THE HAMILTONIAN FORMALISM AND A MANY-VALUED ANALOGUE OF MORSE THEORY

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Introduction

It is now scarcely a matter of dispute that dynamical systems describing real physical processes are, as a rule, Hamiltonian in one sense or another if the dissipation of energy can be disregarded. However, the Hamiltonian formalism may turn out to be non-classical, that is, it may not originate from a Lagrangian formalism as a result of a Legendre transformation. There may not be global canonical coordinates. This refers in the first place to many systems of hydrodynamic origin. Various aspects of the Hamiltonian formalism will be discussed in greater detail in §§ 1, 2. Another aim of this survey is to describe topological methods of search for periodic trajectories. The fact is that the overwhelming majority of non-trivial conservative systems are non-integrable even for two degrees of freedom. After stationary points, periodic solutions are the simplest objects of the qualitative theory of dynamical systems; nevertheless, even the problem of the existence of periodic trajectories is often highly non-trivial and requires the use of topological methods.

The Morse and Lyusternik—Shnirel'man (LSM) theory, which combines the calculus of variations with the topology of function spaces consisting of closed contours (curves) on the relevant *configuration space* (see § 3), is widely known.

However, the use of the LSM theory necessitates the strict requirement of a positive-definite Lagrangian formalism. From this it is clear that in most general Hamiltonian systems not of Lagrangian origin, this theory, generally speaking, cannot be applied. Variational principles on phase trajectories never give rise to positive-definite functionals. Some very interesting systems, which we call Kirchhoff systems, reduce to a problem, mathematically equivalent to the theory of a charged particle in a magnetic field "the Dirac monopole" (see § 4). The following systems are of Kirchhoff type:

(a) the Kirchhoff equation for the motion of a rigid body in an ideal *incompressible fluid* moving under a potential and at rest at infinity;

(b) the equation of motion of a *rigid body* with a fixed point in an axially symmetric strong field;

(c) the Leggett equation for the *magnetic moment* in the low temperature phases of ^{3}He (nuclear magnetic resonance).

In these systems, equations of motion can ultimately be reduced to a principle of extremal action S. But (see § 5) from a global point of view the action S turns out to be a "many-valued" functional on the space of closed contours (smooth curves) on the sphere S^2, which after a reduction plays the role of the configuration space. This means that δS is a single-valued quantity (a 1-form or covector) on the space of contours, but the "integrals over cycles" in the space of contours of δS are nontrivial. Therefore, S is a many-valued functional (for example, on a circle $d\varphi$ is a single-valued 1-form, but φ is a many-valued).

One of the purposes of § 5 is to extend the topological methods of LSM theory to many-valued functionals. This enables us to establish the existence of a large collection of periodic orbits for systems of Kirchhoff type. The results of § 4, 5 are mainly from [1] and [2]. An analogue of Morse theory for many-valued functions (closed 1-forms) on finite-dimensional manifolds is constructed in § 6. The results of this section are from [3].

§ 1. The Hamiltonian formalism. Simplest examples. Systems of Kirchhoff type. Factorization of the Hamiltonian formalism for the B-phase of ^{3}He

From the contemporary point of view, at the basis of the Hamiltonian formalsim lies the concept of a "*Poisson bracket*". Let y^i be local coordinates on a manifold (the "phase-space"); the Poisson bracket of two functions $f(y)$ and $g(y)$ is given by a tensor field $h^{ij}(y)$:

$$(1) \qquad \{f, g\} = h^{ij}(y) \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j}.$$

Here we require the following properties to hold:

(a) bilinearity and skew-symmetry

$$(2) \qquad \{f, g\} = -\{g, f\},$$

(b) the Leibniz identity

$$(3) \qquad \{fg, h\} = g\{f, h\} + f\{g, h\},$$

(c) the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{4}$$

By definition, Hamiltonian systems have the form

$$\dot{f} = \{f, H\}, \tag{5}$$

where f is any function and H is the Hamiltonian.

It can happen that there are non-trivial functions f_q (possibly, defined locally on the manifold) such that

$$\{f_q, g\} = 0 \tag{6}$$

for any function $g(y)$. In this case the Poisson bracket is said to be "degenerate"; the matrix $h^{ij}(y)$ is degenerate. After finding all such quantities $f_l(y)$ then on their common level surface

$$f_l(y) = \text{const} \qquad (l = 1, 2, \dots). \tag{7}$$

the Poisson bracket becomes non-degenerate.

Let z^q be coordinates on the level surface (7). The restriction of the tensor $h^{qt}(z)$ to this surface is non-degenerate, and there is an inverse matrix

$$h_{qp} h^{pt} = \delta_q^t, \tag{8}$$

which determines the 2-form

$$\Omega = h_{qp}(z) \, dz^q \wedge dz^p. \tag{9}$$

From (4) it follows that the form Ω is closed:

$$d\Omega = 0 \leftrightarrow \frac{\partial h_{qp}}{\partial z^t} + \frac{\partial h_{tq}}{\partial z^p} + \frac{\partial h_{pt}}{\partial z^q} = 0. \tag{10}$$

Let us consider the main types of phase spaces.

Type I: The classical Hamiltonian formalism and variational principles.
Suppose that $(y) = (x^1, \dots, x^n, p_1, \dots, p_n)$ and that the matrix h^{ij} is constant and non-degenerate:

$$h^{ij} = h_{ij} = \begin{pmatrix} & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \\ -1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & -1 & & & \end{pmatrix} = \text{const}. \tag{11}$$

The equations (6) take the form $\dot{x}^i = -\partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial x^i$.

The coordinates (x, p) are said to be canonical. Locally they can always be found for non-degenerate Poisson brackets (Darboux's theorem).

If $H(x, p)$ is a Hamiltonian, then we have the Lagrangian $L(x, \dot{x})$, where x is the configuration space coordinate, which can be defined by

$$\dot{x}^i = \frac{\partial H}{\partial p_i}(x, p), \qquad L = p_i \dot{x}^i - H. \tag{12}$$

We assume that the equation $\dot{x}^i = \partial H / \partial p_i$ can be solved for the variables p_i. The Hamiltonian equations (12) are obtained from the variational principle $\delta S = 0$, where

$$(13) \qquad\qquad S = \int L(x, \dot{x}) \, dt.$$

Type II. The Hamiltonian formalism and Lie algebras.[1] We consider now the following (second) case in order of complexity, when the tensor h^{ij} is not constant, but depends linearly on the coordinate (y):

$$(14) \qquad\qquad h^{ij}(y) = C_k^{ij} y^k, \qquad C_k^{ij} = \text{const}.$$

We consider the set L of all linear functions on the phase space, which we denote by L^*. For the basis linear forms (the coordinates y^i) we define the operation of "commutation".

$$(15) \qquad\qquad [y^i, y^j] = C_k^{ij} y^k = \{y^i, y^j\}.$$

From (2) and (4) it follows that the operation (15) turns the linear space L into a Lie algebra for which the dual space L^* is the phase space for the Poisson bracket (14). A bracket of this kind was first considered by Berezin. It was used by Kirillov and Kostant (in the less convenient language of symplectic manifolds) in the theory of infinite-dimensional representations of Lie groups.

Example 1. A basic example of the Hamiltonian formalism of Type I is the phase space $T^*(M)$: the space of covectors (with subscripts) on the manifold M (the configuration space). The manifold M can be infinite-dimensional (a space of fields $q(x)$ in which x is one of the "indices" in the formulae). In the finite-dimensional case there are the local coordinates x^i and the conjugate momenta p_i with the Poisson brackets

$$(16) \qquad\qquad \{x^i, x^j\} = \{p_i, p_j\} = 0, \qquad \{x^i, p_j\} = \delta_j^i$$

and the form

$$\Omega_0 = \sum dx^i \wedge dp_i.$$

In the infinite-dimensional case there are two fields and Poisson brackets of the form

$$(17) \qquad\qquad \begin{cases} \{q^i(x), p_j(y)\} = \delta_j^i \delta(x - y), \\ \{q^i(x), q^j(y)\} = \{p_i(x), p_j(y)\} = 0. \end{cases}$$

Example 2. It is also useful to consider a Poisson bracket of the form (18) with an additional "external field" $F_{ij}(x)$:

$$(18) \qquad\qquad \{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = F_{ij}(x),$$

where the 2-form $F = F_{ij} \, dx^i \wedge dx^j$ is closed:

$$dF = 0.$$

Then we have the 2-form

$$(19) \qquad\qquad \Omega = \sum dx^i \wedge dp_j + \sum F_{ij} \, dx^i \wedge dx^j = \Omega_0 + F.$$

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[1]The third case in order of complexity, when the tensor $h^{ij}(x)$ depends quadratically on x, is also very interesting and has been studied recently (Sklyapin, Faddeev).

The equations of motion with a Hamiltonian $H(x,p)$ and a Poisson bracket (18) are (for $n = 2$ or 3) the equations of motion of a charged particle in an external magnetic field F_{ij} (or an electromagnetic field for $n = 4$). In the domain where $F = dA$, (19) reduces to the standard form (16).

Example 3. Rather more general *a priori* but as a rule reducible to the form (18) are Poisson brackets on the space $T^*(M)$ satisfying the following requirement: any pair of functions (f, g) on the base M (independent of the variables p_i on the *fiber* consisting of all covectors with a lower index) has a vanishing Poisson bracket:

$$(20) \qquad \{f, g\} = 0.$$

We call (20) "variational admissibility" of the Poisson bracket on $T^*(M)$. Clearly, the bracket (18) is variationally admissible. As we know, on sufficiently small domains any (non-degenerate) Poisson bracket reduces to the form (16). Globally this is no longer so; if the form Ω is not exact, then the Poisson bracket does not reduce to the form (16). Variationally admissible Poisson brackets are probably always globally reducible to the form (18), but this has not been proved rigorously; they reduce to the simplest form (16) on any domain when Ω is exact. Let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be local coordinates in a domain U_α such that $\{x^i, x^j\} = 0$ and let $H(x, y)$ be a Hamiltonian. We consider half of the Hamilton equations

$$(21) \qquad \dot{x}^j = \{x^j, H\}.$$

We assume that (21) allows us to express the variables (y^i) uniquely in terms of $(x, \dot{x})$:

$$(22) \qquad y^i = F^i(x, \dot{x}).$$

The other half of the Hamilton equations

$$\dot{y}^j = \{y^j, H\} = G^j(x, y)$$

now reduces by (22) to the second-order system

$$(23) \qquad F^j(x, \dot{x}) = G^j(x, \dot{x}).$$

Let us now construct the "phase Lagrangian" $L(x, y)\, dt = -H\, dt + \omega_\alpha$, where $d\omega_\alpha = \Omega$ in U_α. We express L in terms of $(x, \dot{x})$, using (22).

Lemma. *The equations* (23) *are equations of the extremals for the Lagrangian variational principle* $\delta S = 0$, $S = \int L(x, \dot{x})\, dt$.

These are the elementary properties of variationally admissible Poisson brackets.

We now pass on to discuss examples of Poisson brackets of Type II associated with Lie algebras.

Example 1. Let L be the Lie algebra of the group SO_3. The Killing metric is Euclidean and allows us to identify L with L^*. The Poisson bracket of the basis functions M_i on L^* has the form

$$(24) \qquad \{M_i, M_j\} = \varepsilon_{ijk} M_k, \qquad c_k^{ij} = \varepsilon_{ijk} = \pm 1.$$

The function $M^2 = \sum M_i^2$ is such that

$$(25) \qquad \{M^2, M_i\} = 0 \qquad (i = 1, 2, 3).$$

Hamiltonian systems on L^* have the form

$$(26) \qquad \dot{M}_i = \{M_i, H(M)\}.$$

Let $\Omega^i = \partial H/\partial M_i$; the Killing metric allows us to identify upper and lower indices. The equations (26) reduce to the "Euler equations"

$$(27) \qquad\qquad \dot{M} = [M, \Omega].$$

This conclusion holds for all compact (semisimple) Lie groups on which the Killing metric is Euclidean (pseudo-Euclidean) on the Lie algebra and invariant under inner automorphisms

$$(28) \qquad\qquad L \to gLg^{-1},$$

where g is an element of the Lie group and L its Lie algebra. Arnol'd calls such systems for the groups SO_N "many-dimensional analogues of a rigid body" if the Hamiltonian is a quadratic form on the space of skew-symmetric matrices $(a_{ij}) = (-a_{ji})$, where $M = (M_{ij})$

$$(29) \qquad\qquad H(M) = \sum_{i<j} d_{ij} M_{ij}^2,$$

and

$$(30) \qquad\qquad d_{ij} = q_i + q_j, \qquad q_i > 0.$$

Now it is known that all systems of the form (30) on the Lie aigebra SO_N are completely integrable [46]. Moreover, according to [46], a sufficient condition for integrability is[2] that

$$(31) \qquad\qquad d_{ij} = \frac{\bar{a}_i - \bar{a}_j}{\bar{b}_i - \bar{b}_j}.$$

The idea of [46] is as follows. Under the conditions (31) the Euler equation (27) can be represented as a statinary problem for first-order metric systems in the (x,t)-space admitting an "$L - A$"-pair, or the method of the inverse problem (see [47]). In accordance with the formalism of integration of stationary problems [16] there arises the matrix equation:[3]

$$(32) \qquad\qquad \frac{d}{dx}(M - \lambda a) = [M - \lambda a, \Omega - \lambda b],$$
$$a = (a_{ij}), \quad b = (b_{ij}), \quad a_{ij} = \bar{a}_i \delta_{ij}, \quad b_{ij} = \bar{b}_i \delta_{ij}.$$

The coefficients of the polynomial

$$(33) \qquad\qquad P(\lambda, \mu) = \det(\mu \cdot 1 - M - \lambda a)$$

are integrals of (27) "in involution", that is, have zero Poisson brackets between pairs. A complete set of formulae of the motion can be obtained in terms of the θ-function associated with the Riemann surface $P(\lambda, \mu) = 0$, starting from the methods of [16] and ending in [49] for first-order matrix systems. We recall that

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[2]If $\bar{a}_i = \bar{b}_i^2$, then we have (30), $q_i = \bar{b}$. The Liouville integrability for SO_4 under the condition (30) was first established in [51] and [52]. However, the connection with the method of the inverse problem and the theory of θ-functions of Riemann surfaces remained unknown; for this reason they did not succeed in obtaining an explicit integration, even in this simplest case.

[3]To a reader unfamiliar with the method of the inverse problem (see [47]) the emergence of equations of the type (32) may seem incomprehensible; in this case, to understand what follows he must start directly from (32) as a formal identity whose verification represents no difficulty once it is written down.

the Poisson bracket (15) is invariant only under the transformations (28). For the classical Euler equations for the free rotation of a rigid body we have

$$(34) \qquad G = \mathrm{SO}_3, \qquad H = \sum \frac{a_i M_i^2}{2}, \qquad \Omega = \frac{\partial H}{\partial M},$$

where Ω is the angular velocity of the body and M is the angular momentum.

Example 2. Some important systems arising in hydrodynamics are connected with the Lie algebra L of the group $E(3)$ of motions of the Euclidean space $\mathbb{R}^3$. This algebra is no longer semisimple. On the phase space L^* there are 6 coordinates $(M_1, M_2, M_3, p_1, p_2, p_3)$ and the Poisson brackets

$$(35) \qquad \{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0.$$

The bracket (35) has two independent functions $f_1 = p^2 = \sum p_i^2$, $f_2 = ps = \sum M_i p_i$ such that

$$(36) \qquad \{f_q, M_i\} = \{f_q, p_i\} = 0 \qquad (q = 1, 2, \quad i = 1, 2, 3).$$

Let $H(M, p)$ be the Hamiltonian. We write $u^i = \partial H / \partial p_i$, $\omega^i = \partial H / \partial M_i$. The Hamilton equations assume the "Kirchhoff" form

$$(37) \qquad \dot{p} = [p \times \omega], \qquad \dot{M} = [M \times \omega] + [p \times u].$$

The equations (37) coincide (for a quadratic Hamiltonian H) with the Kirchhoff equations for the motion of a rigid body in an ideal incompressible fluid at rest at infinity [4]. The motion of the liquid itself is assumed to be of potential form. In this case H is the energy, M and p are the total *angular* and *linear momentum* of the system, the body being identified with a moving system of coordinates rigidly attached to it. The energy H is assumed to be positive and quadratic in both variables (M, p). By transformations of the form (28) H can be brought to the form

$$(38) \qquad 2H = \sum a_i M_i^2 + \sum b_{ij}(p_i M_j + M_i p_j) + \sum c_{ij} p_i p_j.$$

Even in classical hydrodynamics non-trivial integrable cases were discovered of Hamiltonians of the form (38) for the algebra $L = E(3)$. These cases of Clebsch and Steklov do not reduce to an "obvious" group symmetry. We are especially interested in the case of Clebsch, in which the diagonality relations

$$(39) \qquad b_{ij} = 0, \qquad c_{ij} = \bar{c}_i \delta_{ij}$$

hold as well as the "Clebsch relations"

$$(40) \qquad \sum_{i=1}^{i=3} \bar{c}_i a_i (\bar{c}_{i+1} - \bar{c}_{i-1}) = 0 \qquad (i + 3 \approx i)$$

The Lie algebra of SO_4 is obtained from $E(3)$ by "deformation" or, conversely, L can be obtained from SO_4 be "retraction". The precise meaning of this is the following: in SO_4 we choose a basis (e_i', e_i'') such that

$$(41) \qquad \begin{cases} [e_i', e_j'] = [e_i'', e_j''] = \varepsilon_{ijk} e_k', \\ [e_i', e_j''] = \varepsilon_{ijk} e_k'' \end{cases}$$

Changing to a new basis $\bar{e}_i' = e_i'$, $\bar{e}_i'' = \alpha e_i''$ we obtain

$$(42) \qquad \begin{cases} [\bar{e}_i', \bar{e}_j'] = \varepsilon_{ijk} \bar{e}_k, \qquad [\bar{e}_i', \bar{e}_j''] = \varepsilon_{ijk} \bar{e}_k'', \\ [\bar{e}_i'', \bar{e}_k''] = \varepsilon_{ijk} \cdot \alpha^2 \bar{e}_j'. \end{cases}$$

Letting $\alpha \to 0$, we obtain from (42) the relations (35), which define the algebra $L = \mathrm{E}(3)$. We now write in (29)

$$(43) \qquad \begin{cases} d_{i0} = c_i & (i = 1, 2, 3), \\ d_{12} = a_3, \quad d_{23} = a_1, \quad d_{13} = a_2. \end{cases}$$

The quantities (31) are connected by the following relation for SO_4

$$(44) \qquad 0 = \sum_{\substack{i=1 \\ i \approx i+3}}^{i=3} c_i a_i (c_{i+1} - c_{i-1} + a_{i+1} - a_{i-1}).$$

Let us complete the retraction of SO_4 to $L = \mathrm{E}(3)$ according to (41) and (42); we require that the quadratic form (29) has a finite limit under this limit passage. For this, under the conditions (43) it is then necessary that the a_i are finite as $\alpha \to 0$ and the c_i are of order $c_i \sim \bar{c}_i \alpha^{-2}$. As $\alpha \to 0$, we obtain from (44) precisely the Clebsch relation $(40)^4$. So we arrive at the result due to Novikov and Golo:

The recently discovered cases of integrability of systems on SO_4 *are deformations of the classical Clebsch case.*

For diagonal Hamiltonians of the form (39) on $L = \mathrm{E}(3)$ (more precisely, on L^*) that do not satisfy the Clebsch condition, the absence of "superfluous" analytic integrals of motion has recently been proved [50]. Thus, "general" diagonal Hamiltonians on L are non-integrable.

We consider two other applications of (37):

(A) The equations of motion of a rigid body with a fixed point in a strong axially symmetric field with potential $W(z)$ reduce to (37). The corresponding Hamiltonian is

$$(45) \qquad H = \sum a_i M_i^2 / 2 + W(l^i p_i),$$

where l^i is the vector determined by the position of the centre of mass with respect to the principal axes of inertia at the fixed point. The quantities p_i are dimensionless and cannot be interpreted physically as *momenta*. They are the direction cosines of a unit vector, that is,

$$(46) \qquad f_1 = p^2 = 1.$$

(B) The (Leggett) equation of the spin dynamics in the A-phase of the *superfluid* ^{3}He can also be reduced to the form (37); this is the dynamics of the spin variables of the vectors (s, d), where $d^2 = 1$, by analogy with (46). (See the survey by Brinkman and Cross in [5].) On transition to the Leggett equations for nuclear magnetic resonance in the A-phase one must alter the notation (S is the "magnetic moment")

$$(47) \qquad M_i \to s_i, \qquad p_i \to d_i,$$

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[4]We note that the coefficients of the "stationary $L - A$-pair" (32) diverge under the retraction $\alpha \to 0$, although the integrals of the motion converge. In this connection, recently in [53] another matrix representation depending on λ of the Kirchhoff equations for the Clebsch case has been constructed. By way of contrast, this representation does not admit a deformation for $\alpha \neq 0$ in any non-trivial way.

and consider the Hamiltonian of the form

$$(48) \qquad H = \frac{1}{2}as^2 + b\left(\sum s_i d_i\right)^2 + \lambda\left(\sum s_i H_i\right) + W(d).$$

Here a, b, and λ are constants, H^i is the external magnetic field, and the potential W has the form

$$(49) \qquad W(d) = \text{const}(l^i d_i)^2.$$

By the property (36) of the Poisson bracket (35), $f_2 = \sum s_i d_i$ is equivalent to a constant in the equations of motion. Therefore, the second term can simply be deleted from the Hamiltonian:

$$(50) \qquad H \sim H' = \frac{1}{2}as^2 + \lambda\sum s_i H_i + W(d).$$

The quantity d, the spin part of the so-called "order parameter", is a unit vector, $d^2 = 1$, as was mentioned above.

We consider a very important example (though not associated with Lie algebras).

There is another phase, the B-phase, of ^{3}He, in which the Leggett equation takes a form that is not similar to the classical top (see, for example, the survey by Brinkman and Cross in [5]).

In a state of hydrodynamic equilibrium and with non-zero spin, the state in the B-phase is defined by a pair comprising a rotation matrix $R = (R_{ij}) \in SO_3$ and a "*magnetic moment*" $s = (s_i)$ $(i = 1, 2, 3)$.

The variables s_i are coordinates in the dual space of the Lie algebra of SO_4 similar to the angular momentum components M_i. The standard Poisson brackets for $T^*(SO_3)$ in the variables (s_i, R_{jk}) can be written:

$$(51) \qquad \begin{cases} \{s_i, s_j\} = \varepsilon_{ijk}s_k, \quad \{R_{ij}, R_{kl}\} = 0, \\ \{s_i, R_{jl}\} = \varepsilon_{ijk}R_{kl}. \end{cases}$$

The Leggett Hamiltonian in the B-phase in an external magnetic field has the form

$$(52) \qquad H = \frac{1}{2}as^2 + b\sum s_i F_i + V(\cos\Theta),$$

where a and b are constants, $F = (F_i)$ is the external field, and

$$(53) \qquad V(\cos\Theta) = \text{const}\left(\frac{1}{2} + 2\cos\Theta\right)^2,$$

R_{ij} is the rotation through the angle Θ around the axis n_i, $n^2 = 1$:

$$R_{ij} = \cos\Theta\delta_{ij} + (1 - \cos\Theta)n_in_j + \sin\Theta\,\varepsilon_{ijk}n_k,$$
$$(54) \qquad\qquad 1 + 2\cos\Theta = R_{ii} = \text{Sp}\,R.$$

After the substitution

$$(55) \qquad as_i = \omega_i, \qquad \Omega_{jk} = \varepsilon_{jki}\omega_i = (\dot{R}R^{-1})_{jk}$$

we obtain a Lagrangian system in the variables $(R_{ij}, \dot{R}_{ij})$ on $T^*(SO_3)$ where the kinetic energy is defined by the 2-sided invariant Killing measure, and the potential $V(\cos\Theta)$ is invariant under the inner automorphisms

$$(56) \qquad R \to gRg^{-1}, \quad s \to gs, \quad g \in SO_3.$$

If the field $F = (F_i)$ is constant, then the Lagrangian is invariant under the one-parameter group of transformations (56), where g belongs to the group of rotations around the axis of F. Let $F = (F, 0, 0)$.

When $F = 0$, the system admits the group SO_3 of transformations (56) and is completely integrated in [6]. The transformations (56) generate the conserved vector (when $F = 0$):

$$(57) \qquad A = (A_j) = (1 - \cos \Theta) \left[n \times \left(\cot \frac{\Theta}{2} S + [n \times S] \right) \right]_j,$$

with the same Poisson brackets as for the usual angular momentum:

$$(58) \qquad \begin{cases} \{A_i, A_j\} = \varepsilon_{ijk} A_k, \\ \left\{ A_i, \frac{1}{2} a s^2 + V(\cos \Theta) \right\} = 0, \end{cases}$$

As Golo has shown [7], when $F = 0$, the variables s^2 and Θ in the Hamiltonian generate a closed algebra of Poisson brackets $\{s^2, s_\parallel, \Theta\}$, where

$$(59) \qquad \begin{cases} s_\parallel = \sum s_i n_i, \\ \{s^2, \Theta\} = 2 s_\parallel, \qquad \{s_\parallel, \Theta\} = 1, \\ \{s^2, s_\parallel\} = \dfrac{1 + \cos \Theta}{\sin \Theta} (s^2 - s_\parallel^2), \end{cases}$$

The quantity $A^2 = \sum A_i^2 = (1 - \cos \Theta)(s^2 - s_\parallel^2)$ has vanishing Poisson brackets with all generators of this subalgebra

$$(60) \qquad \{A^2, s^2\} - \{A^2, s_\parallel\} - \{A^2, \Theta\} = 0.$$

In a non-zero magnetic field $(F, 0, 0)$ there remains only one integral apart from the energy[5]

$$(61) \qquad \{A_1, H\} = 0.$$

The system becomes non-integrable. In this case it seems to be possible to complete (globally) the procedure of "factorization of the Hamiltonian formalism" and to reduce the system to 2 degrees of freedom.

The integral A_1 generates the group (56), where g is a rotation about the axis $n = (1, 0, 0)$. The invariant variables under this subgroup are

$$(62) \qquad s_2, \ s_\parallel, \ \Theta, \ u_1, \ s_1, \ \tau = s_2 n_3 - n_2 s_3$$

with the purely geometrical constraint

$$(63) \qquad s^2 \tau^2 = (s^2 - s_1^2)(s^2 - s_\parallel^2) - (s^2 n_1 - s_1 s_\parallel)^2.$$

It is easy to check that the variables (62) form a closed algebra of Poisson brackets, containing the Hamiltonian H (52) and having the functional dimension 5. The quantity A_1 in this algebra has vanishing brackets with all the variables;

$$(64) \qquad 0 = \{A_1, s^2\} = \{A_1, s_\parallel\} = \{A_1, \Theta\} = \{A_1, n_1\} = \{A_1, s_1\}.$$

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[5]For large fields $F \to \infty$ system has been studied in [8] with viscosity taken into account.

Therefore, by imposing the condition $A_1 = \text{const}$ we can, as before, formally use the formulae for the Poisson brackets of the quantities (62), which arise from (51). Under the condition $A_1 = \text{const}$ we choose as basis the following variables:

$$(65) \qquad (A^2, s_\parallel, \Theta, n_1), \qquad n_1 = n.$$

Their brackets have the form

$$(66) \qquad \begin{cases} \{s_\parallel, \Theta\} = 1, \qquad \{\Theta, n\} = 0, \\ \{A^2, s^2\} = \{A^2, \Theta\} = \{A^2, s_\parallel\} = 0, \\ \{A^2, n\} = \sqrt{\dfrac{1}{2}(1 - n^2)A^2 - \dfrac{1}{4}A_1^2} \end{cases}$$

Thus, the canonical variables can be chosen in the form

$$(67) \qquad \begin{cases} x^1 = \Theta, \quad \xi_1 = p_\Theta = s_\parallel, \\ x^2 = n, \quad \xi_2 = p_n = \sqrt{\dfrac{2A^2}{1 - n^2} - \dfrac{A_1^2}{(1 - n^2)^2}}. \end{cases}$$

The Hamiltonian becomes

$$(68) \quad H = \frac{1}{2}a\left[p_\Theta^2 + \frac{1 - n^2}{2(1 - \cos\Theta)}\left(p_n^2 - \frac{A_1^2}{(1 - n^2)^2}\right)\right] +$$
$$+ bF\left(np_\Theta + \frac{1 - n^2}{2}\sin\Theta\, p_n + A_1^2\frac{2 - \sin^2\Theta}{2(1 - \cos\Theta)}\right) + V(\cos\Theta).$$

We now introduce the spherical coordinates

$$(69) \qquad \Theta = 2\chi, \qquad n = n_1 = \sin\varphi$$

and go over to the Lagrangian formalism. We obtain

$$(70) \qquad L = 2a(\dot{\chi}^2 + \sin^2\chi\,\dot{\varphi}^2) - \widetilde{A}_1\dot{y}^1 - \widetilde{A}_2\dot{y}^2 - U(y),$$

where $y_1 = \chi$, $y_2 = \varphi$,

$$\widetilde{A}_1 = 2b\sin\varphi, \qquad \widetilde{A}_2 = 8bF\cos\varphi\sin^3\chi\cos\chi,$$
$$(71) \qquad U = V(\cos\Theta) + aA_1^2/4\sin^2\chi\cos^2\chi + bFA_1(1 - \sin^2\chi\cos^2\chi)/2\sin^2\chi -$$
$$- b^2F^2(\sin^2\varphi + 4\cos^2\varphi\sin^3\chi\cos\chi)/2.$$

Thus, we have obtained a system in a domain in the sphere S^2 with the usual metric, in which there is an effective magnetic field and a scalar potential. When $A_1 \neq 0$, this system cannot be extended to the whole sphere, since it is singular at $\varphi = 0, \pi$.

If $A_1 = 0$, then the system is defined on the whole sphere except at the poles, where it has singularities. We note that the great circle $\varphi = 0, \pi$ corresponds to the axis $n = (\pm 1, 0, 0)$; the rotations around this axis correspond to the group of symmetries of the system.

Finding the stationary points of the Hamiltonian system (68) presents no difficulty. They are given by the equations

$$(72) \qquad \frac{\partial H}{\partial\Theta} = \frac{\partial H}{\partial n} = \frac{\partial H}{\partial p_\Theta} = \frac{\partial H}{\partial p_n} = 0.$$

The stationary solutions of (72) are periodic solutions of the original Leggett equations. These exact solutions were not known previously, as Fomin has told the author.

Since we have treated in the text (above) the equations for a magnetic moment under homogeneous nuclear magnetic resonance (NMR) in the superfluid ^{3}He, it is appropriate to recall the definition of the A- and the B-phases. From the microscopic theory of super-conductivity of Bardeen—Cooper—Schriffer, Bogolyubov, Gor'kii, and Anderson one deduces for a coupling with moment $l = 1$ that the superfluid ^{3}He can be described macroscopically in the stationary state according to the Ginzburg—Landau scheme by a (3×3) complex matrix $A_{qj}(x, y, z)$, where the index $q = 1, 2, 3$ refers to the (internal) "spin" space, while the index $j = 1, 2, 3$ refers to physical space. The field $A_{qj}(x, y, z)$ is called the "order parameter". It must minimize the free energy functional, which depends on the temperature, the magnetic field, the pressure, and the other external parameters:

$$(73) \qquad F\{A\} = \int_{R^3} (F_{\text{grad}} + V) \, d^3 x,$$

where

$$(74) \qquad F_{\text{grad}} = \gamma_1(\partial_k \bar{A}_{qj} \partial_k A_{qj}) + \gamma_2(\partial_k \bar{A}_{qj} \partial_j A_{qk}) + \gamma_1(\partial_k \bar{A}_{qk})(\partial_i A_{qi}).$$

In the absence of a magnetic field (assuming also that the dipole energy is small) the potential V has the form

$$(75) \quad V = \alpha \, \text{Sp}(\bar{A}^T A) + \beta_1 |\, \text{Sp} \, AA^T|^2 + \beta_2(\text{Sp}(\bar{A}^T A))^2 +$$
$$+ \beta_3 \, \text{Sp}[(A^T \bar{A})(\overline{\bar{A} A^T})] + \beta_4 \, \text{Sp}[(\bar{A}^T A)^2] + \beta_5 \, \text{Sp}[(\bar{A}^T A)(\overline{\bar{A}^T A})].$$

The exact values of the parameters α, β, γ are undetermined and can vary together with the parameters of the system (the temperature etc.).

The concept of "phase" is defined in the spatially homogeneous state $F_{\text{grad}} = 0$ by minimizing the function $V(A_{qj})$. It is difficult to classify the "phases", that is, the minima of V for all values of the parameters α and β are an unsolved problem. In any case, potentials of the form (75) are invariant under the action of the group

$$(76) \qquad \begin{cases} G = \text{U}_1 \times \text{SO}_3 \times \text{SO}_3, \\ gA = e^{i\varphi} R_1^{-1} A R_2, \\ g = (e^{i\varphi}, R_1, R_2). \end{cases}$$

Therefore, the minima of the potential are manifolds on which G acts. In the case of "general position" they are homogeneous spaces of G. However, there is an important example of the A_1-phase close to the critical temperature when the pressure and the field are small which is defined by a degenerate minimum (by a non-homogeneous submanifold M_{A_1}, in the matrix space

$$(77) \qquad M_{A_1} = \{A_{qj} = \Delta \cdot (d_q e_j^{(1)} + \bar{d}_q e_j^{(2)})\}, \qquad d_q = d'_q + i d''_q,$$
$$e_j^{(\alpha)} = e_j'^{(\alpha)} + e_j''^{(\alpha)}, \quad |e'^{(\alpha)}|^2 = |e''^{(\alpha)}|^2 = 1, \quad (e'^{(\alpha)}, e''^{(\alpha)}) = 0 \quad (\alpha = 1, 2).$$

The more popular A- and B-phases are defined by the G-homogeneous matrix manifolds M_A and M_B consisting of matrices of the form

(78)
$$
\begin{cases}
M_A = \{A_{qj} = 2\Delta d_q(e'_j + ie''_j)\}, & |d|^2 = 1, \\
|e'|^2 = |e''|^2 = 1, & (e', e'') = 0, \quad \Delta = \text{const}, \\
M_A = (S^2 \times \text{SO}_3)/\mathbb{Z}^2
\end{cases}
$$

or $e^{(1)} = e^{(2)} = e = e' + ie''$, $d = \bar{d}$;

(79)
$$
M_B = \{A_{qj} = \Delta/\sqrt{3} \cdot R_{qj}e^{i\varphi}\}, \qquad R \in \text{SO}_3,
$$
$$
\Delta = \text{const}, \qquad M_B = \text{SO}_3 \times \text{U}_1.
$$

Passing on to states depending on (x, y, z) we consider "quasi-homogeneous" states, where the deviation of the field $A_{qj}(x, y, z)$, from a spatially-homogeneous state can be disregarded locally, and we may assume that every $A_{qj}(x, y, z)$ lies in a "phase-manifold" $M_{A_1}, M_A, M_B, \dots$, but changes from point to point. Now $F_{\text{grad}} \neq 0$, although the whole field is regarded as having values only in the phase manifold.

The Euler—Lagrange equation $\delta F_{\text{grad}} = 0$ for fields with values in the manifolds M_{A_1}, M_A, M_B etc., which define the state of the system, are called the Ginzburg—Landau equations. States depending only on the single variable z, "planar textures", lead for the B-phase to the usual equation of the Euler top (here it is even symmetrical). For the A-phase the equations of planar textures are more complex; they have been fully integrated in [43], where one can find references on ^{3}He (see also [5] and [6]). In a magnetic field, in a state with non-zero spin, the functional of free energy becomes more complex; there arises a new variable, the "magnetic moment" S whose dynamic (see above) is used in the so-called "nuclear magnetic resonance".

The planar textures in the A_1-phase are not known, and it would be interesting to study them. In the manifold M_{A_1} there is a singular submanifold

(80)
$$
W: \ e'^{(1)} \times e''^{(2)} = e'^{(2)} \times e''^{(2)}, \qquad W \subset W_{A_1}
$$

The submanifold W has codimension 3, although it is given by two equations in the 8-dimensional manifold M_{A_1}. We have become accustomed to the fact that the number of "Goldstone perturbations" is equal to the dimension of degeneration, that is, the dimension of the vacuum manifold. The dimension of M_{A_1} is 8. However, at points of W the number of Goldstone modes turns out to be 9, as Volovik and Fomin have communicated to me. In the given case, the dimension 9 coincides with the dimension of the "tangent space" to M_{A_1} at points of W in the sense of algebraic geometry. Apparently, the number of Goldstone modes always coincides with the dimension of the tangent space of algebraic geometry. In all previously known cases in field theory the vacuum manifold was homogeneous and hence non-singular.

§ 2. The Hamiltonian formalism of systems of hydrodynamic origin

In this survey we do not discuss any new results on hydrodynamic systems (with the exception of the Kirchhoff system already introduced in § 1), and this section is purely methodological in character. The Hamiltonian formalism has already been worked out long ago in the language of the so-called "Clebsch variables" for various types of ideal fluids (see below). However, as will become clear, these field variables cannot always be introduced, and if they can, then frequently only locally. Here

the Clebsch variables are extremely unstable under a change of the type of the system: the addition to the system of a superfluous field (for example, even the transition from an incompressible fluid to a weakly compressible one in which the density and entropy are the new field variables) leads to a non-local and by no means small change in the Clebsch variables. Besides, in several cases the number of field variables is odd. In the latter case one must introduce superfluous fictitious degrees of freedom to define the Clebsch variables; they represent a system with a large, complicated and poorly understood "calibrated" freedom. Consequently, an invariant exposition of the Hamiltonian formalism of hydrodynamic systems is useful. In the incompressible case such an invariant exposition can be found in [19], but its language (that of "symplectic manifolds") seems to be artificially complicated and inconvenient compared with the language of "Poisson brackets". The situation is more complex for compressible fluids and further systems (see below).

The underlying Lie algebra L for hydrodynamic systems over which the subsequent superstructure will be erected, is the algebra of vector fields (we do not yet specify the domain of definition). For vector fields $v^i(x)$, $w^i(x)$ in an n-dimensional space the commutator is

$$(1) \qquad [v, w]^i(x) = v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}.$$

Here the pairs (x, i) (the point x and the index i) act as a single "index". The operation must be expressed in terms of the structure constants in the form

$$(2) \qquad [v, w]^i(x) = \int dy\, dz\, C^i_{jk}(x, y, z) v^i(y) w^k(z).$$

Comparing (1) and (2) we obtain

$$(3) \qquad C^i_{jk}(x, y, z) = \delta^i_j \delta(z - x) \partial_k^{(y)}(y - z) - \delta^i_k \delta(y - x) \partial_j^{(z)} \delta(z - y)$$

$$\partial_j^{(x)} = \frac{\partial}{\partial x^j}, \qquad \int \partial_j^{(z)} \delta(z - x) f(z) = -\frac{\partial f}{\partial z^j}(x).$$

The variables $p_i(x)$ *conjugate* to the velocity components on the dual space L^* to the vector fields $v^i(x)$ must be such that

$$(4) \qquad \int p_i(x) v^i(x)\, d^n x$$

is scalar under change of variables. This means that the variables $p_i(x)$ are densities of covectors, which under changes of variables are additionally multiplied by the Jacobian (we call them momentum densities). According to § 1 (14), the Poisson brackets are of the form

$$(5) \quad \{p_j(y), p_k(z)\} = \int C^i_{jk}(x, y, z) p_i(x)\, d^n x =$$

$$= p_k(y) \partial_j^{(y)} \delta(y - z) - p_j(z) \partial_k^{(z)} \delta(z - y).$$

Here is an important example, the case $n = 1$. Then we obtain

$$(6) \qquad \{p(y), p(z)\} = p(y) \delta'(y - z) - p(z) \delta'(z - y).$$

By making the substitution $p = u^2$ we arrive at the standard Poisson bracket (Gardner, Zakharov, Fadeev) occurring in the theory of the KdV (Korteweg—de Vries) equation, see [9], [10], [11]:

$$\{u(x), u(y)\} = \delta'(x - y), \tag{7}$$

For in the KdV theory it is precisely the quantity $I_0 = \int u^2 dx$ that plays the role of the momentum (see [11]). The Poisson bracket of two functionals has the form

$$\{J, I\} = \int \frac{\delta J}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta I}{\delta u(x)} dx. \tag{8}$$

Since the operator $\partial/\partial x$ has the constants as non-trivial kernel, there is a quantity $I_{-1} = \int u \, dx$ such that

$$\{J, I_{-1}\} = 0 \tag{9}$$

for any functional J. The KdV equation itself is given by the Hamiltonian

$$I_1 = H = \int \left(\frac{u'^2}{2} + u^3 \right) dx, \tag{10}$$

$$\dot{u} = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)} = 6u u_x - u_{xxx}.$$

It is curious that one of the phenomena of the integrability of the KdV equation by the *method of the inverse problem* is the presence of another local Poisson bracket of the two functionals [12] and even of a family of brackets with the operators $A + \lambda \partial/\partial x$;

$$\begin{cases} A = -\dfrac{\partial^3}{\partial x^3} + 2 \left(u\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial x}u \right), \\ \{J, I\}_2 = \displaystyle\int \dfrac{\delta J}{\delta u(x)} A \dfrac{\delta I}{\delta u(x)} dx. \end{cases} \tag{11}$$

The operators $A + \lambda \partial/\partial x$ are obtained from A by the substitution $u \to u + \text{const}$. The KdV equation itself has the following form in the new Hamiltonian structure:

$$\dot{u} = A \frac{\delta I_0/2}{\delta u(x)}, \qquad I_0 = \int u^2 dx. \tag{12}$$

A further investigation of systems that are Hamiltonian for a family of Poisson brackets can be found in [15].

Note 1 (Adler, Manin, Lebedev). We mention (although this adds nothing new to the construction of solutions of non-linear equations by the inverse problem method) that from the purely algebraic point of view the integrable systems can be interpreted, starting from standard properties of "transformation operators" [48], as systems on phase spaces of type L^* for Lie algebras of Volterra integral ("upper triangular") operators L with the corresponding Hamiltonian formalism (see [13], [14]); the set of Poisson brackets arising here was already known earlier [11]). However, this algebraic interpretation does not completely cover the algebraic essence of the Hamiltonian formalism in the method of the inverse problem.

Note 2 (Bogoyavlenskii, Novikov). It is appropriate to note here another interesting phenomenon arising in KdV theory: the connection between the Hamiltonian formalisms of stationary and non-stationary problems for Hamiltonian systems given

by the Poisson bracket (7); suppose that we are given a system

$$(13) \qquad u_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}, \qquad H = \int P(u, u_x, \dots, u^{(n)}) \, dx,$$

where the Hamiltonian H has the form (13) and P is a polynomial with constant coefficients, and an integral of it, that is, a functional of the same form $J = \int Q(u, u_x, \dots, u^{(n)}) \, dx$ such that $\{J, H\} = 0$. We consider the stationary equation $u_t = 0$ or

$$(14) \qquad \frac{\delta(H + \lambda I_{-1})}{\delta u(x)} = 0.$$

Since $\{J, H\} = 0$, we have for any function $u(x)$ the identity

$$(15) \qquad \left(\frac{\partial}{\partial x} \frac{\delta J}{\delta u(x)} \right) \left(\frac{\delta(H + \lambda I_{-1})}{\delta u(x)} \right) = \frac{\partial}{\partial x} T_\lambda(u, u_x, \dots).$$

Therefore, T_λ is an integral of (14). For the translation group $J = I_0$ the integral T_λ is the energy for (14).

We consider a *flow* with the Hamiltonian J in the Poisson bracket (8) that commutes with the initial flow (13):

$$(16) \qquad u_\tau = \frac{\partial}{\partial x} \frac{\delta J}{\delta u(x)}.$$

Proposition. *The restriction of the flow (16) to the finite-dimensional phase space of the stationary system (14) is also Hamiltonian in the new bracket and is generated in Hamiltonian fashion by T_λ (see [11], [16], [17]).*

Apparently this is true for a wide class of Poisson brackets (see [18] and a number of other papers quoted there).

We return to systems of hydrodynamic type. In the algebra of vector fields L in a Euclidean space (in which are distinguished the Euclidean metric and the element of volume, namely, the mass density, which is assumed to be constant) we specify the subalgebra of *divergence-free* fields $L_0 \subset L$ by

$$(17) \qquad \partial_i v^i = 0.$$

As is easy to see, the dual space L_0^* is obtained by

$$(18) \qquad L_0^* = L^*/(\partial_i \varphi).$$

By (17), the momentum densities $p_i(x)$ give trivial linear forms on L_0 if $p_i = \partial_i \varphi$:

$$(19) \qquad 0 = \int p_i v^i \, d^n x = \int v^i \, \partial_i \varphi \, d^n x = - \int \varphi \, \partial_i v^i \, d^n x.$$

The hydrodynamical Euler equation for an ideal incompressible fluid, as a Hamiltonian system, can be written (see [19]) in the space $L_0^* = L^*/\partial_i \varphi$ with a Hamiltonian of the form

$$(20) \qquad H = \int \rho \, \frac{v^2}{2} \, d^n x, \quad \rho = \text{const}, \quad \partial_i v^i = 0, \quad p_i = \rho v^i$$

and the Poisson brackets (5). These equations are always written on the complete space L^*, which in this case is equivalent to the space of velocities:

$$(21) \qquad \begin{cases} \rho v_t^i = \{p_i, H\} + \partial_i p, \\ \partial_i v^i = 0. \end{cases}$$

The terms $\partial_i p$ come from the transition from L_0^* to L^*, where quantities of the form $\partial_i \varphi$ are equivalent to zero. Here, the pressure p is in principle determined by $\partial_i p$ only. On the space L_0^* we can write the Poisson bracket in the form

(22)
$$\begin{cases} \{v_i(x), v_j(x)\} = \dfrac{1}{\rho}(\partial_i v_j - \partial_j v_i)\delta(x - y), \\ p_i = \rho v_i, \end{cases}$$

where $\rho = \text{const}$.

In fact, in the presence of boundary conditions the velocity $v_j(x)$ or an incompressible fluid is determined by the *vortex* $\Omega_{ij} = \partial_i v_j - \partial_j v_i$ under the condition $\partial_i v_i = 0$.

Example. The case $n = 2$. When $n = 2$ the vortex Ω_{12} reduces to a single scalar function $\Omega_{12} = f(x)$. Thus, for $n = 2$ the Poisson bracket (22) reduces to a Poisson bracket for scalar functions $f(x)$. It has the form

(23)
$$\{f(x), f(y)\} = \partial_1 f \partial_2 \delta(x - y) - \partial_2 f \partial_1 \delta(x - y).$$

However, a Hamiltonian H of the form (20) becomes complicated in the "vortex" variables. For the "finite-dimensional" case we have a set of discrete *vortices* ($x = x^1$, $y = x^2$):

(24)
$$\Omega_{12} = f(x, y) = \sum_{\alpha=1}^{N} q_\alpha \delta(x - x_\alpha)\delta(y - y_\alpha).$$

For such f, assuming the q_α to be constant, we obtain from (23) the usual $2N$-dimensional phase space with the canonical variables

(25)
$$x_1, \ \ldots, \ x_N, \ p_1 = y_1, \ \ldots, \ p_N = y_N$$

(that is, the coordinates x and y are canonically conjugate in the plane). The Hamiltonian of the system of vortices has the form

(26)
$$H = \sum_{\alpha > \beta} q_\alpha q_\beta \log \sqrt{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}.$$

For a 3-dimensional incompressible fluid one can introduce the canonical "Clebsch variables" locally, starting from the representation

(27)
$$\begin{cases} p_i = \psi \partial_2 \varphi \quad (\mathrm{mod}\ \rho \partial_i f), \\ p_i = \rho v_i, \qquad \rho = \text{const}, \\ \dfrac{1}{\rho} d\psi \wedge d\varphi = \Omega_{ij}\, dx^i \wedge dx^j = d(v_i\, dx^i), \\ \{\psi(x), \psi(x')\} = \{\varphi(x), \varphi(x')\} = 0, \\ \{\psi(x), \varphi(x')\} = \delta(x - x'). \end{cases}$$

Since $\partial_i p_j - \partial_j p_i = \rho \Omega_{ij}$, we find that the vortex lines are given by

(28)
$$\varphi = \text{const}, \qquad \psi = \text{const},$$

because
$$\rho \Omega_{ij}\, dx^i \wedge dx^j = d\psi \wedge d\varphi.$$

Thus, we arrive at the conclusion: the canonical Clebsch variables can be introduced globally if (and only if) the form Ω_{ij} can be decomposed into the product of two 1-forms; the decomposition gives a mapping of the domain under investigation into

a two-dimensional space (for example, into a domain of a plane, a sphere, or a torus) such that the vortex lines are inverse images of points. Hence we conclude:

if the vortex lines are entangled and form a complex dynamical system, then the Clebsch variables cannot be introduced globally.

We now return to the Lie algebra L of all smooth vector fields. We consider a simple example: the simplest Hamiltonian on the Lie algebra L of all vector fields and the phase space L^* without derivative of momenta is the Hamiltonian of non-interacting sound waves, of the "background gas"

$$(29) \qquad\qquad H = \int c(x)|p|\, d^n x.$$

In this case the Hamilton equations are easily integrated; for $c = \text{const}$ the solution is given by the standard substitution

$$(30) \qquad\qquad v(x,t) = v_0(x - tv(x,t)),$$

where

$$v(x,t) = cp/|p|, \qquad v_0 = v(x,0).$$

The solution (30) means that the particles conserve momentum and their motion is free and rectilinear; if $c(x) \neq \text{const}$, then the motion is also free, but proceeds along a geodesic of the metric $g_{ij} = c(x)\delta_{ij}$, similarly to *Fermat's principle*.

In spite of its evident meaning, the formula (30) contains topologically non-trivial possibilities, if we wish to know the solution $v(x,t)$. For any x and t there is the mapping $F_{x,t}: S^{n-1} \to S^{n-1}$ given by $F_{x,t}(m) = v_0(x - mt)$ for a unit vector m. The fixed points of

$$(31) \qquad\qquad m = v_0(x - mt) = F_{x,t}(m)$$

also give the solution $m = v(x,t)$. If $p(x)$ vanishes nowhere for $t = 0$, then the degree $\deg F_{x,t}$ is always 0.

The Hamiltonian formalism for an ideal compressible fluid cannot be realized on the algebra L; this is a special case of the Hamiltonian formalism for fluids with internal degrees of freedom. Two of the more complicated systems of this kind are the magnetohydrodynamics, where the magnetic field is "frozen" into the particles of the fluid [21], and also the superfluid ^{4}He, which has an internal degree of freedom of quantum provenance [22]. A number of more complicated systems are now known (spin glasses, rigid bodies with dislocations and disclinations, and anisotropic phases of the superfluid ^{3}He; see [23]–[25]). Of course, in real systems in addition to the Hamiltonian part there are "viscous" terms in the equations. However, even when these are large, the approximate Hamiltonian formalism enables us to predict correctly (we hope) the structure of the equations of motion themselves for which in certain cases, for example ^{3}He, there is as yet no alternative.

Example 1. *A classical compressible fluid.* However, we are now interested in the fact that even an ordinary compressible fluid has such internal degrees of freedom: the mass density ρ and the entropy density s, and if we wish to include them, we have to extend the Lie algebra of vector fields. To the vector fields v^i we add another pair of fields v^ρ and v^s with commutators of the form

$$(32) \qquad [(v, v^\rho, v^s), (w, w^\rho, w^s)] = ([v,w],\; v^i\partial_i w^\rho - w^i\partial_i v^\rho,\; v^i\partial_i w^s - w^i\partial_i v^s).$$

We denote the algebra (32) by $L_{\rho,s}$, and the variables in the dual space $L_{\rho,s}^*$ by ρ (mass density) and s (entropy density) with Poisson brackets (the velocities are here the covectors $v_i = p_i \rho^{-1}$):

$$(33) \qquad \begin{cases} \{p_i(x), \rho(y)\} = \rho(x)\partial_i \delta(y - x), \\ \{p_i(x), s(y)\} = s(x)\partial_i \delta(y - x), \\ \{\rho(x), \rho(y)\} = \{s(x), s(y)\} = \{\rho(x), s(y)\} = 0, \\ \{v_i(x), v_j(y)\} = \dfrac{1}{\rho}\, \Omega_{ij}(x)\delta(x - y). \end{cases}$$

Let $H = \int (1/2\rho p^2 + \varepsilon(\rho, s))\, d^n x$ be this energy. The Euclidean metric contained in the Hamiltonian permits us to identify upper and lower indices.

The quantities $M = \int \rho\, d^n x$ and $S = \int s\, d^n x$ have vanishing Poisson brackets with all functionals ("trivial" conservation laws). The Poisson brackets (33) were chosen essentially so that that mass and entropy are transported with the particles, in contrast to the energy, which is conserved only globally. For $n = 2$ we can introduce the canonical "Clebsch variables" (evidently globally):

$$(34) \qquad \begin{cases} p_i = \rho \partial_i \varphi + s \partial_i \psi, \\ \{\rho, s\} = \{\varphi, \psi\} = \{\rho, \psi\} = \{s, \varphi\} = 0, \\ \{\rho(x), \varphi(y)\} = \{s(x), \psi(y)\} = \delta(x - y). \end{cases}$$

For $n = 3$ there are three cases:

(a) An irrotational barotropic *flow*, where the vortex is zero and the entropy is redundant as a field variable. The Clebsch variables are

$$p_i = \rho \partial_i \varphi,$$
$$(35) \qquad \{\rho, \rho\} = \{\varphi, \varphi\} = 0, \quad \{\rho(x)\varphi(y)\} = \delta(x - y).$$

(b) A barotropic flow (the entropy is not a field variable)

$$(36) \qquad p_i = \rho \partial_i \varphi + \alpha \partial_i \beta, \qquad \Omega_{ij} = d(\alpha \rho^{-1}) \wedge d\beta,$$

where α is conjugate to β and φ to p. For the same reasons as above (see (28)) a global introduction of Clebsch variables is, in general, not possible.

(c) General flows. Here the canonical Clebsch variables contain the "redundant" field variable

$$(37) \qquad \begin{cases} p_i = \rho \partial_i \varphi + s \partial_i \psi + \alpha \partial_i \beta, \\ \Omega_{ij}\, dx^i \wedge dx^j = d\left(\dfrac{s}{\rho}\right) \wedge d\psi + d\left(\dfrac{\alpha}{\rho}\right) \wedge d\beta. \end{cases}$$

It would be useful to calculate the degree of many-valuedness of the representation (37) and to clarify the extent to which it holds globally. Let us consider a simpler example.

We recall that in two-dimensional barotropic flow the Clebsch variables also contain the redundant field variable

$$(38) \qquad \begin{cases} p_i = \rho \partial_i \varphi + \alpha \partial_i \beta, \\ \Omega_{12}\, dx^1 \wedge dx^2 = d\left(\dfrac{\alpha}{\beta}\right) \wedge d\beta. \end{cases}$$

For this reason, on the space of field variables $\alpha/\rho(x)$, and $\beta(x)$ there acts a "calibrating group" (that is, a group of transformations of the plane $\mathbb{R}^2$ depending on

$\alpha \rho^{-1}$ and conserving their exterior product: the element of area (or 2-form)). This group preserves the representations (38).

Example 2. *The superfluid* ^{4}He. The equations of hydrodynamics (without dissipation far from the *point* of transition) for the superfluid ^{4}He can be written down in the same variables p_i, ρ, and s together with the "superfluid velocity" $v_{si} = \partial_i \varphi$ (see [22]). The Poisson brackets have the form (33), where additionally the brackets for all quantities with the variable φ are:

(39)
$$\begin{cases} \{\varphi(x), s(y)\} = 0, \quad \{\varphi(x), \rho(y)\} = \delta(x - y), \\ \{p_i(x), \rho\varphi(y)\} = \rho\varphi(x)\partial_i\delta(y - x), \quad \{\varphi, \varphi\} = 0. \end{cases}$$

As previously, the energy acts as Hamiltonian. The Hamiltonian is given in the form

(40)
$$H = \int \left[\frac{\rho v_s^2}{2} + p_{0i}v_s^i + \varepsilon_0(\rho, s, p_0) \right] d^n x.$$

It is assumed that p_0 is proportional to $v_n - v_s$:

(41)
$$\begin{cases} p_0 = \rho_n(v_n - v_s) = p - \rho v_s, \\ v_n^i - v_s^i = \dfrac{\partial \varepsilon_0}{\partial p_{0i}}, \quad v_s^i = \partial_i \varphi. \end{cases}$$

The quantities ρ_n and ρ_s are called the densities of the normal and of the superfluid components of the fluid. The momentum is $p = \rho_n v_n + \rho_s v_s$. We introduce the "Clebsch variables" as usually,

$$p_i = \rho \partial_i \varphi + s \partial_i \psi + \alpha \partial_i \beta.$$

It would make sense to investigate the question of global impediments to the introduction of Clebsch variables in more detail.

Various more complicated versions of equations of "superfluid" systems and other anisotropic fluids can be found in the surveys [24] and [25].

Note. By analogy with § 1, (14), we can write down the Poisson brackets for the algebra of vector fields in an "external" magnetic field given by a 2-form $F = F_{ij}\, dx^i \wedge dx^j$. These brackets are defined by the extended Lie algebra L_F, in which the commutator of the basic vector fields $e_i = \partial/\partial/x^i$ is given in the form (of an e-extension)

(42)
$$[e_i, e] = 0, \qquad [e_i, e_j] = F_{ij}(x)e.$$

For the fields $v = v^i e_i + \psi e$ and $w = w^i e_i + \varphi e$ we obtain

(43)
$$[v, w] = (v^i \partial_j w^i - w^j \partial_j v^i)e_i + (F_{ij}w^i v^j + w^i \partial_i \psi + v^i \partial_i \varphi)e.$$

To this algebra there correspond in L_i^* the conjugate variables (p_i, q) and the Poisson brackets

(44)
$$\begin{cases} \{p_i(x), q(y)\} = q(x)\partial_i\delta(y - x), \\ \{q(x), q(y)\} = 0, \\ \{p_i(x), p_i(y)\} = p_j(x)\partial_i\delta(y - x) - p_i(y)\partial_j\delta(x - y) + F_{ij}(x)\delta(x - y). \end{cases}$$

Example 3. As already mentioned, in magnetohydrodynamics the magnetic field is not external but is "frozen" into the particles of the fluid (as always, the Hamiltonian coincides with the energy, including the magnetic energy); the Poisson brackets have another form: the Poisson bracket of the momentum densities conserve the

form (5), while that of the magnetic field itself with momenta are such that the field is transported by the particles, as with ρ and s. This means that the flux through an arbitrary fluid surface remains unchanged (only the surface itself is transported) (see [21]). Such brackets with momenta can be introduced for differential forms of any rank: if $H_{i_1,\ldots,i_k}(x)$ is a skew-symmetric tensor of any rank (the k-form $H = H_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$), then the bracket has the form

$$(45) \qquad \{H(x), H(y)\} = 0,$$
$$\{p_i(x), H(y)\} = H(x)\partial_i\delta(y-x) - \partial_i \wedge (H(x)\delta(x-y)),$$

where the operation $\partial_i \wedge \ldots$ on skew-symmetric tensors (forms) of rank k gives a skew-symmetric tensor of rank $k+1$. For example,

1) $\{p_i(x), \varphi(y)\} = \varphi(x)\partial_i\delta(y-x) - \partial_i(\varphi\delta) = (-\partial_i\varphi)\delta(x-y)$ where φ is a scalar;

$$(46) \qquad 2)\ \{p_i(x), \rho(y)\} = \rho(x)\partial_i(y-x),$$

where ρ is a form of rank n (scalar density);

3) $\{p_i(x), A_j(y)\} = A_j(x)\partial_i\delta(y-x) - \partial_i(A_j\delta) + \partial_j(A_i\delta)$, where A_j is a covector;

4) $\{p_i(x), H_{jk}(y)\} = H_{jk}(x)\partial_i\delta(y-x) - \partial_i(H_{jk}\delta) + \partial_j(H_{ik}\delta) - \partial_k(H_{ij}\delta)$, $H_{jk} = -H_{kj}$.

§ 3. What is Morse (LSM) theory?

The general Morse theory [26] deals with the solution of the following problem: given a finite- or infinite-dimensional space M (*manifold*) on which there is given a function (functional) $S \colon M \to R$.

Fundamental problem of Morse theory. How are the stationary points $dS = 0$ (or $\delta S = 0$ for functionals) connected with the topology of the manifold M?

If the critical points are non-degenerate, that is if $\delta^2 S$ is non-degenerate at critical points (as one says, there are no "zero modes"), then the "index" (the Morse index) is the number of negative squares of the form $\delta^2 S$ if this is finite.

Morse theory (in its classical version) is constructed under the following assumptions:

(a) all the critical points are non-degenerate, and the Morse indices are finite;

(b) all the domains $S \leqslant$ const for the function S are relatively compact (the Arzélà principle); this means that a sequence of points x_i such that $S(x_i) < C$ has a limit point in M.

Under these hypotheses the following inequality is established: the number $M_i(S)$ of critical points of index i is not less than the Betti number (the rank of the *homology group*) of M:

$$(1) \qquad M_i(S) \geqslant b_i(M).$$

The mechanism by which this inequality arises is very simple. Each critical point x of index k has a "*surface of most rapid discharge*", that is, a map of the disk D^k (the open ball of dimension k, where $\sum_{\alpha=1}^{k}(y^\alpha)^2 < 1$):

$$f \colon\ D^k \to M.$$

A function S that is bounded on the disk D^k can have only one critical point: one maximum at the centre 0, where $f(0) = x$. The map f should be continued "downwards" through the levels of the function in such a way that the image of the boundary $f(\delta D^k)$ falls into the union of "surfaces of fastest discharge" of the

various critical points x_q, where a) $S(x_q) < S(x)$ and b) the indices of all the points x_q are less than k.

Thus, the function S generates a *cell partition* of the manifold M, where the number of cells of dimension k is equal to the number of critical points of index k for S.

Since all k-dimensional cycles can be formed from the cells of a cell partition, the rank of the group of cycles (and of the homology group: its factor group by the boundary) does not exceed the number of cells:

$$M_k(S) \geqslant b_k(M).$$

If M is finite-dimensional, then the "Poincaré—Morse theorem" holds:

$$\sum_{i \geqslant 0} (-1)^i M_i(S) = \sum (-1)^i b_i(M) = \chi(M).$$

where $\chi(M)$ is the Euler—Poincaré characteristic and $M_i(S)$ the number of cells.

In passing through the critical value $c = c_k$ the level surface $V_c\{S = c\}$ and the domain $W_c\{S \leqslant c\}$ undergo the operations of *reconstruction* (it is assumed that there is only one critical point on $S = c_k$):

(a) $W_{c_k+\varepsilon} = W_{c_k-\varepsilon}$ plus "a *handle* of index k",

(b) $V_{c_k+\varepsilon}$ is the "Morse reconstruction" of the manifold $V_{c_k-\varepsilon}$.

The operations of "attaching a handle" and of "Morse reconstruction'" have great significance in topology itself. (There are manifold invariants that are finer than the Betti numbers, which enable us to give a lower bound for the number of critical points of S, even when δS is a degenerate form. These are the so-called "Lyusternik—Shnirel'man" categories. We do not define these invariants here (see [28]).) In the case of "general position" all the critical points are non-degenerate. Also useful is the case (which arises quite frequently, especially when there is symmetry),

$$Q_k \subset M, \qquad \delta S = 0.$$

Suppose that a) l_k is the dimension of the critical manifold Q_k; b) that the form $\delta^2 S$ is non-degenerate on planes normal to the submanifolds Q_k, and that it has a finite number k of negative squares (the Morse index). Then there is an inequality for the numbers determined by the homology of the set of critical points

$$(2) \qquad\qquad M_j(S) = \sum_k b_{j-k}(Q_k) \geqslant b_j(M),$$

where b_j is the Betti number in the homology mod 2 (under certain hypotheses of orientability this is also true for the ranks of the homology groups with arbitrary coefficients).

Such is "Morse theory" on compact or open manifolds *without boundary*. For manifolds with an *edge* Morse theory can be extended naturally when the whole boundary is a level surface $S = c$ and near the boundary $S < c$.

Example 1 (This observation is apparently due to Maxwell). In a mountainous island the number of peaks minus the numbers of colls plus the number of depressions is 1 (the peaks, colls, and depressions are critical points of the function "height").

For $\chi(D^2) = 1$, D^2 being the island whose boundary is the sea, that is, a level surface of the height function $g \geqslant 0$.

Example 2. On a closed orientable surface of genus $g \geqslant 0$ a function always has at least one minimum and at least one maximum. The number of critical points of saddle type is not less than $2g$ if they are all non-degenerate. If the degeneration is resolved, then for $g > 0$ the function may have in all three critical points: a minimum, a maximum, and a "degenerate saddle". A non-constant function on a closed surface (with $g > 0$) cannot have fewer than three critical points.

We do not discuss here the various purely topological applications of Morse theory in the theory of finite-dimensional smooth manifolds: in the problem of calculating the *homotopy groups* of Lie groups [26], [29], in techniques used in the classification of wide classes of manifolds: in the first place, of manifolds of spherical type [30]–[32], then of arbitrary *simply-connected manifolds* [33], [34], and also of some non-simply connected manifolds [35]–[37].

Initially we are interested in a functional $S(\gamma)$ on some class or another of contours on a finite-dimensional manifold M^n, say, without boundary. The classical Poincaré—Birchhoff—Lyusternik—Shnirel'man—Morse theory considers in the first place the positive functional consisting of length in the Riemannian metric $g_{ij}(x)$:

$$(3) \qquad l(\gamma) = \int_\gamma dl = \int_\gamma \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, dt$$

or the more general positive functional of "*Finsler*" *length*

$$(4) \qquad l_F(\gamma) = \int_\gamma F(x,\dot{x})\, dt > 0,$$

which gives rise to a Banach space structure on each n-dimensional tangent plane, where $F(x,\lambda\dot{x}) = \lambda F(x,\dot{x})$ for $\lambda > 0$. Here the x^i are local coordinates on M^n, and the curve γ has the form $x^i(t)$. Frequently the *action functional* of a mechanical system occurs

$$(5) \qquad s(\gamma) = \int_\gamma \left[\frac{1}{2}\, g_{ij}\dot{x}^i\dot{x}^j - U(x)\right] dt$$

We recall the "*Maupertuis—Fermat*" principle [40]: a functional of length, depending on the energy

$$(6) \qquad l^E(\gamma) = \int \sqrt{(E - U(x))g_{ij}\dot{x}^i\dot{x}^j}\, d\tau,$$

has *extremals* that coincide trajectorially with (5). The metric $(E - U)g_{ij}$ is *non-singular* when $E > \max U(x)$. In what follows we require the metric in question to be non-singular and *complete*. For completeness it is sufficient that M^n is compact.

Under these conditions the Arzélà principle holds (the set of curves joining two points and having length $\leqslant c$ is relatively compact; similarly for closed curves of length $\leqslant c$ on a compact manifold M^n). From this there follows the theorem (of Hilbert) that there is a geodesic joining an arbitrary pair of points on a complete Riemannian manifold. The Morse theorem on the finiteness of the number of negative squares and the finiteness of the degrees of degeneracy of the form $\delta^2 S$ for the extremals of a γ-periodic form or one joining a pair of points $x_0, x_1 \in M^n$ holds. Subsequently it will be important for us that this theorem is valid for an arbitrary

functional of the form

$$(7) \qquad S(\gamma) = \int L(x, \dot{x}) \, dt, \qquad \frac{\partial^2}{\partial \dot{x}^i \, \partial \dot{x}^j} \xi^i \xi^j > 0.$$

All functionals of the form (7) have well-defined "level surfaces" $S = \mathrm{const}$ and "lines of steepest descent" (along the gradient ∇S), although not all have the Arzélà principle. The Arzélà principle holds for length (3) functionals on a complete Riemannian manifold. Also Finsler (positive) metrics (4).

Consequently, for functionals of the form (4) the Morse inequalities (1) hold, where M is either a space $\Omega(x_0, x_1)$ of contours, joining two points of M, or the space of closed contours $M = \Omega$.

If the value of $S(\gamma)$ depends on the direction of the curve γ, then we consider the spaces of directed closed curves Ω^+. One must distinguish two cases:

a) M^n is simply-connected. In this case the homotopy groups are

$$(8) \qquad \begin{cases} 1. \ \pi_{i-1}(\Omega(x_0, x_1)) = \pi_i(M^n), \\ 2. \ \pi_i(\Omega^+) = \pi_{i+1}(M^n) + \pi_i(M^n). \end{cases}$$

These equalities are established starting from the two *fibrations* (Serre):

1. $E(x_0) \xrightarrow[p_1]{\Omega(x_0, x_1)} M^n$, where $E(x_0)$ is the contractible space of all paths with origin $\gamma(0) = x_0$, $p_1(\gamma) = \gamma(1) = x_1$.

2. $\Omega^+ \xrightarrow[p_2]{\Omega(x, x)} M^n$, where $p_2(\gamma) = x = \gamma(0) = \gamma(1)$.

There is a *section* $\psi \colon M^n \to \Omega^+$ consisting of one-point curves in Ω^+. In the case of the sphere $M^n = S^n$ the Betti numbers are

$$(9) \qquad b_i(\Omega(x_0, x_1)) = \begin{cases} 1, & i = k(n-1), \\ 0, & i \neq k(n-1). \end{cases}$$

b) M^n is not simply-connected (for example, all surfaces M^2 except for the sphere S^2).

In this case the space $\Omega(x_0, x_1)$ splits into the union

$$(10) \qquad \Omega(x_0, x_1) = \bigcup_\alpha \Omega_\alpha(x_0, x_1)$$

over all homotopy classes $\alpha \in \pi_1(M^n)$ in each of which the functional must have a minimum.

For closed contours we also have the splitting

$$(11) \qquad \Omega^+ = \bigcup_\beta \Omega_\beta^+,$$

where β is a homotopy class of closed paths, that is, a class of conjugate elements in $\pi_1(M^n)$. The minima in each class β correspond to conjugacy classes in π_1. For example, for manifolds M^n with a complete Riemannian metric for which the curvature of any element of area is non-positive

$$R_{ijkl}\xi^i \eta^j \xi^k \eta^l \leqslant 0,$$

the situation is as follows: all stationary points of the length functional $l(\gamma)$ are minima, both for the problem with two ends and for the periodic problem; all the spaces $\Omega_\alpha(x_0, x_1)$, Ω_β^+, $\beta \neq 1$ are *contractible* (homotopically trivial), and each contains one minimum for the length l.

We call attention to certain peculiarities (important in what follows, see § 4) of the periodic problem. We consider the space Ω_+0^+ of curves homotopic to zero ($\Omega_0^+ = \Omega^+$ in simply-connected manifolds). Then the minimum of the functional l is achieved on one-point curves

$$\psi(M^N) = M^N \subset \Omega_0^+.$$

As a consequence of this, not all stationary points can be non-degenerate in the strict sense of the word (see above): we may require all except the single-point extremals of the functional to be non-degenerate. The Morse inequalities (1) must take the following form:

$$(12) \qquad M_i(S) \geqslant b_i(\Omega_0^+, M^n)$$

in the relative homology modulo the single-point curves.

However, here yet another difficulty arises: it is not a priori excluded that all closed extremals except one are multiples of each one of them. This means that we may find only one periodic extremal from Morse theory other than the one-point one.

For $n = 2$ and $M^2 = S^2$ this difficulty was overcome by Lyusternik and Shnirel'man in 1930 (see [28]), who were able to show that for $n = 2$ the number $M_i^n(S^2)$ of *non-self-intersecting* periodic extremals can be estimated from below by the homology (and other topological invariants) of the Lyusternik—Shnirel'man subspace $\hat{\Omega}^+$ of closed non-self-intersecting curves in S^2 completed by the one-point curves (the sign $+$ denotes directed curves)

$$(13) \qquad \begin{cases} S_+^2 \cup S_-^2 \subset \hat{\Omega}^+(S^2) \subset \Omega^+(S^2), \\ M_i^n(S) \geqslant b_i(\hat{\Omega}^+, S_+^2 \cup S_-^2). \end{cases}$$

The space of non-self-intersecting directed curves on the sphere (completed by the one-point curves) contracts modulo single-point curves to the subset of plane sections of the sphere S^2 having the form $S^2 \times I$, where I is the interval $-1 \leqslant \tau \leqslant +1$, and the boundary is formed by the one-point curves

$$(14) \qquad \begin{cases} S_+^2 \cup S_-^2 \subset S^2 \times I \subset \hat{\Omega}^+ \subset \Omega^+, \\ b_i(S^2 \times I, S_+^2 \cup S_-^2) = \begin{cases} 1 & i = 1, 3, \\ 0, & i \neq 1, 3. \end{cases} \end{cases}$$

In the classical papers only the functional of Riemannian length (3) is considered, independently of the choice of direction (there is an invariance $t \to -t$).

Therefore, the subject of study are the spaces of directed closed curves

$$S^2 \subset \hat{\Omega} \subset \Omega(S^2),$$

where $\hat{\Omega}$ are the non-self-intersecting curves. In this case the Betti numbers mod 2 have the form

$$(15) \qquad b_i(\hat{\Omega}, S^2) = 1 \qquad (i = 1, 2, 3)$$

(here the Lyusternik—Shnirel'man category turns out also to be 3).

In this case one can extract from the methods of the LSM theory no less than three closed non-intersecting geodesics (without taking the direction into account; with direction there would be 6).

For functionals of type (4) without the invariance $t \to -t$ the LSM theory gives
from (14) the existence of two non-self-intersecting closed extremals (in the neigh-
bourhood of which, by the Poincaré—Birkhoff—Kolmogorov—Arnol'd—Moser per-
turbation theory for conservative systems with 2 degrees of freedom, there is, in
general position, an infinite number of self-intersecting periodic extremals, if the
initial system is elliptic [38]). On spheres of dimension $n \geqslant 3$ these arguments no
longer work. At present the only rigorously established result is that in general
position there is at least one further periodic extremal that is not a multiple of the
first [39]. On manifolds on which the Betti numbers of the space of paths $b_i(\Omega^+)$
increases as $i \to \infty$, the matter is far simpler: the number of critical points of the
functional $l = S$ is much greater than the number of periodic geodesics that could
be multiples of any finite number of "basic" geodesics (see [44]). However, this
argument fails for the sphere S^n.

Unfortunately, topological methods are, as a rule, not applicable to all natural
functionals whose domain of definition has dimension > 1 (that is, the Euler—
Lagrange equations involve partial derivatives). In some examples the minima
that arise naturally in modern geometry (or in the apparatus of modern physics)
form non-degenerate critical manifolds in each *connected component* of the function
space, and their neighbourhoods are of "good" structure (see [40], II, Ch. 6). How-
ever, the theory of critical points of saddle type and the Morse theory no longer
hold here, as a rule.

§ 4. Equations of Kirchhoff type and the Dirac monopole

Systems of Kirchhoff type were discussed in § 2. These are systems on the phase
space L^* of the Lie algebra L of the group E(3) of motions of $\mathbb{R}^3$. Among them are:
a) the Kirchhoff equations for the motion of a rigid body in an ideal fluid (without
vortices); b) the motion of a top in a gravitational field; c) the (Leggett) system
for the spin dynamics of the superfluid ^{3}He–A.

The phase-variables are (M_i, p_i) $(i = 1, 2, 3)$, the Poisson bracket is given by
§ 1, (35). The Kirchhoff integrals are $f_1 = p^2$ and $f_2 = ps = \sum M_i p_i$ such that
$\{f_q, M_i\} = \{f_q, p_i\} = 0$ $(q = 1, 2, \; i = 1, 2, 3)$. The Poisson bracket on the level
surface $p^2 = \text{const} \neq 0$ and $ps = \text{const}$ can be found from the same formulae (35)
in § 1.

It is easy to see that the level surface f_1, f_2 for $f_1 = p^2 \neq 0$ is topologically
equivalent to the tangent manifold $T^*(S^2)$ of the two-dimensional sphere S^2 given
by the equation $p^2 = \text{const}$. The variables in the tangent space are given by

$$(1) \qquad\qquad \sigma_i = M_i - \gamma p_i, \qquad \gamma = s/p,$$

so that

$$(2) \qquad\qquad \sum \sigma_i p_i = 0.$$

According to (35) of § 1, the coordinates p_i have zero Poisson bracket $\{p_i, p_j\} = 0$
on S^2. Therefore, the Poisson bracket on $T^*(S^2)$ turns out to be variationally
admissible (see § 1) and must reduce to the form (18) of § 1. The corresponding

change (see [1]) has the form

(3)
$$\begin{cases} -\pi/2 \leqslant \Theta \leqslant \pi/2, \quad 0 < \psi \leqslant 2\pi, \\ p_1 = p\cos\Theta\cos\psi, \quad p_2 = p\cos\Theta\sin\psi, \quad p_3 = p\sin\Theta, \\ \sigma_1 = p_\psi\tan\Theta\cos\psi - p_\Theta\sin\psi, \quad \sigma_3 = -p_\psi, \\ \sigma_2 = p_\psi\tan\Theta\sin\psi + p_\Theta\cos\psi, \quad \sigma_i = M_i - sp^{-1}p_i. \end{cases}$$

It is easy to verify that from (3) it follows that

(4)
$$\begin{cases} \{\Theta,\psi\} = \{p_\Theta,\psi\} = \{p_\psi,\Theta\} = 0, \\ \{\Theta,p_\Theta\} = \{\psi,p_\psi\} = 1, \\ \{p_\Theta,p_\psi\} = \cos\Theta. \end{cases}$$

The corresponding 2-form is

(5)
$$\begin{cases} \Omega = d\Theta \wedge dp_\Theta + d\psi \wedge dp_\psi + s\cos\Theta\, d\Theta \wedge d\psi, \\ x^1 = \Theta, \quad x^2 = \psi, \quad \xi_1 = p_\Theta, \quad \xi_2 = p_\psi, \\ \Omega = dx^\alpha \wedge d\xi_\alpha + \cos x^1\, dx^1 \wedge dx^2. \end{cases}$$

Thus, the Poisson bracket is explicitly reduced to the form (18) of § 1 where the ξ_α for $\alpha = 1$ and 2 are the momenta.

In these variables the Hamiltonian $H(M,p)$ of equations of Kirchhoff type (see (38) and (45) of § 1) has the form

(6)
$$H = \frac{1}{2}g^{\alpha\beta}\xi_\alpha\xi_\beta + A^\alpha\xi_\alpha + V(x^1, x^2)$$

for a rigid body in a fluid. Here

(7)
$$\begin{cases} \sum a_i\sigma_i^2 = g^{\alpha\beta}\xi_\alpha\xi_\beta > 0, \quad \sigma_i = M_i - sp^{-1}p_i, \\ A^\alpha\xi_\alpha = s\left(\sum a_ip_ip^{-1}\sigma_i\right) + p\left(\sum_{i,j} b_{ij}(\sigma_ip_jp^{-1} + \sigma_jp_ip^{-1})\right), \\ 2V = s^2\left(\sum a_ip_i^2p^{-2}\right) + 2ps\left(\sum_{ij} b_{ij}p_ip_jp^{-2}\right) + p^2\left(\sum c_{ij}p_ip_jp^{-2}\right). \end{cases}$$

By virtue of homogeneity, the Hamiltonian H depends only on sp^{-1}.

By substituting for σ_i and p_i in the expressions (3); $\Theta = x^1$, $\psi = x^2$, $p_\Theta = \xi_1$, $p_\psi = \xi_2$, we obtain the final formulae

(8)
$$H = \frac{1}{2}g^{\alpha\beta}\xi_\alpha\xi_\beta + A'^\alpha\xi_\alpha + V'$$

for the top. Here the $g^{\alpha\beta}(x)$ are the same, $g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma$. For A' and V' we have

(9)
$$\begin{cases} A'^\alpha\xi_\alpha = s\left(\sum a_i\sigma_ip_ip^{-1}\right), \\ 2V' = s^2\left(\sum a_ip_i^2p^{-2}\right) + 2W(l^ip_i). \end{cases}$$

In addition, $p^2 = 1$. Therefore, the Hamiltonian depends only on the level $f_2 = S$

(10)
$$H = \frac{1}{2}a(\dot{\Theta}^2 + \cos^2\Theta\dot{\psi}^2) + A''^\alpha\xi_\alpha + V''$$

for the Leggett equations in ^{3}He–A. Here, always $p^2 = 1$ (p is given by d);

$$(11) \qquad \begin{cases} A^\alpha \xi_\alpha = \lambda = \left(\sum \sigma_i H_i \right), \\ V'' = \lambda \left(s \sum p_i H_i \right) + W(p_1, p_2, p_3), \qquad p^2 = 1. \end{cases}$$

The Hamiltonian H depends on the parameter $s = f_2$.

Thus we reach the following conclusion.

Conclusion. Equations of Kirchhoff type reduce to a system mathematically equivalent to a classical charged particle moving on a sphere S^2 with Riemannian metric $g_{\alpha\beta}(x)$ in a potential field $U(x)$, and also in an effective magnetic field $F_{12}(x)$. In spherical coordinates (Θ, ψ) this magnetic field has the form

$$U(x) = V(x) - \frac{1}{2} g_{\alpha\beta} A^\alpha A^\beta,$$

$$(12) \qquad F_{12} = s \cos\Theta + \partial_1 A_2 - \partial_2 A_1, \quad A_\alpha = g_{\alpha\beta} A^\beta, \quad s = f_2 f_1^{-1/2}.$$

We note that the form $A_\alpha \, dx^\alpha$ is defined globally on the sphere S^2. For the flow we obtain

$$(13) \qquad \iint_{S^2} F_{12} \, d\Theta \wedge d\psi = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} s \cos\Theta \, d\Theta \wedge d\psi = 4\pi s.$$

Thus, when $s \neq 0$, the effective magnetic field is always non-zero and represents a (non quantized) "Dirac monopole". Here s is the level of the Kirchhoff integrals. For $s = 0$ the "magnetic field", if it does not vanish, has zero flow through S^2.

Note. When $s = 0$ for the top (45) of § 1, there arises a mechanical system of traditional type on the sphere S^2 (with a non-zero effective magnetic field).[6] This result was obtained by another method somewhat earlier by Kozlov and Kharlaniov (see [41], Ch. 6): these authors then turned to the classical LSM theory to find periodic solutions. As indicated in § 3, here there arises the Maupertuis—Fermat functional of type (6) in § 3, which for $s = 0$ and $E > \max U$ is the length in a certain Riemannian metric. Since the length functional does not depend on the direction, it follows from LSM theory that there exist no fewer than 6 periodic motions (geometrically 3 curves) that are non-self-intersecting on the sphere S^2.

§ 5. Many-valued functionals and an analogue of Morse theory. The periodic problem for equations of Kirchhoff type. Chiral fields in an external field

We have reduced equations of Kirchhoff type to the theory of a charged particle on the sphere S^2 (with some metric) in a scalar potential field and an effective "magnetic field" with non-zero total flow $4\pi s$, that is, a "Dirac monopole". The magnetic field $F = F_{12} \, dx^1 \wedge dx^2$ is a closed, but not necessarily exact 2-form on S^2 (for $s \neq 0$) (see [1], [2]).

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[6] In its physical meaning the problem of a top in an axially symmetrical field is not associated with the Lie algebra $L = L(3)$. This problem is naturally depicted as a Lagrangian system on SO$_3$. Its factorization and transition to $T^*(S^2)$ with a certain symplectic structure is discussed, though not investigated further, in [45]. As the author of the book [45] has communicated to me, the erroneous assertion that the resulting symplectic structure is equivalent to the standard structure on $T^*(S^*)$ has been removed in the English translation.

It is useful to generalize this situation: let M^n, $n > 1$, be a manifold with a metric $g_{\alpha\beta}$, let U be a scalar function (potential) and F a 2-form (magnetic field), not necessarily exact. We consider a domain $Q \subset M^n$ such that F is exact on Q:

$$(1) \qquad \begin{cases} F = d\omega_Q = d(A_\alpha^Q \, dx^\alpha), \\ -F_{\alpha\beta} = \partial_\beta A_\alpha^Q - \partial_\alpha A_\beta^Q. \end{cases}$$

Let γ be a curve located entirely in Q. Then we can define the action for it:

$$(2) \qquad S_Q(\gamma) = \int_\gamma \left[\frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta - A_\alpha^Q \dot{x}^\alpha - U \right] dt.$$

By the Maupertuis—Fermat principle, these same trajectories of motion (up to parametrization) can be obtained for a fixed energy from the functional

$$(3) \qquad l_Q^E(\gamma) = \int_\gamma \left[\sqrt{(E - U) g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} - A_\alpha^Q \dot{x}^\alpha \right] d\tau.$$

This can be done for any domain Q in which $F = d\omega_Q$. We fix a 1-form ω_Q for all possible domains Q on which the 2-form F is exact. If γ lies entirely in both domains Q_1 and Q_2, then:[7]

$$(4) \qquad F = d\omega_{Q_1} = d\omega_{Q_2},$$

$$S_{Q_1}(\gamma) - S_{Q_2}(\gamma) = l_{Q_1}^E(\gamma) - l_{Q_2}^E(\gamma) = \int_\gamma (\omega_{Q_1} - \omega_{Q_2}).$$

The value of the integral remains unchanged under any deformation γ_λ of the curve $\gamma = \gamma_0$, assuming that γ is periodic (or under any deformation γ_λ of γ_0 with the same end-points if γ has such):

$$(5) \qquad 0 = \frac{d}{d\lambda} \int_{\gamma_\lambda} (\omega_{Q_1} - \omega_{Q_2}) = \frac{d}{d\lambda} [l_{Q_1}^E(\gamma_\lambda) - l_{Q_2}^E(\gamma_\lambda)],$$

since the form $\omega_{Q_1} - \omega_{Q_2}$ is closed.

From this we obtain the following conclusion.

Conclusion. The set of local actions l_Q^E (or S_Q) for all domains Q defines a "many-valued functional" on the function spaces: a) of the closed contours (the directed curves Ω^+); b) of the paths joining two points, $\Omega(x_0, x_1)$. Here we assume that $E > \max U(x)$. This means that the infinite-dimensional 1-form δl^E is everywhere uniquely determined and closed, but its "path integral", in general, determines a many-valued function on Ω^+ or $\Omega(x_0, x_1)$.

Near any extremal this function may be assumed to be unique. The Morse index theorem and all other "good" local properties hold for the functionals (3) in so far as the condition (7) of § 3 is satisfied. For example, it is clear that one-point curves give a local minimum of the functionals (3). This is a very important fact for our purposes.

The many-value function (functional) l^E becomes single-valued after transition to a certain covering with infinitely many sheets:

$$(6) \qquad \hat{\widehat{\Omega}} \to \Omega^+, \qquad \widetilde{\Omega} \to \Omega(x_0, x_1),$$

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[7]This situation essentially arose in connection with arguments (see [42]) for the construction of the quantum amplitude $\exp\{iS\}$ as a single-valued functional under the condition that the *flow* of the magnetic field of the "Dirac monopole" is integer-valued.

by defining on the covering space a single-valued function $l^E(S)$, running through all the values

$$-\infty < l^E < \infty, \qquad -\infty < S < \infty$$

(on $\hat{\Omega}$ or $\widetilde{\Omega}$). Of course, no analogue of the "Arzélà principle" of § 3 can hold.

When the magnetic field F_{ij} is an exact 2-form, then the functional S or l^E is defined everywhere $(Q = M^n)$ and is therefore single-valued. Nevertheless, this functional can turn out to be non-positive. In this case also the Arzélà principle fails for $-\infty < S < \infty$, $-\infty < l^E < \infty$.

Example 1. Let $\mathbb{R}^2(x, y)$ be a plane with the Euclidean metric and $F \neq 0$ a homogeneous magnetic field, directed along the z-axis $\perp \mathbb{R}^2$. All orbits of motion of a charged particle are circles (with a definite direction, depending on the sign of F). The radius of the (Larmor) orbits has the form

$$(7) \qquad\qquad\qquad r^2 = \text{const} \cdot E/F^2$$

(the constant involves the charge, the mass, and c). From this it follows that if the distance between x_0 and x_1 is sufficiently large, then there is no extremal in $\Omega(x_0, x_1)$. The reason is that the functional l^E on curves γ with large area is not positive (although l^E is single-valued).

Example 2. Let S^2 be the sphere with the standard metric and F a magnetic field invariant under all the motions from SO_3. For fixed energy E and large fields F the Larmor radius $r^2 \sim EF^{-2}$ becomes arbitrarily small. The problem is exactly integrable: as for the plane $\mathbb{R}^2$ all the orbits are closed. By arguments similar to those of Example 1 we arrive at the conclusion that there is a pair of points x_0, x_1 on S^2 such that the many-valued functional l^E has no extremal in $\Omega(x_0, x_1)$.

We note that in Example 2 the manifold (the sphere S^2) is compact, but the functional is many-valued.

The periodic problem of the variational calculus in this case differs strongly, on the whole, from the problem with two fixed end-points. In the periodic case, the Maupertuis—Fermat functional l^E when δl^E is an everywhere defined 1-form on Ω^+ always has "trivial" critical points: these are the one-point curves, which form a submanifold of local minima $M_n \subset \Omega^+(M^n)$. On any sheet of the covering $\hat{\hat{\Omega}}_q \to \Omega^+$ the complete inverse image

$$(8) \qquad\qquad q^{-1}(M^n) = \bigcup_j M_j^n = M_0 \cup M_1 \cup M_{-1} \cup \ldots$$

gives a manifold of local minima of l^E that is single-valued on $\hat{\hat{\Omega}}$.

Let us join by a homotopy two manifolds of local minima, say, M_0^n and M_1^n; that is, we construct a map of the cylinder (I is the interval $0 \leqslant \tau \leqslant 1$)

$$(9) \qquad\qquad\qquad f \colon M^q \times I \to \hat{\hat{\Omega}}$$

over any q-dimensional cycle $M^q \subset M^n$.

At the boundary we impose the condition

$$(10) \qquad\qquad\qquad \begin{cases} f(x, 0) = M_0^q \subset M_0^n, \\ f(x, 1) = M_1^q \subset M_1^n, \end{cases}$$

In particular, for $q = 0$ we obtain a map of the interval $I \to \hat{\hat{\Omega}}$; for $q = n$ we obtain a map $M^n \times I \to \hat{\hat{\Omega}}$ of the cylinder over M^n, provided that it is compact and closed.

When we restrict the functional l^E to $M^q \times I$ and begin to move the map f "downwards" along the gradient ∇l^E, then the ends $f|_{\tau=0}$ and $f|_{\tau=1}$ do not move (they even occur in the local minima); we see that somewhere "in the middle" l^E has a maximum (x_f, τ_f) on $M^q \times I$ for any map f; we have the obvious inequality

$$l^E(x_f, \tau_f) > l^E(x, \tau)|_{\tau=0,1}.$$

Since the values on the boundaries do not depend on f, we arrive at the following conclusion:

Theorem. *In general position, every basic cycle in the homology group $H_q(M^n)$ generates a critical point of the many-valued functional l^E of Morse index $i = q + 1$. For arbitrary energy E, l^E has at least one critical point (the non-trivial periodic extremal) that is not a singleton. It is assumed that the Maupertuis metric $(E - U)g_{ij}$ has no singularities $E > \max U(x)$ and is complete on the compact manifold M^n (this is always so, for example, when $M^n = S^2$).*

The same arguments can also be applied to the case of the two-ended problem, for a many-valued functional l^E on $\Omega(x_0, x_1)$. Here one has to assume that either $x_0 = x_1$ is any fixed point $x_0 \in M^n$, or that x_0 and x_1 are so close that there is a unique "short" extremal (locally minimal) from x_0 to x_1 that can be denoted by $[x_0, x_1]$. In this case the manifold of local minima consists of the single point $[x_0, x_1] \in \Omega(x_0, x_1)$. So we obtain the result: in addition to $[x_0, x_1]$ there is also a "long" extremal of index 1.

Now let $x_0 = x_1$. We have obtained a "long" extremal $\gamma(x_0)$ with an "angle" at x_0. We consider the scalar function $l^E(\gamma(x_0)) = \psi(x_0)$ (which is easily seen to be single-valued) on M^n, where $\gamma(x_0)$ is the "short" extremal.

Proposition. *If ψ is smooth, then its critical points on M^n are also periodic extremals.*

Conjecture. *The periodic extremals obtained from cycles on M^n of one-point curves cannot all concide geometrically (that is, they cannot all be multiples of one of them).*

This condition is easily met in the dimension $n = 1$ on the sphere S^2, by using the Lyusternik—Shnirel'rnan space of smooth non-self-intersecting curves $\hat{\Omega}^+(S^2)$, completed by the one-point curves. Since $\hat{\Omega}^+$ is simply-connected, the many-valued functional l^E on this space reduces to a single-valued one; however, this functional extends up to the "boundary" (the set of one-point curves) with two different values (the set of single-pointed curves "bifurcates" into two pieces). As was shown above (§ 3, (14)), homotopically we have

$$\hat{\Omega}^+ \sim S^2 \times I, \quad -1 \leqslant \tau \leqslant +1,$$

$S_+^2 \cup S_-^2$ are the one-point curves (the boundary).

More precisely, we must consider the subspace $\hat{\hat{\Omega}}^+$ in the covering space $\hat{\hat{\Omega}} \to \Omega^+$ where $\hat{\Omega}^+ \subset \hat{\hat{\Omega}}$. Two copies of the one-point curves ($S_+^2 \cup S_-^2$) are contained in the

closure of $\hat{\Omega}^+$. We normalize the value of l^E so that

(11) $$l^E(S^2_+) = 0, \qquad l^E(S^2_-) \geqslant 0.$$

For equations of Kirchhoff type we obtain from (4.14?) that

$$l^E(S^2_-) = 4\pi|s|.$$

Even among the non-self-intersecting curves we obtain two extrema of indices 1 and 3 by using two saddles as in (9) and (10) above:

(12) $$\begin{cases} f\colon\ I \to \Omega^+, \quad f(0) \in S^2_+, \quad f(1) \in S^2_-, \\ f\colon\ S^2 \times I \to \hat{\Omega}^+, \quad f(x,0) = S^2_+, \quad f(x,1) = S^2_-, \end{cases}$$
$$0 \leqslant \tau \leqslant 1.$$

Thus, in this case we obtain two non-self-intersecting periodic extremals γ_1 and γ_2 of Morse indices 1 and 3, respectively.

For a single-valued functional l^E in the presence of a magnetic field we have $s = 0$; all the results remain valid.

Thus, we have obtained the following result:

Theorem 1. *For all the values of the parameters $f_1 = p^2 \neq 0$ and $f_2 = ps$ for which the Hamiltonians are defined of systems of Kirchhoff type* (4), (6)–(10), *reduced to the sphere S^2, and for all energies $E > \max U$ the system has at least two periodic orbits* (*non-self-intersecting in p*), *depending on the parameters E, s, and p. If they are non-degenerate, then their Morse indices are 1 and 3, respectively.* (*In fact, for the motion of a rigid body in a fluid because of the homogeneity of the Hamiltonian this dependence reduces to the variables E and sp^{-1}. For a top and the Leggett system we must set $p = 1$.*)[8]

As Arnol'd has told the author, from Poincaré's paper of 1905 (for the quotation, see [28]) one can extract an idea whose natural development makes it possible to prove the existence of a closed extremal in a magnetic field in a number of cases. Poincaré's arguments, when translated into modern language, have the following meaning: we consider the sphere S^2 with some Riemannian metric g_{ab}. Among all curves γ bounding a given area A we look for the shortest (the isoperimetric problem). We denote this[9] by γ^*_A. It is easy to see from arguments with Lagrangian multipliers that γ^*_A is a closed extremal of a (formal) charged particle in some magnetic field F proportional to the element of area $F = \lambda\,d^2\sigma$ of the metric g_{ab} with some (so far undetermined) $\lambda = \lambda(A)$. When A is increased from zero to the whole area of the sphere S^2, then $\lambda(A)$ increases from $-\infty$ to $+\infty$, as is easy to verify. Hence, by continuity, $\lambda(A_0) = 0$ for some A_0. From this we also obtain the fact that in any "constant" magnetic field (that is, $F = \lambda\,d^2\sigma$ for all λ) there is at least one non-self-intersecting extremal. By a trivial generalization of this argument, one has to consider for any given 2-form ω on S^2 the problem of finding the shortest $\gamma^{\min}(A, \omega)$ in the set of non-self-intersecting curves γ for which $\int_U \omega = A$, $\gamma = \partial U$. By changing A one can apparently obtain closed extremals in the magnetic field $\lambda\omega$

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[8]As is shown in [2], § 5, for a broad class of Hamiltonians there is a $\mathbb{Z}_2$-symmetry enabling us to find periodic motions, unknown classically, among plane sections of a sphere, by looking for the extremum of action as a function of a single variable.

[9]Anosov has pointed out that although the isoperimetric problem is present in Poincaré's work, Arnold's idea is lacking; that is, the further arguments with the change of the parameter A from 0 to the area of the sphere; it would be useful to make this argument rigorous.

for any $-\infty < \lambda < \infty$. In any case this is true for forms like $f(x)\,d^2\sigma$, where $f > 0$ and $d^2\sigma$ is the element of area.

However, it is more convenient, as Arnol'd has suggested, to act dually. We fix the length of a curve $l(\gamma) = L$ homotopic to zero in any complete Riemannian metric on a simply-connected manifold M^n, and we fix a closed 2-form Ω. We look for a curve γ such that the quantity $I(\gamma) = \int_\sigma \Omega$ fit has a maximum or a minimum (a local maximum or even a stationary point if $H_2(M^n, R) \neq 0$ and the homology class $[\Omega] \neq 0$, although here it is already difficult to prove an existence theorem). If the class $[\Omega] \neq 0$, then the whole of this construction must be carried out on the space of pairs $(\gamma, n) \in \hat{P}_L$, where γ is a closed null-homotopic curve of fixed length L, and n is the homotopy class of the membrane σ, $\partial\sigma = \gamma$. The space of pairs $\hat{P}_L$ is an infinitely-sheeted covering of the space P_L of curves γ of the given length, $\hat{P}_L \to P_L$, but for small lengths L the covering is trivial;

$$\hat{P}_L = P_L \times \mathbb{Z} \quad \text{as} \quad L \to 0;$$

as $L \to \infty$, this covering becomes non-trivial.

The space P_L itself is compact. Therefore, if the class $[\Omega] = 0$ or the covering $\hat{P}_L$ is trivial, then there is always a maximum of $I(\gamma)$. Suppose that it is attained at $\gamma_L^* \in P_L$. Using the previous arguments, and changing L from 0 to ∞, we obtain a closed periodic extremal in any magnetic field proportional to Ω. Things are more complicated if the homology class is $[\Omega] \neq 0$. For a certain "critical" $L = L_0$ the covering P_L becomes non-trivial.[10] However, on the two-dimensional sphere S^2 one can use non-self-intersecting curves γ which bound only two membranes $\partial\sigma_1 = \gamma$, $\partial\sigma_2 = \gamma$, $\sigma_1 \cup \sigma_2 = S^2$.

In this case there is always a maximum; with a change of parameter the previous arguments reduce to the theorem on a single periodic extremal. The "finite-dimensional" model of the present arguments with the space of closed curves and its subspaces of curves of length L is as follows: given a manifold M (open, of large dimension); suppose that on it there are given

a) a smooth function $l(x) \geqslant 0$ such that the domains $l \leqslant L$ and the level surfaces $l = L$ are all compact,

b) a closed 1-form $\hat{\omega}$, $d\hat{\omega} = 0$.

We investigate the critical points of the form $\omega = dl + \hat{\omega}$. For this purpose we consider the family of forms $\omega_L = \omega$ on the level $l = L$. If the form ω_L on the level $l = L$ is exact: $\omega_L = d\varphi_L$, then we consider the maximum γ_L^* of φ_L on the level $l = L$. By varying L, $\infty > L \geqslant 0$, we find the critical points of the form $\lambda(L)\,dl + \hat{\omega}$. We recall that for $L = 0$ we must obtain not an isolated minimum, but a whole non-degenerate manifold of minima since in the case of curves we obtain all the one-point curves. All other critical points may be assumed to be non-degenerate. Moreover, in the domains $l \geqslant L$ for small $L \sim e$ the form $\omega = dl + \hat{\omega}$ must be (locally) the gradient of a function $f = l + \varphi$, where $\varphi \sim \varepsilon^2$, that is, f has a local minimum on the whole set $L = 0$ of minima of $l(x)$. As $L \to 0$, we see that $\lambda(L) \to 0$.

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[10]In this case the critical value $L = L_0$ is a stationary point of the functional $L(\gamma)$ of index 1^0; it is appropriate to conjecture that: the minimal (maxima) γ_L^*, generate "short" closed orbits in the magnetic field $\lambda^{-1}(L)\Omega$ where $\lambda(L)$ runs through *all* the values from 0 to ∞ as L changes from 0 to L_0.

We now assume that the functional l^E of the Kirchhoff problem (6)–(10) of § 4 on the space of non-self-intersecting curves, normalized by the conditions (11), turns out to be everywhere positive.

According to Stokes' formula, the magnetic part of the functional l^E for a non-self-intersecting curve γ is always bounded above by the area of the domains on the sphere which it bounds. Hence, l^E is always semi-bounded on the subspace $\hat{\Omega}^+(S^2)$:

$$(13) \qquad\qquad l^E(\gamma) > \text{const} > -\infty.$$

If there is a curve γ such that $l^E(\gamma) < 0$ (under the conditions (11) for one-point curves), then there is a minimum $\gamma_{\min} \in \hat{\Omega}^+$, that is, one more "superfluous" periodic extremal. Apart from the minimum there is also a "superfluous" saddle γ_1 where $l^E(\gamma_1) > 0$ in the space $\hat{\Omega}^+$.

Example. We consider the Kirchhoff problem (6)–(10) of § 4; at energies close to the maximum of the potential $E \sim \max U$, we look for "small" test curves γ_ε surrounding the maximum point x_0, $U(x_0) = \max$, such that

$$(14) \qquad\qquad l^E(\gamma_\varepsilon) < 0, \qquad \varepsilon \to 0.$$

Let $x^1 = \Theta$ and $x^2 = \psi$ be the coordinates (5) of § 4, where the Lagrangian and the magnetic field have the form (6)–(10) of § 4 and $g_{\alpha\beta}$ is a Riemannian metric.

If the effective magnetic field $F = F_{12}$ exceeds a certain "threshold", then there are small test curves γ_ε, such that $l^E(\gamma_\varepsilon) < 0$ for energies E sufficiently close to the critical energy:

$$(15) \qquad\qquad 9{,}2\,\lambda_{\max} < 4|F_{12}(x_0)|\det g^{\alpha\beta}(x).$$

where $\lambda_{\max}$ is the largest eigenvalue of the form $(-\partial^2 U/\partial x^\alpha \partial x^\beta)$:

$$(16) \qquad\qquad \det\left(-\frac{\partial^2 U}{\partial x^\alpha \partial x^\beta} - \lambda_{\max} g_{\alpha\beta}\right)_{x=x_0} = 0$$

(see [2]). The conclusion that there is a saddle when inequalities of the type (15) hold is valid in all dimensions, in contrast to the existence of a minimum.

For purely methodological purposes it is useful to consider an example that is unrelated to equations of Kirchhoff type. Suppose that in the (x, y)-plane $\mathbb{R}^2$ there is a magnetic field $F(x, y) = F(x + T_1, y) = F(x, y + T_2)$ with two periods, directed along the z-axis. We consider the classical motion of a charged particle in this field (a generalization of Larmor orbits). Suppose that the average field is non-zero

$$(17) \qquad\qquad \bar{F} = \frac{1}{T_1 T_2}\int_0^{T_1}\int_0^{T_2} F\,dx \wedge dy \neq 0.$$

We form the Maupertuis—Fermat functional, when $Q = \mathbb{R}^2$. We have a single-valued functional $l^E(\gamma)$ on the space $\hat{\Omega}_0^+$ of all smooth closed non-self-intersecting curves, directed in the same sense as the motion along a Larmor orbit in the homogeneous field $\bar{F}$. The functional l^E is non-positive (for $F = \bar{F}$ this was shown in Example 1 above). For circles γ_r of radius r we have

$$(18) \qquad\qquad \begin{cases} l^E(\gamma_2) < 0, & r \to \infty, \\ l^E(\gamma_2) > 0, & r \to 0. \end{cases}$$

We consider the function $\psi(r) = l^E(\gamma_r)$ on the half-line. Using the periodicity of F, we identify the curves $\gamma_1 \sim \gamma_2$ if they differ by a shift through a vector of the lattice (mT_2, nT_2). Then the one-point curves form a torus. We apply the same arguments as above (see (11), (12)) taking instead of the interval I a map of the half-line

$$f \colon M^m \times R^+ \to \hat{\Omega}_0^+.$$

Here M^q is a cycle on the torus T^2 ($q = 0, 1, 2$). We have four cycles: a point, two neighbourhoods, and the whole torus. The map f is subject to the boundary conditions:

(a) $f(x, 0) = M_0^q \subset T^2$ are one-point curves;

(b) for large $\tau \to \infty$ the images $f(x, \tau)$ consist of curves γ such that $l^E(\gamma) < 0$.

By analogy with the preceding we establish in the case of general position the existence of four "Larmor" orbits for any energy $E > 0$ with Morse indices (1, 2, 2, 3).

Other examples and the development of an analogue of Morse theory for closed 1-form can be found in [1]–[3].

Let us now introduce the class of *chiral fields* [3] among the generalized "external fields"; it is natural to correct "many-valued functionals" with these fields, not unlike those arising on the space of contours for the Dirac monopole.

The definition of a non-linear chiral field is as follows (see, for example, [40], II, Ch. 6): let N^q and M^n be arbitrary Riemannian manifolds; let $S_0(f)$ be a functional defined on the map $f \colon N^q \to M^n$. Usually, $N^q = \mathbb{R}^q$ or $N^q = S^q$. If $N^q = \mathbb{R}^q$, then we require that at infinity the field $f(x)$ tends to a constant, $f(x) \to y_0 \in M^n$ as $|x| \to \infty$. Here $S_0(f)$ has the form of a Dirichlet functional that is quadratic in the derivatives of f, possibly with some additions. The principal chiral fields arise when $M^n = G$ is a Lie group. In field theory one considers the case when the metric on G is invariant on both sides (the Killing metric); the metrics on $\mathbb{R}^q$ or S^q are also assumed to be standard. The standard "chiral Lagrangian" has the form

$$(19) \qquad S_0(f) = \int_{N^q} \mathrm{Sp}(A_\mu A^\mu)\sqrt{g}\, d^q x,$$

where $g_{\mu\nu}$ is a metric on N^q (we note that in the theory of the Ginzburg—Landau equation for the superfluid $^3\mathrm{He}$–A or $^3\mathrm{He}$–B, more complex Lagrangians arise for chiral fields; for references, see [43]).

Let Ω be an additional closed $(q+1)$-form on M^n (the "external field"):

$$(20) \qquad\qquad\qquad d\Omega = 0.$$

We take a *covering* of M^n by domains

$$M^n = \bigcup_\alpha W_\alpha$$

(with continuously many domains) such that

1. Ω is exact, $\Omega = d\psi_\alpha$, on each W_α;

2. the image of each map $f \colon N^q \to M^n$ lies entirely in some domain W_α.

The "local action functionals" are defined by

$$(21) \qquad\qquad S_\alpha(f) = S_0(f) + \int_{(N^q, f)} \psi_\alpha.$$

In the intersections, if

$$f(N^q) \subset W_\alpha \cap W_\beta,$$

then

$$(22) \qquad S_\alpha - S_\beta = \int_{(N^q, f)} (\psi_\alpha - \psi_\beta),$$

where $\psi_\alpha - \psi_\beta$ is a closed q-form in $W_\alpha \cap W_\beta$. By analogy with § 5, where $q = 1$, $N^q = S^1$, $M^n = S^2$ (and $M^n = \mathbb{R}^2$) we obtain the following lemma:

Lemma. *The set of functionals $S_\alpha(f)$ defines a closed 1-form δS on the functional space F of admissible map $f\colon N^q \to M^n$. This 1-form δS determines a many-valued functional S, which is single-valued on an infinitely-sheeted covering $\hat F \to F$.*

In the quantization of such fields one has to require that the quantity ("amplitude") $\exp(iS)$ is a single-valued functional on F. From this it follows that Ω has integer-valued integrals along $(q + 1)$-cycles in M^n.

Example. Let $q = 2$, $N^q = \mathbb{R}^2$ or S^2 with the standard metric, and let M^n be a compact Lie group with the Killing metric. The functional $S_0(f)$ is taken in the form (19); Ω is a two-sided invariant 3-form on the group $G = M^n$ (such a form always exists; when $G = \mathrm{SU}_2$ then Ω is the volume element).

Problem. Does the Euler—Lagrange equation $\delta S = 0$ have a solution by the inverse problem method (see [11], Ch. III for ordinary chiral fields with $\Omega = 0$)?

Remark. We obtain a curious example for the so-called relativistic "strings" in Minkowsky space $M^n = \mathbb{R}^{3,1}$ or Euclidean space $M^4 = \mathbb{R}^4$, where $q = 2$, $N^2 = \mathbb{R}^2$ (or S^2 or D^2) and $f\colon N^2 \to \mathbb{R}^4$. Let $S_0(f) = \iint_{N^2} \sqrt{g}\, d^2x$ where $g = \det g_{ij}$, g_{ij} being the metric on N^2 induced by the embedding f. For any closed 3-form on some domain in $\mathbb{R}^4$ we again obtain the "many-valued functional"

$$(23) \qquad S(f) = S_0(f) + \iint_{(N^2, f)} \psi, \qquad d\psi = \Omega.$$

If Ω is defined everywhere in $\mathbb{R}^4$ except for isolated singularities at the points $(x_1, \dots, x_k)$, then there is a set of integrals over small spheres S_i^3 $(i = 1, \dots, k)$ around these points

$$(24) \qquad \varkappa_1 = \int_{S_i^3} \Omega.$$

From the requirement that $\exp(iS)$ is single-valued it follows that $(2\pi)^{-1}\varkappa_i$ is an integer. This situation is similar to the "Dirac monopole", but here it is more natural to call the singular points x_i "instantons", since they are localized in $\mathbb{R}^4$. If $\Omega \to 0$ sufficiently rapidly as $|x| \to \infty$, then $\sum \varkappa_i = 0$.

The class of many-valued functionals for chiral fields introduced above can be extended naturally. Let $E \xrightarrow{p} N^q$ be a smooth fibration (or a direct product) with fiber M^n. In the case of a direct product, $E = M^n \times N^q$. Let $S_0(f)$ be a single-valued functional on the sections $f\colon N^q \to E$, $p \circ f = 1$, and let Ω be a closed $(q + 1)$-form on the manifold E, $d\Omega = 0$. The subsequent definition of the "many-valued functional" $S\{f\} = S_0\{f\} + \int_{N^q f} d^{-1}(\Omega)$ is a word-for-word repetition of (20)–(22) above with the obvious change that the W_α are domains on E. In the special case mentioned above Ω was a form on the fiber M^n, and the W_α where domains on M^n, which naturally generate "cylindrical" domains and forms on $E = M^n \times N^q$, independent of the basis N^q. The following interesting problem

was solved in [54], [55]: let $P(f) = 0$ be a differential equation of the sections of the fibration E, $f \colon N^q \xrightarrow{p} E$, $p \circ f = 1$, such that the "local" expression $P(f)$ is formally the variational derivative of some functional. When is the operator $P(f)$ globally the variational derivative of a functional (which, of course, is assumed to be single-valued), that is, $P(f) = \delta S/\delta f$?

In [54], [55] an "obstruction" $\alpha[P] \in H^{q+1}(E, R)$, $\alpha[P_1 + P_2] = \alpha_1 + \alpha_2$ was constructed such that $\alpha = 0$ is equivalent to the global existence of $S\{f\}$. By comparing this with our construction of "many-valued functionals", we obtain the following proposition: all locally Lagrangian systems of differential equations reduce to the variational derivatives of many-valued functionals of the form $S_0\{f\} + \int_{(N^q,f)} d^{-1}(\Omega)$, $d\Omega = 0$.

For the direct product $E = N^q \times M^n$ the simplest natural class of examples of such "external fields", that is, of $(q+1)$-forms Ω, is given by products of closed forms of the base and the fiber: $x^1, \ldots, x^q$ are local coordinates in the base N^q, and $\varphi^1, \ldots, \varphi^n$ in the fibre M^n;

$$(25) \qquad \Omega_{k,l} = \Omega = \omega'_k \wedge \omega''_l, \qquad k + l = q + 1,$$
$$d\omega'_k = d\omega''_l = 0,$$

where ω'_k is a form on N^q and ω''_l a form on M^n.

Example 1. Let $l = 1$, $k = q$, $\omega''_1 = dU(\varphi)$ (locally), and let $\omega'_k = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^q$ be the volume element on N^q. Then

$$(26) \qquad S = S_0\{f\} + \lambda \int_{(N^q,f)} U(\varphi)\sqrt{g}\, d^n x.$$

An external field $\Omega_{q,1}$ of this kind reduces to a potential $U(\varphi)$ on M^n or the covering $\hat{M} \to M^n$.

Example 2. Let $k = 2$, $l = q - 1$, $\omega'_2 = dA$, where $A = A_\alpha(x)\, dx^\alpha$ is a vector potential. In this case

$$S\{f\} = s_0\{f\} + \lambda \int_{N^q} (A_\alpha dx^\alpha) \wedge f^*\omega''_{q-1}.$$

The field $\Omega'_{2,q-1}$ can represent a pair: the "magnetic field" $H_{\alpha,\beta} = \partial_\alpha A_\beta - \partial'_\beta A_\alpha$ for $q = 2, 3$ and another field ω''_{q-1} of φ, interacting with the chiral field $f \colon N^q \to M^n$.

Example 3. Let $k = 1$, $l = q$, $\omega'_1 = E_\alpha\, dx^\alpha = dU(x)$. In this case E_α can be an electrical field.

In the simplest interesting cases we have

a) $N^q = \mathbb{R}^q$ (the field f tends to f_0 as $|x| \to \infty$) or $N^q = T^q$ (periodic boundary conditions), the metric is $g_{\alpha\beta} = \delta_{\alpha\beta}$.

b) $M^n = S^1$ (the field f of "sine-Gordon" type) or, more generally, $M^n = S^n$. Let ω''_{q-1} be the element of area.

The magnetic field in Example 2 interacts with the chiral field of unit tangent vectors for $N^q = \mathbb{R}^q, T^q$. The basic examples of the functionals S_0 are as follows:

I. The Dirichlet integral on the sections $(x, f(x)) \subset N^q \times M^n$. Let $g_{\alpha\beta}$ be the metric in N^q and G_{ab} in M^n. We put

$$(27) \qquad S_0\{f\} = \int_{N^q} \left[\lambda + g^{\alpha\beta}(x) G_{ab}(f(x)) \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta}\right] \sqrt{g}\, d^n x.$$

More generally, the Dirichlet integral plus a potential:

$$(28) \qquad S_0\{f\} = \int_{N^q} \left[g^{\alpha\beta}(x) G_{ab}(f(x)) \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta} + V(\varphi) \right] \sqrt{g}\, d^n x.$$

where $V(\varphi)$ is a single-valued scalar (in Example 1 it can happen that $dV(\varphi)$ is closed, but not an exact 1-form).

II. Sectional volume. In the same notation

$$(29) \qquad S_0\{f\} = \int_{N^q} \sqrt{h}\, d^n x, \qquad h = \det h_{\alpha\beta}(x),$$

where $h_{\alpha\beta} = g_{\alpha\beta} + G_{ab}(f(x)) \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta}$ is the induced metric on the section $\tilde{f} \colon N^q \to N^q \times M^n$ generated by the map $f \colon N^q \to M^n$.

For functionals $S_0\{f\}$ of type I and II on the set of null-homotopic maps $N^q \to M^n$ the following proposition holds: $f_0 = \text{const}$ is a local minimum of the functional $S_0\{f\}$, then f_0 is also a local minimum of the functional $S\{f\} = S_0\{f\} + \lambda \int_{(N^q, f)} d^{-1}(\Omega)$. Let $S\{f_0\} = 0$. If $S\{f\}$ is essentially many-valued or single-valued but non-positive (that is, $S\{f\} < 0$ for some null-homotopic f), then $S\{f\}$ has a "saddle" extremal. The proof of this statement is similar to that of the corresponding theorem for closed curves, where $q = 1$.

§ 6. Many-valued functions on finite-dimensional manifolds. An analogue of Morse theory

On the manifold M^n we specify a closed 1-form ω; there is an (infinitely-sheeted) covering $\hat{M} \xrightarrow{p} M^n$ such that the form $p^*\omega$ is the differential of a function (the simplest example is $\omega = d\varphi$ on $\mathbb{R}^2 \setminus 0 = M^n$, where $\hat{M}$ is the Riemann surface of the logarithm):

$$(1) \qquad p^*\omega = dS.$$

We call S a "many-valued function" on M^n. In fact, we consider only the case when all the critical points are either non-degenerate or form non-degenerate critical manifolds (see § 3). We also assume that S has a well-defined "gradient discharge" that is, on M any compact space under descent along the gradient ∇S either approaches a critical point or passes successively "downwards" through all levels of S.

Problem. To construct an analogue of Morse theory for an estimate of the number of stationary points of a many-valued function S (that is, of a closed 1-form ω) of any Morse index i. We denote the number of stationary points of Morse index i by $m_i(S)$ (or $m_i(\omega)$), $p^*\omega = dS$.

In the group $H_1(M, \mathbb{Z})$ we can choose a basis $\gamma_1, \ldots, \gamma_k, \gamma_{k+1}, \ldots, \gamma_N$ such that

$$(2) \qquad \oint_{\gamma_j} \omega = \begin{cases} 0, & j \geqslant k+1, \\ \varkappa_j \neq 0, & j \leqslant k, \end{cases}$$

and the numbers $\varkappa_j$ for $j = 1, \ldots, k$ are rationally (or integrally) independent. The number $k - 1$ is called the "degree of irrationality" of ω. The monodromy group of the minimal covering $p \colon \hat{M} \to M^n$, turning ω into a differential of a single-valued

function $dS = p^*\omega$ is precisely equal to $\mathbb{Z}^k$ the free Abelian group with k generators $t_1, \ldots, t_k$ acting by shifts on $\hat{M}$:

$$t_j \colon \ \hat{M} \to \hat{M}.$$

In fact, the irrationality exponent is a point of the projective space

$$\varkappa = (\varkappa_1 : \varkappa_2 : \cdots : \varkappa_k) \in \mathbb{RP}^{k-1}.$$

A particularly simple and interesting case is $k = 1$, when ω (possibly after multiplication by a factor) gives an element of the first integral cohomology group $[\omega] \in H^1(M^n, \mathbb{Z})$. In this case, $\exp(2\pi i S)$ is a well-defined complex-valued function of modulus 1, that is,

(3) $$f = \exp(2\pi i S) \colon \ M^n \to S^1.$$

The problem of constructing an analogue of Morse theory for the critical points of such maps appears to be absolutely classical; this problem has never previously (up to 1981) been studied in the literature. We consider the following case. If there are no critical points, then the map f defines a fibration with base $B = S^1$. A cyclic $\mathbb{Z}$-covering $\hat{M} \xrightarrow{p} M^n$ is constructed as follows: we realize the cycle $D[\omega] \in H_{n-1}(M^n, \mathbb{Z})$ by the submanifold N^{n-1}, where D is the Poincaré duality operator. By cutting the manifold along the cycle N^{n-1} we obtain a membrane W^n with two edges $\partial W = N^{n-1} \cup N_1^{n-1}$, diffeomorphic to N^{n-1}. We take infinitely many of copies of this membrane $W \approx W_i$ with boundaries $\partial W_i = N_{i,0} \cup N_{i,1}$, diffeomorphic to N^{n-1}. We paste them to each other along the boundary and according to the number of components of the boundary

(4) $$\hat{M} = \bigcup W_i, \qquad N_{i+1,0} = N_{i,1}, \qquad -\infty < i < \infty.$$

The manifold $N^{n-1} = N_0^{n-1}$ may be assumed to be a level surface of the function S (or the complete inverse image of a point under the map $f = \exp(2\pi i S)$). The monodromy operator acts as follows:

(5) $$t \colon \ W_i \to W_{i+1}, \qquad N_{i,0} \to N_{i,1} = N_{i+1,0}.$$

In accordance with general principles, S must generate a cell complex (see § 3). However, in our case the most important requirement on which the usual Morse theory is based is not satisfied: this theory requires that the domains of lesser values $S \leqslant a$ are relatively compact, both in the finite-dimensional and infinite-dimensional case. In our case this is not true. However, in our case from each critical point of index ; the "surface of most rapid descent" (or, if necessary, its smallest displacement) which can naturally be regarded as a "cell", emerges "downwards" through the levels. However, this "cell" can be pulled through the levels of S as far as $-\infty$; infinitely many such "cells" of dimension $i - 1$ can be contained in its algebraic boundary. Under the shift $t \colon \hat{M} \to \hat{M}$ the functions S goes over into itself with the addition of a constant, taking critical points into critical points. Thus, we conclude that a) every critical point determines a free generator in the complex in question; b) the boundary of a cell can be an infinite linear combination of cells of this complex, lying "lower" in the levels of S, that is, emanating from ∞ only to one side in $\hat{M}$; c) all the "cells" are obtained from finitely many of all possible base shift through elements t^m of $\mathbb{Z}$ acting on $\hat{M}$.

We introduce the ring of Laurent series of the form

(6)
$$\sum_{j > \mathrm{const} > -\infty} m_j t^j \in K$$

with integer coefficients m_j that vanish for sufficiently large negative j. We denote this ring by $K = \hat{\mathbb{Z}}^+[t, t^{-1}]$. We regard the cell complex generated by a many-valued function on the manifold M^n or a function S on the covering $\hat{M} \to M^n$ as a free complex of finitely generated K-modules C (since the number of critical points is finite). The complex C has the form

$$0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0.$$

where ∂ is a K-module homomorphism. We note that in contrast to the usual Morse theory it can happen that $C_0 = C_n = 0$. Furthermore, on any manifold M^n there is a closed 1-form of any non-trivial cohomology class $[\omega] \in H^1(M^n, \mathbb{R})$ such that there are no local minima and maxima at all (that is, $C_0 = C_n = 0$).

For the skew products of M^n with the base S^1 there is a form ω without critical points, that is, $C_n = C_{n-1} = \cdots = C_1 = C_0 = 0$.

Lemma. *The homology of the complex of K-modules C, generated by any smooth closed 1-form ω is homotopy invariant.*

Without proving this simple lemma, we see that the invariants of these homology groups can be used to obtain analogues of the Morse inequalities for the case of many-valued functions generating maps into the circle

$$\exp(2\pi i S)\colon \; M^n \to S^1.$$

The ring K is homologically one-dimensional (if the coefficients m_j of the series are elements of a field, then the corresponding analogue of K is also a field). Consequently, submodules of free modules are always free. This enables us to choose free bases in the groups (modules) of "cycles" $\mathbb{Z}_k = \mathrm{Ker}\,\partial \subset C_n$ and "boundaries" $B_k = \mathrm{Im}\,\partial \subset C_n$. The difference in rank of these modules is called the "Betti number" and is denoted by $b_k(M^n, a)$ where $a = [\omega] \in H^1(M^n, \mathbb{Z})$.

The analogues of the torsion numbers $q_k(M^n, a)$ are defined as follows: we can choose free bases of the module $\mathbb{Z}_k(e_1, \ldots, e_N)$ and the submodule $B_k(e'_1, \ldots, e'_L)$, where $N - L = b_k$, such that:

(7)
$$e'_j = \left(n_j + \sum_{k \geqslant 1} n_{jk} t^k \right) e_j + \sum_{i > L} q_{ij}(t) e_i,$$

moreover:

   1) the number n_j is divisible by n_{j+1};
   2) the degrees of all the terms $q_{ij}(t)$ of the series are non-negative;
   3) the numbers $q_{ij}(0) \neq 0$ are also divisible by n_j for all i and j (if the series does not vanish identically).

The total number of indices j such that $n_j \neq 1$ is called the torsion number and is denoted by $q_k(M^n, a)$. The number $q_k + b_k$ coincides with the minimal number of generators of the module $H_k = \mathbb{Z}_k / B_k$.

Theorem. *The following analogues of the Morse inequalities hold for the numbers $m_i(S)$ (or $m_i(\omega)$) of critical points of index i for a map in the neighbourhood of $\exp(2\pi i S)$ or for a closed 1-form ω, where $[\omega] = a \in H^1(M^n, \mathbb{Z})$:*

(8)
$$m_i(S) \geqslant b_i(M^n, a) + q_i(M^n, a) + q_{i-1}(M^n, a).$$

The proof of this theorem is easily deduced from the preceding.

We note that these analogues of the Morse inequalities are similar to the classical ones, but the topological invariants in them have a more complex geometrical meaning.

For manifolds with $\pi_1(M^n) = \mathbb{Z}$ it makes sense to ask when there is equality in (8), which resembles the familiar Smale theorem on single-valued functions on simply-connected manifolds. One can construct without difficulty a level surface N^{n-1} that is dual to the class $a = [\omega] \in H^{-1}(M^n)$ and is connected and simply-connected (in any case, for $n \geqslant 6$). Next, by using the Smale function on the membrane W^n with two boundaries $\partial W = N_0^{n-1} \cup N_1^{n-1}$, which is obtained from M^n by a section, a level surface N^{n-1} can be "minimally" continued (by using the Smale function on W) to the whole manifold M^n resulting in a form on M^n and a function S on the covering $\hat{M}$. However, this form (or many-valued function) can be far from minimal in its number of critical points. The construction of a minimal 1-form ω requires the choice of an initial manifold $N^{n-1} \subset M^n$ that is "minimal" in a certain sense if this choice is at all possible. It would be interesting to analyse to the end this problem for manifolds with the group $\pi_1 = \mathbb{Z}$.

We make a few remarks concerning the more complicated case $k > 1$, that is when the form ω has at least two rationally independent integrals over one-dimensional cycles $\varkappa_i = \oint_{\gamma_i} \omega, \gamma_i, \ldots, \gamma_k$, where $\gamma_{k+1}, \ldots, \gamma_N$ is a basis of $H_1(M^n, \mathbb{Z})$, $\varkappa_i \neq 0$, $i \leqslant k$, $\sum m_i \varkappa_i \neq 0$, and the m_i are arbitrary integers. Here we have the covering $\hat{M} \xrightarrow{p} M^n$, where $p\omega = dS$, and the monodromy group is free Abelian. We introduce the ring $K_\varkappa$ of series $b \in K_\varkappa$ with integer coefficients

$$(9) \qquad b = \sum_{m=(m_1,\ldots,m_k)} b_m t_1^{m_1} \cdot \ldots \cdot t_k^{m_k}.$$

Here

1. $b_m = 0$ if $\sum m_i \varkappa_i$ is sufficiently large in modulus and negative.
2. "Stability", that is, for any series b there are numbers $\varepsilon > 0$ and N such that $b_m = 0$ if

$$(10) \qquad \sum m_i \varkappa_i^* < -N, \qquad \sum |\varkappa_i^* - \varkappa_i| < \varepsilon.$$

The closed 1-form ω defines a cell complex, regarded as a complex of $K_\varkappa$-modules. The homology of this complex is homotopy invariant and can serve as a basis for constructing inequalities of Morse type. It is interesting to study the way in which the complexes and homology that arise here depend on $\varkappa$ if ω is altered slightly and the critical points remain essentially as before. If ω has no critical points at all, then the manifold M^n has the form

$$M^n = \hat{M}/\mathbb{Z}^k = (\hat{N} \times R)/\mathbb{Z}^k,$$

where $\hat{N}$ is a typical fiber of the fibration $\omega = 0$. All the fibers in this case are identical. From an approximation of ω by closed forms $\omega_j \to \omega$ with rational integrals over cycles without critical points it is clear that $\hat{M}^n$ is a skew product with circular base. The fibers of these skew products are compact manifolds N_j^{n-1} that are factors of $\hat{N}$:

$$\hat{N} \to N_j^{n-1},$$

That is, $\hat{N}$ is a regular covering over N_j with monodromy group $\mathbb{Z}^{k-1}$.

Any Riemannian metric on M^n together with a form ω without critical points generate a discrete "dynamical system", namely, the action ρ of $\mathbb{Z}^k$ on M^n, conserving the fiber: the transformations $\rho(t_j) \in \mathbb{Z}^k$ on $\hat{M}$ are constructed by inversely transferring to the initial fiber $\hat{N} \subset \hat{M}$ the image $t_j(\hat{N})$ of the monodromy map $t_j \colon \hat{M} \to \hat{M}$ along the normals to the fibers in the given metric. In the group $\rho(\mathbb{Z}^k)$ there are subgroups isomorphic to $\mathbb{Z}^{k-1}$ that act discretely on the fibers with a compact factor; their action can be everywhere dense on M^n, as shown by the very simple example $M^2 = T^2$, $\omega = \varkappa_1 \, d\varphi_1 + \varkappa_2 \, d\varphi_2$, $\varkappa_1/\varkappa_2$ is irrational. The whole group $\rho(\mathbb{Z}^k)$ acts non-discretely even on the fibers. What are the ergodic properties of the action of $\rho(\mathbb{Z}^k)$ on M^n? Do they depend on the Riemannian metric? Can we assert that in the "typical" case the orbits of $\rho(\mathbb{Z}^k)$ are everywhere dense on the fibers? The optimal Riemannian metric that can be used here naturally must be such that the distance between neighbouring fibers is constant.

For forms with critical points the geometrical picture becomes substantially more complicated. Let us assume that all the critical points are non-degenerate, therefore, that there are finitely many of them. For simplicity we may assume that the form has no local minima and maxima (such closed forms are always in any cohomology class). Apparently, in the "typical" case the non-singular fibers are everywhere dense. It is an interesting problem to describe the topological properties of non-singular fibers. In a certain natural sense it is a "quasi-periodic manifold".

The simplest non-trivial case is $k = 2$; here the minimal covering $\hat{M} \xrightarrow{p} M^n$ that turns the form ω into the differential of a single-valued function $p^*\omega = dS$ has the monodromy group $G = \mathbb{Z}^2$; the fibers (that is, the surfaces $S = \text{const}$) in the covering $\hat{M}$ are in a certain sense similar to $\mathbb{Z}$-coverings over compact $(n-1)$-manifolds. In any case, these fibers extend to infinity in two directions, topologically speaking, they have two "ends" $(\pm\infty)$ if they are connected around $\pm\infty$. The simplest model of a quasi-periodic manifold that can occur is as follows: there is a finite set of manifolds $W_i n - 1$ with boundaries $\partial W_i^{n-1} = V_{0i}^{n-2} V_{1i}^{n-2}$ $(i = 1, 2, \ldots, m)$. Let $\alpha = (\ldots, i_{-2}, i_{-1}, i_0, i_1, i_2, \ldots)$ be a doubly infinite sequence of m symbols. If this sequence is "admissible", then we can construct an open manifold W_α in the following natural way;

(11) $$W_\alpha = (\cdots \cup W_{i_{-2}} \cup W_{i_{-1}} \cup W_{i_0} \cup W_{i_1} \cup \ldots)$$

with the pastings

(12) $$V_{1,i_j}^{n-2} = V_{0,i_{j+1}}^{n-2}, \qquad -\infty < j < \infty.$$

Admissibility indicates that the pastings (12) are possible.

For $k = 2$ the proposition is that the non-singular fibers $\omega = 0$ (the level surfaces of the many-valued function) can all be obtained by this construction. The singular points, of which there are finitely many, can also be constructed, but in the pasting (11) one of the elements W_{i_s} is replaced in just one place by a manifold with the simplest Morse singularity of index i.

On the covering $\hat{M}$ the fibers $S = \text{const}$ are singular for a countable everywhere dense set of values, (in the "typical" case), and on each singular fiber there is only one critical point of S. Let $S = c$, $c + \varepsilon$ be non-singular fibers and $\varepsilon > 0$ sufficiently small; we can achieve that all the non-singular fibers are pasted together from one and the same elements W_i, but possibly relative to distinct sequences $\alpha = \alpha_c$, $S = c$. The transition $c \to c + \varepsilon$ in the set α_c gives rise to a change in the subset of indices

$\gamma \subset \alpha_c$ at the expense of the Morse reconstruction of critical points in the domain $(c, c + \varepsilon)$. The sequence $\gamma \subset \alpha_i$ has "on average" finitely many elements $n(\gamma)$ on an interval of order $1/\varepsilon$. The relevant "density" $\bar{n}(\gamma) = \varepsilon n(\gamma)$ determines the average number of reconstructions of a level surface in the interval $(c, c + \varepsilon)$. It is natural to introduce the quantities $\bar{n}_i(\gamma)$: the densities of the number of Morse reconstructions of a given index i in the interval $(c, c + \varepsilon)$, $n(\gamma) = \sum_{i=1}^{n-1} \bar{n}_i(\gamma)$ (we recall that by assumption there are no minima and maxima). Thus, for all non-singular fibers $S = c$ the function α_c, together with an indication of the places and type of replacements in an everywhere dense countable set of singular fibers c_j, determines a series of quantities that characterize the family of level surfaces of a many-valued function for $k = 2$, a function that has so far only been studied on the covering $\hat{M}$. The transition to M^n, that is, the factorization over $\mathbb{Z}^2$ with the generators (t_1, t_2) causes new difficulties. We can achieve that the representations of fibers in the form (11) are consistent with a single shift t_1. This has the following significance: the whole manifold $\hat{M}$ is constructed, beginning with a single fiber represented in the form (11), by the sequence of Morse reconstructions of the pieces W_j, according to the combinatorial scheme described above. Here the partition can be made so that under the shift $t_1^{q_1} \colon \hat{M} \to \hat{M}$ for some integer $q_1 \neq 0$ ($\hat{N}_c \to \hat{N}_{c+q_1\varkappa_1}$) the representation of the fiber $\hat{N}_{q_1\varkappa_1+c}$ in the form (11) is obtained from a representation of the fiber $\hat{N}_c$ by some shift of the sequences $\alpha = \alpha_c$ by an integer s_1:

$$(13) \qquad \begin{aligned} \alpha_{c+q_1\varkappa_1} &= s_1(\alpha_c), \\ i_j &\to i_j + s_1. \end{aligned}$$

Into the topological arbitrariness of this construction there enters yet another elementwise diffeomorphism $\psi^{(1)} \colon \hat{N}_c \to \hat{N}_c$, where $\psi_m^{(1)} \colon W_{j_m} \to W_{j_m}$ and the diffeomorphisms $\psi_m^{(1)}$ are compatible at the boundaries. One can choose another partition of the fiber $\{W_l'\}$, where $\hat{N}_c = W_\beta'$, $\beta = (\dots, l_{-1}, l_0, l_1, \dots)$, and construct the whole manifold $\hat{M}$ by analogy with preceding but adapted to the shift t_2: for some integer $q_2 \neq 0$ the fiber $\hat{N}_{c+q_2\varkappa_2}$ is obtained after a series of reconstructions from a representation of the fiber $\hat{N}_c$ in the form

$$(14) \qquad \hat{N}_{c+q_2\varkappa_2} = W_{s_2(\beta)}' = W_{\beta_{c+q_2\varkappa_2}}',$$

and $t_2^{q_2} \colon \hat{N}_c \to \hat{N}_{c+q_2\varkappa_2}$, where $s_2(\beta_c)$ is a shift of the sequences $\beta = \beta_c$ by an integer and possibly an elementwise diffeomorphism $\psi^{(2)}$:

$$\psi^{(2)} \colon \ \hat{N}_c \to \hat{N}_c, \qquad \psi^{(2)} = \bigcup \psi_m^{(2)}, \qquad \psi_m^{(2)} \colon \ W_{l_m}' \to W_{l_m}'.$$

The fact that these two partitions can be consistent is plausible, but not proved: can we choose them so that $W_j = W_j'$? Undoubtedly of great interest is the question when the "quasi-periodic manifold" (11) can be realized as a non-singular everywhere dense fiber: the level surface of a closed 1-form is on the compact manifold M^n. All the constructions easily generalized to the case $k > 2$: the manifolds W_j^{n-1} must be given together with maps into the cube $\varphi_j \colon W_j^{n-1} \to I^{k-1}$ in such a way that the complete inverse images of the faces of the cube give the corresponding partition of the boundary $\partial W_j^{n-1} = \varphi_j^{-1}(\partial I^{k-1})$. The sequences of symbols a must be "admissible" functions on a $(k-1)$-dimensional lattice $j(n_1, \dots, n_{k-1})$

with value in the set of symbols (j) numbering the manifolds W_j^{n-1}. The pasting

$$(15) \qquad W_\alpha = \bigcup_{(n_1,\dots,n_{k-1})} W_{j(n_1,\dots,n_{k-1})}^{n-1}, \quad \alpha = \{(j(n_1,\dots,n_{k-1})\},$$

can be defined as in (11), if a is an admissible distribution of indices (of course, the complete inverse images φ_j^{-1} are pasted together from the adjacent faces of cubes according to the numbering on the lattice). All the problems raised here naturally become more complicated for $k > 2$.

By analogy with the forms without critical points (above), for forms ω with Morse singularities one can also define "almost everywhere" the action of a somewhat smaller group $\rho(\mathbb{Z}^k)$ as follows. Let a and b be integers such that

$$|a\varkappa_1 + b\varkappa_2| < \varepsilon,$$

where $\varepsilon \to 0$ is sufficiently small. The transformation $t_1^a t_2^b$ is such that the image of any fiber $\hat{N}_c$ on $\hat{M}$ turns out to be equal to $\hat{N}_{c+a\varkappa_1+b\varkappa_2}$, and so is uniformly distributed close to $\hat{N}_c$ for any c everywhere on $\hat{M}$. The map $\rho(t_1^2 t_2^b)\colon \hat{N}_c \to \hat{N}_c$ is constructed by means of the composition $t_1^a t_2^b$ with a translation along the normals to the fibers in the given Riemannian metric on M^n. This map is not defined on the set of measure zero consisting of the intersection of fibers "of surfaces of most rapid descent" emerging on both sides from critical points lying between the fibers $\hat{N}_c$ and $\hat{N}_{c+a\varkappa_1+b\varkappa_2}$. (We recall that by assumption the form ω has no minima and maxima, therefore, the set where the map and its inverse are not defined is of measure zero on the fibers; moreover, the critical points between these fibers are "sectionally" distributed as $\varepsilon \to 0$.) The map $\rho(t_1^a t_2^b)$ and its inverse are defined on M^n and are everywhere smooth, except at critical points and those parts of their "surfaces of most rapid volume decrease" that fall into ε-neighbourhoods of critical points. Of course, to define the whole group $\rho(\mathbb{Z}^k)$ one would have to eliminate entirely all these surfaces from M^n.

A closer investigation of the family or level surfaces of closed 1-forms seems to the author to be an extremely interesting (purely topological) problem. Incidentally, in the Hamiltonian formalism, as we have seen, the Hamiltonian is not necessarily single-valued on the symplectic manifold on which the Poisson brackets are defined, but only a closed 1-form. The simplest example of this is the motion of a classical particle in space under the influence of a periodic potential (for some lattice Γ) plus a constant strong field. In this case the Hamiltonian

$$(16) \qquad \begin{cases} H = \dfrac{p^2}{2m} + eU(x) + eE_i x^i, \\[2mm] E_i = \text{const}, \quad U(\vec{x} + \vec{\Gamma}_i) = U(\vec{x}), \end{cases}$$
$$\Gamma = (n_1\vec{\Gamma}_1 + n_2\vec{\Gamma}_2 + n_3\vec{\Gamma}_3)$$

is a 1-form on $T^*(T^3) = \mathbb{R}_{(p)}^3 \times T_{(x)}^3$. If p are the so-called "quasi-momenta", periodic with the inverse lattice Γ^* and $\varepsilon(p)$ is the "dispersion law", then the Hamiltonian, after the inclusion of a ("weak") external electric field, can have the form

$$(17) \qquad H = \varepsilon(p) + eE_i x^i + eU(x),$$

where $U(x)$ is periodic with the lattice Γ or one of its sublattices $\Gamma' \subset \Gamma$, and $\varepsilon(p)$ is periodic for Γ^*. Here $M^n = T_{(p)}^3 \times T_{(x)}^3$. Of course, these examples are somewhat artificial, but we have given them to illustrate the fact that the study of the level

surfaces of 1-forms can be useful from various points of view. It is also possible to include a weak magnetic field where the Hamiltonian of the form (17) is modified and becomes

$$(18) \qquad H = \varepsilon \left(p - \frac{e}{c} A(x) \right) + e(E_i x^i + U(x)),$$

where $A_j(\ldots, x_i + \Gamma_i, \ldots) = A_j(x) + \partial_j f_i$ the f_i being a single-valued function. It is easy to see that the term $\varepsilon(p - \frac{e}{c} A)$ determines a single-valued function on the torus T^6 (or on a finite covering of it) if and only if the flows of the magnetic field $H_{ij} = \partial_i A_j - \partial_j A_i$ across all two-dimensional elementary cells of the lattice Γ are rational multiples of the unit of magnetic flow $2\pi c e^{-1}$ ($\hbar = 1$). The additional distorted potential $U(x)$ is not, as a rule, considered. The "electrical part" of the Hamiltonian $E_i x^i$ gives a closed, but not-exact 1-form dH on the torus $T^3 \times T^3$, which in general has 3 non-commensurable non-zero periods:

$$(19) \qquad dH = \frac{\partial \varepsilon}{\partial p_i} dp_i + e \left(E_i + \frac{\partial U}{\partial x^i} - \frac{1}{c} \frac{\partial \varepsilon}{\partial p_j} \frac{\partial A_j}{\partial x^i} \right) dx^i.$$

The symplectic 2-form has the usual shape $\Omega = dp_i \wedge dx^i$ and determines the Poisson brackets and the Hamiltonian systems. A classical dynamical system with Hamiltonian (18), where $\varepsilon(p)$ is derived from quantum theory, can in principle be of considerable interest in solid state theory whereas the system with the Hamiltonian (16) is interesting only after quantization, where its properties are non-trivial even in the one-dimensional case because of the presence of the 1-form $E_i dx^i$ (of "constant force").

Remark. In 3-dimensional space, in the absence of a constant electric field and weak external periodic potential $E_i = 0$, $U = 0$ we have the semiclassical Hamiltonian $H = \varepsilon(p - \frac{e}{c} A)$ oi a particle with momentum $p = p' + \frac{e}{c} A$, $p' \in T^3_{(p')}$ in a constant (homogeneous) magnetic field $\bar{H}$: strictly speaking, the field $\bar{H}$ is a 2-form

$$\bar{H} = H_1 \, dy \wedge dz + H_2 \, dz \wedge dx + H_3 \, dx \wedge dy$$

such that $d\bar{H} = 0$. In the absence of local currents we have rot $\bar{H} = 0$ or $d(*\bar{H}) = 0$. Of course, this is satisfied for a homogeneous field. The semiclassical "motion" of a particle occurs in the p'-space T^3 under the influence of the field $\bar{H}$ by virtue of the equation $\dot{p}' = \frac{e}{c}[v \times \bar{H}]$ or $\dot{p}'_j = \frac{e}{c} v^k H_{kj}$, $v = \dot{x} = \partial \varepsilon / \partial p'$; where $p' = p - \frac{e}{c} A$. This is the "old momentum" to within inclusion of the field $\bar{H}$; the particle moves along the surfaces $\varepsilon(p') = $ const orthogonally to $\bar{H}$. Thus, the motion is along the level surfaces of the 1-form ($\omega = 0$) on the 2-dimensional manifolds $\varepsilon(p') = $ const, provided that the vector H has two or three pairwise incommemsurable coefficients of inclination with respect to the initial lattice in which $\omega = H_1 \, dp'_1 + H_2 \, dp'_2 + H_3 \, dp'_3$. Although the fiber $\omega = 0$ is, in general, not connected, transitions are possible from one trajectory to another on the same fiber $\omega = 0$ near the critical points. Thus, it is natural to regard the fiber $\omega = 0$ as a single integer and to study (in the irrational case) its quasiperiodic structure discussed above.

Here we have the following proposition:

If the Fermi surface $\varepsilon(p) = \varepsilon_0$ in the torus T^3 has the genus $g \leqslant 1$, then the form co has the degree of irrationality $k - 1 \leqslant 0$; if the genus is $g = 2$, then $k \leqslant 2$; if the genus is $g \geqslant 3$, then $k \leqslant 3$. Here ω is the restriction of the form $*\bar{H}$ to the

Fermi surface and k is the rank of the monodromy group of the covering $\hat{M} \xrightarrow{p} M^2$ such that $p^*\omega = dS$.

The proof uses the fact that $\varepsilon(p)$ is a smooth single-valued function on the torus T^3. Therefore, any Fermi surface $\varepsilon(p) = \varepsilon_0$ bounds a membrane in T^3: and so, the inclusion $j\colon M^2 \to T^3$ is such that $j_*[M^2] = 0$. The covering over M^2 is induced by a $\mathbb{Z}^3$-covering over T^3. The restriction of the cohomology $j^*\colon H^1(T^3, \mathbb{Z}) \to H^1(M^2)$ has an image on which the multiplication of cohomology is trivial, since $j_*[M^2] = 0$. Hence, the rank of the image $j^*H^1 \subset H^1(M^2)$ does not exceed the genus g (but also never exceeds 3), since $\omega = j^*(*\bar{H})$, and the proof is complete.

Therefore, if the genus of the Fermi-surface is $\leqslant 1$, then the trajectories of a semiclassical motion are always closed (compact) in $\mathbb{R}^3$ for the quasi-momenta, although for $g = 1$ the covering itself over the Fermi-surface in $\mathbb{R}^3$ may be an open covering of cylinder type (it cannot be $\mathbb{R}^2$). If the genus is $g = 2$, then $k = 2$ is possible. In this case the level surfaces $\omega = 0$ (orthogonal to the field $\bar{H}$, as a vector in $\mathbb{R}^3$) may be "quasi-periodic" manifolds and extend to $\pm\infty$ in two "asymptotic" directions. We recall that according to Morse theory the function $\varepsilon(p)$ on the torus T^3 must have no less than $\binom{l}{3} = b_l(T^3)$ critical points of index l (if they are non-degenerate), and their number increases in the pressence of a non-trivial finite group of symmetries, which (in general) occurs in crystals. Consequently, Fermi-surfaces of high genus are possible.

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