DIFFEOMORPHISMS OF SIMPLY CONNECTED MANIFOLDS

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We consider smooth simply connected manifolds of dimension $n \geq 5$ which have a fixed orientation. We shall consider two such manifolds to be the same if there exists a diffeomorphism of one onto the other which has degree $+1$ in terms of the given orientations. In the following the term "diffeomorphism" will mean only a diffeomorphism of degree $+1$. We seek to give a description of the class of manifolds $\{M_i^n\}$ having the following properties:

1. $\pi_1(M_i^n) = 0$.

2. M_i^n is homotopically equivalent to M_j^n.

3. There exists a homotopy equivalence $f: M_i^n \to M_j^n$ of degree $+1$ such that $f^* V_N(M_j^n) = V_N(M_i^n)$, where $V_N(M_s^n)$ is the normal bundle of the manifold M_s^n regarded as imbedded in a Euclidean space E^{N+n}, for $N \geq n+3$.

It is known [4] that under these conditions we have a diffeomorphism $M_i^n \times E^N \approx M_j^n \times E^N$.

From the work of Smale [8] and Hirsch, it follows that $M_j^n \times S^{N-1} \approx M_i^n \times S^{N-1}$; however, the manifolds M_i^n and M_j^n may not be diffeomorphic [5]. In the case where the manifolds M_i^n are homotopically equivalent to the sphere S^n, a complete description of them is given in the remarkable works [3, 7, 8]. Other interesting examples of manifolds are discussed in [9, 11].

We select from the class of manifolds $\{M_i^n\}$ which possess properties 1–3 a particular representative M_0^n and discuss the Thom space [10] of its normal bundle $V_N(M_0^n)$ in Euclidean space of dimension $N+n$, which we denote by T_N. The space T_N is obtained from the bundle $V_N(M_0^n)$ of closed balls R^N by identifying the boundary ∂V_N to a point. Now, we have an isomorphism $\phi_N : H_i(M_0^n) \to H_{N+i}(T_N)$ which is due to Thom. We denote by $[M^n]$ the fundamental cycle of the manifold M^n in the given orientation. The following is evident:

Lemma 1. *The cycle $\phi_N([M_0^n]) \in H_{N+n}(T_N)$ is a sphere.*

We denote by $A \subset \pi_{N+n}(T_N)$ the set of elements such that $H(\alpha) = \phi_N([M_0^n])$, where $\alpha \in \pi_{N+n}(T_N)$ and H is the Hurewicz isomorphism. From the theory of transversally regular mappings [10] we easily obtain:

Lemma 2. *To each element $\alpha \in A$ there correspond a manifold $M_\alpha^n \subset E^{N+n}$ and a map $f_\alpha : M_\alpha^n \to M_0^n$ which has degree $+1$ and is such that $f_\alpha^* V_N(M_0^n) = V_N(M_\alpha^n)$. If two manifolds $M_{\alpha,1}^n$ and $M_{\alpha,2}^n$ and maps $f_{\alpha,1}: M_{\alpha,1}^n \to M_0^n$, $f_{\alpha,2}: M_{\alpha,2}^n \to M_0^n$ correspond to the same element $\alpha \in A$, then there exists a manifold $N^{n+1} \subset E^{N+n} \times I(0,1)$ with boundary $\partial N^{n+1} = M_{\alpha,1}^n \cup (-M_{\alpha,2}^n)$ where $M_{\alpha,1}^n \subset E^{N+n} \times 0$, $M_{\alpha,2}^n \subset E^{N+n} \times 1$ and there exists a map $F: N^{n+1} \to M_0^n$ such that $F^* V_N(M_0^n) = V_N(N^{n+1})$, $F/M_{\alpha,1}^n = f_{\alpha,1}$, $F/M_{\alpha,2}^n = f_{\alpha,2}$.*

Let us define the subset $\tilde{A} \subset A$ to consist of those elements $\alpha \in \tilde{A}$ which act as a representative of a manifold $M_\alpha^n \in \alpha$ which is homotopically equivalent to the manifold M_0^n.

Lemma 3. *If a map $g: M_1^k \to M_2^k$ of oriented manifolds has degree ± 1 then the map $g^*: H^*(M_2^k, K) \to H^*(M_1^k, K)$ is an isomorphism onto a direct summand for any field K.*

From Lemmas 2 and 3 follows easily:

Lemma 4. *If $\alpha \in \tilde{A}$ and the manifold $M^n_\alpha \in \alpha$ is homotopically equivalent to the manifold M^n_0 then the map $f_\alpha: M^n_\alpha \to M^n_0$ is a homotopy equivalence and $f^*_\alpha V_N(M^n_0) = V_N(M^n_\alpha)$. If two manifolds $M^n_{\alpha,1} \in \alpha \in \tilde{A}$, $M^n_{\alpha,2} \in \alpha \in \tilde{A}$ possess this property, then the manifold N^{n+1} constructed in Lemma 2 retracts onto each of their boundaries.*

In order to see what the elements of the set $\tilde{A} \subset \pi_{N+n}(T_N)$ are, we will define a fixed $SO(N)$-bundle structure in the normal bundle $V_N(M^n_0)$ of the manifold $M^n_0 \subset E^{N+n}$ which we will call the normal equipment of the manifold M^n_0. This normal equipment of the manifold M^n_0 will induce a normal equipment on the manifolds $M^n_\alpha \in \alpha \in \tilde{A}$ with the help of the mappings $f_\alpha: M^n_\alpha \to M^n_0$. If two equipped manifolds $M^n_{\alpha,1} \in \alpha$ and $M^n_{\alpha,2} \in \alpha$ are equivalent, then the manifold N^{n+1} mentioned in Lemma 2 is also normally equipped and its equipment induces the given equipments on the boundaries.

Lemma 5. *Let a manifold $M^n_\alpha \in \alpha \in \tilde{A}$ and a mapping $f_\alpha: M^n_\alpha \to M^n_0$ be given. Then the set of homotopy classes of normal equipments induced by the mapping f_α from the given equipment on the manifold M^n_0 is in one-one correspondence with the elements of the group $\pi(M^n_\alpha, SO(N))$.*

The group $\pi(M^n, SO(N))$ is evidently a homotopy invariant of the manifold M^n and thus the group $\pi(M^n_0, SO(N))$ operates in a natural way on the set $\tilde{A}$.

Lemma 6. *The orbits of the group $\pi(M^n_0, SO(N))$ operating in the set $\tilde{A}$ contain the same number of elements (and so the set $\tilde{A}$ is finite).*

It is possible to show that the operation of the group $\pi(M^n_0, SO(N))$ on the set $\tilde{A} \subset \pi_{N+n}(T_N)$ is given by the Whitehead homomorphism

$$J: \pi_k(SO(q)) \to \pi_{k+q}(S^q), \qquad q > k+1,$$

the structure of which is studied in dimensions $k = 4s-1$ (see [12]); for $k \not\equiv 0, 1 \pmod 8$ the group $\pi_k(SO(q))$, $k \neq 4s-1$, vanishes (see [1]).

The following two theorems are very important for the basis of our construction.

Theorem 1. *If $n \neq 4k+2$, then we have $\tilde{A} = A$; if $n = 4k+2$, then either $\tilde{A} = A$ or $\tilde{A}$ consists of half the elements of A, and moreover, this depends only on the dimension, not on the manifold M^n_0.*

Theorem 2. *Suppose the manifolds M^n_1 and M^n_2 satisfy the following: 1) $\pi_1(M^n_1) = \pi_1(M^n_2) = 0$; 2) M^n_1 is homotopically equivalent to M^n_2 and there exists a homotopy equivalence $f: M^n_1 \to M^n_2$ such that $f^* V_N(M^n_2) = V_N(M^n_1)$, where $N \geq n+3$, and $V_N(M^n)$ is the normal bundle in Euclidean space $E^{N+n} \supset M^n$; 3) there exists a manifold N^{n+1} with boundary $\partial N^{n+1} = M^n_1 \cup (-M^n_2)$; 4) there exists a map $g: N^{n+1} \to M^n_2$ such that $g/M^n_2 = 1$, $g/M^n_1 = f$. Under these assumptions there exists a Milnor sphere $\tilde{S}^n \in \theta^n(\partial\pi)$ such that $M^n_1 = (M^n_2 \# \tilde{S}^n)$.*

The proof of these theorems is based on the method of killing homotopy groups by the reconstructions of Morse, which has been applied also to the case of π-manifolds by Milnor and Kervaire.

a) For the proof of Theorem 1 we must carry out the killing of part of the homotopy groups of the manifold $M^n_\alpha \in \alpha \in A$. Indeed, we have a map $f_\alpha: M^n_\alpha \to M^n_0$ such that $f^*_\alpha V_N(M^n_0) = V_N(M^n_\alpha)$. Let $x \in \operatorname{Ker} f^{(i)}_{\alpha*}$, $i < n/2$, where $f_{\alpha*}: \pi_i(M^n_\alpha) \to \pi_i(M^n)$ and i is an integer such that $\operatorname{Ker} f^{(j)}_{\alpha*} = 0$ for $j < i$.

It is possible to show that the sphere $S^i \subset M^n_\alpha$ which is a realization of the element $x \in \operatorname{Ker} f^{(i)}_{\alpha*}$ has a trivial normal bundle in the manifold M^n_α. It is possible to carry out Morse's reconstruction in such a way that the normal bundle of the reconstructed manifold in Euclidean space will be "the same" as it was before (the expression is not to be taken literally). The normal equipment of the manifold M^n_α may also intersect on the reconstructed manifold for $i < n/2$. An analysis of the possible reconstructions and of the possibility of carrying over the normal equipment onto the reconstructed manifold for $i = [n/2]$ reduces this problem to the corresponding problem for the case $M_0^n = S^n$ after which we can employ the unpublished results of Milnor-Kervaire. We obtain the desired killing of the groups $\operatorname{Ker} f_{a*}^{(i)}$ for $i \leq n/2$ by applying Lemmas 3 and 4.

b) For the proof of Theorem 2 we must kill the groups $\operatorname{Ker} g_*^{(i)}$, where $g_*^{(i)}: \pi_i(N^{n+1}) \longrightarrow \pi_i(M_2^n)$ in dimensions $i \leq (n+1)/2$. Evidently the properties of N^{n+1} give the direct sums:

$$\pi_i(N^{n+1}) = \pi_i(M_j^n) + \operatorname{Ker} g_*^{(i)}, \quad j = 1, 2;$$
$$H_i(N^{n+1}) = H_i(M_j^n) + \operatorname{Ker} g_*^{(i)}, \quad j = 1, 2,$$
$$H^i(N^{n+1}) = H^i(M_j^n) + \operatorname{Coker} g_{(i)}^*, \; j = 1, 2.$$

Since $H_i(N^{n+1}, M_1^n) = H_i(N^{n+1}, M_2^n) = \operatorname{Ker} g_*^{(i)}$ and $H_i(N^{n+1}, M_1^n) = H^{n+1-i}(N^{n+1}, M_2^n)$ (the latter relation follows from Poincaré duality), we have an intersection matrix $B = (b_{kl})$, $b_{kl} = Z_k \cdot Z_l$, Z_k, $Z_l \in \operatorname{Ker} g_*^{(n+1/2)}$ for n odd. It is easy to show that this matrix is unimodular (and with even integers on the diagonal for $n + 1 = 4k$). For $n = 4k - 1$ we shall call the index of the matrix B the "relative" index and denote it by $\tau(N^{n+1}, M_1^n, M_2^n)$. For $n = 4k - 3$ there may only arise an invariant with values in Z_2. The groups $\operatorname{Ker} g_*^{(i)}$ may be killed by the reconstructions of Morse for $i < (n+1)/2$. In killing the group $\operatorname{Ker} g_*^{(i)}$ for $i = (n+1)/2$ an obstruction may arise which is easily reduced to the group of Milnor $\theta^n(\partial\pi)$ (see [3, 6, 7]). If this obstruction vanishes, then the manifolds M_1^n and M_2^n are evidently l-equivalent and, according to the theorem of Smale [8], diffeomorphic.

We will construct the set $\tilde{A} \subset \pi_{N+n}(T_N)$ in which the group $\pi(M_0^n, SO(N))$ operates. The elements of the factor group $\tilde{A}/\pi(M_0^n, SO(N))$ are not, strictly speaking, in one-one correspondence with the diffeomorphism classes $\bmod\, \theta^n(\partial\pi)$ of the manifolds $\{M_j^n\}$ which are homotopically equivalent to the manifold M_0^n and which have a common normal bundle since there may exist various homotopy equivalences $f_{ij}: M_j^n \longrightarrow M_0^n$ which preserve the normal bundle. Yet the group $\pi^+(M_0^n)$ of homotopy classes of maps of the manifold M_0^n into itself having degree $+1$ and preserving the normal bundle operates in the factor-group $\tilde{\tilde{A}} = \tilde{A}/\pi(M_0^n, SO(N))$. To describe the operation of this group we observe that a map $f: M_0^n \longrightarrow M_0^n$ which preserves the normal bundle may be made to correspond to a map $\tilde{T}_N f: T_N \longrightarrow T_N$ which leaves the set $\tilde{A}$ invariant and is compatible with the operation of the group $\pi(M_0^n, SO(N))$. Let us denote by $\pi^+(T_N)$ the group of homotopy classes of homotopy equivalences of the space T_N into itself which leave invariant the set $\tilde{A}$ and the cohomology class $\phi(1)$; this determines a map $\tilde{T}_N: \pi^+(M_0^n) \longrightarrow \pi^+(T_N)$.

Lemma 7. *The operation of the group $\pi^+(M_0^n)$ on the set $\tilde{\tilde{A}} = \tilde{A}/\pi(M_0^n, SO(N))$ is given by the formula $x(a) = \tilde{T}_N x(a)$, where $x \in \pi^+(M_0^n)$, $a \in \tilde{\tilde{A}}$.*

By geometry it is easy to deduce the following property of the operation of the group $\pi^+(M_0^n)$. Let us denote by $D_a^+ \in \pi^+(M_0^n)$ the subgroup of this group generated by the diffeomorphisms of degree $+1$ of the manifold $M_a^n \in a \in \tilde{A}/\pi(M_0^n, SO(N))$; we then have:

Lemma 8. *If the element $x \in \pi^+(M_0^n)$ belongs to the subgroup D_a^+, then $x(a) = a$.*

Remark. The group D_a^+ depends essentially on the choice of the element $a \in \tilde{A}/\pi(M_0^n, SO(N))$; however, if $a = x(b)$, then there is an isomorphism $D_a^+ = D_b^+$.

We can now formulate our principal result:

Theorem. *Let there be given a collection $\{M_j^n\}$ of simply-connected manifolds of dimension $n \geq 5$ having a common normal bundle in the Euclidean space E^{N+n} of dimension $N + n \geq 2n + 3$. Then there is a map of this collection $\{M_j^n\} \longrightarrow (\tilde{A}/\pi(M_0^n, SO(N)))/\pi^+(M_0^n)$ onto the factor-group of a finite family of elements $\tilde{A} \subset \pi_{N+n}(T_N)$ by the group $\pi(M_0^n, SO(N))$ and $\pi^+(M_0^n)$ which operates naturally in it. Two manifolds $M_{i_1}^n$ and $M_{i_2}^n$ are carried onto the same element by this mapping if and only if* *there exists a Milnor sphere* $\widetilde{S}^n \in \theta^n(\partial\pi)$ *such that* $M^n_{i_1} = (M^n_{i_2} \# \widetilde{S}^n)$.

Example 1 [3, 7, 8]. $M^n_0 = S^n$, $n \neq 4k+2$. $\pi_{N+n}(T_N) = Z + \pi_{N+n}(S^N)$; $\gamma(u_0) = u_0 + h$, where $\gamma \in \pi_N(SO(N))$, $h \in \operatorname{Im} J$; $\gamma/\widetilde{\pi} = 1$; $\widetilde{A}/\pi_n(SO(N)) \approx \pi_{N+n}(S^N)/\operatorname{Im} J$. The group $\pi^+(S^n) = 1$.

Example 2. $M^n_0 = S^n \times S^n$, $n = 2k$. $\pi_{N+n}(T_N) = Z + \pi_{N+2n}(S^N) + \pi_{N+2n}(S^{N+n}) + \pi_{N+2n}(S^{N+n})$; $\widetilde{A}/\pi(M^n_0, SO(N)) = \{1+\alpha\}$, where $\alpha \in \pi_{N+2n}(S^N)/\operatorname{Im} J + \pi_{N+2n}(S^{N+n})/\operatorname{Im} J + \pi_{N+2n}(S^{N+n})/\operatorname{Im} J$. The elements of the set $\widetilde{A}/\pi(M^n_0, SO(N))$ are determined by the three invariants: (α, β, β'); by the elements of the groups $\pi_{N+i}(S^N)/\operatorname{Im} J$, where $i = 2n$ or $i = n$. Evidently $\pi^+(M^n_0) = Z_2 + Z_2$. It is possible to select elements $x, y \in \pi^+(M^n_0)$ such that $x(\alpha, \beta, \beta') = (\alpha, -\beta, -\beta')$, $y(\alpha, \beta, \beta') = (\alpha, \beta', \beta)$.

Example 3. Let M^n_0 satisfy $n = 8$, $\pi_i(M^8_0) = 0$, $i < 4$, $\pi_4(M^8_0) = Z$. Then we have: $\pi^+(M^8_0) = 1$; $\pi(M^8_0, SO(N)) = Z_2$; $\pi_{N+8}(T_N) = Z + Z_2 + Z_2$; $\widetilde{A} = \{1+\alpha\}$, $\alpha \in Z_2 + Z_2 = \pi_{N+8}(S^N)$. Thus the set $\widetilde{A}/\pi(M^8_0, SO(N))$ contains two elements $\{1+\alpha_1\}$, $\{1+\alpha_2\}$.

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