THE CARTAN-SERRE THEOREM AND INTRINSIC HOMOLOGY

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Contents

§ 1. The Cartan–Serre theorem

1. **Homological properties of Eilenberg–MacLane complexes.**

Theorem 1.1. *If the abelian group π is trivial, finite, or finitely generated, then the homology groups $H^k(K(\pi,n))$ for all k and n are trivial, finite, or finitely generated, respectively.*

The proof is by induction on n. For $n=1$, the theorem follows from the examples and Theorem 1 in D.B. Fuks' lecture. We assume that the assertion is true for n and prove it for $n+1$.

Consider the space of paths on $K(\pi,n+1)$ beginning at a fixed point x_0. Putting each path into correspondence with its endpoint, we obtain the Serre fibering

$$E \xrightarrow[p]{\Omega} K(\pi,n+1), \tag{1}$$

in which E is homotopically trivial and the fibre Ω is homotopy-equivalent to $K(\pi,n)$.

For fiberings we have:

Lemma 1.1. *If $E \xrightarrow[p]{F} B$ is a fibering with simply-connected base and the cohomology group of two of the three spaces E, B, F are trivial, finite, or finitely generated, then the cohomology groups of the third space have the corresponding property.*

This is easily obtained from the properties of the spectral sequence of a fibering.

Applying the lemma to the fibering (1), we obtain the result of the theorem for $n+1$. This proves Theorem 1.1. Similarly we can prove:

Theorem 1.2. *The ring $H^*(K(Z,p),Q)$ is a polynomial algebra over Q for even p, and an exterior algebra over Q for odd p, with one generator of dimension p.*

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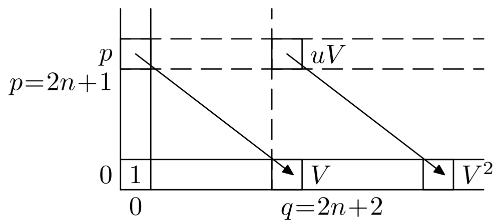
The proof is analogous to that of Theorem 1.1, but we have to use the *following diagram*, which is obtained from the Leibniz formula for the differentials of the spectral sequence of the fibering (1):



$$u \in H^{2n+1}(Z, 2n+1; Q),$$
$$v^i \in H^{(2n+2)i}(Z, 2n+2; Q),$$
$$d_{2n+2}u = v,$$
$$d_{2n+2}uv^i = v^{i+1},$$
$$d_i = 0, \quad i \neq 2n+2,$$
$$E_2^{p,q} = $$
$$= H^p(Z, 2n+1; Q) \otimes H^q(Z, 2n+2).$$

The passage from $K(Z, 2n+2)$ to $K(Z, 2n+3)$ is similar.

2. **Application of Eilenberg–MacLane complexes to homotopy problems.**

Theorem 1.3. *The homotopy groups of a simply-connected space X are finitely generated if its homology groups are finitely generated*; *they are finite if the homology groups are finite, and trivial if the homology groups are trivial.*

Proof. Let $\pi_i(X) = 0$ $(i < k,\ k \geq 2)$, $\pi_k(X) = \pi$ $(i > k)$. By the Hurewicz theorem, $\pi_k(X) = H_k(X)$. Consider $K = K(\pi, k)$ and construct a mapping $f\colon X \to K$ inducing the isomorphism $f^*\colon \pi_k(X) \to \pi_k(K) = \pi$.

We convert f into a fibering in Serre's sense (see the Appendix to this section) $f\colon X \xrightarrow{F} K$. From the exact sequence of the fibering it follows that $\pi_i(F) = 0$ for $i \leq k$ and $\pi_i(F) = \pi_i(X)$ for $i \geq k+1$, and this isomorphism is established by f_*:

$$\cdots \to \pi_i(X) \xrightarrow{f_*} \pi_i(K) \xrightarrow{\partial} \pi_{i-1}(F) \to \pi_{i-1}(X) \to \cdots$$

From Theorem 1.1 we find that X and $K = K(\pi, k)$ have together the properties of the condition of Theorem 1.3. Hence the same is true for the fibre F (Lemma 1.1). But by the Hurewicz theorem $\pi_{k+1}(F) = H_{k+1}(F)$. Also $\pi_{k+i}(F) = \pi_{k+i}(X)$ for all $i \geq 1$. Hence Theorem 1.3 is true for $i = 1$, and thus by induction on i and passing from X to the fibre we can complete the proof of the theorem. □

Corollary 1.1. *The homotopy groups of a finite simply-connected complex are finitely generated, and are finite if the homology groups are finite.*

(This is not true for complexes that are not simply-connected: the reader is recommended to examine the example of $\pi_2(S^2 \vee^1)$, where $\vee$ denotes the one-point union or "bouquet".)

Theorem 1.4 (Cartan, Serre)**.** *Let X be a simply-connected space with finitely generated homology groups. Let the algebra $H^*(X, Q)$ be the tensor product of an exterior algebra on odd-dimensional generators $\{l_{ij}\}$ of dimension i $(1 \leq j \leq r_i)$ and a polynomial algebra on even-dimensional generators $\{\lambda_{ik}\}$ of dimension i $(1 \leq k \leq r_{R^0})$.*

Then the group $\pi_l(X) \otimes Q$ has exactly r_l generators $l^*_{l,1}, \dots, l^*_{l,r_l}$. In addition, if the elements $l^*_{l,k}$ are regarded as elements of the homology group $H_l(X, Q)$, then they have zero scalar product with all cohomology classes that can be non-trivially expressed as a product, and the matrix of their scalar products with the multiplicative generators of dimension l of the algebra $H^*(X, Q)$ is non-singular.

Proof. For each multiplicative generator of $H^*(X, Q)$ we pick a complex $K(Z, l)$ of the same dimension l, and multiply them together for all l. We get a complex $Y = \prod K(Z, l_i)$ with the same cohomology algebra $H^*(Y, Q) \approx H^*(X, Q)$.

From the basic properties of $K(\pi, n)$ there is a mapping $f\colon X \to Y$ inducing this ring isomorphism. If X is $(k-1)$-connected, then for $l = k$ we could at once take $K(\pi_k(X), k)$ (this will be better for our purposes) and assume that $\pi_k(Y) = \pi_k(X)$, where the isomorphism is set up by f. We assume that $f\colon X \to Y$ is a fibering $f\colon X \xrightarrow{F} Y$ (see Appendix).

Then $H^*(F, Q) = 0$ and $\pi_1(F) = 0$, where $H^i(F)$ is finitely generated. The latter fact follows from the exact homotopy sequence of the fibering and from Lemma 1.1 and Theorem 1.1. The equation $H^*(F, Q) = 0$ follows from the following simple lemma:

Lemma 1.2. *Let $E \xrightarrow[p]{F} B$ be a Serre fibering (with simply-connected base) in which the homology groups of E, F, and B are finitely generated. Let $p^*\colon H^*(B) \to H^*(E)$ be an isomorphism. Then $H^*(F) = 0$.*

The proof follows from the spectral sequence as in Lemma 1.1.

Now note that by Theorem 1.3 all the homotopy groups of the fibre F are finite. Hence $\pi_i(X) \otimes Q \approx \pi_i(Y) \otimes Q$.

Thus, the groups $\pi_i(X) \otimes Q$ are calculated. As for the assertion about scalar products with cocycles, it follows from the corresponding fact for $Y = \prod K(Z, l_i)$ where it is obvious.

Theorem 1.4 is now proved. □

Note 1.1. Let $X = S^{2n-1}$. Then $H^*(S^{2n-1}, Q)$ is the exterior algebra with one generator of dimension $2n-1$. So we find that all the homotopy groups of S^{2n-1} are finite, apart from $\pi_{2n-1}(S^{2n-1}) = Z$ (Serre).

Note 1.2. Let $X = S^{2n}$; then the algebra $H^*(X, Q)$ does not satisfy our conditions. Consider ΩS^{2n}; obviously $\pi_i(S^{2n}) = \pi_{i-1}(\Omega S^{2n})$. We know that $H^*(\Omega S^{2n}, Q)$ is an algebra with two generators, one exterior in dimension $2n-1$, and one polynomial in dimension $4n-2$. We find that all the homotopy groups $\pi_i(S^{2n})$ for $i \neq 2n, 4n-1$ are finite, and $\pi_{4n-1}(S^{2n}) \otimes Q = Q$, that is, $\pi_{4n-1}(S^{2n}) = Z +$a finite group (Serre).

Note 1.3. Let $X = BSO_n, BU_n, BSp_n$, here again Theorem 1.4 is applicable. It follows from the Cartan–Serre theorem that bundles over S^k are determined "modulo finlteness" by classes — for BSO the Pontryagin or Euler–Poincaré classes, for BU the Chern classes, and for BSp the symplectic classes of Borel–Hirzebruch.

Recall that the classes are the multiplicative generators of the ring $H^*(BG, Q)$, where $G = SO_n, U_n, Sp_n$ and BG is the classifying space.

Note 1.4. X is a complex and $\pi_i(X) = 0$, $i < n$. Then $\pi_i(X) \otimes Q = H_i(X, Q)$ for $i < 2n-1$, the "stable case".

Appendix. The conversion of a mapping into a fibering. Consider a mapping of complexes (or of Hausdorff spaces) $f\colon X \to Y$, and construct the "mapping cylinder" $C_f = X \times I \cup_f Y$, where I is the interval $[0, 1]$ and $f\colon X \times 1 \to Y$. Obviously C_f is contractible to Y and $X \subset C_f$, so that the inclusion $X \subset C_f$ is homotopic to f.

Consider paths starting in X and ending at some point of $C_f \sim Y$. Denote the space of these paths by E. Obviously E can be contracted to $X \subset E$ (one-point paths). We have defined a Serre fibering

$$E \xrightarrow[p]{F} C_f,$$

where p maps each path to its endpoint. However, E is contractible to X and C_f to Y, so that p is homotopic to f. Hence we can say that E is X and C_f is Y (we are working only up to homotopy, so it is valid to identify E with X, C_f with Y and p with f).

Thus, we have replaced a mapping by a fibering.

§ 2. The Cartan–Serre theorem for vector bundles

Let X be a finite complex and η a (complex) vector bundle over X. We have:

Theorem 2.1. *If the Chern character* $\operatorname{ch}\eta \in H^*(X, Q)$ *is trivial, then there exists an* n *such that the Whitney sum* $\underbrace{\eta \oplus \cdots \oplus \eta}_{n}$ *is trivial.*

Proof. If X is a sphere S^k and $\eta_1, \eta_2 \colon S^k \to BU$ are two bundles over S^k, then the sum $\eta_1 \oplus \eta_2$ corresponds to the sum of elements $\eta_1 + \eta_2 \in \pi_k(BU)$. Here we assume that the bundles are "stable", that is, the dimension of the fibre is sufficiently large, since this is tacitly contained in the hypothesis if n is chosen to be large.

In fact, the sum $\oplus$ is induced by the inclusion $U_l \times U_s \subset U_{l+s}$ by diagonal blocks

$$\left(\begin{array}{c|c} U_i & 0 \\ \hline 0 & U_s \end{array}\right)$$

Also it is easy to see that two mappings

$$p_1 \colon U_l \times U_s \to U_{l+s},$$
$$p_2 \colon U_l \times U_s \to U_{l+s},$$

where

$$p_1(a, b) = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right)$$

and

$$p_2(a, b) = \left(\begin{array}{c|c} ab & 0 \\ \hline 0 & E \end{array}\right)$$

are homotopic.

Here p_2 is defined for $l = s$, which we may assume by increasing l and s.

Lemma 2.1. *Let* G *be a topological group and* $f \colon G \times G \to G$ *its multiplication. Then* $\alpha + \beta = f_*(\alpha, \beta)$*, where* $\alpha, \beta \in \pi_k(G)$ *and* $\pi_k(G) \times \pi_k(G) \approx \pi_k(G \times G)$.

The proof of this lemma is left to the reader.

We now use the isomorphism $q \colon \pi_i(G) \approx \pi_{i+1}(BG)$. Let $\eta_1, \eta_2 \in \pi_k(BU) \approx \pi_{k-1}(U)$, we assume that η_1 and η_2 are bundles over S^k. Then we have

$$\eta_1 \oplus \eta_2 = qp_1(q^{-1}\eta_1, q^{-1}\eta_2) = qp_2(q^{-1}\eta_1, q^{-1}\eta_2) = q(q^{-1}\eta_2 + q^{-1}\eta_2) = \eta_1 + \eta_2,$$

from the fact that p_1 and p_2 are homotopic and Lemma 2.1.

It follows that the sum $\oplus$ coincides with the ordinary one for "stable bundles" (of high dimensions). From the Cartan–Serre theorem we know that a bundle over S^k is determined by its Chern class "modulo finiteness" relative to the usual sum (for $G = U_n$). Having established that the Chern class and character only differ by a non-zero (rational) factor for spheres and that the operations $\oplus$ and $+$ coincide, we obtain our assertion for spheres.

For an arbitrary complex X we can assume that the bundle η is trivial on the $(k-1)$-skeleton. Choose n so that $\eta \oplus \cdots \oplus \eta$ is trivial on the k-skeleton. But if the $(k-1)$-skeleton is shrunk to a point, the k-skeleton becomes a bouquet of spheres S^k and the assertion has already been proved for these.

By induction on k, we obtain our theorem. □

The second assertion is proved similarly to the first, but instead of "differences" we must consider the "obstruction" to the extension of a napping.

Theorem 2.2. *Let $\alpha \in H^*(X, Q)$ and $H^{(0)}(X, Q) = Q$ (scalars). There exist numbers N_1 and N_2 such that the element $N_1 + N_2\alpha$ is the Chern character $\operatorname{ch}\eta$ of a bundle over X, provided, of course, the decomposition of α into homogeneous components $\alpha_i \in H^i(X, Q)$ does not contain elements of odd dimension (for $G = U$), or elements of dimension divisible by 4 (for $G = SO, Sp$).*

The proof of Theorem 2.2 runs parallel to that of Theorem 2.1; Theorems 2.1 and 2.2 were first published in a closely related form by Dold.

In the large, we can prove the following theorem.

Theorem 2.3. *Consider the rings $K(X) = K_C(X)$ and $K(X) = K_R(X)$. Then there are natural isomorphisms*:

$$\operatorname{ch}\colon K_C(X) \otimes Q \leftrightarrow \sum_{i \geq 0} H^{2i}(X, Q),$$

$$\operatorname{ch}\colon K_R(X) \otimes Q \leftrightarrow \sum_{i \geq 0} H^{4i}(X, Q),$$

where the additive group of the ring $K(X)$ is finitely generated.

The proof follows from Theorems 2.1 and 2.2 and the multiplicative property of ch. The assertion that the group of $K(X)$ is finitely generated follows from the fact that $\pi_i(BG)$ is finitely generated for $G = U$, SO and an argument similar to the proof of Theorem 2.1, not "modulo finiteness", but "modulo being finitely generated".

Appendix. 1) Bott periodicity:

$$\left.\begin{aligned} \Omega^2 U = U, \quad \Omega^2 BU = BU,\\ \Omega^4 Q = Sp, \quad \Omega^8 Q = Q,\\ \Omega^2 Sp = Q, \quad \Omega^2 Sp = Sp \end{aligned}\right\} \quad \text{up to homotopy type}$$

2) $K_C(X \times S^2) \approx K_C(X) \otimes K_C(S^2)$, $K_C(S^2) = Z + Z$ (one Z is the scalars, the other has trivial multiplication);

$$K_C(S^8) \approx K_R(S^\infty).$$

3) $\operatorname{ch}\colon K_C(S^{2i}) \to H^*(S^{2i}, Z)$ is an isomorphism. Since $\operatorname{ch} = \sum \operatorname{ch}^i$ and $\operatorname{ch}^n = \pm\frac{1}{(n-1)!} C_n + \cdots$, the class C_n of a bundle over S^{2n} is divisible by $(n-1)!$ (and a bundle with such a class exists).

4) $K_R(X \times S^8) \approx K_R(X) \otimes K_R(S^8)$; $\operatorname{ch}\colon K_R(S^{8n}) \to H^*(S^{8n}, Z)$ is an isoinorphism; $\operatorname{ch}\colon K_R^0(S^{8n+4}) \to H^{*8n+4}(S^{8n+4}, Z)$ is a monomorphism whose image is divisible only by 2.

The Pontryagin class p_k of a bundle over S^{4k} is divisible by $a_k(2k-1)!$, where $a_k = 1$ or 2 (1 for *even* k and 2 for odd).

5) Now consider the "complete" k-functor $K^* = \sum K^i(X)$. We have already defined $K^*(X,Y)$ and the exact sequence of the pair $X \supset Y$. K^* is a ring and $K^{-i}(X,Y) = K^0(E^i X/Y)$.

Let $X = P$ (a point).

Theorem 2.4. *There exists a spectral sequence of rings*

$$E_r = \sum_{r \geq 2} E_r^{p,q}, \quad d_r \colon E_r^{p,q} \to E^{p+r,q-r+1}$$

such that

a) $\{E_r, d_r\}$, $K^*_\Lambda(X)$, $\Lambda = R, C$,

b) $E_2^{p,q} = H^*(X, K^q_\Lambda(P))$,

c) $\sum E_\infty^{p,q}$ *is associated with* $K^n(X)$.

6) The form of the functor of a point $K^n_\Lambda(X)$.

a) $\Lambda = C$; K^0_C is the scalars; $K^i_C = 0$ for odd i and K_C^{-2} is generated by the element u, and K_C^{-2n} by the element u^n.

b) $\Lambda = R$: K^0 is the scalars; generators: $h \in K_R^{-1}$, $u \in K_R^{-2}$, $v \subset K_R^{-8}$ (ring generators); relations: $2h = 0$, $h^3 = 0$, $hu = 0$, $u^2 = 4v$.

The functor K^*_R is periodic "modulo" 8.

§ 3. T-regularity and Thom complexes

The aim of this section is the study of "cobordism" by algebraic methods, following Thom's plan.

1. **The concept of t-regularity.** Let M^n be a manifold and W a submanifold of codimension p (in the smooth sense). We can assume M^n is a manifold only in a neighbourhood of W (this remark is essential in applications). Let R^p_x denote a "normal plane" to W at $x \in W$: $R^p_x = R^n_x / R^{n-p}_x$, where R^n_x and R^{n-p}_x are tangent planes to M^n and W. The projection $R^n_x \xrightarrow{q} R^p_x$ is denoted by q.

Definition 3.1. A mapping $f \colon V^m \to M^n$, smooth in the neighbourhood $f^{-1}(W)$, is called *t-regular* on W if for each point $y \in f^{-1}(W)$ the linear mapping $q \circ df_y \colon R^m_y \to R^p_x$ is an epimorphism.

A more general concept of t-regularity can be formulated in the "jet-bundle" $J(X,Y)$ in the sense of Ehresmann; it can also be carried over to infinite-dimensional manifolds (Abraham) (all known forms of reduction to general position fall under these "t-regularities", but this is not necessary for our purpose).

Lemma 3.1 (Thom)**.** *Every mapping can be arbitrarily closely approximated by a t-regular mapping* (*in any smooth metric of mappings*)*.*

Properties of a t-regular mapping f.

1) The inverse image $V_f = f^{-1}(W) \subset V^m$ is a submanifold of codimension p; the mapping $f|V_f \to W$ is smooth and "non-singular in the direction of the normal".

2) The normal bundle of V_f in V is induced by $f \colon V_f \to W$ from the normal bundle of W in M^n.

f induces a mapping of these bundles.

2. **Thom complexes.** Let W be a manifold and η a vector bundle over W. Let E_η be the space of the bundle of unit discs in each fibre. Obviously dE_η is a sphere bundle and $E_\eta \supset W$ as a surface of zero vectors.

Definition 3.2. The factor-space E_η/dE_η, with the distinguished embedding of the zero section $W \subset T_\eta$, is called the *Thom complex* T_η of η.

(The whole of dE_η is one point in T_η.)

Example 3.1. W is a point P. The bundle η is trivial, $\eta = R^n$. E_η is the disc D^n; $T_\eta = D^n/dD^n = S^n$ ("Pontryagin Thom complex").

Example 3.2. W is BG for the group $G \subset O(m)$ and η is the universal G-bundle with fibre R^n. We assume that BG is a manifold of sufficiently high dimension.

Then E_η is the space of the universal bundle of the disc D^n and dE_η is a sphere bundle. Put

$$MG = T_\eta = \frac{E_\eta}{dE_\eta} \supset BG.$$

For example, $G = O(n), SO(n), U(n), SU(n), Sp(n)$, $e \subset O(n)$.

For $G = e$ we have a "Pontryagin Thom complex" $S^n = Me$. We can usefully introduct $MSpin$ for the usual embedding $Spin(n) \subset SO_{2n}$.

Example 3.3. $W = BG \times M^n$ and the bundle η over W is induced by the projection $p\colon W \to BG$ from the universal G-bundle of Example 3.2.

The corresponding Thom complex will be denoted by $MG(M^n)$.

Example 3.4. W is an n-dimensional manifold lying in R^{n+N} or S^{n+N} and η is its normal O-bundle with fibre R^N. The space E_η is a neighbourhood of W in R^{n+N} and the complex T_η is a sphere S^{n+N} with its complement in E_η shrunk to a point.

We have only introduced the most important forms of the Thom complex, each of which has a substantial theory associated with it; Example 3.1 is connected, with homotopy groups of spheres and smoothnesses on Pontryagin spheres (Milnor, Kervaire); Example 3.4 is connected with the problem of diffeomorphisms of simply-connected smooth manifolds (Novikov). Here we shall need the Thom complexes of Example 3.2 ("the original Thom complexes"), connected with the theory of intrinsic hornology ("cobordism") and of Example 3.3 (Atiyah, Conner, Ployd), connected with the "bordism" theory necessary for the index problem (in Atiyah and Singer' s work for $G = SO$, $M^n = BU$).

3. **Cohomology properties of Thom complexes.**

Lemma 3.2. a) *There is a natural isomorphism* $\phi\colon H^i(W, Z_2) \to H^{i+n}(T_\eta, Z_2)$ *where* η *is an* O_n*-bundle over* W.

b) *If* η *is an* SO*-bundle over* W*, there are isomorphisms*

$$\phi\colon H^i(W, Z) \to H^{i+n}(W, Z),$$
$$\phi\colon H^i(W, Q) \to H^{i+n}(W, Q).$$

The proof follows from the fact that the cells in T_η are products of cells σ in W and the fibre $D^n \backslash dD^n$, and the boundary is shrunk to a point. For O-bundles similar considerations lead to Lemma 3.2 modulo 2 because of the possibility of "confusing" orientations.

Lemma 3.3. *Let $G = SO(2n)$ or $G = U(n)$, $SU(n)$. Consider the inclusion $j\colon BG \subset MG$ and the Thom isomorphism $\phi\colon H^i(BG) \to H^{2n+i}(MG)$. Then the composition $j^*\phi$ is the isomorphism of multiplication by the Euler class $W_{2n} \in H^{2n}(BG, Q)$ for $G = SO_{2n}$, and by the Chern class $C_n \in H^{2n}(BG, Q)$ for $G = U_n, SU_n$. (For $G = O_{2n}$, it is also W_{2n} but only* mod 2.)

The proof follows from the universal bundle $BSO_{2n-1} \xrightarrow{S^{2n-1}} BSO_{2n}$ for $G = SO_{2n}$, and $BU_{n-1} \xrightarrow{S^{2n-1}} BU_n$, $BSU_{n-1} \xrightarrow{S^{2n-1}} BSU_n$ for $G = U, SU$. We have to consider the cohomology of MG as the cohomology of the space $E_\eta \bmod dE_\eta$ which is the sphere bundle mentioned above, and note that it can be shrunk to the base of η; and this kills precisely the ideal generated by W_{2n} or C_n in H^* in the spectral sequence of the above sphere bundle.

The exact cohomology sequence of the pair (E_η, dE_η) leads to the required result.

Note 3.1. It is easy to define a multiplication $H^i(X, Y) \otimes H^j(X) \to H^{i+j}(X, Y)$ for any pair $X \supset Y$. We apply this to $K = E_\eta$, $Y = dE_\eta$. We define the element $\phi(1) \in H^n(E_\eta, dE_\eta)$ as the cocycle taking the value 1 on each fibre. For BG we have

$$j^*\phi(1) = W_{2n}, \quad G = SO_{2n},$$
$$j^*\phi(1) = C_n, \quad G = U_n, SU_n.$$

There is the simple formula:

$$\phi(x) = \phi(1)X \in H^{i+n}(T_\eta), \quad x \in H^i(W).$$

4. **Homotopy groups of the Thom complex MSO_{2n}.** From Note 1.4 we get the following theorem.

Theorem 3.1. *$\pi_{i+2n}(MSO_{2n}) \otimes Q$, $i < 2n-1$, has a system of generators in one-to-one correspondence with polynomials in the Pontryagin classes of dimension i. This means that $\pi_{i+2n}(MSO) \otimes Q = 0$ for $i \not\equiv 0 \bmod 4$ and is determined by these Pontryagin polynomials "modulo finiteness" for $i \not\equiv 0 \bmod 4$.*

The proof is easily obtained from the Thom isomorphism

$$\phi\colon H^i(BSO, Q) \to H^{2n+1}(MSO_{2n}, Q)$$

and the form of the cohomology $H^*(BSO_{2n}, Q)$ — polynomials in the Pontryagin classes and the Euler–Poincaré class of dimension $2n$. Then Note 1.4 is applied to the Cartan–Serre theorem.

We now consider the Thom complex of Example 3.3 (for the index problem); $W = BSO_{2n} \times BU_m$, $\eta = p^*\eta_{SO_{2n}}$, where η_{SO} is a universal SO-bundle.

We write $T_\eta = MSO_{2n}(BU_m)$ (m large).

Theorem 3.2. *$\pi_{2n+i}(MSO_{2n}(BU_m)) \otimes Q$ has a system of generators of the following form: polynomials in $p_k \in H^*(BSO_{2n}, Q)$ and $c_j \in H^*(BU_m, Q)$, that is, products of a polynomial in $p_k - x$ and a polynomial in $c_j - x$.*

The proof is the same as that of Theorem 3.1.

§ 4. Cobordism

1. **The cobordism group and ring.** Let $G_{SO} = \sum_{K\geq 0} G^K_{SO}$ be the set of all oriented compact (not necessarily connected) manifolds with or without boundary; G_{SO} is a group relative to the disjoint sum.

There is a ring structure in G_{SO} induced by the cartesian product of manifolds; there is also a boundary operator $d\colon G^n_{SO} \to G^{n-1}_{SO}$ in G_{SO}, induced by taking the boundary of a manifold (the cycles are closed manifolds). We denote the product of $a, b \in G_{SO}$ by ab; obviously $d^2 = 0$ and $d(ab) = (da)b \pm a(db)$.

We obtain the cohomology ring $H^*(G_{SO}, d)$, denoted by $\Omega_{SO} = \sum_{i\geq 0} \Omega^i_{SO}$, where $\Omega^i_{SO}\Omega^j_{SO} \subset \Omega^{i+j}_{SO}$.

The ring Ω_{SO} is called the "cobordism" ring (*intrinsic homology*).

Other forms of the cobordism ring:

1) Ω_O — an manifolds without regard to orientation.

2) Ω_U — manifolds with complex structure in the normal bundle under embedding in a space of sufficiently high dimension.

Similarly $\Omega_{SU}, \Omega_{Sp}, \Omega_{Spin}, \Omega_l$ — the ring of stable homotopy groups of spheres.

3) $\Omega^{(U)}_{SO}$ is constructed from pairs (M, η), where M is an oriented manifold and η is a complex bundle over it, $\eta \in K_C(M)$.

At present the following rings are completely known: $\Omega_O, \Omega_{SO}, \Omega_U$ (Thom, Rokhlin, Milnor, Averbrukh, Novikov); fairly well known are Ω_{SU} (Novikov, Conner, Floyd), the tensor products of $\Omega_{Sp}, \Omega_{Spin}, \Omega^{(U)}_{SO}$ with a field of characteristic $\neq 2$, $\sum_{i\leq 22} \Omega^i_l$ (many authors), and also of course, their tensor products with the rationals Q, as will be seen here for Ω_{SO} and $\Omega^{(U)}_{SO}$ (as a consequence of the Cartan–Serre theorem and the connection between cobordism and homotopy).

We shall now study the connection between cobordism and the homotopy groups $\pi_{n+i}(MSO_n)$, $i < n-1$, for the group Ω^i_{SO}.

We have the important

Theorem 4.1 (Thom, and for Ω^i_l Pontryagin)**.** *The groups Ω^i_{SO} are isomorphic to $\pi_{n+i}(MSO_n)$ for $i < n-1$.*

Proof. Let $\alpha \in \pi_{n+i}(MSO_n)$ and $f\colon S^{n+i} \to MSO_n$ be a mapping in the class α. We make f t-regular on the submanifold $BSO_n \subset MSO_n$. The corresponding mapping (now t-regular) is denoted by g. It is obviously homotopic to f, since they are close. Put $M^i_g = g^{-1}(BSO_n)$, where $M^i \subset S^{n+i}$ (the embedding is smooth).

If f_1 and f_2 are homotopic (and represent α), there is a homotopy $F\colon S^{n+i} \times I \to MSO_n$, $F|S^{n+i} \times 0 = f_1$ and $F|S^{n+i} \times 1 = f_2$. Suppose that f_1 and f_2 are already t-regular. We make F t-regular in such a way that it is unchanged on $S^{n+i} \times 0$ and $S^{n+i} \times 1$. The t-regular homotopy so obtained between f_1 and f_2 is denoted by $G\colon S^{n+i} \times I \to MSO_n$.

Put $N^{i+1}_G = G^{-1}(MSO_n)$. Obviously $dN^{i+1}_G = M^i_{f_1} \cup (-M^i_{f_2})$. We obtain a well-defined mapping $\pi_{n+i}(MSO_n) \to \Omega^i_{SO}$. It is easy to verify that it is additive.

Let M^i be a manifold with orientation. Embed it in S^{n+i}. We get a normal bundle ν with group SO_n. We obtain its classifying mapping $M^i \to BSO_n$. The space of the bundle ν (a neighbourhood of M^i in S^{n+i}) is mapped into the space E_η of the universal bundle η over BSO_n with fibre a closed disk. Obviously the mapping of the fibres can be normalized so that the boundary of ν goes into dE_η. Since $T_\eta = MSO_n = E_\eta/dE_\eta$, the boundary of the neighbourhood W_i in S^{n+i} goes into the point (dE_η) in T_η. We extend trivially the mapping into the neighbourhood of M^i to the whole sphere S^{n+i}, taking all the exterior of this neighbourhood to the point (dE_η). We get a mapping $f(M^i)\colon S^{n+i} \to MSO_n$. This mapping (its homotopy class) does not depend on the embedding $M^i \subset S^{n+i}$ for $n \gg i$; similarly we construct a homotopy film by film, where the film $N^{i+1} \subset S^{n+i} \times I$

goes orthogonally to the boundaries $S^{n_i} \times 0$ and $S^{n+i} \times 1$. We obtain an inverse mapping $\Omega^i_{SO} \to \pi_{n+i}(MSO_n)$, $i \ll n$. It is easily verified that this mapping is the inverse of the previous one, which proves the theorem. (Various incidental details are omitted here: such as getting an exact agreement with the inverse in the first construction, the normalization at the corners of the inverse image of the boundaries, etc.) □

Similarly it can be proved that the other forms of cobordism coincide itb the homotopy of the corresponding Thom complexes. It can also be shown that the multiplicative structures in Ω_{SO} (and the other Ω_G) are induced by certain purely homotopic constructions on the Thom complexes, but we shall not need this for purely "rational" problems.

Corollaries:

Corollary 4.1. *Ω^i_{SO} is finite for $i \equiv 0 \bmod 4$.*

Corollary 4.2. *$\Omega^{4k}_{SO} \otimes Q$ has rank equal to the number of polynomials in the Pontryagin classes of dimension $4k$ (that is, as many as the Pontryagin numbers of a four-dimensional manifold).*

The Pontryagin numbers are additive and defined for Ω^{4k}_{SO}. Therefore we have the conclusion: every linear form on Ω^{4k}_{SO} with values in Z (or in Q) is a linear combination of "basis forms", the Pontryagin numbers.

Corollary 4.3. *The structure of the ring $\Omega_{SO} \otimes Q = \sum_{k \geq 0} \Omega^{4k}_{SO} \otimes Q$.*

Polynomials in the classes p_k are, as already stated, symmetric functions in the squares of the "*Wu* generators" $t_1, \dots, t_N$, where

$$p_k = \sum t^2_{i_1} \dots t^2_{i_k}.$$

Let $\omega = (a_1, \dots, a_s)$, $a_i > 0$ (ω is a partition of k into summands a_i, disregarding order).

Put $P_\omega = \sum t_i^{2a_1} \dots t_{i_s}^{2a_s}$, where P_ω is a function in P_i, $i \leq k$. (A new basis for polynomials in the Pontryagin classes.)

The Whitney formula (for the sum of bundles)

$$P_\omega(\zeta \oplus \eta) = \sum_{\substack{(\omega_1,\omega_2)=\omega \\ \omega_1 \neq \omega_2}} P_{\omega_1}(\zeta)P_{\omega_2}(\eta) + P_{\omega_2}(\zeta)P_{\omega_1}(\eta) + \sum_{(\omega_1,\omega_2)=\omega} P_\omega(\zeta)P_{\omega_1}(\eta)$$

Put $P_{(k)} = \sum t_i^{2k}$.

There is the following simple lemma:

Lemma 4.1. *The Pontryagin class $P_{(k)}$ (or the Pontryagin number $(P_{(k)}, M^{4k})$) is trivial for all manifolds of the form $M^{4l_1} \times M^{4l_2}$, $l_1 + l_2 = k$, $l_1 > 0$, $l_2 > 0$.*

Lemma 4.2. *For a sequence of manifolds $\{M^{4k}_k\}$ $(k = 1, 2, \dots)$ to represent a system of polynomial generators of $\Omega_{SO} \otimes Q$ it is necessary and sufficient that the Pontryagin number $(P_{(k)}, M^{4k}_k)$ is non-trivial for all $k = 1, 2, \dots$*

In particular, if there is such a set of manifolds, it follows that $\Omega_{SO} \otimes Q$ is a polynomial ring.

Lemma 4.2 follows from Lemma 4.1 and the fact that because of the Whitney formulae the polynomials in such manifolds are independent in $\Omega^{4i}_{SO} \otimes Q$ and generate the whole group. The value of the rank of Ω^{4i}_{SO} was calculated earlier; we shall now give a lower bound and find the polynomial generators.

Theorem 4.2. *The manifolds CP^{2i} $(i = 1, 2, \dots)$ form a system of polynomial generators of the ring $\Omega_{SO} \otimes 1Q$. In addition, $(P_{(i)}, CP^{2i}) = 2i = 1$.*

Proof. Let $\tau(CP^n)$ be the (complex) tangent bundle, O_1 the one-dimensional trivial complex bundle and ν_1 a one-dimensional complex bundle (or SO_2-bundle) such that $C_1(\nu_1) = x$, x a basis element of $H^2(CP^n)$.

Earlier we proved the formula

$$\tau(CP^n) \oplus O_1 = \underbrace{\nu_1 \oplus \cdots \oplus \nu_1}_{n+1 \text{ terms}}.$$

Since $P(\nu_1) = 1 + p_1(\nu_1) = l + x^2$, we have $P(\tau(CP^n)) = P(CP^n) = (1 + x^2)^{n+1}$. From the definition and meaning of the *Wu* generators $t_1, \dots, t_{2i+1}$ for the bundle $\tau(CP^{2i}) \oplus O_1$, we get

$$t_j = x (j = 1, \dots, 2i + 1),$$

$$P_{(2i)} = \sum_{j=1}^{2i+1} t_j^{2i} = \sum_{j=1}^{2i+1} x^{2i}.$$

Hence

$$(P_{(2i)}, CP^{2i}) = ((2i + 1)x^{2i}, CP^{2i}) = 2i + 1.$$

The theorem is now proved. □

Conclusions.

1) The *signature of the manifold* M^{4k} is the index of the quadratic form $(y^2, [M^{4k}])$ on the group $H^{2k}(M^{4k}, R)$.

The signature is a linear form on Ω^{4k}_{SO} (see Appendix).

Therefore the signature $\sigma(M^{4k})$ is a linear form L_k in the Pontryagin numbers of M^{4k} with rational coefficients, independent of the manifold (for $k = 1$, the Thom–Rokhlin theorem, for $k > 1$, Thom).

For example, $\sigma = \frac{1}{3}p_1$, $p_1(CP^2) = 3$; for M^{4k} (Thom, Rokhlin) it is easy to find that for $k = 2$ $(\sigma(CP^4) = \sigma(CP^2 \times CP^2) = 1)$:

$$\sigma = \frac{1}{45}(7p_2 - p_1^2) \quad \text{(Thom)}.$$

A general form for $L_k(p_1, \dots, p_k)$ was found by Hirzebruch (also by using polynomial generators of CP^{2i}). Let

$$Q(z) = \frac{\sqrt{z}}{\tan h\sqrt{z}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k z^k$$

(B_k is the Bernoulli number, $B_k > 0$ and $\neq \frac{1}{2}$, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$.)

As usual we put $p_i = \sum_{i=1}^{N} t_{j_1}^2 \dots t_j^2$ (N very large) and $L = \sum_{k \geq 0} L_k z^k$, $L_0 = 1$; then $L = \prod_{i=1}^{N} Q(t_i^2 z)$, $N \to \infty$. It is easy to calculate $L_1 = \frac{1}{3}p_1$, $L_2 = \frac{1}{45}(7p_2 - p_1^2)$,

$$L_3 = \frac{1}{27 \cdot 5 \cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3).$$

2) For the ring $\Omega^{(U)}_{SO} \otimes Q$, which is needed for the index, we have the following answer in exact analogy to that for $\Omega_{SO} \otimes Q$: $\Omega^{(\omega)}_{SO} \otimes Q$ is a polynomial algebra with generators of the following forms:

1. $(CP^{2i}, 1)$, where $1 \in K^0_G(CP^{2i})$ is a one-dimensional trivial U_1-bundle.

2. (S^{2i},η), where $\eta \in K_C^0(S^{2i})$ is an element such that $\operatorname{ch}\eta = u \in H^{2i}(S^{2i},Z)$ (basis element). The homomorphism $I\colon \Omega^{(U)} \to Z$ such that $I(CP^{2i},1) = 1$ and $I(S^{2i},\eta) = 2$ can be calculated, by analogy with with the L-series of Hirzebruch, using $\operatorname{ch}\eta$ of the bundle η over W^{2i}, and the "Todd genus" from the Pontryagin classes of the manifold (or, which comes to the same thing, from the Chern classes of the complexitication of its tangent bundle).

Appendix. The intrinsic homology invariance of the signature and the Pontryagin numbers (the Pontryagin theorem).

1. The Pontryagin numbers. Let $M^n \subset R^{n+k}$, $k \gg n$ and $M^n = dN^{n+1}$. Place N^{n+1} in R^{n+k} so that it approaches the boundary $R^{n+k}\times 0$ orthogonally. The normal bundle of N^{n+1} in $R^{n+k}\times I$ induces the normal bundle of M^n in R^{n+k}. There is a classifying map $f\colon N^{n+1}\to BSO_k$, where $f|dN^{n+1}$ is the classifying map for $M^n = dN^{n+1}$ of the normal bundle to the boundary. The Pontryagin number (of the normal bundle) is by definition, $(f_*[M^n], p_{i_1}\circ\cdots\circ p_{i_s})$, where $p_i \in H^*(BSO,Q)$. But since $f|M^n$ extends to $f\colon N^{n+1}\to BSO$, we have $f_*[M^n] = 0$, and all the numbers are zero. By the Whitney formula, the Pontryagin polynomials of the tangent and normal bundles are mutual inverses (their sum is the trivial bundle, and therefore the same is true for the tangent Pontryagin numbers.

2. The signature (Rokhlin, Wu, Thom). Let $M^{4k} = dW^{4k+1}$. Consider the exact cohomology sequence

$$H^{2k}(W) \xrightarrow{i_*} H^{2k}(M^{4k}) \xrightarrow{\delta} H^{2k+1}(W,M).$$

Let μ denote a basis element of the group $H^{4k}(M^{4k}) = Z$, with $(\mu[M^{4k}]) = 1$. Then for any $y\in H^{2k}(M^{4k})$ we have $y^2 = \lambda\mu$, where $\lambda = (y^2,[M^{4k}])$. If $y \in \operatorname{Im} i^*$, then $y^2 = 0$, since $\delta\mu \neq 0$, and if $y = i^*z$, then $y^2 = i^*z^2$.

Therefore $(y^2,[M^{4k}]) = 0$, if $y\in\operatorname{Im} i^*$. Because of the Poincaré duality law for manifolds with boundary the honomorphisms i^* and δ are associated. Therefore the image $\operatorname{Im} i^*$ has halt the dimension of $H^{2k}(M^{4k})$ and the whole dimension is even. The from $(y^2,[M^{4k}])$ is identically zero on the subspace $\operatorname{Im} i^*$ of half the dimension, and the form is non-singular. Hence its signature is 0. By Poincaré, it coincides with the form induced by intersections of cycles.

§ 5. Some applications of the Hirzebruch formula

In the previous section we proved the Hirzebruch formula

$$(L_k(p_1,\dots,p_k), M^{4k}) = \sigma(M^{4k}),$$

where $L = \prod_{i=1}^N Q(t_i^2 z)$, $L = \sum L_k z^k$ and

$$p_k = \sum t_i^2\circ\cdots\circ t_{i_k}^2,\quad Q(z) = \frac{\tan h\sqrt{z}}{\sqrt{z}},$$

with $L_1 = \frac{1}{3}p_1$, $L_2 = \frac{1}{45}(7p_2 - p_1^2)$.

1. **Milnor's example of a smooth structure on the seven-sphere.** Consider an SO_4-bundle with fibre a closed ball D^4 and base S^4. The boundary of this bundle is an SO_4-bundle of spheres S^3. Such bundles are determined by two integer parameters since $\pi_3(SO_4) = Z + Z = \pi_4(BSO_4)$. These integer parameters are

1) the Pontryagin class $p_1 \in H^*(BSO_4)$,
2) the Euler class $\chi \in H^4(BSO_4)$.

Lemma 5.1. *There exists an SO_4-bundle ν over S^4 with given numbers $\chi = (\chi(\nu), [S^4])$, $p_1 = (p_1(\nu), [S^4])$ if and only if $2\chi - p_1 \equiv 0 \bmod 4$.*

Note that $p_1 \bmod 2$ is W_2^2, and therefore p_1 is always even. If $\chi = 0$, then a non-vanishing vector field can be constructed in this bundle. Therefore the bundle with $\chi = 0$ reduces to SO_3. It is easy to show that the class p_1 of the bundle is divisible by 4. There exists a unique Hopf fibering with fibre S^3 over S^4 and total space S^7, for which p_1 is 2 and $\chi = 1$. (This fibering is obtained by using quaternions.) The unique linear relation satisfying these conditions on χ and p_1 is $2\chi - p_1 = 4k$. However, all this follows simply from the homotopy structure of BSO_4 in dimensions ≤ 4.

Consider bundles with $\chi = 1$.

Lemma 5.2. *The space of a bundle of spheres S^3 over S^4 with $\chi = 1$ is homotopy equivalent to S^7.*

The proof is left to the reader as an exercise in the definition of and elementary homotopy theory.

Bundles with $\chi = 1$ have $p_1 = 4k + 2$, where k is arbitrary.

Theorem 5.1 (Milnor)**.** *If $\frac{45+(4k+2)^2}{7}$ is not an integer, then the space of the sphere bundle with fibre S^3, base S^4, $\chi = 1$, $p_1 = 4k + 2$ is not diffeomorphic to S^7, although it has the homotopy type of S^7.*

Note 5.1. (In fact, Milnor explicitly showed that on such manifolds M^7 there is a function f with two non-singular stationary points, and therefore they are all piecewise-smoothly (piecewise-linearly) homeomorphic[1] to S^7.)

Proof. Consider the space E of a bundle ν of discs D^4 over S^4 with $\chi = 1$, $p_1 = 4k + 2$. We denote it by E, where dE is the manifold required. The cycle $S^4 \subset E$ has self-intersection 1, by definition of χ. Assume that dE is diffeomorphic to S^7. Consider the smooth manifold $M^8 = E \cup_h D^8$, where h is a diffeomorphism $S^7 \to dE$.

The following are obvious:

a) $H_i(M_8) = 0$ ($i \neq 0, 4, 8$), $\quad H_4(M_8) = Z$;

b) $\sigma(M^8) = 1$; $\quad p_1(M^8) = p_1(\nu) + p_1(S^4) = p_1(\nu) = (4k + 2)u$, where u is a basis element of $H^4(M^8) = Z$.

From the Hirzebruch formula we have

$$p_2 = \frac{1}{7}(45\tau + p_1^2) = \frac{1}{7}(45 + (4k + 2)^2).$$

By hypothesis, this number is not an integer, which contradicts the fact that p_2 is always an integer, by definition. □

Example 5.1. For $k = 1$, the number $\frac{1}{7}(45 + (4k + 2)^2)$ is not an integer.

2. **The piecewise-linear invariance of the Pontryagin classes (the Thom–Rokhlin–Shvarts theorem).** Let M_i^n be a smooth manifold and $W^{4q} \times R^{n-4q} \subset M^n$ be a smooth embedding. Then there is the simple formula

$$(L_q(p_1, \dots, p_q), i_*[W^{4q}]) = \sigma$$

[1]Smale and Wallis have proved that such functions always exist on homotopy spheres of dimensions ≥ 5.

where $i_*[W^{4q}]$ is the cycle in M^n corresponding to W^{4q}, and L_q is the Hirzebruch polynomial in the Pontryagin classes of M^n. The formula holds because $p_i(W^{4q}) = i^* p_i(M^n)$ in view of the triviality of the normal bundle to W^{4q} in M^n and the Hirzebruch formula for W^{4q}.

Since $\sigma(W^{4q})$ has a meaning even if W^{4q} is not smooth, this formula can be used in reverse to define the classes, or rather L_q not p_q.

a) Let W^{4q} be an h-manifold (a complex whose local homology groups are those of the sphere S^{4q-1}) and $h\colon W^{4q} \times R^{n-q} \overset{i}{\subset} M^n$ be a polyhedral embedding, where M^n is also an h-manifold.

Put $(L_k(M^n), i_*[W^{4q}]) = \sigma(W^{4q})$.

b) Justification of the definition — its existence and validity: M^n is an h-manifold, $x \in H_{4k}(M^n)$ is a cycle (if $4k > \frac{1}{2}n$, consider $M^n \times S^N$ for large N instead of M^n).

Lemma 5.3. *There exists a number λ such that there is a mapping $f\colon M^n \to S^{n-4k}$, where $f^*\mu_{n-4k} = \lambda Dx$ (D is the Poincaré duality operator), and μ_{n-4k} is a basis element of $H^{n-4k}(S^{n-4k})$.*

The lemma follows easily from the fact that the stable homotopy groups of spheres are finite and the obstruction to the extension of a mapping are finite, as in the Cartan–Serre theorem for vector bundles.

Lemma 5.4. *If $n - 4k > \frac{1}{2}n + 1$ and there are two mappings $f_1, f_2\colon M^n \to S^{n-4k}$ such that $f_1^*\mu_{n-4k} = f_2^*\mu_{n-4k}$, then there is a number δ such that the δ-fold sums of the mappings are homotopic.*

Lemma 5.4 is similar to Lemma 5.3. The sum of the mappings is defined because of the "stability" if $n-4k > \frac{1}{2}n+1$. Also S^{n-4k} is the Thom complex of the trivial bundle, and therefore we can stick together manifolds — the counter-images of points $Be \subset S^{n-4k}$ (see §2 on Thom complexes) — that have dimension $4k < \frac{1}{2}n$, in view of which we can regard them as non-intersection (as if M^n were smooth). Since this is a fact that is purely about homotopy, the smoothness of M^n is not necessary.

Lemma 5.5. *Let $\sigma^{n-4k} \subset S^{n-4k}$ be a simplex of maximal dimension, y_0 its centre and $\tilde\sigma$ its interior. If $f\colon M^n \to S^{n-4k}$ is a simplicity mapping, then $f^{-1}(\tilde\sigma) = f^{-1}(y) \times \tilde\sigma$ in the piecewise-linear sense, and $f^{-1}(y)$ is an h-manifold.*

Now let $x \in H_{4k}(M^n)$ and $f\colon M^n \to S^{n-4k}$, where $f^*\mu_{n-4k} = \lambda Dx$. We put $(L_k(M^n), x) = \frac{1}{\lambda}\sigma(f^{-1}(y))$, where $f^{-1}(y)$ is an h-manifold and y is an interior point of an $(n-4k)$-dimensional simplex in S^{n-4k}.

From Lemma 5.3 it is possible to construct such a mapping f, and from Lemma 5.4 the definition is valid (two cobounding h-manifolds have the same signature; the membrane arises as the counter-image under the homotopy).

Note 5.2. The classes p_{4k} are determined conversely by the L_k, and therefore we have a definition of $p_k \in H^{4k}(M^n, Q)$. For example, $p_1 = 3L_1$, $p_2 = \frac{1}{7}(45L_2 + 9L_1^2)$. In general, the p_k do not now have integer scalar products with cycles; see Milnor's example.

Note 5.3. For $n = 4k + 1$, the definition gives a wider result: the class $p_{4k}(M^{4k+1})$ is a topological invariant (Rokhlin's theorem, which preceded the general Rokhlin–Shvarts theorem). Recently it has been proved that this class is a homotopy invariant (Novikov).

Note 5.4. The integer Pontryagin classes are not even combinatorial invariants (Milnor for the 7-torsion class $p_2 \in H^8(M^n, Z)$). This is connected with the existence of "bad smooth structures" on S^7 and, hence, with the fact that the combinatorial Pontryagin class $p_2 \in H^8(M^n, Q)$ not belong to the integer lattice (the class $7p_2$ probably does belong to it).

Note 5.5. It is known that apart from the Hirzebruch formula $(L_k, M^{4k}) = \sigma$ there is no relation of homotopy invariance for rational Pontryagin classes of simply-connected manifolds (Dold, Milnor, Novikov, Browder, Tamura).

Note 5.6. The central problem is the invariance of the rational Pontryagin classes relative to continuous homeomorphisms (topological Invariance). Only very recently it was proved that certain classes were topologically invariant in cases when they were known not to be homotopy invariant.