of which has fewer selfintersections; denote them by f_1' and f_2', where $\hat{f}_{t_1} = f_1' f_2'$. The projection f_{t_1} of $\hat{f}_{t_1}$ likewise splits up into the product of the projections, $f_{t_1} = f_1 f_2$ and both f_1 and f_2 are null-homotopic on A_{t_1}. In turn, both curves

f_1 and f_2 have single-valued projections into closed pieces of every curve f_t, for $0 \le t \le t_1$. Denote these projections along normals to the left by $f_1^{(t)}$ and $f_2^{(t)}$, respectively, with $f_t = f_1^{(t)} f_2^{(t)}$ for $t \le t_1$. Take the moment of time t' closest to t_1 such that one of these pieces, either $f_1^{(t')}$ or $f_2^{(t')}$, is not null-homotopic on the corresponding leaf $A_{t'}$. Say this is $f_1^{(t')}$. Such a moment exists, since for $t = 0$ the product $f_1^{(0)} f_2^{(0)} = f$ is not null-homotopic. Observe that $f_1^{(t)}$ and $f_2^{(t)}$, for $t \le t_1$, have fewer selfintersections than f_0. Hence, replacing f_0 by $f_1^{(t')}$ and iterating the process with the same $t_1 > t'$, we arrive after finitely many steps (since the number of selfintersections strictly decreases) at the required number $0 \le t_0 < t_1$ and the curve $f'_0 \subset A_{t_0} = A'_0$ such that, for $t > 0$, its displacements f'_t to the leaves $A'_t = A_{t_0+t}$ are covered on $\hat{A}'_t = \hat{A}_{t_0+t}$ by curves $\hat{f}'_t$ which have no selfintersections (and finitely many tangential breaks at angles $\neq 0$) and which are obviously the boundaries, on $\hat{A}'_t = R^2$, of disks $\hat{D}_t \colon D^2 \to R = \hat{A}'_t$ projecting into regular disks $D_t \colon D^2 \to A'_t$ of the required type. This completes the proof of the lemma. □

In what follows, we shall work with the leaf A'_0 and prove it is compact. The compactness of A_0 will then follow from the fact that A'_0 can be chosen arbitrarily close to A_0. Thus we have on A'_0 a curve f'_0 and a "normal fence" $F' \colon S^1 \times I \to M^3$, where we may now suppose that the height of the fence F' is small (but fixed), and that the parameter t on the segments $\lambda \times I$ is so chosen, from 0 to 1, that $F'|S^1 \times t$ lies on the leaf A'_t for $0 \le t \le 1$ and bounds a regular disk $D_t \colon D^2 \to A$ without any breaks on the boundary.

We proceed now to the next stage of the proof. We try to displace the disk $D_1 \colon D^2 \to A'_1$ a little to the left of the leaf A'_1 along the normal to the foliation. Properly speaking, we consider the dynamical system on M^3 of the form $i = -N(x)$, where $-N(x)$ is the normal vector, in the backward direction, at the point x. The semitrajectory of this vector field which begins at the point $D_1(y)$, $y \in D^2$, and is parametrized by arclength, will be denoted by $\lambda_y(s)$, $s \ge 0$, where $\lambda_k(0) = D_1(y)$. For small $s \ge 0$, the point $\lambda_y(s)$ lies on a "left-displaced" disk $D_t \colon D^2 \to A'_t$, for a t which depends smoothly on the point $y \in D^2$ and the parameter $s \ge 0$ and which is close to 1; we denote this number t by $\mu(y, s)$. It is obvious that $\mu(y, 0) = 1$ and the function $\mu(y, s)$ is always positive and decreases monotonically with increasing s.

We now try to continue the motion of the disk D_1 to the left and correspondingly extend the function $\mu(y, s)$ to larger values of s. It is obvious that the function $\mu(y, s)$ is smooth wherever it is defined. Define a number $s(y)$ as follows:

1) For $0 \le s < s(y)$, the function $\mu(y, s)$ is defined.

2) For $s = s(y)$ the function $\mu(y, s)$ is not defined.

We have then the following

Lemma 7.3. a) *For every point $y \in \partial D^2$, the number $s(y)$ is finite and equal to the length of the normal segment $F'(q \times I)$, where $F'(q \times 1) = y$. For all points $y \in D^2$ sufficiently close to the boundary ∂D^2, the function $s(y)$ is finite and depends smoothly on y, and the points $\lambda_y(s(y))$ all lie on the same leaf A'_0.*

b) *For any point $y \in D^2$, we have $\lim_{s \to s(y)} \mu(y, s) = 0$, independently of whether $s(y)$ is finite or infinite.*

Proof. Of part a), it is obvious that, for $y \in \partial D^2$, the number $s(y)$ is finite and no larger than the length of the normal segment $F'(q \times I)$, since the curve f'_0 is not null-homotopic on A'_0. Of part b), it is obvious that for any point $y \in D^2$ the limit

$\lim_{s\to s(y)} \mu(y,s)$ exists and is nonnegative; denote it by $p(y) \geq 0$. We show first that $p(y) = 0$. Suppose $p(y) > 0$. A disk $D_{p(y)}$ in any case exists, since $p(y) > 0$, and can be displaced "backward," i.e., to the right with respect to the leaves, to yield a disk $\tilde{D}_{p(y)+\epsilon}\colon D^2 \to A'_{p(y)+\epsilon}$ spanning the same curve $f'_{p(y)+\epsilon} = \tilde{D}_{p(y)+\epsilon}|\partial D^2$. On the other hand, the motion "forward" with respect to s, for $s \to s(y)$, has yielded from the disk D_1 a disk $\tilde{\tilde{D}}_{p(y)+\epsilon}$ with the same boundary, for $\epsilon > 0$. Together, the regular disks $\tilde{D}_{p(y)+\epsilon}$ and $\tilde{\tilde{D}}_{p(y)+\epsilon}$ determine a mapping $S^2 \to A'_{p(y)+\epsilon}$. If now $A'_{p(y)+\epsilon} = S^2$ or P^2, it is then a compact leaf with finite group π_1, and therefore all leaves are compact and M^3 is a fiber bundle with fiber S^2 or P^2, which is impossible since $\Pi_1^j(A'_0) \neq 0$. Hence the universal covering space $\hat{A}'_{p(y)+\epsilon}$ is R^2, and the curve $\hat{f}'_{p(y)+\epsilon}$ is smoothly imbedded in R^2 and is the boundary on R^2 of only one disk $D^2 \subset R^2$; hence the regular covering mappings $\hat{\tilde{D}}_{p(y)+\epsilon}$, $\hat{\tilde{D}}_{p(y)+\epsilon}\colon D^2 \to \hat{A}'_{p(y)+\epsilon}$ are diffeomorphisms $\hat{\tilde{D}}_{p(y)+\epsilon}\colon D^2 \to D^2 \subset R^2$ and $\hat{\tilde{\tilde{D}}}_{p(y)+\epsilon}\colon D^2 \to D^2 \subset R^2$ onto the same disk $D^2 \subset R^2$; and hence these mappings differ only by a smooth change of coordinates on the disk D^2. It follows that the motion of the disk along the normal $\lambda_y(s)$ can be continued to $s(y)$, contradicting the definition of $s(y)$. Hence we have always $p(y) = \lim_{s\to s(y)} \mu(y,s) = 0$. It now follows that for all points $y \in \partial D^2$ the number $s(y)$ is equal to the length of the segment of the normal $F'(q \times I)$, where $F'(q \times 1) = y$. The smoothness and finiteness of the function $s(y)$ near the boundary are also obvious; all the points $\lambda_y(s(y))$ lie on the leaf A'_0 near the curve f'_0. This completes the proof of the lemma. □

We now prove the following lemma.

Lemma 7.4. *There exists a point $y_0 \in D^2$ such that the number $s(y_0)$ is infinite.*

Proof. Suppose the contrary: that at all points $y \in D^2$ the number $s(y)$ is finite. We already know that this is true near the boundary of the disk D^2 and that there the function $s(y)$ is smooth and the points $\lambda_y(s(y))$ all lie on the leaf A'_0. If $s(y)$ is everywhere finite, then the points $\lambda_y(s(y))$ lie on the leaf A'_0 and $s(y)$ is a smooth function. This leads to a contradiction, since the mapping $D^2 \to A'_0$ taking y into $\lambda_y(s(y))$ is smooth and is bounded by the curve $f'_0 \subset A'_0$. Suppose $s(y)$ is finite at y. We show it is finite at nearby points. The point $\lambda_y(s(y))$ lies on a leaf $B \subset M^3$, and the points $\lambda_y(s)$, for s close to $s(y)$, lie on leaves B_s close to B on the right. The points $\lambda_y(s)$ lie on disks $D_{\mu(y,s)}$ near B. An ϵ-neighborhood of the point $y \in D^2$ goes into a δ-neighborhood of the point $\lambda_y(s) \in B_s$. Around the point $\lambda_y(s)$ close to $\lambda_y(s(y)) \in B$, the normal projection of the leaf B_s onto B_t is a diffeomorphism, with derivatives bounded above and below for $s \leq t \leq s(y)$, because of the compactness of M^3 and the smoothness of the foliation. A point $z \in D^2$ close to y projects from B_s into a point of the leaf B_t; and if y is interior to the image $D_{\mu(y,s)}$, z will be interior to $D_{\mu(y,s)}$ after this projection, for $t < s(y)$. Hence, as $z \to y$, we have $\lim s(z) \leq s(y)$.

The converse is proved similarly. We have now proved the continuity of the function $s(y)$, as well as the fact that the points $\lambda_y(s(y))$ lie on a single leaf. This completes the proof of the lemma. □

Now make a fixed selection of a point $y_0 \in D^2$ for which $s(y_0) = \infty$. Denote by R^+ the half-line $s \geq 0$. The family of disks D_t, for $0 < t \leq 1$, determines a mapping $G\colon D^2 \times R^+ \to M^3$, where $G(y,s) = D_{\mu(y_0,s)}(y)$, and the mapping G is regular. Essential to our argument is the following

Lemma 7.5. *The image of the regular mapping $G(D^2 \times R^+)$ does not contain any point of the curve $f_0' \subset A_0'$.*

Proof. The height of the "right normal fence" $F'\colon S^1 \times I \to M^3$ has already been chosen so that the curves $f_t' = F'|S^1 \times t$, for $t > 0$, have no point in common with f_0', in accordance with Lemma 7.1. Suppose there exists a point $y \in D^2$ such that $D_{\mu(y_0,s)}(y)$ is contained in the image f_0'. Then y is an interior point of the disk D^2. Moreover, since the curves f_0' and $f'_{\mu(y_0,s)}$ fail to intersect in M^3, it follows that on the universal covering space $\hat{A}'_{\mu(y_0,s)}$ of the leaf $A'_{\mu(y_0,s)}$, the nonclosed curve $\hat{f}_0''$, which covers f_0' has no point in common with the boundary of the (imbedded) covering disk

$$\hat{D}_{\mu(y_0,s)}\colon D^2 \xrightarrow{\subset} \hat{A}_{\mu(y_0,s)} = R^2,$$

and may be taken to be interior to this disk $D^2 \subset R^2$. Now, the curve $f_0' \subset A_0' = A'_{\mu(y_0,s)}$ represents a nonzero element a of the group $\pi_1(A_0')$, the latter being a discrete group of transformations of the plane $R^2 = \hat{A}_0' = \hat{A}'_{\mu(y_0,s)}$. Denote the initial point of the curve $\hat{f}_0' \subset D^2 \subset R^2$ by a and the terminal point by $\alpha(a)$. The sequence $a, \alpha(a), \dots, \alpha^k(a), \dots$ must all lie interior to the disk $D^2 \subset R^2$, which is bounded by the curve

$$\hat{f}'_{\mu(y_0,s)} \subset \hat{A}'_{\mu(y_0,s)} = \hat{A}_0' = R^2,$$

since otherwise one of the curves $\alpha^k(\hat{f}_0')$ would intersect $\partial D^2 = \hat{f}'_{\mu(y_0,s)}$. But because $\pi_1(A_0')$ acts discretely on $R^2 = \hat{A}_0'$, there exist integers $k_1 \neq k_2$ such that $\alpha^{k_1}(a) = \alpha^{k_2}(a)$, or $\alpha^{k_1-k_2}(a) = a$; hence $\alpha^{k_1-k_2} = 1$. But the fundamental group of an open surface has no torsion. This contradiction proves the lemma. □

For our subsequent purposes we need also the somewhat stronger

Lemma 7.6. *There exists a number $\epsilon > 0$ such that the image $G(D^2 \times R^+)$ does not intersect the ϵ-neighborhood of the curve f_0' on the leaf A_0'.*

Proof. By Lemma 7.5, the image $G(D^2 \times R^+)$ at least does not intersect the curve f_0'. If in the image $G(D^2 \times R^+)$ there occur points $x_1, \dots, x_n, \dots$ arbitrarily close to the curve f_0' and lying on the leaf A_0', then for sufficiently large n the point x_n will be strictly interior to some disk $D_{\mu(y_0,s)}$, since the boundary of the disk, namely the curve $f'_{\mu(y_0,s)}$ lies by Lemma 7.1 at a finite distance, bounded below by a number independent of s, from f_0'. Furthermore, if x_n is sufficiently close to f_0', then f_0' itself intersects the interior of the disk $D_{\mu(y_0,s)}$ contradicting Lemma 7.5. □

We now denote by $V_\epsilon \subset A_0'$ a fixed such ϵ-neighborhood of the curve f_0' on the leaf A_0'.

Consider an ω-limit point $z \in M^3$ of the normal $\lambda_{y_0}(s)$ of the foliation; this means that there exists a sequence of numbers $s_1, \dots, s_n, \dots \to \infty$ such that $z_n = \lambda_{y_0}(s_n) \to z$ as $n \to \infty$. We may suppose that all the points z_n lie on the same leaf as z. Denote this leaf by B. All the disks $D_{\mu(y_0,s_n)}$ lie on B. We have then

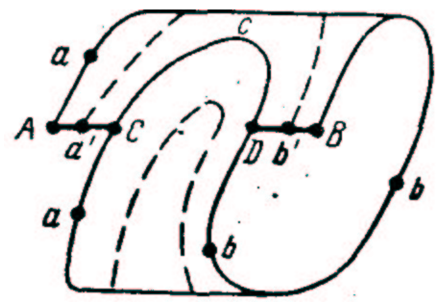

FIGURE 10

Lemma 7.7. *For all sufficiently large n and small $\epsilon > 0$, the point $z \in B$ lies, together with its ϵ-neighborhood, in the interior of every disk $D_{\mu(y_0,s_n)}$.*

Proof. If every ϵ-neighborhood of the point $z \in B$ contained points belonging to the boundaries of the disks $D_{\mu(y_0,s_n)}$ arbitrarily large n, then close to the point z there would be points of the curves $f'_{\mu(y_0,s_n)}$. In this case, after passing through a point z_n close to z, we should find ourselves in a short time interval within an ϵ-neighborhood of the curve $f'_0 \subset A'_0$, while the point z_n lies strictly interior to the disk $D_{\mu(y_0,s_n)}$ by definition of y_0, thus contradicting Lemma 7.6, since $s(y_0) = \infty$ and any ϵ-neighborhood of the curve $f'_0 \subset A'_0$ would contain points of the disks D_t, $0 < t \leq 1$. This proves the lemma. □

We shall suppose now that the values of n are sufficiently large to satisfy the conditions of Lemma 7.7. As an easy consequence of the preceding we have then

Lemma 7.8. *For any n, there exists an m, depending on n, so large that the disk $D_{\mu(y_0,s_n)}$ is entirely contained in the interior of $D_{\mu(y_0,s_m)}$.*

Proof. An ϵ-neighborhood of the point z on the leaf B belongs to every disk $D_{\mu(y_0,s_k)}$, for large k. Furthermore, for given n, there exists a k such that the boundaries $\partial D_{\mu(y_0,s_n)} = f'_{\mu(y_0,s_n)}$ and $D_{\mu(y_0,s_m)} = f'_{\mu(y_0,s_m)}$ do not intersect for $m > k$, by Lemma 7.1. If all the disks $D_{\mu(y_0,s_m)}$, for $m > k$, were interior to $D_{\mu(y_0,s_n)}$, then their boundaries would sooner or later be very close to each other in the metric of the leaf, again contradicting Lemma 7.1. This proves the lemma. □

The pair of integers n, m of Lemma 7.8 determine a regular mapping

$$G_{m,n} = G/D^2 \times I(s_n, s_m),$$

where on the upper base $G_{m,n}|D^2 \times s_m$ coincides with $D_{\mu(y_0,s_m)}$, and on the lower base $G_{m,n}|D^2 \times s_n$ coincides with $D_{\mu(y_0,s_n)}$ and the mapping of the upper base induces a mapping of the lower onto a certain smaller disk $D^2 \subset D^2 \times s_m$ strictly interior to the latter. The lateral sides go into the family of curves f'_t, for $\mu(y_0, s_m) \leq t \leq (y_0, s_n)$, which lies on the "normal fence" F' near the leaf A'_0 and is the closer to A'_0, the larger the integers n and m. Thus, if we first identify the lower base $D^2 \times s_n$ with the corresponding part of the upper (since the mapping $G_{m,n}|D^2 \times s_n$ can be factored as the composition of the inclusion $h_{m,n} \colon D^2 \times s_n \subset D^2 \times s_m$ and the mapping $G_{m,n}|D^2 \times s_m$), we obtain an induced mapping $\bar{G}_{m,n} \colon D^2 \times S^1 \to M^3$. [Cf. Figure 10, which is a representation of a "two-dimensional model," with $\overrightarrow{AaCcDbB}$

representing $D^2 \times s_m$ and $\overrightarrow{CcD}$ representing $D^2 \times s_n$. The intervals $\overrightarrow{Aa'C}$ and $\overrightarrow{Db'B}$ represent the family of curves f'_t for $\mu(y_0, s_m) \leq t \leq \mu(y_0, s_n)$.] The mapping $\bar{G}_{m,n}$ is everywhere smooth except for the two "break curves" $f'_{\mu(y_0,s_n)}$ and $f'_{\mu(y_0,s_m)}$; the annulus between these two goes into the family of curves f'_t for the values of t indicated. The rest of the boundary goes into the single leaf B containing the disk $D_{\mu(y_0,s_m)}$.

Finally, we have

Lemma 7.9. *No closed transversal to the foliation passes through the leaf A'_0.*

Proof. Let l be a closed transversal passing through A'_0. We shall suppose that it passes through a point on the curve $f'_0 \subset A'_0$ and is inclined to the leaves at an angle $> \alpha > 0$.

Pick n and m so large that the transversal l, after passing through a point of the curve f'_0, penetrates rather quickly within the image $\bar{G}_{m,n}(D^2 \times S^1)$; to effect this, the interval $[\mu(y_0, s_m), \mu(y_0, s_n)]$ and number $\mu(y_0, s_n)$ itself must be small relative to the angle α of minimum inclination of the transversal l to the leaves. But having entered within the image $\bar{G}_{m,n}(D^2 \times S^1)$, the transversal must at least once exit from it, in order to close up at the point of the curve $f'_0 \subset A'_0$, since $\bar{G}_{m,n}(D^2 \times S^1) \subset G(D^2 \times R^+)$ and the image of G, by Lemma 7.5, contains no points of the curve f'_0. But in view of the regularity of the mapping G, and therefore of $\bar{G}_{m,n}$, the boundary of the image of $\bar{G}_{m,n}$ lies entirely in the image $\bar{G}_{m,n}(\partial D^2 \times S^1)$ of the boundary, which consists only of a piece of the leaf B and the family of curves f'_t for $\mu(y_0, s_m) \leq t \leq \mu(y_0, s_n)$. Hence, at the boundary of the image of $\bar{G}_{m,n}$, the normal to the foliation points toward the interior of the image at all points except for the small family of curves f'_t; and obviously the transversal cannot exit through the latter either, in view of its narrow width relative to the angle of inclination α. We are thus led to a contradiction, since the transversal l cannot exist from the image of $\bar{G}_{m,n}$ to close up at a point of the curve f'_0. This proves the lemma. □

We may now bring the proof of Theorem 7.1 to a conclusion. By Lemma 1.2, the leaf A'_0 is compact, and by part b) of Theorem 1.1, it lies on the boundary of a connected component. But by Lemma 7.2, A'_0 can be chosen arbitrarily close to the leaf A_0. Hence the leaf A_0 is itself compact: through it, obviously, there passes no closed transversal. If now $A'_0 \neq A_0$, then A'_0 can be chosen arbitrarily close to A_0, and the region between A_0 and A'_0 is diffeomorphic to $A_0 \times I$; so that A_0 and A'_0, for A'_0 close to A_0, are diffeomorphic. But this contradicts the existence on A_0 of an element $\alpha \neq 1$ in the group $\Pi^1_1(A_0)$.

Thus, $A'_0 = A_0$, and the leaf A_0 itself lies on the boundary of a connected component of the foliation.

This completes the proof of the theorem.

Remark 7.1. If the group $H_2(M^3)$ is trivial, for example if $\pi_1(M^3)$ is finite or if $M^3 = S^3$, then the closed leaf so obtained is a torus, since its Euler characteristic is zero.

Remark 7.2. For $n > 3$, Theorem 7.1 remains valid if the group Π^j_1 is replaced by $\Pi^j_{n-2}(A)$ and the additional requirement is made that the nonzero element $\alpha \in \Pi^j_{n-2}(A)$ be realizable by an imbedded sphere $S^{n-2} \subset A$. The proof is entirely similar.

Comparing the result of Theorem 7.1 with § 6, we obtain the following conclusion:

If a manifold M^3 has an orientable foliation with only one connected component containing more than a single leaf (e. g., if there are no closed leaves at all), then either the universal covering $\hat{M}$ of M^3 is contractible and the leaves on $\hat{M}$ are planes R^2, or else $\hat{M} = S^2 \times R$ and the leaves on $\hat{M}$ are spheres S^2.

§ 8. The closed leaf theorem (conclusion)

We shall clarify here the structure of the closed leaf of § 7, and of the part of the foliation bounded by it on the right (for $j = 1$) or on the left (for $j = 2$). We shall take $j = 1$. Under the same conditions as in Theorem 7.1, we have

Theorem 8.1. *The closed leaf A_0 obtained in Theorem 7.1 is the only boundary leaf of the connected component of the foliation on M^3 which lies to the right of A_0, and the limit set of any leaf A_t, for small $t > 0$, is precisely A_0.*

Proof. As shown in § 7, the leaf A'_0 coincides with the leaf $A = A_0$ and is the limit leaf of a leaf $A_{\mu(y_0,s)}$, where the latter is the union of the images of the disks $D_{\mu(y_0,s_k)}$ for some increasing sequence of integers k.

Lemma 8.1. *Every leaf A_t, for sufficiently small $t > 0$, is the union of an increasing sequence of disks D_{t_k}, where $t_k \to 0$ as $k \to \infty$, and the leaf A_0 is a limit leaf for A_t.*

Proof. The mapping $G_{m,n}|D^2 \times I(s_n, s_m)$ was constructed in the preceding section, under which the image of this upper base entirely encloses the lower, and which is induced by the mapping $G\colon D^2 \times R^+ \to M^3$. If the image of $G|D^2 \times s_m$ entirely encloses $G|D^2 \times s_n$, then the disk $G|D^2 \times (s_n + s)$, for small s, is entirely swallowed up by the disk $G|D^2 \times (s_m + s)$. Let s vary from s_n to s_m. Then the disk $G|D^2 \times (s_m + s)$ will always enclose the disk $G|D^2 \times (s_n + s)$ as long as their boundaries fail to intersect. But their boundaries are part of the family of curves f'_t; and if the interval (s_n, s_m) is big enough, we may suppose the boundaries never intersect for all $s_n \le s \le s_m$, by Lemma 7.1. Having reached s_m, we may keep going. We thus obtain that for all $s \ge s_n$ the disk $D_{\mu(y_0,s)}$ is contained in a larger disk on the same leaf; and so we may construct a sequence of disks exhausting the leaf A_t, for any small $t > 0$, with the boundaries of the disks converging to the curve $f'_0 \subset A_0 = A'_0$. This proves the lemma. □

Lemma 8.2. *The leaf A_0 is the only limit leaf for every leaf A'_t, for small $t > 0$.*

Proof. A_0 is compact. Consequently, choosing an initial point on the curve $f'_0 \subset A_0$, we may cover it by a compact family of smooth closed curves starting at the point $x_0 \in f'_0$, which have the length $|l(x)| < l$ bounded above, where $l(x)$ means the curve of the family that passes through the point x on A_0. On the leaf A_t, for small t, consider the disk D_t with boundary f'_t near f'_0. Take a curve $l(x) \subset A_0$ and, starting from the point $x_0 \in f'_0$, "cover" the curve $l(x)$ on the leaf A_t close to it by a curve $l(t, x)$; this is possible for all small t because $|l(x)| < l$. After traveling along $l(x)$ from x_0 to x_0, we arrive on the leaf A_t either at the same point or at another. If at the same point, then along $l(x)$ the leaf A_t does not recede from A_0. If at another point, and the latter is closer to A_0 than the initial one, then we may regard the terminal point of the curve $l(t, x)$ as lying on a curve f'_{t_1} on the same leaf $A_t = A_{t_1}$, where f'_{t_1} has no points in common with f'_t. We can then continue to cover $l(x)$ from the terminal point of the curve $l(t,x)$, and the curve so obtained will approach the leaf A_0, being at all times at a distance from A_0 uniformly small relative to t. If, on the other hand, the terminal point of the curve $l(t,x)$ is farther from A_0 than the initial point, we must proceed in the opposite direction and go through the same covering process, unrestrictedly many times. The whole of the leaf A_t is obviously covered by the curves of this sort with the exception of the disk D_t, and the whole exterior of D_t on A_t is thus uniformly close, with respect to t, to the leaf A_0. Since D_t is compact, the desired result follows. This completes the proof of the lemma. □

We may proceed now to prove the theorem. If the boundary of the component in question had another piece—say, a leaf C—then the leaves of the component which lie near C would have C in their limit set. It would follow that there exists a leaf somewhere within the component, which has a limit point in common with the leaves A_t and contains in its limit set something else besides A_0, contradicting Lemma 8.2. This proves the theorem. □

A consequence of Theorem 8.1 is

Corollary 8.1. *The leaf A_0 is a torus T^2.*

Proof. Since A_0 is the entire boundary of the connected component, A_0 is homologous to zero in M^3, bounding a region in M^3 on which there exists the vector field of normals to the foliation. Hence the Euler characteristic $\chi(A_0)$ is zero. Since $n-1=2$, we obtain that $A_0 = T^2$. [9] □

Indeed, we may now prove the following assertion.

Theorem 8.2. *The connected component of the foliation which lies to the right of the leaf $A_0 = T^2$ is diffeomorphic to $D^2 \times S^1$, and the foliation on $D^2 \times S^1$ is homeomorphic to the Reeb foliation.*

Proof. The curve f'_0 on the leaf $A_0 = T^2$ represents a nonzero element α of the group $\pi_1(T^2) = Z + Z$. It $\alpha = \beta^m$, then the element $\beta \in \pi_1(T^2)$ is likewise a nonlimit cycle, since the group $P_1(A_0)$ has no torsion; displace the curve $g\colon S^1 \to A_0$, which represents β, to the right of A_0. Then for small $t > 0$, the curve g_t (the "displacement" of g) lies on the leaf A_t, and its multiple g_t^m is null-homotopic on A_t, since g_t^m is homotopic to f'_t. But the groups $\pi_1(A_t)$ have no torsion for $n-1=2$, and hence the curve g_t itself is null-homo topic on A. Thus we may suppose that a is a generator of the group $\pi_1(T^2)$. The element $\alpha \in \Pi_1^1(A_0) \subset \pi_1(T^2)$ can then be realized by an imbedded curve $S^1 \subset A_0$. It follows that the displacements f'_t may likewise be taken as imbedded without selfintersections on the leaves A_t. Hence all the disks $D_t\colon D^2 \to A_t$ may also be taken as smoothly imbedded for all t, and the curves f'_t as mutually nonintersecting for all t. We now retrace the whole argument of § 7 in the light of these remarks. Thus, we may take every leaf A_t (for $t > 0$) to be the union of imbedded disks $D_{t_1} \subset \cdots \subset D_{t_k} \subset \dots$, where each is interior to the one following, and the boundaries are the circles f'_{t_k}. Hence every leaf A_t is an R^2. Moreover, the mapping $\bar{G}_{m,n}$ is as imbedding $D^2 \times S^1 \subset M^3$, with a "break" along the curves $f'_{\mu(y_0,s_n)}$ and $f'_{\mu(y_0,s_m)}$, and the boundary of the image is close to the leaf A'_0 (cf. Figure 10). Hence the underlying space of the connected component

[9]Orientability of M^3 is assumed here.

FIGURE 11

is $D^2 \times S^1$, and the foliation is constructed in exactly the same way as the Reeb foliation (§ 4). It may differ from the Reeb foliation only in the rate of convergence of the leaves A_t to the leaf A_0, and may therefore be nondiffeomorphic to the Reeb foliation, but is always homeomorphic to it (a diffeomorphism with discontinuity in the derivatives along the leaf A_0).

This proves the theorem. □

§ 9. SOME COROLLARIES. ANALYTIC FOLIATIONS

We have thus proved the following assertion.

Theorem 9.1. *Let $n = 3$ and let the universal covering manifold $\hat{M}$ be noncontractible. Then either any orientable foliation on M^3 has a Reeb component $D^1 \times S^1 \subset M^3$, with a torus $T^2 \subset M^3$ as compact boundary leaf, or else $\hat{M} = S^2 \times R$ and the covering leaves in $\hat{M}$ are spheres S^2.*

The proof follows from the results of §§ 6, 7, 8. We pass now to some corollaries.

1. Consider the special case of a three-dimensional sphere S^3 . Then a set of solid toruses $(D^2 \times S^1)_i$ with Reeb foliations is contained in any smooth foliation on S^3. We shall say that the set is unknotted if one of the circles $(0 \times S^1)_i$ of the set is itself unknotted and can be spanned by a disk $D^2 \subset S^3$, $\partial D^2 = (0 \times S^1)_i$, not intersecting the others.

Theorem 9.2. *The entire set of Reeb components of any smooth foliation is always knotted in S^3.*

Proof. If the set is unknotted in the sense described, we can select from the set an unknotted component $(D^2 \times S^1)_i$ which is unlinked with the others, and on its boundary leaf, a torus T^2, consider one of the generating cycles of the group $\pi_1(T^2)$, realized by an imbedded circle $S^1 \subset \partial(D^2 \times S^1)_i$ homotopic to zero in the complement $S^3 \setminus (0 \times S^1)_i$. Indeed, it is homotopic to zero in the sphere from which have been removed all the remaining Reeb components, since they may all together be enclosed in a disk $D^3 \subset S^3$ not intersecting the component $(D^2 \times S^1)_i$. By part 3) of Theorem 6.1 and the theorems of §§ 7–8, there then exists a Reeb component of the foliation other than those removed. We are thus led to a contradiction, and this proves the theorem. Examples of this knottedness have already been given in § 4. The simplest axial systems for Reeb components are indicated in Figure 11. □

An obvious consequence of the theorem is

Corollary 9.1. *If a foliation on S^3 has only one compact leaf $T^2 \subset S^3$, bounding an unknotted Reeb component, then the foliation is homeomorphic to the Reeb foliation on S^3.*

Corollary 9.2. *A smooth foliation on a solid torus which has ho Reeb components strictly interior to it is itself homeomorphic to the Reeb foliation.*

Proof. A meridian of the torus $T^2 = \partial(D^2 \times S^1)$ is homotopic to zero in $D^2 \times S^1$; hence by part 3) of Theorem 6.1 and §§ 7, 8, there must exist a Reeb component in the interior, or else the foliation itself is a Reeb foliation, as was to be proved. □

2. Consider, by itself, a Reeb foliation on $D^2 \times S^1$ and any vector field transverse to it.

Lemma 9.1. *A vector field transverse to a Reeb foliation on $D^2 \times S^1$ has a periodic trajectory, which is homotopic, and indeed isotopic, to the circle $0 \times S^1$ traversed once.*

Proof. We shall suppose that on the boundary $T^2 = \partial(D^2 \times S^1)$ the field is directed toward the interior of $D^2 \times S^1$ and that it is inclined to the leaves at an angle $> \alpha > 0$, where α is fixed. On any leaf $A = R^2 \subset D^2 \times S^1$ of the foliation, consider the imbedded disks D_{t_k} constructed as in the proof of Theorem 8.2, whose boundaries lie near the curve $f'_0 \subset T^2$, which we may take to be a meridian of the torus (homotopic to zero in $D^2 \times S^1$). Choose t_k so small that a trajectory of the field which begins near the boundary will have withdrawn after a fixed time δ to a distance $> t_k$ from the boundary, where t_k is the maximum width of the "normal fence" from the curve f'_0 to the curve f'_{t_k}. Consider the trajectories of the field beginning at points of the disk D_{t_k}. They proceed transversely to the leaves and do not approach to within a distance t_k of the boundary. Hence they return, after one circuit, to the disk D_{t_k}, and we obtain a mapping $D_{t_k} \to D_{t_k}$ taking the disk strictly interior to itself. The mapping has a fixed point, and this yields the required periodic trajectory. Because of the transversality of the motion, the trajectory is isotopic to $0 \times S^1$. □

From Theorems 9.1, 9.2 and Corollary 9.2, we now obtain

Theorem 9.3. *A vector field transverse to any smooth foliation on the three-dimensional sphere or the solid torus always has a periodic trajectory. On the sphere S^3 it has, in fact, a nontrivial knotted family of periodic trajectories* (*see Figures* 11a *and* 11b).

3. We shall now consider analytic foliations. Following [4], we use of the analyticity only the nonexistence of one-sided limit cycles on the leaves. It is known that every foliation on a manifold M^n, $n \geq 3$, with finite group $\pi_1(M^n)$ has a one-sided limit cycle. We shall apply the results of § 8 to this question for $n = 3$.

Lemma 9.2. *If a smooth foliation with finitely many closed leaves on a manifold M^3, of which the universal covering space is noncontractible, has no one-sided limit cycles, then the foliation is homeomorphic to one obtained from another smooth foliation by "turbulization"* (§ 4).

Proof. By the results of § 8, the foliation on M^3 has a Reeb component $D^2 \times S^1 \subset M^3$; on its boundary $T^2 \subset M^3$, let $\alpha \in \Pi_1^1(T^2)$ be a basis cycle which is not a limit cycle on the right. Then by assumption it is also not a limit cycle on the left. The second basis element $\beta \in \pi_1(T^2)$ is a right-hand, and therefore also a left-hand, limit cycle. Denote the left-displacements of the meridian $S^1 \subset T^2$, which represents α, by S^1_t, $t > 0$. If we make a circuit along a parallel representing β, the curves S^1_t go into curves $S^1_{\lambda(t)}$, where $\lambda(t) \neq t$, for small $t > 0$, since there are only finitely many closed leaves.

Suppose $\lambda(t) < t$. Construct a small "left normal fence" along the whole leaf T^2 such that its upper base $T^2 \times 1 \subset T^2 \times I$ is transverse to the leaves. This can be done because of the monotonicity of the function $\lambda(t)$. The intersections of $T^2 \times 1$ with the leaves are closed curves, namely meridians of the torus $T^2 \times 1$, and we need only fill them in each by a disk D^2, i.e., paste back $D^2 \times S^1$ with an altered foliation. Smoothing it out if necessary, we obtain a foliation from which which the original is recoverable by "turbulization" (up to homeomorphism, but not diffeomorphism). This proves the lemma. □

The lemma easily implies

Theorem 9.4. *If a three-dimensional manifold M^3 has an analytic foliation, then either the universal covering space is contractible and the covering leaves are planes R^2, or else $\hat{M} = S^2 \times R$.*

For the proof, it suffices to apply Lemma 9.2 repeatedly so as to eliminate each Reeb component in turn.

4. If a manifold M^n is provided with k commuting vector fields, then together they determine an operation of the group R^k on M^n, and if the fields are independent at every point, the corresponding foliation has no singularities. If $k = n$, it is evident that $M^n = T^n$ (the torus).

The situation is more complicated when $k = n - 1$. We have then an $(n-1)$-dimensional foliation on M^n, of which a leaf may be $R^{n-1}, S^1 \times R^{n-2}, \dots, T^i \times R^{n-i-1}, \dots, T^{n-1}$. By the *rank* of a compact manifold M^n is meant the maximum number of linearly independent commuting vector fields. Denote the rank by $\mathrm{rk}(M^n)$. Suppose $n \geq 3$. Then we have

Lemma 9.3. *If the group R^{n-1} operates smoothly and without singularities on M^n, determining a foliation, then the groups $\Pi^j_1(A)$ are trivial for all leaves $A \subset M^n$.*

For $n = 3$ this lemma was communicated to the author by V. I. Arnol'd.

Proof. Suppose that, on the contrary, there exists a nonzero element $\alpha \subset \Pi^j_1(A)$ for a leaf $A \subset M^n$, which is realized by an imbedded circle $S^1 \subset A$ (for $n = 3$, we may suppose that $A = T^2$ and α is a generator of $\pi_1(T^2)$). The leaf A is a quotient by part of a lattice, and the group $\pi_1(A)$ is free abelian. We may suppose that α is a generator. The element α operates on R^{n-1}; let ξ_1 be the vector on R^{n-1} joining 0 to $\alpha(0)$, and let l be the line (one-parameter subgroup) determined by it. This one-parameter subgroup operates on R^{n-1} by parallel translations, and thus operates on M^n. On the leaf A, the subgroup l determines the cycle $\alpha \in \Pi^j_1(A)$ and operates periodically. Suppose $j = 1$. Then the cycle $\alpha \in \Pi^1_1(A)$ displaces to the right of A, remaining on a nearby leaf A_t $(t > 0)$ but becoming null-homotopic, and the subgroup $l \subset R^{n-1}$ operates on A_t, and after time $= 1$ returns close to the initial point, the more closely for smaller t.

Case $n = 3$**.** Consider the curve l_t on the leaf A_t for $t > 0$, which is close to a segment of an orbit of the subgroup $l \subset R^{n-1}$ and is obtained from the periodic trajectory of l on A by a small displacement to the right. The curve l_t on A_t may be taken to be smoothly imbedded, since $A = T^2$ by the results of §§ 6–7, and bounds a disk on A_t by assumption, but the rotation index of the field l (regarded only on $A_t = R^2$) along l_t is equal to 1, since the field l differs only slightly from the tangent field to l_t. This is impossible, since the field l extends to the whole interior of the disk D_t^2 spanning l_t; so we have arrived at a contradiction. This proves the lemma for $n = 3$.

Case $n \geq 3$**.** Consider the whole family of one-parameter subgroups on R^{n-1} (of parallel translations); one of them is l, and all the others also operate on M^n, the operation depending smoothly on the variation of the subgroup. The subgroup l itself operates periodically on A, but nonperiodically on A_t for $t > 0$, since under right-displacement the cycle $\alpha \in \Pi_1^1(A)$ becomes null-homotopic on A_t for $t > 0$. Displace the initial point x_0 of the periodic trajectory $l|A = A_0$ representing α to a nearby leaf, obtaining a point $x_t \in A_t$; and begin varying the subgroup l, starting the trajectory of a nearby one-parameter subgroup at the same point $x_t \in A_t$. If t is sufficiently small, then close to l there exists a one-parameter subgroup of R^{n-1} which, beginning at x_t, returns to x_t in a time close to 1 (depending on t), since this is true for $t = 0$, and since the trajectories of nearby one-parameter subgroups on A_t return, after making a circuit in a time close to 1, near to the starting point x_t in the metric of the leaf A_t, the curve $\alpha \in \Pi_1^1(A)$ is not a limit cycle to the right. Hence one of the one-parameter subgroups on A_t, close to l during a unit time interval, determines a cycle on A_t which is homotopic to the displacement of $\alpha \in \Pi_1^1(A)$ to the leaf A_t, but is not null-homotopic on A_t, since A_t is a quotient by part of a lattice.

Hence the element $\alpha \in \pi_1(A)$ cannot belong to the group $\Pi_1^1(A)$; and we have arrived at a contradiction. This proves the lemma. □

Comparing the lemma with the results of § 6, we obtain the following theorem.

Theorem 9.5. a) *If M^3 is an orientable closed manifold such that its universal covering space is noncontractible, then* $\operatorname{rk} M^3 = 1$*. In particular,*

$$\operatorname{rk} S^3 = 1, \quad \operatorname{rk} S^2 \times S^1 = 1, \quad \operatorname{rk}(M_1^3 \# M_2^3) = 1,$$

if $M_i^3 \not\sim S^3$ $(i = 1, 2)$.

b) *If $\pi_1(M^n)$ is finite or if $\pi_2(M^n) \neq 0$, then* $\operatorname{rk} M^n \leq n - 2$ *for $n \geq 3$. In particular,*

$$\operatorname{rk} S^2 \times T^{n-2} = n - 2, \quad \operatorname{rk} M^3 \times T^{n-3} = n - 2,$$

if $\hat{M}^3$ is noncontractible.

Added in proof. Theorem 9.5 for the cases S^3 and $S^2 \times S^1$ has also been obtained by Lima and Rosenberg; and the following result follows easily from Theorem 5.1 of this paper and a theorem of Sacksteder contained in a manuscript sent to the author: *if R^{n-1} operates smoothly (of class ≥ 2) and nonsingularity on M^n, and M^n does not have the homotopy type of a fiber bundle with the torus T^k as base and T^{n-k} as fiber, then the operation of R^{n-1} has a compact leaf T^{n-1}.*

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