

**ON POISSON BRACKETS COMPATIBLE WITH ALGEBRAIC  
GEOMETRY AND KORTEWEG–DE VRIES DYNAMICS ON THE  
SET OF FINITE-ZONE POTENTIALS**

A. P. VESELOV AND S. P. NOVIKOV

**I. Some information regarding finite-zone potentials.** [1]. A finite-zone or quasiperiodic (in general, complex) potential  $u(x)$  of the one-dimensional operator  $L = -d^2/dx^2 + u(x)$  possesses a “Bloch” eigenfunction  $\psi_{\pm}(x, \lambda)$  meromorphic in  $\lambda$  on a Riemann surface of genus  $g$  which is two-sheeted over the  $\lambda$  plane. In the periodic case  $\psi$  is the eigenvector of translation by a period. With the normalization  $\psi_{\pm}(x_0, \lambda) \equiv 1$  there are  $g$  poles  $\gamma_q(x_0) = (\tilde{\gamma}_q, \pm)$  and  $g$  zeros  $\gamma_q(x)$ ,  $q = 1, \dots, g$ . The surface  $\Gamma$  can be written in the form

$$(1) \quad y^2 = R(\lambda) = \prod_{j=0}^{2g} (\lambda - \lambda_j), \quad \lambda_j \neq \lambda_k.$$

For real  $L$  we have  $\lambda \in \mathbb{R}$ ,  $\lambda_0 < \lambda_1 < \dots < \lambda_{2g} < \infty$ , and the points  $\lambda_j$ ,  $\lambda_{2g+1} = \infty$  are end points of the lacunae of the spectrum.

For us the following facts are important.

1. A finite-zone potential is completely determined by an independent collection of data; a nonsingular Riemann surface (1) and a collection of distinct points  $\gamma_1, \dots, \gamma_g \in \Gamma$ . The collection of all finite-zone potentials is a  $(3g + 1)$ -dimensional complex space  $K_g$  in which there is distinguished a “real part”  $K_g^{\mathbb{R}} \subset K_g$  where  $\lambda_i \in \mathbb{R}$ , and  $\gamma_q = (\tilde{\gamma}_q, \pm)$ ,  $\tilde{\gamma}_q \in \mathbb{R}$ . For a real, smooth potential,  $\gamma_q \in [\lambda_{2q-1}, \lambda_{2q}]$  or  $\gamma_q \in a_q$  where  $a_q$  is a cycle on  $\Gamma$  over a lacuna.

2. The Abel transformation assigns to the symmetrized collection of points  $\gamma = (\gamma_1, \dots, \gamma_g)$  a point of the complex torus  $T^{2g} = J(\Gamma)$  (the Jacobi variety), i.e., a vector of the complex space  $\mathbb{C}^g$  up to the lattice generated by  $2g$  vectors. The image  $A(\gamma)$  in  $\mathbb{C}^g$  moves linearly as the potential varies according to all higher KdV equations.

3. In the real, smooth case  $\lambda_i \in \mathbb{R}$  and  $\gamma_q \in a_q$ ; a real torus  $T^g \in J(\Gamma)$  is distinguished which can be geometrically represented as  $T^g = A(a_1 \times \dots \times a_g)$ . On the torus  $T^g$  there are coordinates  $(\phi_1, \dots, \phi_g)$  defined mod  $2\pi$  in which the dynamics of the KdV equation and its higher analogues are linear:

$$(2) \quad \begin{aligned} d\phi_q/d\tau_m &= \omega_{mq}(\lambda_0, \lambda_1, \dots, \lambda_{2g}), \quad d\lambda_s/d\tau_m = 0, \\ \oint_{a_{q_1}} d\phi_{q_2} &= 2\pi\delta_{q_1 q_2}, \quad m = 0, 1, 2, \dots \end{aligned}$$

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The coordinates  $\phi_1, \dots, \phi_g$  are uniquely determined, admitting only changes of the form

$$(3) \quad \phi_j \rightarrow \phi_j + \phi_{j0}(\lambda_0, \lambda_1, \dots, \lambda_{2g}).$$

## II. The Poisson brackets. The most important examples.

**Example 1.** All the higher KdV equations are Hamiltonian systems (see [2] and [3]) on the space of rapidly decreasing or periodic functions with a given period (and also on the space of quasiperiodic functions with a given period group). The Hamiltonian structure can be restricted to the set  $K_g$  with coordinates  $\lambda_0, \dots, \lambda_{2g}, \gamma_1, \dots, \gamma_g$ . The annihilator (center) of this bracket is generated functionally by the quantities

$$F_0 = \bar{u}, \quad F_q = T_q(\lambda_0, \lambda_1, \dots, \lambda_{2g}), \quad q = 1, 2, \dots, \\ g: (F_l, \lambda_j)_1 = \{F_l, \gamma_q\}_1 = 0,$$

where

$$\bar{u} = \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L u(x) dx.$$

and  $T_q$  are the periods of the potential. Moreover, we have

$$\{\lambda_{j_1}, \lambda_{j_2}\}_1 = \{\gamma_{q_1}, \gamma_{q_2}\}_1 = 0,$$

and the Hamiltonians of the higher KdV equations can be expressed in terms of  $\lambda_0, \dots, \lambda_{2g}$ .

**Example 2.** The finite-zone potentials satisfy the commutativity equation  $[L, A] = 0$ , where  $A$  is an operator of order  $2g + 1$  having the Lagrangian form (see [1], [4] or [5])

$$[L, A] = 0 \leftrightarrow \delta J / \delta u = 0,$$

$$J(u, u_x, \dots, u^{(g)}, c_1, c_2, \dots, c_{g+1}) = I_g + \sum_{i=1}^{g+1} c_i I_{g-i},$$

where  $I_{-1}, I_0, \dots, I_g$  are the integrals of Kruskal and others (see [1]),

$$c_1 = \sum_{j=0}^{2g} \lambda_j, \quad c_k = \sum_{i_1 < i_2 < \dots < i_k} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}.$$

Thus, the commutativity equations are Hamiltonian systems relative to the standard Poisson bracket generated by the variational problem (6). The quantities  $c_i$  form the annihilator of this bracket on  $K_g$ :

$$\{c_i, \lambda_j\}_2 = \{c_i, \gamma_q\}_2 = 0.$$

Moreover, in a similar way

$$\{\lambda_{j_1}, \lambda_{j_2}\}_2 = \{\gamma_{q_1}, \gamma_{q_2}\}_2 = 0.$$

All the higher KdV equations are Hamiltonian systems in the bracket  $\{\cdot, \cdot\}_2$  with Hamiltonians which can be expressed explicitly in terms of  $\lambda_0, \dots, \lambda_{2g}$  (see [7] and [6]).

**Example 3.** An important Poisson bracket is generated by the cryptoisomorphism (of Moser and Trubowitz, see [9] and [8]) between the KdV dynamics on the space  $K_g$  and the classical Neumann systems for the motion of a particle on the sphere  $S^g \subset \mathbb{R}^{g+1}$  under the action of a constraint and a quadratic potential  $U(\vec{x}) = \sum_0^g d_s x_s^2$  depending on  $g + 1$  parameters  $d_0, \dots, d_g$ . After this isomorphism the

standard Poisson bracket on  $T^*(S^g)$  induces a bracket  $\{\cdot, \cdot\}_3$  on  $K_g$  such that all the higher KdV equations are Hamiltonian and

$$\{\lambda_{2j}, \lambda_i\}_3 = \{\lambda_{2j}, \gamma_q\}_3 = 0, \quad \lambda_{2j} = d, \quad \{\lambda_{j_1}, \lambda_{j_2}\}_3 = \{\gamma_{q_1}, \gamma_{q_2}\}_3 = 0.$$

The Hamiltonians of all higher KdV equations in this bracket are explicitly computed in [9].

We shall consider some other examples below.

**III. The main results.** Suppose that there is a submanifold  $K'_g \subset K_g$  given by equations only in the variables  $\lambda_j$ , while the dimension of  $K'_g$  is not less than  $2g$ ,

$$\dim K'_g = 2g + r, \quad r \geq 0.$$

On  $K'_g$  we shall consider Poisson brackets  $\{\cdot, \cdot\}$  possessing certain of the following properties (“analytic brackets”),

a)  $\{\lambda_{j_2}, \lambda_{j_2}\} = \{\gamma_{q_1}, \gamma_{q_2}\} = 0.$

b) There exists a function  $Q(\lambda_0, \dots, \lambda_{2g}, \lambda)$ ,  $z^{-1} = \sqrt{\lambda}$  which is analytic in  $z$  in a punctured neighborhood of  $z = 0$  and such that

$$\{Q(\lambda_0, \lambda_1, \dots, \lambda_{2g}, \gamma_{q_1}), \gamma_{q_2}\} = \delta_{q_1 q_2}.$$

**Definition 1.** We call a Poisson bracket possessing property a) *compatible with algebraic geometry*.

**Definition 2.** We call a Poisson bracket *compatible with KdV dynamics* if it possesses property a) and all the higher KdV equations are Hamiltonian systems. It is assumed that their Hamiltonians  $h_m(\lambda_0, \dots, \lambda_{2g})$  can be chosen so that the series

$$h_m(\lambda_0, \lambda_1, \dots, \lambda_{2g}, z) = 2 \sum_{m \geq 0} \left(\frac{z}{2}\right)^{2m+3} h_m(\lambda_0, \lambda_1, \dots, \lambda_{2g})$$

has nonzero radius of convergence. (It is likely that the last condition is not essential, but without it the proofs become more involved.)

**Definition 3.** A Poisson bracket is said to be *analytically compatible with the real structure* if it possesses properties a) and b), while  $Q(\lambda_0, \lambda_1, \dots, \lambda_{2g}, \lambda) d\lambda$  can be extended as a single-valued meromorphic 1-form without poles on  $a_j$  to a covering  $\hat{\Gamma} \xrightarrow{P} \Gamma$  of the Riemann surface  $\Gamma$  where all the cycles  $a_q$  remain closed in  $\hat{\Gamma}$ , and for  $\lambda_j \in \mathbb{R}$  it is required that

$$Q(\lambda_0, \lambda_1, \dots, \lambda_{2g}, \bar{\lambda}) = \bar{Q}(\lambda_0, \lambda_1, \dots, \lambda_{2g}, \lambda).$$

We have the following simple result.

**Lemma 1.** *If a Poisson bracket is compatible with algebraic geometry, and the Hamiltonian  $H$  depends only on  $\lambda_0, \dots, \lambda_{2g}$ , then the Hamiltonian system is linearized by the Abel transformation.*

In correspondence with general Liouville theory [10] we can go over to action-angle variables. The angles have already been described above (see (2)); the arbitrariness (3) makes it possible to satisfy the condition  $\{\phi_{q_1}, \phi_{q_2}\} = 0.$

**Lemma 2.** *If a Poisson bracket is analytically compatible with the real structure, then the action variables  $J_q$  conjugate to the angles  $\phi_1, \dots, \phi_g$  can be expressed solely in terms of  $\lambda_j$  and have the form*

$$J_q = \frac{1}{2\pi} \oint_{a_q} Q(\lambda) d\lambda, \quad z = \lambda^{-1/2}.$$

We shall consider the examples of brackets presented above.

**Example 1.** It is shown in [11] that  $Q(\lambda) = \pm 2ip(\lambda)$ , where  $p(\lambda)$  is the “quasimomentum” defined by the Abelian differential  $dp$ :

$$\oint_{a_q} dp(\lambda) = 0, \quad dp(\lambda) = \frac{dz}{z^2} + 0(1), \quad z = \lambda^{-1/2}.$$

If  $u(x)$  is periodic, then

$$\psi_{\pm}(x + T, \lambda) = \exp(\pm ip(\lambda)T) \psi_{\pm}(x, \lambda).$$

Expanding the quantity  $p(\lambda)$  in  $z$  at  $\lambda = \infty$ , we obtain the Hamiltonians of the higher KdV equations with index  $m$  as the coefficients of  $z^{2m+3}$  (see [1]), and

$$h(z) = -2 \sum_{m \geq 0} \left(\frac{z}{2}\right)^{2m+3} h_i(\lambda_0, \lambda_1, \dots, \lambda_{2g}) = 2p(z) - \frac{2}{z} + z\bar{u}.$$

**Example 2.** Using the results of [12], we obtain

$$Q(\lambda) = \sqrt{-R(\lambda)}, \quad R(\lambda) = \prod_{i=0}^{2g} (\lambda - \lambda_i).$$

**Example 3.** Comparing with [9], [8] and [12], we obtain

$$Q(\lambda) = \frac{1}{2} \sqrt{-R(\lambda)} \prod_{i=0}^{2g} (\lambda - \lambda_{2g})^{-1}.$$

**Lemma 3.** *If a Poisson bracket is compatible with KdV dynamics and possesses property b), then the difference  $Q - h = P$  is a Laurent series in  $z$  and  $z^{-1}$  with coefficients in the annihilator:*

$$Q - h = \sum_{-\infty < \alpha < \infty} p_{\alpha}(\lambda_0, \lambda_1, \dots, \lambda_{2g}) z^{\alpha} = P(\lambda_0, \lambda_1, \dots, \lambda_{2g}, z),$$

$$\{p_{\alpha}, \lambda_j\} = \{p_{\alpha}, \gamma_q\} = 0.$$

*Proof.* We consider the time derivatives of  $\gamma_q^{(m)}$  on the basis of the higher KdV equations with index  $m$  and the series

$$(4) \quad \{h, \gamma_q\} = 2 \sum_{m \geq 0} \left(\frac{z}{2}\right)^{2m+3} \gamma_q^{(m)}.$$

It follows from the results of [12] that the series (4) possesses the property

$$\{h(\lambda_0, \lambda_1, \dots, \lambda_{2g}, z(\gamma_{q_1})), \gamma_{q_2}\} = 2 \sum_{m \geq 0} (2^{-1} z(\gamma_{q_1}))^{2m+3} \gamma_{q_2}^{(m)} = \delta_{q_1 q_2}.$$

Hence  $\{P(\lambda_0, \dots, \lambda_{2g}, z(\gamma_{q_1})), \gamma_{q_2}\} = 0$ .

On the level surfaces of the annihilator this Poisson bracket is nondegenerate. From the foregoing it therefore follows that

$$P(\lambda_0, \dots, \lambda_{2g}, \gamma_i) = G_i(a, \gamma_1, \dots, \gamma_g)$$

where  $a(\lambda_0, \dots, \lambda_{2g})$  belongs to the annihilator. From the form of  $G_i = P(\lambda_0, \dots, \lambda_{2g}, \gamma_i)$  and the independence of the groups of variables  $\lambda_j$  and  $\gamma_q$  on the full phase space it follows that  $G_i = G(a, \gamma_i)$ . Finally, again using the independence of  $\lambda_j$  and  $\gamma_i$ , we make the reverse substitution  $\gamma_i \rightarrow \lambda$  and obtain finally

$$P(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{2g}, \lambda) = Q - h = G(a, \lambda),$$

where  $a(\lambda_0, \dots, \lambda_{2g})$  belongs to the annihilator. The proof of the lemma is complete.  $\square$

Summarizing the preceding lemmas, we obtain the following result.

**Theorem.** *If a Poisson bracket is compatible with KdV dynamics and is analytically compatible with the real structure, then the following assertions are true:*

a) *The action variables conjugate to the angles on the compact real torus  $T^g = A(a_1 \times \dots \times a_g) \subset J(\Gamma)$  have the form*

$$(5) \quad J_q = \frac{1}{2\pi} \oint_{a_q} Q d\lambda = -\frac{1}{2\pi} \oint_{a_q} \lambda dQ.$$

b) *In the expansion*

$$Q = \sum_{\alpha} \frac{z^{\alpha}}{2} g_{\alpha}(\lambda_0, \dots, \lambda_{2g}),$$

*of  $Q$  in the Laurent series in  $z = \lambda^{-1/2}$  near  $z = 0$ , the coefficients  $h_m = \frac{1}{2}g_{2m+3}$ ,  $m \geq 0$ , are the successive Hamiltonians of the higher KdV equations; the remaining  $g_{\alpha}$  belong to the annihilator of the Poisson bracket.*

*Remark.* It follows from (5) that in Example 1, where  $Q = \pm 2ip(\lambda)$ , the action variables (and hence the quasiclassical Bohr–Sommerfeld quantization conditions) can be expressed in terms of integrals of the type of the Peierls functions in the “jelly” model (see [13]) but over forbidden zones with imaginary quasimomentum.

#### IV. Other examples of Poisson brackets.

**Example 4.** The restriction to  $K_g$  of the bracket of [14] is not unique; here  $O_{\alpha, \beta}(\lambda) = \pm 2ip(\lambda)(\alpha\lambda + \beta)^{-1}$ ; the annihilator of the Poisson brackets obtained consists of all periods  $T_q(\lambda_0, \dots, \lambda_{2g})$  and any quantity functionally independent of the coefficients of the series  $Q_{\alpha, \beta}(\lambda)$  and all the  $T_q$  (for example,  $\sum_0^{2g} \lambda_j$ ). These brackets possess all the “good” properties.

**Example 5.** Already in [1] for  $g = 2$  a cryptoisomorphism of the KdV dynamics with the dynamics of a solid body in the integrable case of Kovalevskaja was discussed; this induces on  $K'_2$  new Poisson bracket. Here

$$Q(\lambda) = \sqrt{-R(\lambda)}(\lambda - \lambda_3)^{-1}(\lambda - \lambda_4)^{-1} = z^{-1} + 3Hz + \dots,$$

where  $H = \frac{1}{3} \sum \lambda_i$  is the gyroscope Hamiltonian giving the dynamics corresponding to translation in  $x$  (see, for example, [15]). This bracket is already not compatible with KdV dynamics (see Lemma 3).

Other algebro-geometric brackets on  $K_2$  or  $K'_2 \subset K_2$  can be obtained by proceeding from the integrable cases of Clebsch and Steklov in hydrodynamics and of Gorjacev and others in the theory of a solid body with a fixed point [15].

*Remark.* A parallel theory can also be developed for the nonlinear Schrödinger equation, the Toda lattice, etc. The analytic properties of  $Q(\lambda)$  are to be altered with the consideration that  $\lambda = \infty$  consists of 2 points. For the nonlinear Schrödinger equation the number of phase variables increases. The special case of the nonlinear Schrödinger equation and the sine-Gordon equation, where reality conditions are nontrivial, will be investigated in a subsequent paper by B. A. Dubrovin and S. P. Novikov.

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LANDAU INSTITUTE OF THEORETICAL PHYSICS, ACADEMY OF SCIENCES OF THE USSR