HOMOTOPY PROPERTIES OF THE GROUP OF DIFFEOMORPHISMS OF A SPHERE

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We denote by $\operatorname{diff} M^n$ and $\operatorname{diff}^0 M^n$, respectively, the group of orientation-preserving diffeomorphisms of a smooth manifold M^n and its arcwise connected component of the identity in the C^∞-topology. Let W be a smooth manifold of class C^∞.

Definition 1. A mapping $f\colon W \to \operatorname{diff} M^n$ is called *smooth of class* C^r $(r \geq 0)$ if the mapping $F(f)\colon W \times M^n \to W \times M^n$ such that $F(f)(x, y) = (x, f_x(y))$, where $x \in W$, $y \in M^n$, is a diffeomorphism of class C^r.

One easily proves the following

Lemma 1. *Every mapping* $f\colon W \to \operatorname{diff} M^n$ *may be approximated arbitrarily closely by a smooth mapping of class* C^∞.

Our ultimate purpose is to study the groups

$$\operatorname{diff} S^{n-1}, \quad \operatorname{diff}^0 S^{n-1}, \quad \operatorname{diff} D^n, \quad \operatorname{diff}^0 D^n, \quad \bar{K}^n = K^n \cap \operatorname{diff}^0 D^n,$$

where K^n is the group of diffeomorphisms of the disk D^n which are stationary on the boundary $S^{n-1} = \partial D^n$. There is determined the fiber space in the sense of Serre:

$$\operatorname{diff}^0 D^n \xrightarrow[p]{\bar{K}^n} \operatorname{diff}^0 S^{n-1},$$

where p is the natural projection. The orthogonal group SO_n is imbedded in a natural way in the group $\operatorname{diff}^0 S^{n-1}$. Milnor [6–8] has established for certain dimensions n the nontriviality of the groups $\pi_0(\operatorname{diff} S^{n-1})$, and even of the group $\pi_0(\operatorname{diff} S^{n-1})/p_* \pi_0(\operatorname{diff} D^n)$, which has been more or less completely computed for $n \geq 6$. We shall study the groups $\pi_i(\operatorname{diff}^0 S^{n-1})/p_* \pi_i(\operatorname{diff}^0 D^n)$ for certain values of n and for $i > 0$. To this end we study first the relation between the Whitehead homomorphism $J\colon \pi_i(SO_N) \to \pi_{N+i}(S^N)$ and composition multiplication in the ring $G = \sum_{i \geq 0} G_i$ of stable homotopy groups of spheres, $G_i = \pi_{N+i}(S^N)$, $N > i+1$. We denote by $J_i \subset G_i$ the image $J\pi_i(SO_N)$, and by $\tilde{\theta}_i \subset G_i$ the subgroup of G_i whose elements are representable by framed* manifolds combinatorially equivalent to the sphere S^i. It is evident that $J_i \subset \tilde{\theta}_i$. It is known that $\tilde{\theta}_i = G_i$ for $i \not\equiv 2 \bmod 4$ and for $i = 10$, and that $G_i/\tilde{\theta}_i$ contains exactly two elements for $i = 2, 6, 14$ and no more than two elements in the remaining cases (these results are due to Kervaire, Milnor and Smale [5, 6, 10, 11]). By use of the technique of Morse modifications and of frame carry-over [3, 5], it is comparatively simple for us to prove the following theorem.

*Translator's note. The Russian term "osnaščennyĭ" (literally: "equipped," "rigged" and here translated "framed") is the same as in, e.g., Pontrjagin, Trudy Mat. Inst. Steklov. 45 (1955) (Amer. Math. Soc. Transl. (2) 11 (1959), 1), where a "framed manifold" M^n is a manifold imbedded in E^{n+N} together with a particular field of normal N-frames (N linearly independent normal vectors); this implies that the normal bundle is trivial. In Dokl. Akad. Nauk SSSR 143 (1962), 1046 (cf. correction in RŽMat. 1962 #10A233), the present author uses the same term, with no restriction on the normal bundle, to mean an automorphism of the SO_N-bundle structure which is the identity on the base M^n.

Theorem 1. *Let* $\alpha \in G_i$, $\beta \in G_j$, $i > 0$, $j > 0$. *Then* $\alpha \circ \beta \in \widetilde{\theta}_{i+j}$ *for all pairs* (i, j) *except for the cases* $i = j = 1, 3, 7$.

The Morse modification depends, as is well known, on an imbedding of a sphere $S^i \subset M^n$ with trivial normal bundle and on an element $h \in \pi_i(SO_{n-i})$. Application of the Morse modification results in a manifold $M^n(S^i, h)$.

It is easy to prove

Lemma 2. *Let* $M^n = S^i \times S^j$. *Then the manifold* $M^n(S^i, h)$ *is diffeomorphic to* S^n.

The proof is based on the fact that every diffeomorphism $\widetilde{h}$: $S^i \times D^j \to S^i \times D^j$, such that $\widetilde{h}(x, y) = (x, \widetilde{h}_x(y))$, extends to a diffeomorphism $\widetilde{\widetilde{h}}$: $S^i \times S^j \to S^i \times S^j$ such that $\widetilde{\widetilde{h}} \mid S^i \times D^j = (\widetilde{h}_x \in SO_j)$.

From Lemma 2 follows

Lemma 3. *Let* $\alpha \in J_i$, $\beta \in J_j$. *Then* $\alpha \circ \beta \in J_{i+j}$, *except for the cases* $i = j = 1, 3, 7$.

Now denote by $B(M^n) \subset G_n$, for a π-manifold M^n, the set of elements $\alpha \in G_n$ representable by the manifold M^n with some frame. Using Morse modifications, we have from Lemma 2

Lemma 4. $B(S^i \times S^j) = J_{i+j}$, *except for the cases* $i = j = 1, 3, 7$.

From results of Haefliger [1, 2], Smale [11] and Kervaire [4] one easily derives

Lemma 5. *Let* $\widetilde{S}^j$ *be a smooth* π*-manifold homeomorphic to the sphere, and* $i > j/2 + 1$. *Then the manifold* $S^i \times \widetilde{S}^j$ *is diffeomorphic to the direct product* $S^i \times S^j$.

The proof of Lemma 5 is obtained from the theorems of Haefliger on approximation of topological by smooth imbeddings, of Smale on J-equivalence and of Kervaire on the normal bundles of homotopy spheres in euclidean space.

From Lemmas 2–5 follows

Theorem 2. *Let* $\alpha \in J_i$, $\beta \in \widetilde{\theta}_j$ *for* $i > j/2 + 1$. *Then* $\alpha \circ \beta \in J_{i+j}$ *except for the cases* $i = j = 1, 3, 7$.

When the conditions of Theorem 2 are not satisfied, i. e., if $i \leq j/2 + 1$, it is possible to have $J_i \circ \widetilde{\theta}_j \not\subset J_{i+j}$, for suitable choice of the dimensions i, j. Suppose $\beta \in \widetilde{\theta}_j$ and the element β is represented by a framed homotopy sphere $\widetilde{S}^j_\beta$ with a certain frame, where the sphere $\widetilde{S}^j_\beta$ is determined uniquely modulo $\theta^j(\partial\pi)$. As is known, $\widetilde{S}^j_\beta$ may be split up as the union of two disks $\widetilde{S}^j_\beta = D^j \underset{q_\beta}{\cup} D^j$, where $q_\beta \in \text{diff}\, S^{j-1}$, i. e., $q_\beta \in \pi_0(\text{diff}\, S^{j-1})$ and defines an element $\widetilde{q}_\beta \in \pi_0(\text{diff}\, S^{j-1})/p_* \pi_0(\text{diff}\, D^j)$.

The following lemma is very important for our purposes.

Lemma 6. *Let* $\alpha \in J_i$, $\alpha \circ \beta \notin J_{i+j}$. *Then there exists a smooth mapping* h: $S^i \to SO_j$ *such that the diffeomorphism* $F_\beta(h)(x, y) = (x, [q_\beta h_x q_\beta^{-1}](y))$ *does not extend to a diffeomorphism* $S^i \times D^j \to S^i \times D^j$, *where*

$$F_\beta(h): S^i \times S^{j-1} \to S^i \times S^{j-1}.$$

Proof. Consider the direct product $S^i \times \widetilde{S}^j_\beta$ and assign to it two different frames: the trivial one, and one corresponding to the element $\alpha \circ \beta \notin J_{i+j}$. We construct two Morse modifications and pass to the manifolds $M^n(S^i, h_1)$ and $M^n(S^i, h_2)$, where h_i: $S^i \to SO_j$ and $n = i + j$. The mapping h_1 is chosen so as to be able to carry over to $M^n(S^i, h_1)$ the trivial frame, and h_2 so as to be able to carry over to $M^n(S^i, h_2)$ the frame representing the element $\alpha \circ \beta$. Both manifolds $M^n(S^i, h_1)$ and $M^n(S^i, h_2)$ are homotopy spheres, but $M^n(S^i, h_1) \in \theta^n(\partial\pi)$ and $M^n(S^i, h_2) \notin \theta^n(\partial\pi)$. Therefore the mapping $h = h_1 h_2^{-1}$ is such that the diffeomorphism $F(h)\colon S^i \times D^j \to S^i \times D^j$, where $F(h)(x, y) = (x, h_x(y))$, does not extend to a diffeomorphism $S^i \times \widetilde{S}^j_\beta \to S^i \times \widetilde{S}^j_\beta$. This is equivalent to saying that the diffeomorphism $F_\beta(h)\colon S^i \times S^{j-1} \to S^i \times S^{j-1}$, where $F_\beta(h)(x, y) = (x, [q_\beta h_x q_\beta^{-1}](y))$, does not extend to $S^i \times D^j$ This proves the lemma.

From Lemmas 5 and 1 we obtain the following

Corollary 1. *Let* $\alpha \in J_i$, $\beta \in \widetilde{\theta}_j$, $\alpha \circ \beta \notin J_{i+j}$. *Then there exist a diffeomorphism* $q_\beta\colon S^{j-1} \to S^{j-1}$, $q_\beta \notin \operatorname{diff}^0 S^{j-1}$, *and an element* $h \in \pi_i(SO_j)$ *such that the element* $q_\beta h q_\beta^{-1} \in \pi_i(\operatorname{diff}^0 S^{j-1})$ *does not belong to* $p_* \pi_i(\operatorname{diff}^0 D^j)$ *(we may suppose that* $q_\beta \in \pi_0(\operatorname{diff} S^{j-1})$*).*

To apply these last results it is necessary to know the structure of the groups G_i, the multiplication $G_i \circ G_j$, the image $\operatorname{Im} J$ and the subgroups $\widetilde{\theta}_i$. We exhibit a table of these groups for $i \leq 14$ and a table of the multiplication $G_i \circ G_j$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
G_i	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{240}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_6	Z_{504}	0	Z_3	Z_2
$\widetilde{\theta}_i$	Z_2	0	Z_{24}	0	0	0	Z_{240}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_6	Z_{504}	0	Z_3	0
J_i	Z_2	0	Z_{24}	0	0	0	Z_{240}	Z_2	Z_2	0	Z_{504}	0	0	0

We can take generators $x_i^{(p)}, y_i^{(p)}, z_i^{(p)} \in G_i$ (p a prime), viz., $x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_3^{(3)}, x_6^{(2)}, x_7^{(2)}, x_7^{(3)}, x_7^{(5)}, x_8^{(2)}, y_8^{(2)}, x_9^{(2)}, y_9^{(2)}, z_9^{(2)}, x_{10}^{(2)}, x_{10}^{(3)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(7)}, x_{13}^{(3)}, x_{14}^{(2)}$, such that:

1. $2x_1^{(2)} = 0$, $2x_2^{(2)} = 0$, $8x_3^{(2)} = 0$, $3x_3^{(3)} = 0$, $2x_6^{(2)} = 0$, $16x_7^{(2)} = 0$, $3x_7^{(3)} = 0$, $5x_7^{(5)} = 0$, $2x_8^{(2)} = 0$, $2y_8^{(2)} = 0$, $2x_9^{(2)} = 0$, $2y_9^{(2)} = 0$, $2z_9^{(2)} = 0$, $2x_{10}^{(2)} = 0$, $3x_{10}^{(3)} = 0$, $8x_{11}^{(2)} = 0$, $9x_{11}^{(3)} = 0$, $7x_{11}^{(7)} = 0$, $3x_{13}^{(3)} = 0$, $2x_{14}^{(2)} = 0$.

2. $x_1^{(2)2} = x_2^{(2)}$, $x_1^{(2)3} = 4x_3^{(2)}$, $x_3^{(2)2} = x_6^{(2)}$, $x_1^{(2)} x_7^{(2)} = x_8^{(2)}$, $x_1^{(2)2} x_7^{(2)} = x_3^{(2)3} = x_9^{(2)}$, $x_1^{(2)} y_8^{(2)} = y_9^{(2)}$, $x_1^{(2)2} y_8^{(2)} = 0$, $x_1^{(2)3} x_7^{(2)} = 0$, $x_1^{(2)} z_9^{(2)} = x_{10}^{(2)}$, $x_1^{(2)2} z_9^{(2)} = 4x_{11}^{(2)}$, $x_7^{(2)2} = x_{14}^{(2)}$, $x_{11}^{(2)} x_3^{(2)} = 0$, $x_3^{(3)} x_{10}^{(3)} = x_{13}^{(3)}$.

3. $x_1^{(2)}, x_3^{(2)}, x_3^{(3)}, x_7^{(2)}, x_7^{(3)}, x_7^{(5)}, x_8^{(2)}, x_9^{(2)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(7)} \in \operatorname{Im} J$, while the remaining generators do not belong to $\operatorname{Im} J$.

4. All the generators except $x_1^{(2)2} = x_2^{(2)}$, $x_3^{(2)2} = x_6^{(2)}$ and $x_7^{(2)2} = x_{14}^{(2)}$ belong to the subgroups $\widetilde{\theta}_i$.

Furthermore, as regards the p-components $G_i^{(p)}$ of the groups, it is known that:

1) $G_{2p-3}^{(p)} = Z_p = J_{2p-3}^{(p)}$ (generator $x_{2p-3}^{(p)}$);

2) $G_{2p(p-1)-2}^{(p)} = \widetilde{\theta}_{2p(p-1)-2}^{(p)} = Z_p$ (generator $x_{2p(p-1)-2}^{(p)}$), and for $p > 2$ the group $J_{2p(p-1)-2}^{(p)} = 0$;

3) the elements $x_{2p-3}^{(p)} \circ x_{2p(p-1)-2}^{(p)k} \notin \operatorname{Im} J$ for $k \leq p-2$, and $x_{2p-3}^{(p)} \circ x_{2p(p-1)-2}^{(p)p-1} \neq 0$, $x_{2p(p-1)-2}^{(p)p} \neq 0$.

(Concerning the results on multiplication in homotopy groups of spheres, cf. [9].)*

It remains now, using the preceding results and the data on the groups G_n, $\tilde{\theta}_n$, J_n and multiplication $G_i \circ G_j$, to find cases of nontriviality for the groups $A_{i,j} = \pi_i(\operatorname{diff}^0 S^{j-1})/p_* \pi_i(\operatorname{diff}^0 D^j)$.

Theorem 3. *The groups* $A_{i,j} = \pi_i(\operatorname{diff}^0 S^{j-1})/p_* \pi_i(\operatorname{diff}^0 D^j)$ *have the following form:*

1) $A_{1,8} \supset Z_2$;

2) $A_{1,9} \supset Z_2$;

3) $A_{2p-3,\, 2kp(p-1)-2k} \otimes Z_p \supset Z_p + \cdots + Z_p$ ($p-1$ *terms*), $p \geq 3$, *for* $k \leq p-2$; $A_{3,10} \otimes Z_3 \supset Z_3 + Z_3$ *for* $p = 3$.

The proof of Theorem 3 follows at once from the lemmas and the structure of the ring $G = \Sigma G_i$. Since $\pi_1(SO_n) = Z_2 (n > 2)$, we have $h \in Z_2$ (cf. Lemma 6) and the element β has order 2, $\beta \in G_8(G_9)$. Therefore $q_\beta h q_\beta^{-1} \in \pi_1(\operatorname{diff}^0 S^7)$ also has order 2 and $q_\beta^2 \in \operatorname{diff}^0 S^7$. Therefore the group $\pi_1(\operatorname{diff}^0 S^7) \supset Z_2 + Z_2$ with generators h and $q_\beta h q_\beta^{-1} \notin \operatorname{Im} p_*$. Therefore $A_{1,8} \supset Z_2$. Similarly for $A_{1,9}$. This proves items 1) and 2). We prove item 3). Note that $\pi_{2p-3}(SO_j) = Z$ for $j > 2p-2$. Let h be a generator of $\pi_{2p-3}(SO_j)$ and $\beta = x^{(p)^k}_{2p(p-1)-2}$, $k \leq p-2$. Then $h, q_\beta h q_\beta^{-1}, \cdots$ $\cdots, q_\beta^{p-1} h q_\beta^{1-p}$ are distinct elements of $\pi_{2p-3}(\operatorname{diff}^0 S^{j-1})$, $j = 2kp(p-1) - 2k$, and all have infinite order. But relations are possible of the form $\lambda_1 ph = \lambda_2 p(q_\beta h q_\beta^{-1}) = \cdots = \lambda_p p(q_\beta^{p-1} h q_\beta^{1-p})$, whence the desired result. This proves the theorem.

As usual, denote by B_G the classifying space of a group G.

Corollary 2. *The classifying space* $B_{\operatorname{diff} S^{n-1}}$ *is not homotopically simple for* $n = 8, 9$, $2kp(p-1) - 2k$, $k \leq p-2$; *namely: the group* π_1 *operates nontrivially on the respective groups* $\pi_2(B_{\operatorname{diff} S^7})$, $\pi_2(B_{\operatorname{diff} S^8})$, $\pi_{2p-2}(B_{\operatorname{diff} S^{2kp(p-1)-2k-1}})$, $k \leq p-2$.

From the Serre fibering $\operatorname{diff}^0 D^n \xrightarrow[p]{\overline{K}^n} \operatorname{diff}^0 S^{n-1}$, where $\overline{K}^n = K^n \cap \operatorname{diff}^0 D^n$, we obtain

Corollary 3. a) *There exists a diffeomorphism* $F: D^n \to D^n$ *such that* $F \in \operatorname{diff}^0 D^n$, $F|\partial D^n = 1$, F *is nonisotopic to the identity in the group* K^n *for* $n = 8, 9$; b) *the groups* $\pi_{2p-4}(K^{2k(p-1)p-2k}) \neq 0$ $(p > 2)$ *for* $k \leq p-2$ (p *a prime*).

Corollary 4. *There exist sphere bundles over spheres with structure group* $\operatorname{diff}^0 S^{n-1}$ *which are equivalent to orthogonal bundles in the group* $\operatorname{diff} S^{n-1}$ *but not in the group* $\operatorname{diff}^0 S^{n-1}$.

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*We note that an earlier computation gave $G_{14} = \pi_{N+14}(S^N) = Z_2 + Z_2$. This result is false. The author has shown that $G_{14} = Z_2$, by using the fact that $J_3 \circ J_{11} \subset J_{14} = 0$ and that $G_{14} = J_7 \circ J_7 \cup J_3 \circ J_{11}$.

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