# MULTIVALUED FUNCTIONS AND FUNCTIONALS. AN ANALOGUE OF THE MORSE THEORY

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**I.** Let $M$ be a finite or infinite dimensional manifold and $\omega$ a closed 1-form, $d\omega = 0$. Integrating $\omega$ over paths in $M$ defines a "multivalued function" $S$ which becomes single valued on some covering $\hat{M} \xrightarrow{\pi} M$ with a free abelian monodromy group: $dS = \pi^*\omega$. The number of generators of the monodromy group is equal to the number of rationally inde- pendent integrals of the 1-form $\omega$ over integral cycles in $M$.

**Problem.** To construct an analogue of Morse theory for the multivalued functions $S$. That is, to find a relationship between the stationary points $dS = 0$ of different index and the topology of the manifold $M$.

**II.** Natural examples of multivalued functionals were examined and used in [1] and [2] to investigate periodic solutions of equations of Kirchhoff type, of a top in gravitational and other fields, and also of a charged particle in a magnetic field if there exists a "Dirac monopole". In this case the multivalued functional was defined on the space $M = \Omega^+(M^n)$ of smooth closed directed curves or on the space $\Omega(M^n, x_0, x_1)$ of curves joining the points $x_0, x_1 \in M^n$, or on some subspace of one of these spaces (such as, when $n = 2$, the subspace of curves without self-intersections). We shall define a more general class of examples, of this type. Suppose that we are given two smooth manifolds $(N^q, M^n)$, where $N^q$ is compact, and a functional $S_0$ on the space $F$ of smooth maps $f\colon N^q \to M^n$. Choose a closed $(q+1)$-form $\Omega$ on $M^n$ and a covering $M^n = \bigcup_\alpha U_\alpha$ with following properties (see [1] for $q = 1$):

a) the image of each map $f\colon N^q \to M^n$ lies wholly in some $U_\alpha$ where $\alpha$ depends on $f$; and

b) the form $\Omega$ is exact on $U_\alpha\colon \Omega = d\psi_\alpha$.

We obtain a multivalued functional $S$ as follows. For every map $f\colon N^q \to U_\alpha$ set

$$S^\alpha(f) = S_0(f) + \int_{(N^q,f)} \psi_\alpha. \tag{1}$$

If $f(N^q)$ lies in the intersection $U_\alpha \cap U_\beta$, then

$$S^\alpha(f) - S^\beta(f) = \int_{(N^q,f)} (\psi_\alpha - \psi_\beta)$$

where $d(\psi_\alpha - \psi_\beta) = 0$.

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By the same token, the 1-form $\delta S$ is well defined on the entire infinite-dimensional functional manifold $F$, and generates a single valued function on some covering $\hat{F} \to F$.

For $q = 1$ and $n = 2, 3$, if $S_0$ is the action or Maupertuis–Fermat functional for a mechanical system consisting of a charged particle, then the 2-form directly coincides with the additional magnetic field (or electromagnetic field in the 4-dimensional formalism when $n = 4$). If $S_0$, in the absence of the field $\Omega$, is positive definite and homogeneous of degree 1, then the subspace of one-point curves $M^n \subset \Omega^+(M^n)$ represents the local minima of $S$ on every sheet of the covering $\hat{F}$ over $F = \Omega^+$. This property played a major role in [1] and [2] in extracting information about the stationary points of S from the topology of the manifold $M^n$.

The following fact was essentially established (although it was not formulated in complete generality) in [2].

**Theorem 1.** *Let $S_0$ be the length functional in some complete (positive) Riemannian metric on a simply connected manifold $M^n$. In every "magnetic field" $\Omega$ on $M^n$ (that is, for any closed 2-form $\Omega$ on $M^n$) the functional $S$ of the form (1), single valued or multivalued, has at least one periodic extremal.*

For manifolds which are not simply connected, we also considered in [1] and [2] the functional $S$ on the closed curves homotopic to zero. Here, it is possible that the form $\Omega$ is not cohomologous to zero in $H^2(M^n, R)$. Then $\pi^*\Omega \sim 0$ in $H^2(\hat{M}, R)$, where $\hat{M} \xrightarrow{\pi} M$ is the universal covering. Thus, the functional $S$ is single valued on the space of curves homotopic to zero in $M^n$.

**Problem.** [1] To find a class of groups $\pi_1$ (this class will contain, in particular, all groups near to free abelian groups) such that on the space of directed closed curves $\Omega_0^+(M^n)$ homotopic to zero the functional $S$ is not everywhere positive (that is, for some "large" curves $\gamma$ the "magnetic" part of $S$ is larger in absolute value than the length $S_0(\gamma)$). Suppose that $\Omega$ is not cohomologous to zero in $H^2(M^n, R)$. In this case, there always is a "saddle" periodic extremal situated in the "interval" of one point curves in the curves where $S < 0$. (See [2] for the torus.)

We now present a curious example for $q = 2$. Consider a "chiral field" $g(x)$ on the sphere $S^2$ with values in a compact simple group (for example, $G = SU_2$). There always exists a canonical two-sided invariant 3-form $\Omega$ on $G$, $d\Omega = 0$. Let $S_0(g)$ be the usual chiral Lagrangian (the "Dirichlet integral") on the maps $g\colon S^2 \to G$ (see [3], pages 741–742), induced by the Killing metric on $S^2$ and on $G$. As above, we get a functional $S$ of the form (1) on the space of maps $g\colon S^2 \to G$. From the requirement that $\exp(iS)$ be single valued (this is necessary for quantization) we find that $\Omega$ is an integral cohomology class in $H^3(G)$. (For $q = 1$ this was discussed in [6] in the context of quantization of a charged particle in the field of a "Dirac monopole".)

**III.** The problem of constructing an analogue of Morse theory for closed 1-forms $\omega$ on a compact finite-dimensional manifold $M$ is helpful from a methodological point of view (and technically simple), if the integrals of $\omega$ over cycles are whole

[1]The topology of the problem is determined by the group $\pi_1$ and a distinguished element $\alpha = [\Omega]$ of $H^2(\pi_1, R)$.

numbers. In this case we have a $\mathbb{Z}$-covering $\hat{M} \xrightarrow{\pi} M$ such that $\pi^*\omega = dS$, where $\exp(iS)$ is single valued on $M$ and gives the map $f = \exp(iS)\colon M \to S^1$. If $M$ is not a fibre bundle with base $S^1$, the map $f$ necessarily has critical points. The structure of "surfaces of steepest descent" for critical points of a function $S$ on a covering define in the usual way a cell complex $C$, invariant with respect to the action of a generator $t\colon \hat{M} \to \hat{M}$ of the group $\mathbb{Z}$. At first glance, the complex $C$ can be considered as a complex of $\mathbb{Z}[t,t^{-1}]$-modules, as is always the case for the usual complexes of chains on regular coverings.

It is appropriate, however, to call attention to the fact that the homology of interest to us is the "semi-open" homology $H_*(\hat{M}, \infty_+)$ where $\infty = \infty_+ \cup \infty_-$. In connection with this, we introduce the ring $\mathbb{Z}[t,t^{-1}]$ consisting of formal Laurent series, infinite in the positive powers, with integral coefficients:

$$q(t,t^{-1}) = \sum_{j \geq j_0} n_j t^j \in \hat{\mathbb{Z}}[t,t^{-1}]. \tag{2}$$

The following lemma shows that the complex $C$ (although it reduces to a complex of $\mathbb{Z}[t,t^{-1}]$ modules) should be expanded to a complex of $\hat{\mathbb{Z}}[t,t^{-1}]$-modules.

**Lemma 1.** *The homology of the complex $C$ of finite-dimensional free $\hat{\mathbb{Z}}[t,t^{-1}]$-modules generated by the critical points of the map $f = \exp(iS)\colon M \to S^1$ on $\mathbb{Z}$-coverings $\hat{M}$, is homotopy invariant and coincides with the homology of $\hat{M}$ with respect to the right infinite ("semiopen") modules: $H_i(C) = H_i(\hat{M}, \infty_+)$.*

The proof does not present any difficulty.

Every submodule of the free finite-dimensional $\hat{\mathbb{Z}}[t,t^{-1}]$-module $F_N$ with $N$ generators $(e_1, \dots, e_N)$ is itself free with number of generators $L \leq N$ and has a basis $e'_1, \dots, e'_L$ of the form (after a rearrangement of $e_1, \dots, e_N$)

$$e'_j = \left( n_j + \sum_{k \geq 1} n_{jk} t^k \right) e_j + \sum_{i > L} q_{ij} e_i \tag{3}$$

such that the number $n_j$ is divisible by $n_{j+1}$.

Via the shift $e_j \to \tilde{e}_j = t^q e_i$, for $i > L$, we can make the powers of all the terms of the series $q_{ij}$ nonnegative in the new basis $\tilde{e}_i$ and ensure, moreover, that for any $i > L$ there exists $j_i$ such that $q_{i,j_i}(0) \neq 0$. By further changes of basis over the ring $\hat{\mathbb{Z}}[t,t^{-1}]$ (retaining the previous properties) we can arrange that the $q_{ij}(0)$ are divisible by $n_j$ for all $i$ and $j$ (this is similar to the reduction of the basis in subgroups of a free abelian group).

Consider the quotient module $H = F_N/F_L$.

**Definition 1.** a) The number $N - L$ is called the "*rank*" of the module $H$: $r(H) = N - L$.

b) The numbers $n_j$ (if they are not equal to 1) are called the *torsion exponents* of the module $H$. The total number of indices such that $n_j \neq 1$ is called the *torsion number* and denoted by $q(H)$.

We consider the free complex of $\hat{\mathbb{Z}}[t,t^{-1}]$-modules generated by the functions $S$ on $\hat{M}$ (it is trivial that there exist $S$ such that $C_0 = C_n = 0$ as distinct from the usual cell complexes):

$$
(4)\qquad \begin{gathered} 0 \to C_n \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0, \\ H_0 = H_n = 0, \quad H_j(C) = H_j(\hat{M}, \infty_+). \end{gathered}
$$

**Definition 2.** By the *Morse numbers* we mean the ranks $r(C_j) = M_j$, which coincide with the number of critical points of the map $f\colon M \to S^1$ of index $j$.

We introduce the analogues, which stem from Definition 1, of the usual notions:

(a) the *Betti number*: $b_i^+(M) = r(H_i)$,

(b) the *torsion number*: $q_i^+(M) = q(H_i)$,

where $H_i = H_i(C) = H_i(\hat{M}, \infty_+)$.

The following theorem holds.

**Theorem 2.** *The Morse numbers $M_i(f)$ of the critical points of a multivalued function, i.e. a map $f\colon M \to S^1$, satisfy the following inequalities (in the nondegenerate case):*

$$M_i(f) \geq b_i^*(\hat{M}) + q_i^+(\hat{M}) + q_{i-1}^+(\hat{M}).$$

*Remark.* It seems that all other authors, even those who considered manifolds which were not simply connected (see [4] and [5]), have attempted to construct an analogue of the usual Morse theory only for single valued functions. In this case it suffices to work with a complex of free $\mathbb{Z}[\pi_1]$-modules on the universal covering. It is interesting that, even for the group $\pi_1 = \mathbb{Z}$, we get another coefficient ring $\hat{\mathbb{Z}}[t, t^{-1}]$ in the "multivalued case".

**IV.** We consider the general case when the number of rationally independent integrals over a basis of 1-cycles is greater than or equal to 1. Denote the set of integrals by $(\kappa_1 : \dots : \kappa_k) = \kappa$ and let $\Gamma_k = \mathbb{Z} \times \dots \times \mathbb{Z}$ be the monodromy group on the basis $(t_1, \dots, t_k)$. The multivalued function $S$ on $\hat{M} \to M$ defines a complex $C$ of $\mathbb{Z}[\Gamma_k]$-modules. Let $\phi^i$ be coordinates on $R^k$ and consider the set of hyperplanes $\sum \kappa_i \phi^i = \text{const}$. Here $S = \sum k_i \hat{f}_*(\phi^i)$, $f\colon M \to T^k$, and $\hat{f}\colon \hat{M} \to R^k$. The cover $\pi$ is defined with the help of a subgroup $T \subset \pi_1(M)$ such that $(\omega, \gamma) = 0$ whenever $\gamma \in T$ and $\pi_1(M)/T = \Gamma_k$. The cycles $\gamma_1, \dots, \gamma_k$ in the quotient group are such that $(\omega, \gamma_i) = \kappa_i$.

We define the completion $\hat{\mathbb{Z}}_\kappa^+[\Gamma_k]$ of the group ring as follows. The elements $q \in \hat{\mathbb{Z}}_\kappa^+[\Gamma_k]$ consist of series with integral coefficients in the variables $(t_1, \dots, t_k, t_1^{-1}, \dots, t_k^{-1})$ such that:

a) $q = \sum q_{m_1,\dots,m_k} t_1^{m_1} \circ \dots \circ t_k^{m_k}$, where all the sets $(m_1, \dots, m_k)$ are integral and lie on one side of some level surface: $\sum \kappa_i m_i < B(q) < \infty$; and

b) for any pair $B_1 < B_2$ the number of sets such that $B_1 < \sum \kappa_i m_i < B_2$ is finite.

**Lemma 2.** *The homology of the complex $C$ with coefficients in the ring $\hat{\mathbb{Z}}_\kappa^+[\Gamma_k]$ (the semi-open homology) is homotopy invariant. For all $\kappa'$ sufficiently close to an initial $\kappa \in RP^{k-1}$ consisting of integrals of the form $\omega$ along 1-cycles, there exists a closed 1-form $\omega'$ close to $C^\infty$ and with periods $\kappa'$, the Morse numbers for which coincide with the Morse numbers of the critical points of the form $\omega$.*

For integral sets $\kappa'$ the ring $\hat{\mathbb{Z}}_{\kappa'}^+[\Gamma_k]$ reduces to the group ring of the group $\mathbb{Z}$ with coefficients in the ring $\mathbb{Z}[\Gamma_{k-1}]$ augmented by series in positive powers:

$$(\mathbb{Z}[\Gamma_{k-1}])^+[t, t^{-1}] = \hat{K}^+[t, t^{-1}]$$

**Conclusion.** The Morse type inequalities for 1-forms (of multivalued functions) with monodromy group $\Gamma_k$, $k \geq 1$, bound from below the ranks of complexes of free $\hat{K}^+[t, t^{-1}]$-modules whose homology is homotopy invariant. We note that the 0-dimensional homology is always equal to zero for the complexes which interest us. This is different from the classical Morse theory for single valued functions ($k = 0$) satisfying the "Arzela principle" (the sets of lesser values are always assumed to be relatively compact).

**Problem.** To investigate the dependence of the semi-open homology of $\hat{M}$ on $\kappa$.

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