SOME PROPERTIES OF $(4k+2)$-DIMENSIONAL MANIFOLDS

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This paper is related to the writer's work in [8] on diffeomorphism of simply-connected manifolds and to Kervaire's work on the existence of nonsmoothable manifolds of dimension 10 [4]. The main part of the paper, which is devoted to properties of 10-dimensional manifolds, will also make essential use of ideas due to Milnor and to the present writer [5, 9] on generalized cobordism rings. We recall (see [8]) that in studying the homotopy group $\pi_{N+n}(T_N)$ of the Thom space T_N of the normal bundle of a manifold M^n $(n=4k+2)$, we single out the subset $A \subset \pi_{N+n}(T_N)$ consisting of those $\alpha \in A$ such that $H(\alpha)=\phi[M^n]$, where $\phi\colon H_k(M^n) \to H_{k+N}(T_N)$ is the Thom isomorphism, and $H\colon \pi_j(X) \to H_j(X)$ is the Hurewicz homomorphism. We proved that a t-regular representative $f_\alpha\colon S^{N+n} \to T_N$ of an element $\alpha \in A$ can be chosen such that the manifold $f_\alpha^{-1}(M^n)=M_\alpha^n$ has the following properties:

1. $f_{\alpha *}\colon H_i(M_\alpha^n) \to H_i(M^n)$ is an isomorphism for $i \neq 2k+1$.
2. $\operatorname{Ker} f_{\alpha *} = Z + Z \subset H_{2k+1}(M_\alpha^n)$.
3. The Hurewicz homomorphism $H\colon \operatorname{Ker} f_{\alpha *} \to \operatorname{Ker} f_{\alpha *} \subset H_{2k+1}(M_\alpha^n)$ is an isomorphism.
4. If a cycle $x \in \operatorname{Ker} f_{\alpha *}$ is realized by an embedded sphere $S^{2k+1} \subset M_\alpha^n$ and $n \neq 6, 14$, then the normal bundle $\nu(S^{2k+1}, M_\alpha^n)$ of S^{2k+1} in the manifold M_α^n depends only on the element x, belongs to the group Z_2, and defines a mapping $\phi\colon \operatorname{Ker} f_{\alpha *} \to Z_2$ such that

$$\phi(x+y)=\phi(x)+\phi(y)+x\cdot y \bmod 2.$$

If x, y is a basis for the group $\operatorname{Ker} f_{\alpha *}$, then we let $\phi(\alpha)=\phi(x)\phi(y) \in Z_2$.

Theorem 1. *The invariant $\phi(\alpha)$ is independent of the choice of the representative f_α satisfying conditions 1–4 above.*

If $n=6, 14$, the definition of the invariant has to be modified. Instead of using the normal bundle of S^{2k+1} in M_α^n (under heading 4 above), we use the "framed"* structure of the embedded sphere, and refer to [10], for the exact formulation of the definition. In this case we denote the invariant by $\psi(\alpha) \in Z_2$. All algebraic properties of $\phi(\alpha)$ carry over to $\psi(\alpha)$ in the cases $n=6, 14$.

Theorem 1'. *The invariant $\psi(\alpha)$ is singlevalued and well-defined.*

The proof of Theorem 1' is identical to that of Theorem 1.

We let $\tilde{A} \subset A$ denote the subset of $A \subset \pi_{N+n}(T_N)$ which consists of those $\alpha \in \tilde{A}$ such that $\phi(\alpha)=0$ $(n \neq 6, 14)$, or $\psi(\alpha)=0$ $(n=6, 14)$. The following results are easy corollaries of the definitions of ϕ and ψ and Theorem 1.

Corollary 1. *When $n=6, 14$, the set $\tilde{A}$ contains exactly half of the elements of A (for any manifold M^n).*

Corollary 2. *If $M^n = M_1^n \# M_2^n$, then $\tilde{A}$ coincides with A for M^n if and only if $\tilde{A}$ coincides with A for M_1^n and M_2^n.*

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* Translator's note: See the gloss on this term in Soviet Mathematics 4 (1963), p. 27 (footnote).

Now we want to study the distribution of the values of the invariant ϕ on the set A. For simplicity we assume that M^n satisfies $H^{2k+1}(M^n, Z) \otimes Z_2 = 0$ and $n = 4k + 2$, $k \neq 0, 1, 3$.

We note that $\pi_{N+n}(T_N) = Z + \tilde{\pi}$, where $\tilde{\pi}$ is a finite group. The set A consists of all elements of the form $1 + \gamma$, where $1 \in Z$ and $\gamma \in \tilde{\pi}$. Let us assume from now on that a particular direct sum decomposition of $\pi_{N+n}(T_N)$ has been chosen.

Theorem 2. *The following formula holds*:

$$\phi(1 + \gamma + \delta) = \phi(1 + \gamma) + \phi(1 + \delta) + \phi(1 + 0), \text{ where } \gamma, \delta \in \tilde{\pi}.$$

The proof is simple and depends on writing $1 + \gamma + \delta = (1 + \gamma) + (1 + \delta) - (1 + 0)$, which allows one to realize $1 + \gamma + \delta$ by a "good" representative map $f\colon S^{N+n} \to T_N$ and then use modifications on the complete preimage $f^{-1}(M^n)$.

In our case we can choose the splitting $\pi_{N+n} = Z + \tilde{\pi}$ in such a way that $\phi(1 + 0) = 0$, so that by putting $\bar{\phi}(\gamma) = \phi(1 + \gamma)$ we get a homomorphism $\bar{\phi}\colon \tilde{\pi} \to Z_2$. This gives

Corollary 3. *If* $H^{2k+1}(M^n, Z) \otimes Z_2 = 0$, *then the set* $\tilde{A}$ *either contains half of* A *or coincides with* A.

What we have done so far gives us enough information about the relations between $\tilde{A}$ and A in dimensions $n = 4k + 2$ for $k = 1, 3$, and we also have a certain amount of information on higher dimensions. The first really nontrivial case is $k = 2$ $(n = 10)$, which we now concentrate on. One can see that this case is nontrivial from the fact that, on the sphere of this dimension $\tilde{A}$ and A coincide, and it is totally unclear what the situation is for other manifolds. Our goal will be to generalize the invariant $\Phi(M^{10}) \in Z_2$ defined for 4-connected 10-dimensional manifolds by Kervaire [4], and then to apply this invariant to solve some problems.

Since the cohomology operation $\mathrm{Sq}^2\mathrm{Sq}^4\colon H^5(X, Z) \to H^{11}(X, Z_2)$ is identically zero (thanks to the relation $\mathrm{Sq}^2\mathrm{Sq}^4 = \mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1$), there is a "secondary" cohomology operation $\Phi\colon \mathrm{Ker}\,\mathrm{Sq}^4 \to \mathrm{Coker}\,\mathrm{Sq}^2$ which is defined on $\mathrm{Ker}\,\mathrm{Sq}^4 \subset H^5(X, Z)$.

Lemma 1. *The operation* Φ *has the property*:

$$\Phi(x + y) = \Phi(x) + \Phi(y) + xy.$$

The proof is quite simple.

Lemma 2. *If* $\pi_1(M^{10}) = 0$ *and* $w_2(M^{10}) = 0$ *for a topological manifold* M^{10}, *then the operation* $\Phi\colon H^5(M^{10}, Z) \to H^{10}(M^{10}, Z_2) = Z_2$ *is always defined and singlevalued.*

Lemma 3. *Under the conditions of Lemma 2 the operation* Φ *defines a singlevalued homomorphism* $\Phi\colon \mathrm{Tor}\, H^5(M^{10}, Z) \to H^{10}(M^{10}, Z_2)$.

This lemma follows easily from Lemmas 1 and 2.

From now on we shall restrict ourselves to manifolds which satisfy the following conditions:

1. $\pi_1(M^{10}) = 0$.
2. $w_2(M^{10}) = 0$.
3. The homomorphism $\Phi\colon \mathrm{Tor}\, H^5(M^{10}, Z) \to H^{10}(M^{10}, Z_2)$ is trivial.

We now define the "generalized Kervaire invariant" for topological manifolds satisfying conditions 1–3: a) let $x_1, \cdots, x_{2l}$ be a basis for the group $H^5(M^{10}, Z)/\mathrm{Torsion}$ such that $x_{2i-1}x_{2i} \neq 0$ for $1 \leq i \leq l$ and $x_k x_s = 0$ otherwise; b) by condition 3, the operation Φ is defined on the group $H^5(M^{10}, Z)/\mathrm{Torsion}$ and takes values in Z_2; the sum $\Phi(M^{10}) = \sum_{i=1}^{l} \Phi(x_{2i-1})\Phi(x_{2i})$ is independent of the above basis and is called the "generalized Kervaire invariant"; c) the invariant $\Phi(M^{10})$

is a homotopy invariant of the manifold.

We now have the following important lemma.

Lemma 4. *If* M^{10} *is a smooth manifold and is the boundary of a smooth oriented manifold* W^{11} *with* $w_2(W^{11}) = 0$, *then* $\Phi(M^{10}) = 0$.

In analogy with [5, 9], we consider the spinor cobordism ring $V_{\text{Spin}} = \Sigma V^i_{\text{Spin}}$, $V^i_{\text{Spin}} = \pi_{N+i}(M\,\text{Spin}\,N)$, where $M\,\text{Spin}\,N$ is the Thom complex of the spinor group. As is well known, the tangent bundle (or stable normal bundle) of a manifold can be reduced to the spinor group if $w_1(M^n) = w_2(M^n) = 0$. Lemma 4 therefore implies the following lemma.

Lemma 5. *The Kervaire invariant defines a singlevalued homomorphism* $\Phi: \tilde{V}^{10}_{\text{Spin}} \to Z_2$, *where* $\tilde{V}^{10}_{\text{Spin}} \subset V^{10}_{\text{Spin}}$.

Proof. The invariant Φ is obviously additive. If a manifold determines the zero element of V^{10}_{Spin}, then the manifold must be the boundary of something in the next dimension, W^{11}, such that $w_2(W^{11}) = 0$, and therefore $\Phi = 0$. However, on account of restriction 3, the invariant Φ may possibly not be defined for all elements of the group V^{10}_{Spin}. The lemma is proved.

We shall now give without proof a number of results on the ring $V_{\text{Spin}} = \Sigma V^i_{\text{Spin}}$.

I. The groups V^i_{Spin} for $i \leq 10$ are as follows:

i	0	1	2	3	4	5	6	7	8	9	10
V^i_{Spin}	Z	Z_2	Z_2	0	Z	0	0	0	$Z+Z$	Z_2+Z_2	$Z_2+Z_2+Z_2$

II. Generators for the groups V^k_{Spin} for $k \leq 10$ can be chosen as:

$$1 \in V^0_{\text{Spin}},\quad x_1 \in V^1_{\text{Spin}},\quad x_1^2 \in V^2_{\text{Spin}},\quad x_4 \in V^4_{\text{Spin}},$$
$$x_8 \in V^8_{\text{Spin}},\quad y_8 \in V^8_{\text{Spin}},\quad 4y_8 = x_4^2,\quad x_1x_8 \in V^9_{\text{Spin}},$$
$$x_1y_8 \in V^9_{\text{Spin}},\quad x_1^2x_8 \in V^{10}_{\text{Spin}},\quad x_1^2y_8 \in V^{10}_{\text{Spin}},\quad Z_{10} \in V^{10}_{\text{Spin}}.$$

III. The element $x_1 \in V^1_{\text{Spin}}$ is represented by a circle $S^1 \subset R^{N+1}$ with nontrivial framed structure.

IV. The group V^8_{Spin} is generated by the following manifolds: a) the quaternion projective plane $P^2(Q)$; b) Milnor's 8-dimensional 3-connected almost parallelizable manifold M^8_0 of index $I(M^8_0) = 8 \cdot 28$.

V. The generator $Z_{10} \in V^{10}_{\text{Spin}}$ is represented by a manifold M^{10} which has $w_4w_6(M^{10}) \neq 0$.

The subgroup $V^1_{\text{Spin}}\, V^1_{\text{Spin}}\, V^8_{\text{Spin}} \subset V^{10}_{\text{Spin}}$ is determined by the condition $w_4w_6 = 0$.

The results in I–V can be proved as in the papers by Milnor [5] and the author [9] on generalized cobordism rings; the main tools are the Adams spectral sequence [1] and the A-genus [2].

Lemma 6. *The homomorphism* $\Phi: V^{10}_{\text{Spin}} \to Z_2$ *is zero on the subgroup* $V^1_{\text{Spin}}\, V^1_{\text{Spin}}\, V^8_{\text{Spin}} \subset V^{10}_{\text{Spin}}$.

The proof of this lemma is nontrivial and makes essential use of the information in heading IV above on the geometrical generators of the group $V^8_{\text{Spin}} = Z + Z$. The hardest part is the analysis of the element represented by the manifold $P^2(Q) \times S^1 \times S^1$. Essentially what one has to do is carry out explicit Morse modifications over one-dimensional cycles in the manifolds $M^8_0 \times S^1 \times S^1$ and $P^2(Q) \times S^1 \times S^1$.

The following is an easy consequence of these lemmas:

Theorem 3. *The invariant* $\Phi(M^{10})$ *is a singlevalued function of the residue* $w_4w_6(M^{10})$, *and* $\Phi(M^{10}) = 0$ *if* $w_4w_6(M^{10}) = 0$, *for a smooth manifold* M^{10}.

Thus $\Phi = \Phi(w_4w_6)$, for smooth manifolds.

Remark. The author conjectures that $\Phi(w_4w_6) = 0$ for smooth manifolds. To prove this, it would be sufficient to construct a smooth manifold M^{10} with $w_4w_6(M^{10}) \neq 0$ and $\Phi(M^{10}) = 0$.

Theorem 4. *If the invariant* $\Phi(M^{10})$ *is defined on a smooth manifold* M^{10}, *then this manifold has* $\widetilde{A} = A$, *i.e.,* $\phi(\alpha) \equiv 0$ *for all* $\alpha \in A$.

Proof. Let α be such that $\phi(\alpha) = 1$. Pick a representative $M_\alpha^{10} = f_\alpha^{-1}(M^{10})$ having properties 1–4 as indicated at the beginning of this paper. Clearly $\phi(M_\alpha^{10}) = \phi(M^{10}) + 1$, and $w_4w_6(M_\alpha^{10}) = w_4w_6(M^{10})$. Since M_α^{10} is a smooth manifold, this contradicts Theorem 3.

Let M^{10} be a topological manifold (or, more generally, a polyhedron satisfying Poincaré duality). Proceeding along the lines of [8] and Browder's paper [3], one can prove the following assertion:

Theorem 5. *If the invariant* Φ *is defined for the polyhedron* M^{10}, *then the following two conditions are necessary and sufficient for* M^{10} *to have the homotopy type of a smooth manifold:* a) $\Phi(M^{10}) = \Phi(w_4w_6)$; b) *there is an* SO_N*-bundle* ν *over* M^{10} *such that* $\phi[M^{10}] \in H_{N+10}(T_N)$ *is a spherical cycle, where* T_N *is the Thom complex of the bundle* ν. (ν *is the normal bundle of the desired smooth manifold.*)

Note that in [4] Kervaire constructed a 4-connected manifold satisfying b) but not a). Thus both conditions are essential. In the range $5 \leq n \leq 17$, the only dimension which presents any difficulty is $n = 10$ (the cases $n = 6$, 14 are simple). When $n = 4k + 2$ with $k \geq 4$, new difficulties arise. The author conjectures that when k is even the invariant Φ can be generalized on the basis of the relation $\mathrm{Sq}^2\mathrm{Sq}^{2k} = \mathrm{Sq}^{2k+2} + \mathrm{Sq}^{2k+1}\mathrm{Sq}^1$ in the Steenrod algebra, and an appropriate study of the ring V_{Spin}.

We shall now give some results on spinor cobordism.

Lemma 7. *If* $\pi_1(M^n) = 0$ *and* $w_2(M^n) = 0$, *then the stable normal* SO_N*-bundle (tangent bundle) can be reduced to the group* Spin N, *and in a unique way. If* $n \geq 3$, *then every element of the group* V_{Spin}^n *is represented by a simply-connected manifold.*

Consider the natural homomorphism ("removing the frame") $p: G_i \to V_{\mathrm{Spin}}^i$, where $G_i = \pi_{N+i}(S^N)$. The following important lemma holds.

Lemma 8. *If* $3 \leq i \leq 8$, *the image of the homomorphism* $p: G_i \to V_{\mathrm{Spin}}^i$ *is zero. For* $i = 9, 10$, *the image of* p *is isomorphic to* Z_2.

For the proof, one takes the Milnor manifold M_0^8 mentioned above, and makes modifications in the manifolds $M_0^8 \times S^1$, for $i = 9$, and $M_0^8 \times S^1 \times S^1$, for $i = 10$, carrying over the nontrivial "spinor frames." As a result, the corresponding elements of the groups V_{Spin}^9 and V_{Spin}^{10} will be realized by homotopy spheres.

Theorem 6. *There are smooth manifolds of the homotopy types of the 9-sphere and the 10-sphere which are not boundaries of any smooth manifolds with vanishing Stiefel class* $w_2 = 0$.

Corollary. *In dimensions 9 and 10, membership of a smooth simply-connected manifold in a spinor cobordism class is not a combinatorial invariant (in contrast with cobordism* *with respect to the groups O and SO), and is not determined by the homotopy type and the tangent bundle.*

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Received 13/JUNE/63

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Translated by:
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