TWO-DIMENSIONAL SCHRÖDINGER OPERATORS IN PERIODIC FIELDS

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Abstract. A class of problems connected with the description of the motion of an attracted quantum particle in possibly time-dependent, periodic, external fields is studied on the basis of a development of the method of the inverse problem.

INTRODUCTION

Linear operators with periodic coefficients arise in many problems both of classical mechanics and mathematics and modern mathematical physics; it would be superfluous to enumerate them. An interesting class of "periodic" problems is that connected with the description of the motion of a nonrelativistic quantum particle (for example, an electron although, with the exception of Sec. 5, the spin does not play a role in our considerations) in periodic external fields—electric and magnetic—which are possibly time-dependent. In this case the Schrödinger operator has the form

$$L = i\partial/\partial t - L_0,$$
$$L_0 = \frac{1}{2}(i\partial/\partial x^\alpha + eA_\alpha(x))^2 + e\varphi \tag{1}$$

(here we have set $n = \hbar = c = 1$), e is the charge of the particle, the spin is equal to zero, $\alpha = 1, 2, 3$, $\varphi = A_0$, and $A = (A_0, A_1, A_2, A_3)$ is the vector potential of the electromagnetic field F_{ab}

$$F_{ab} = \partial_a A_b - \partial_b A_a, \quad \partial_a = \partial/\partial x^a, \quad a, b = 0, 1, 2, 3 \tag{2}$$

or

$$F_{0\alpha} = E_\alpha, \quad E_\alpha = \partial_0 A_\alpha - \partial_\alpha \varphi,$$
$$\frac{1}{2}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta} = H^\gamma, \quad \alpha, \beta = 1, 2, 3, \tag{2'}$$

and E and H are the electric and magnetic field intensities. By changing the vector potential $A_a \to A_a + \partial_a \psi$, the Schrödinger operator L can be chosen with periodic coefficients relative to a lattice Γ in $R^4(x, y, z, t)$ if and only if the following two conditions are satisfied:

(a) the electromagnetic field F_{ab} is periodic relative to the latter Γ, $t = x^0$;

UDC. 517.957+512.7.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Vol. 23, pp. 3–33, 1983.

(b) the integrals of the electromagnetic field—the 2-form $F = F_{ab}\,dx^a \wedge dx^b$—over all two-dimensional elementary cells of the lattice Γ are equal to zero (i.e., the field is "topologically trivial," as they say).

If these conditions are satisfied, then, by choosing an appropriate vector potential, we obtain a linear operator $L(x, i\partial/\partial x)$ with periodic coefficients where $L = i\partial/\partial t - L_0$. We are interested in either the eigenvalue problem (the stationary Schrödinger equation) if F_{ab} does not depend on t

$$L_0\psi = \varepsilon\psi, \tag{3}$$

or a solution of the time-dependent Schrödinger equation

$$L\psi = 0. \tag{4}$$

To simplify the formulas, we shall henceforth assume that the lattice Γ is rectangular, i.e., the coefficients of the operator L are periodic in each of the variables x^a with period T_a, $a = 0, 1, 2, 3$.

For both Eqs. (3) and (4) we study the "Bloch" solution

$$\psi(\vec{x} + \Gamma_a) = \exp\{ip_a T_a\}\psi(\vec{x}) \tag{5}$$

where $\Gamma_a = T_a e_a$, e_a is the unit vector of the x^a axis in the case of a rectangular lattice $\Gamma = \sum n_a \Gamma_a$, and n_a are integers.

A considerable part of the present survey—Chaps. I and II—involves the study of the properties of the "quasimomentum" p_a and finding exactly solvable cases reducing to the classical theory of Riemann surfaces and the θ-functions connected with them; the latter involves developing the ideas of the method of the inverse problem that arise in the "one-dimensional" case.

Chapter III is devoted to periodic problems in the case where the magnetic field is topologically nontrivial, i.e., condition b) (above) is not satisfied.

Here there is a sharp distinction between cases where the flows through elementary two-cells of the field intensity are irrational, in which there is practically no theory, and the rational case in which it is possible to obtain a number of results.

CHAPTER 1

GENERAL PROPERTIES OF LINEAR OPERATORS WITH PERIODIC COEFFICIENTS. INTEGRABLE PROBLEMS IN THE ONE-DIMENSIONAL CASE

1. Bloch Functions. The Dispersion Law. Operators of General Position

We consider a linear operator L of order m in n-dimensional space $(x^1, \dots, x^n)$ with smooth, possibly matrix-valued coefficients which are periodic in all variables:

$$\begin{gathered} L = \sum_{i_1 + \cdots + i_n < m} a_{i_1 \dots i_n}(x)\partial_{i_1} \dots \partial_{i_n} = L(x, \partial). \\ \partial_j = \partial/\partial x^j, \quad L(\dots, x^i + T_i, \dots, \partial) = L(x, \partial). \end{gathered} \tag{1}$$

Definition. A Bloch solution ψ (or a Floquet solution) of the equation $L\psi = 0$ or $L\psi = \varepsilon\psi$ is one such that

$$\hat{T}_k\psi = \psi(\dots, x^k + T_k, \dots) = \mu_k\psi(x), \quad \mu_k = \text{const}, \tag{2}$$

i.e., it is an eigenvector of all translation operators $\hat{T}_k$.

We denote the eigenvalues μ_k of the operators $\hat{T}_k$ by

$$\mu_k = \exp\{ip_kT_k\}. \tag{3}$$

The numbers p_k are called the components of the quasimomentum. By the very definition these numbers are not uniquely determined: p_k is equivalent to $p_k + 2\pi T_k^{-1}$.

For a self-adjoint elliptic operator L, because of the discreteness and finite multiplicity of the spectrum on compact manifolds—in the present case in a bundle over the torus T^n after prescribing the quasimomenta $(p_1, \dots, p_n)$ as boundary conditions (i.e., the rules for gluing together with the bundle)—we obtain a countable number of solutions

$$L\psi = \varepsilon_j\psi, \quad \varepsilon_j(p_1, \dots, p_n) = \varepsilon, \quad L = \int_{T^n} L_p\, d^np, \tag{4}$$

where the quasimomentum $p = (p_k)$, because of the equivalence $p_k \sim p_k + 2\pi n_k T_k^{-1}$ (n_k is any integer), runs over the torus T_*^n of the lattice Γ^* reciprocal to the lattice Γ in x-space. These eigenvectors are dense in $\mathcal{L}_2(R^n)$.

Hypothesis. There exists a collection of solutions $\psi_j(x, p)$ of Eq. (2) for all complex $p \in C^n$ and a countable set of indices j which depends on x as a set of parameters. This collection is such that $\psi_j(x, p)$ is a meromorphic function on the complex "manifold of quasimomenta" W^n—the multidimensional graph of the dispersion law $\varepsilon_j(p)$ whose points are pairs (p, j). For real $p \in R^n \mod \Gamma^*$ the functions ψ_j generate all Bloch eigenfunctions of the operator L acting in the Hilbert space $\mathcal{L}_2(R^n)$. The "dispersion law" $\varepsilon = \varepsilon_j(p)$ is defined as a meromorphic function on the manifold of quasimomenta

$$\varepsilon\colon W^n \to C. \tag{5}$$

In the self-adjoint case, obviously, the dispersion law $\varepsilon_j(p)$ is real for real p, since the operator $L_p \oplus L_{-p}$ is a self-adjoint operator on a compact manifold.

If the operator L is purely real, then the simple involution

$$\psi^*(-p) = \psi(p), \quad \varepsilon(-p) = \varepsilon(p) \tag{6}$$

acts on the set of Bloch functions; here ψ^* are the complex conjugate quantities, and p is real. In the set of all linear operators with periodic coefficients there are "operators of general position." We assume that elliptic operators of general position possess the analytic properties formulated in the hypothesis (above), although this has not been rigorously proved. It is clear that self-adjoint elliptic operators of general position apparently possess a dispersion law $\varepsilon_j(p)$ with the following properties: for real p the condition $\varepsilon_j(p) = \varepsilon_k(p)$ defines a submanifold A_{kj} (possibly with singularities) of codimension 3 in the space of real quasimomenta p, i.e., on the torus T^n_*. This also pertains to purely real operators L of general position: they also have a manifold of coalescence of roots $\varepsilon_j(p) = \varepsilon_k(p)$ of codimension 3; the dispersion laws possess the involution (6).

Simply speaking, this means that the condition of coalescence of branches has the same structure as for a "general" n-parameter family of the family of Hermitian operators $\hat{H}_p = L_p \oplus L_{-p}$. The submanifold of coalescence of branches A_{kj} at a general nonsingular point $p_0 \in A_{kj}$ possesses a topological invariant: we shall consider the whole picture on a three-dimensional plane $R^3_0 \perp A_{kj}$ normal at a point $p_0 \in R^3_0$. At all points $p \in R^3_0$ except for p_0 itself (and points near p_0) the branches are distinct $\varepsilon_k(p) \neq \varepsilon_j(p)$. The eigendirections $\psi_j(p)$ and $\psi_k(p)$ defined one-dimensional complex bundles over the sphere $S^2_\varepsilon\colon |p - p_0| = \varepsilon$. We denote these bundles by $\eta^\varepsilon_j, \eta^\varepsilon_k$. In the case of general position the Chern classes are (the sum of the bundles $\eta^\varepsilon_j \oplus \eta^\varepsilon_k$ is extended to the disk)

$$c_1(\eta^\varepsilon_j) = -c_1(\eta^\varepsilon_k),$$

where $c_1(\eta^\varepsilon_j)$ is a basis class of the cohomology of the sphere $H^2(S^2_\varepsilon, Z)$.

Each branch of the dispersion law $\varepsilon_j(p)$ defines a one-dimensional bundle η_j [with the help of the direction $\psi_j(p)$] over the following complex (base)

$$B_j = T^n_* \setminus \left(\bigcup_{k \neq j} A_{kj} \right),$$

i.e., where the branch $\varepsilon_j(p)$ does not coalesce with any branch. The bundle η_j has some Chern class $c_1(\eta_j) \in H^2(B_j, Z)$. The collection of classes $c_1(\eta_j)$ is a topological invariant of the collection of dispersion laws, i.e., of the spectrum (see [25]).

For us the cases $n = 2$, 3 are of primary importance. In some cases these properties of operators of general position will be justified (for $n = 2$ and nonzero flux of the magnetic field see Sec. 4). For $n = 2$ in general position the dispersion laws do not coalesce: $\varepsilon_j(p) \neq \varepsilon_k(p)$ for all k, j, p.

For $n = 2$ we consider a general one-parameter family of operators L_τ depending on the parameter τ. Coalescence of the branches occurs for isolated values of the parameter $\tau = \tau_0$ at isolated points p_0:

$$\varepsilon_j(p_0, \tau_0) = \varepsilon_k(p_0, \tau_0).$$

Because of what has been said above, when the parameter τ passes through this point the dispersion laws "collide," as it were, and one imparts to the other a part of the Chern class

$$c_1(\eta_j, \tau > \tau_0) = c_1(\eta_j, \tau < \tau_0) + c,$$
$$c_1(\eta_k, \tau > \tau_0) = c_1(\eta_k, \tau < \tau_0) - c,$$

where c is a basis class in $H^2(T^2_*, Z)$ (see [25]).

2. Algebraic Operators of Rank $l > 1$. Examples. Nonlinear Systems

The theory of one-dimensional operators with periodic coefficients is well developed especially for operators of second order. In the one-dimensional case all the definitions are especially simple and need no involved justification.

Let $L = \sum_{i \le n} a_{n-i}(x)\partial^i$, where a_0 is a constant matrix with distinct eigenvalues and a_j are periodic matrix-valued $m \times m$ functions with periodic T. The equation $L\psi = \varepsilon\psi$ under the condition $\psi(x+T) = \mu_j(\varepsilon)\psi(x)$, $j = 1, 2, \ldots, N = nm$, defines a Riemann surface Γ (the manifold of quasimomenta) which is N-sheeted over the ε-plane. Here $\mu_j = \exp(ip_jT)$.

Definition 1. The operator L is called a "finite-zone" operator if the Riemann surface Γ of the Bloch function $\psi_j(x, \varepsilon)$ (also called the "spectrum" of the operator L) has finite genus.

The ordinary spectrum of the operator L in the Hilbert space $\mathcal{L}_2(R)$ is found there where the Bloch function $\psi_j(x, \varepsilon)$ is bounded on the entire line, i.e., where $|\mu_j(\varepsilon)| = 1$ for at least one j. Thus, the finite-zone property in the sense of Definition 1 for operators of second order with a periodic coefficient (potential) is equivalent to the finiteness of the number of lacunae in the spectrum (on the ε-line), while for operators of higher order it is a stronger condition. For second-order $L = -\partial^2 + u(x)$ the branch points of the surface Γ coincide with the end points of the lacunae (the forbidden zones) of the spectrum in $\mathcal{L}_2(R)$, and hence the surface Γ precisely determines the spectrum of the operator (see [6] for operators L of second order and [17] for operators of higher order).

Definition 2. The operator L is called algebraic if there exists a nontrivial differential operator A such that $[A, L] = 0$.

In itself the concept of an algebraic operator does not require periodicity of the coefficients. Already in the 1920s a formal classification theory was constructed in certain cases without discussion of the domain and the function class of the coefficients. For algebraic operators a collection of joint solutions

$$L\varphi = \varepsilon\varphi, \quad A\varphi = \gamma\varphi, \qquad \varphi = \varphi(x, \varepsilon, \gamma) \tag{1}$$

is meromorphic on some Riemann surface $\tilde{\Gamma}\colon \varphi(\varepsilon, \gamma) = 0$, where φ is a polynomial.

There is hereby a polynomial algebraic relation between the operators determining defining a Riemann surface Γ of finite (Burchnall, Chaundy [4, 5]):

$$\varphi(L, A) = 0, \qquad (\varepsilon, \gamma) = \mathcal{P} \in \tilde{\Gamma}, \quad \varphi = \varphi(x, \mathcal{P}). \tag{2}$$

Definition 3. The number of linearly independent solutions of (1) for a fixed "general" point $\mathcal{P} \in \tilde{\Gamma}$ is called the rank of the commuting pair A, L.

Scalar commuting pairs of operators of "general position" having relatively prime orders and always rank 1 were classified in [38, 39] by Burchnall and Chaundy. In the note [35] Baker pointed out that the general eigenfunction $\varphi(x,\mathcal{P})$ in this case (using material of [36] summarizing a number of deep results of the XIX century, in particular, some functional constructions of Gordan and Clebsch) can, in principle, be computed in terms of θ-functions. However, this was not done concretely, and the theory itself was forgotten because of the lack of connections and applications. Effective explicit formulas for φ and for the coefficients L were obtained much later after the appearance of the "method of the inverse problem" (see the surveys [6, 17]). Especially important are formulas for the coefficients of L which can be obtained independently of φ, are more simple in structure than formulas for φ, and are important in applications to nonlinear systems of KdV type. Sometimes (especially for rank $l > 1$ below) the formulas for φ cannot be obtained, but it is possible to obtain formulas for the coefficients.

In connection with the theory of the KdV equation it was discovered in [26] in 1974 that for the important operator $L = -\partial^2 + u(x)$ of second order (the Sturm–Liouville, Schrödinger, and Hill operators) with a periodic potential the algebraic property (here the rank is always equal to 1) implies the finite-zone property of the Bloch function $\Gamma = \tilde{\Gamma}$, and a program was formulated for finding exact solutions of nonlinear equations of KdV type on this basis. Final equivalence of the algebraic property for rank 1 and the finite-zone property for operators of order 2 (in the sense of the spectral theory for operators with periodic and quasiperiodic coefficients) was established in [3], while in the general case it was established in [16]. S. P. Novikov, B. A. Dubrovin, V. B. Matveev, Its, and Lax (and somewhat later, McKean and van Moerbeke) in a cycle of works (see the survey [6]) realized this program completely and created the theory of "algebrogeometric" or "finite-zone" solutions of the KdV equation and its analogues. For $2+1$ systems tins was done by Krichever [15] who successfully algebraicized the theory and also completed the investigation of commuting operators of rank 1 of "general position" (see [16]), where the Riemann surface $\tilde{\Gamma}$ is nonsingular, including infinitely distant points.

In [18] Krichever developed a classification theory of scalar operators of any rank $l > 1$. We normalize a solution $\varphi(x,\mathcal{P})$ of Eqs. (1) by the condition

$$\begin{gathered}\varphi_{k-1}(\varphi_j), \quad j = 1, \dots, l, \\ \left(\frac{d\varphi_j}{dx^{k-1}}\right)_{x=x_0} = \delta_{jk}.\end{gathered} \tag{3}$$

Then $\varphi(x,\mathcal{P})$ possesses the following analytic properties:

1. φ has lg simple poles $\gamma_1(x_0), \dots, \gamma_{lg}(x_0)$,

$$\varphi \sim \varphi_{qj}(x,x_0)/(z-\gamma_j)$$

2. Among the residues $\varphi_{qj}(x,x_0)$ all are proportional to one with constant coefficients (not depending on x)

$$\varphi_{qj} = \alpha_{qj}(x_0)\varphi_{lj}, \quad q = 1, \dots, l-1. \tag{4}$$

We note that the collection of "Turin parameters" (α_{qj}, γ_j) has a natural interpretation in the classification theory of vector bundles over $\tilde{\Gamma}$.

3. If $z = k^{-1}$ is a local parameter on $\tilde{\Gamma}$ near the infinitely distant point $\mathcal{P}_0 = \infty$ $(z = 0)$, then the following asymptotics hold:

$$\begin{gathered}
\varphi(x, \mathcal{P}) = \left(\xi_0 + \sum_{i \geq 1} \xi_i(X) k^{-i}\right) \varphi_0, \\
\varphi_{0x}\varphi_0^{-1} = \varkappa + \begin{pmatrix} 0 & & & 0 \\ u_0(x), & \ldots, & u_{l-2}(x), & 0 \end{pmatrix}, \\
\xi_0 = (1, 0, \ldots, 0), \quad \varkappa = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ k & 0 & 0 & \ldots & 0 \end{pmatrix}, \\
\varkappa^l = k \cdot 1
\end{gathered} \tag{5}$$

where $u_0, \ldots, u_{l-2}$ are arbitrary functions of x.

According to [18], the analytic properties 1, 2, 3, the constants (α_{qj}, γ_j) in the general number $l^2 g$, and the arbitrary functions $u_0, \ldots, u_{l-2}(x)$ collectively determine the function $\varphi(x, \mathcal{P})$ and the commuting pair $[A, L] = 0$ of rank l of general position by means of a certain Riemann problem on the surface $\tilde{\Gamma}$ near the point $\mathcal{P}_0 = \infty$ which depends on the parameter x. In principle, this makes it possible to recover $\varphi(x, \mathcal{P})$, but it does not give effective formulas for their coefficients. It is not even clear for what choice of "data of the inverse problem," i.e., ($\tilde{\Gamma}$, $\mathcal{P}_0$, $z = k^{-1}$, γ_j, α_{qj}, $u_0(x), \ldots, u_{l-2}(x)$), operators L with periodic (quasiperiodic) coefficients are obtained.

Concerning the connection of the algebraic property of an operator L with its periodic coefficients and its spectral properties (i.e., the Bloch function ψ) Novikov [27] established the following theorem: for operators L of general position of rank l with periodic coefficients the algebraic property is equivalent to the fact that the Riemann surface Γ of the Bloch eigenfunction $\psi(x, Q)$ is an l-sheeted ramified covering over the Burchnall–Chaundy surface $\tilde{\Gamma}$ of a common eigenfunction $\varphi(x, \mathcal{P})$ of two commuting operators ($\tilde{\Gamma}$ always has finite genus). For matrix operators this theory was completed by P. G. Grinevich

$$\Gamma \xrightarrow{l} \tilde{\Gamma} \xrightarrow{N/l} C,$$

where C is the plane of the parameter ε and $L\Psi = \varepsilon\Psi$.

For $l = 1$ the surfaces Γ and $\tilde{\Gamma}$ coincide. For $l > 1$ the meaning of this theorem is that the transcendental surface is l-sheeted over the algebraic curve $\tilde{\Gamma}$; it can thus be said that the theory of algebraic operators of any order $N = nm$ and rank l with periodic coefficients is equivalent to the theory of a pencil of operators of order l with spectral parameter on an algebraic curve $\tilde{\Gamma}$, i.e., the order is sharply reduced at the expense of complicating the spectral parameter.

In a special case this was first investigated by Novikov and Grinevich in [28] although, of course, various degenerate examples may be found in various areas, and they were known earlier (for example, in the theory of the sine-Gordon equation the operator L of order 4 (see [13]) reduces to a representation [31] on the Riemann surface of the function $\varepsilon + \varepsilon^{-1}$, lowering the order to 2). This theorem was preceded by a series of investigations [20, 21] on effective solution of the problem of computing the coefficients of commuting operators $[L, A] = 0$ of rank $l > 1$ without resorting to computation of the eigenfunction φ. Moreover, applications to the construction of exact solutions of the KP equation known in mathematical physics (a two-dimensional analogue of KdV) required the introduction of a "multiparameter" vector analogue of the rank l of the Baker–Akhiezer function $\varphi(x, \mathcal{P})$, where $x = (x^1, \dots, x^k)$, ($k = 3$, $x^1 = x$, $x^2 = y$, $x^3 = t$ for KP).

The analytic properties 1 and 2 formulated above remain as before, but x and x_0 become vectors; in place of property 3 we have the following property:

3′. The asymptotic expressions (5) hold; the matrices $A_j = \varphi_{0x}j\varphi_0^{-1}$ are polynomials in k and satisfy the compatibility conditions

$$\partial_p A_j - \partial_j A_p = [A_j, A_p]. \tag{6}$$

For applications to the theory of the KP equation a vector-valued function $\varphi(x, y, t, \mathcal{P})$ with the following asymptotics is required:

$$A_1 = \varkappa + O(1), \quad A_2 = \varkappa^2 + O(1), \quad A_3 = \begin{cases} \varkappa^3 + O(1), & l \geq 3, \\ \varkappa^3 + O(k), & l = 2. \end{cases} \tag{7}$$

Here the functional parameters $(u_0, \dots, u_{l-2})$ in the matrix A_1 must be determined from the compatibility equations (6). For $l = 2$ Eqs. (6) reduce to the ordinary KdV equation for the quantity $u_0(x, t)$ which does not depend on y. For $l = 3$ the two functions u_0, u_1 combine to form a solution of the Boussinesq equation in (x, y), and they do not depend on t (see [20, 19]).

The method of the works [20–22] is based on the fact that the quantities $(\alpha_{qj}(x_0), \gamma_j(x_0))$—the "Turin parameters"—satisfy a collection of equations in x_0. For rank $l = 2$, genus $g = 1$, and three variables x, y, t important in the KP theory these equations can be solved explicitly in terms of x, y; from the solution we deduce, in particular, the explicit form of the coefficients of commuting operators of rank $l = 2$ (order $L = 4$, order $A = 6$) [20] $L = (\partial^2 + u)^2 + \lambda\,\partial/\partial x + \partial/\partial x\,\lambda + \mu$. All operators L of genus $g = 1$ and rank $l = 2$ with coefficients rational in x were indicated in [2]; for genus $g = 1$ and rank $l = 3$ all solutions were found in [24]. Formulas close to [2] but without clear formulation of the problem were indicated in [40]. For the dependence on t from the equations for the parameters α_{qj}, γ_j the following equation is obtained $u = u\{c\}$, $\lambda = \lambda\{c\}$, $\mu = \mu\{c\}$:

$$c_t = \frac{1}{4}c_{xxx} + \frac{3}{8c_x}(1 - c_{xx}^2) - \frac{1}{2}Q(c)c_x^3, \tag{8}$$

where the quantity $u(x, y, t)$ satisfies the KP (Kadomtsev–Petviashvili) equation. Here $\zeta(z)$ is the familiar Weierstrass function (see [1]), $\varphi = \zeta(-2c) + \zeta(c - y) + \zeta(c + y)$, $Q = \varphi_c + \varphi^2$, $Q_y \equiv 0$.

We introduce matrices depending elliptically on the spectral parameter (for their exact form see [2])

$$B_1 = \hat{\varphi}_x \hat{\varphi}^{-1}, \quad B_3 = \hat{\varphi}_t \hat{\varphi}^{-1},$$
$$\hat{\varphi} = \begin{pmatrix} \varphi(x, \mathcal{P}) \\ \varphi^{(l-1)}(x, \mathcal{P}) \end{pmatrix}, \quad l = 2, \tag{9}$$

where φ is the Baker–Akhiezer vector of genus 1 and rank l (above), and $x = (x, t)$. By definition, since the matrix $\hat{\varphi}$ exists, we have

$$[\partial/\partial x - B_1, \partial/\partial t - B_3] = 0. \tag{10}$$

The operator $\partial - B_1$ is an operator of order 4 reduced to a pencil of operators of second order with spectral parameter on a Riemann surface. The spectral theory of such operators with periodic coefficients is discussed in [28].

It is interesting that in recent works [29] a classification of scalar nonlinear equations of the form

$$c_t = \text{const} \cdot c_{xxx} + f(c, c_x, c_{xx}), \tag{11}$$

has been obtained which possess "concealed symmetry." In particular, it follows from [29] that Eq. (8) is the only equation of the form (11) not reducing to the ordinary KdV equation by transformations of "Miura type" $w(c, c_x, c_{xx}, \dots)$ and possessing "concealed symmetry," i.e., an infinite series of conservation laws (or flows commuting with the original equation).

From a purely mathematical point of view Eq. (8) describes deformations of a commuting pair of operators L, A of genus 1 and rank 2.

Equation (8) reduces by the substitution $v = \zeta(c)$ to the algebraic form

$$v_t = \frac{1}{4} v_{xxx} + \frac{3}{8v_x}(v_{xx}^2 - P_3(v)),$$
$$p_3(v) = 4v^3 - g_2 v - g_3. \tag{12}$$

It is shown in [28] that smoothness of the coefficients of the operator L of rank 2 of the form $L = (\partial^2 + u)^2 + 2v$ $(\lambda = 0)$ is equivalent to the smoothness of the right side of Eq. (12), i.e., at points $v_x = 0$ it is necessary that $v_{xx}^2 = P_3(v)$.

CHAPTER 2

TWO-DIMENSIONAL SCHRÖDINGER OPERATOR WITH PERIODIC COEFFICIENTS (ZERO FLUX OF THE MAGNETIC FIELD)

3. Analytic Properties. The Complex Manifold of Quasimomenta and Constant Energy. The Two-Dimensional Analogue of Finite-Zone Operators

Our purpose is to study the direct and inverse problems of spectral theory for the Schrödinger operator

$$L = -\frac{1}{2}\left(\frac{\partial}{\partial x} - eA_1\right)^2 - \frac{1}{2}\left(\frac{\partial}{\partial y} - eA_2\right)^2 + eU \tag{1}$$

with smooth, periodic coefficients. The Bloch eigenfunction $\psi(x, y, p_1, p_2)$ with energy $\varepsilon_j(p_1, p_2)$

$$L\psi = \varepsilon_j\psi, \quad T_1\psi = \psi(x + T_1, y), \quad T_1\psi = \psi \exp\{ip_1T_1\},$$
$$T_2\psi = \psi(x, y + T_2) = \exp\{ip_2T_2\}\psi \tag{2}$$

was defined in Sec. 1. Following [5], we first determine what analytic properties are to be expected from the "spectral data" in this case. Suppose that both quasimomenta (p_1, p_2) tend in modulus to ∞, i.e., their imaginary parts tend to infinity. The function ψ can be represented in the form

$$\psi = \exp\{ip_1x + ip_2y\}V(x, y), \tag{3}$$

where V is a periodic function. If V varies considerably more slowly than exponential, then the variable character of the coefficients of the operator L has a weak influence. Replacing the coefficients of L by constants, we obtain two possible asymptotics:

$$1.\ \psi \sim c_1 \exp\{ik_1(x + iy)\}(1 + O(k_1^{-1}));$$
$$2.\ \psi \sim c_2 \exp\{ik_2(x - iy)\}(1 + O(k_2^{-1})); \tag{4}$$
$$\varepsilon = \varepsilon_0 \approx p_1^2 + p_2^2, \quad k_{1,2} = p_1, \quad p_2 \approx \pm p_1 i.$$

Naturally, for variable coefficients the coefficients c_1 and c_2 depend on x, y, but they do not depend on p_1, p_2.

The transformation $\psi \to f(x)\psi$ is equivalent to the gradient transformation of the vector potential $A \to A + i\Delta \ln f/e$; therefore, it may be assumed with no loss of generality that $c_1 \equiv 1$. We denote by $z, \bar{z}$, respectively, the quantities $x + iy$, $x - iy$. Following [5], on the basis of experience studying finite-zone, one-dimensional operators, we introduce the following class of "finite-zone" two-dimensional Schrödinger operators with periodic coefficients relative to one energy level $\varepsilon = \varepsilon_0$.

Definition. The Schrödinger operator (1) is called a finite-zone operator of rank 1 if there is an energy level $\varepsilon = \varepsilon_0$ such that there exists a Bloch solution $L\psi = \varepsilon_0\psi$, $\psi = \psi(x, y, p_1, p_2)$ where the quasimomenta (p_1, p_2) run over a Riemann surface Γ of finite genus $g < \infty$. It is assumed that the function $\Psi(x, y, p_1, p_2)$ possesses the following analytic properties on Γ: $x = (x, y)$.

1. There is distinguished a pair of "infinitely distant points" $\mathcal{P}_1, \mathcal{P}_2 \in \Gamma$ with local parameters $w_1 = k_1^{-1}$, $w_2 = k_2^{-1}$. The function $\psi(x, \mathcal{P})$ is meromorphic away from the points $\mathcal{P}_1, \mathcal{P}_2$ on Γ and has exactly g poles of first order whose position does not depend on x.

2. At the points $\mathcal{P}_1, \mathcal{P}_2$ there are the asymptotic expressions (4):

$$\psi = \exp(ik_1z)(1 + O(k_1^{-1})), \quad \mathcal{P} \to \mathcal{P}_1,$$
$$\psi = c(x)\exp(ik_2\bar{z})(1 + O(k_2^{-1})), \quad \mathcal{P} \to \mathcal{P}_2. \tag{4'}$$

A function $\psi(x, y, \mathcal{P})$ on the surface Γ possessing analytic properties 1, 2 is called a two-point analogue of the Baker–Akhiezer function of rank 1. Construction of this function and obtaining explicit formulas for it present no difficulties; this is done in analogy with the classical one-point Gordan–Clebsch–Baker–Akhiezer function used by I. M. Krichever in the theory of commuting scalar operators of rank 1 and the theory of the LP equation (see Sec. 2 and the survey [17]). The function ψ constructed satisfies the Schrödinger equation

$$L\psi = 0, \tag{5}$$

where

$$L = -\partial^2/\partial z\,\partial\bar z + A\,\partial/\partial\bar z + v, \quad A = \partial \ln c/\partial z. \tag{6}$$

Thus, the operator L constructed is a finite-zone Schrödinger operator relative to a single energy level. For the coefficients we have the formulas

$$\begin{aligned} L &= (i\partial/\partial x - A_1)^2 + (i\partial/\partial x - A_2)^2 + u(x,y),\\ u &= -2\partial^2/\partial z\,\partial\bar z \ln\theta(\vec U_2 z + \vec U_2\bar z + \vec U_0) + C(\Gamma),\\ A &= \partial/\partial z \ln\frac{\theta(\vec U_1\bar z + \vec U_2\bar z + \vec U_0 + U_{01})}{\theta(U_1 z + \vec U_2\bar z + \vec U_0 + U_{02})}, \end{aligned} \tag{7}$$

where $\theta(\eta_1,\dots,\eta_g)$ is the standard θ-function (of Riemann) with zero characteristics defined on the Riemann surface Γ. $\vec U_1, \vec U_2$ are the vectors of b-periods of normalized differentials ω_1,ω_2 of second kind (with zero a-periods) with singularities of the type $(dw_i w_i^{-2} + \text{regular})$ at points $\mathcal{P}_1$ and $\mathcal{P}_2$, where w_i is a local parameter. The θ-function formulas are discussed in more detail in the survey [4]. Of course, an explicit formula can also be written out for the Bloch function ψ of the single energy level $\varepsilon = \varepsilon_0$.

The Simplest Example. Aside from the considerations presented above, the analytic properties of "finite-zone" operators L can be inferred from an analysis of the following example: let $L = L_{(1)} + L_{(2)} = -\partial_1^2 + u(x) - \partial_2^2 + v(y)$, where $u(x)$ and $v(y)$ are finite-zone one-dimensional potentials. The complete Bloch function has the form $\psi = \psi_1\psi_2$ on the manifold of quasimomenta $W = \Gamma_{(1)} \times \Gamma_{(2)}$, where $\Gamma_{(1)}$ and $\Gamma_{(2)}$ are the Riemann surfaces ("spectra") of the operators $L_{(1)}$ and $L_{(2)}$ which have genera $g_1 \geq 0$, $g_2 \geq 0$. It is easy to see that the genus of the surface $\varepsilon = \varepsilon_{(1)} + \varepsilon_{(2)} = \varepsilon_0$ can be computed by the formula

$$g(\Gamma_\varepsilon) = 2(g_1 + g_2) + 1. \tag{8}$$

It can be verified directly that the number of poles of the function ψ for a "typical" value of ε is exactly equal to the genus $g(\Gamma_\varepsilon)$. In this example the operator L is a "finite-zone" operator relative to any energy ε, but this is a rare, degenerate case.

It is interesting to consider still another degenerate example. Suppose that the magnetic field and the electric potential are periodic functions of the variable y only (and the mean value of the magnetic field is equal to zero): $L = -(\partial_1 - if(y)^2 - \partial_2^2 + u(y))$. We seek ψ in the form

$$\psi = \exp(ip_1 x)\,\psi_0(p_1, p_2, y), \tag{9}$$

where p_1, p_2 are the components of the quasimomentum and

$$L_0\psi_0 = [-\partial_2^2 + u(y) + (p_1 - f(y))^2]\psi_0 = \varepsilon\psi_0. \tag{10}$$

For a fixed quasimomentum $p_1 = p_1^0$ we can obtain a Riemann surface of finite genus if $(p_1^0 - f)^2 + u$ is a finite-zone potential; this is possible even for two values of the quasimomentum p_1; we recall that we must recover the two functions f, u. This shows that a formulation of the inverse problem is possible not with a fixed energy but with a fixed component of the quasimomentum $p_1 = p_1^{(\alpha)}$, $\alpha = 0, 1$. This formulation of the question generalizes more naturally to higher dimensions.

Analogues of commutative rings of operators are also connected with the analogues considered here of "finite-zone" operators ($n = 2$). Namely, the operator $L = -(\partial_1 - A_1)^2 - (\partial^2 - A_2)^2 + u$ can be considered only on the zero level $L\psi = 0$. Suppose that the differential operators L_1, L_2 commute on the set of all formal solutions of the equation $L\psi = 0$. Then the set of relations

$$[L_i, L_j] = B_{ij} L \tag{11}$$

is satisfied, where $i, j = 0, 1, 2$, $L = L_0$, and B_{ij} are differential operators.

Definition. A collection of operators $L = L_0, L_1, L_2$ satisfying relations (11) we call a "weakly commuting" family.

Of course, definitions of families of any rank $l \geq 1$ are possible. An operator $L = L_0$ of such a family, if it is periodic or quasiperiodic, is a finite-zone operator with respect to the zero energy level (see [21]). Operators of rank $l = 1$ (for nonsingular Riemann surfaces Γ) are obtained from the construction presented above and provide a collection of "data of the inverse problem": a Riemann surface Γ, i.e., a complexified Fermi surface $\varepsilon(p_1, p_2) = 0$, a pair of distinguished "infinitely distant" points $\mathcal{P}_1, \mathcal{P}_2 \in \Gamma$ with local parameters $w_1 = k_1^{-1}$, $w_2 = k_2^{-1}$, and a collection of g distinct points $Q_1, \dots, Q_g$—the poles of ψ.

A classification of weakly commuting families of any rank was obtained in [21]. It would be useful for $g = 1$, $l = 2$ to obtain explicit effective formulas for the coefficients of the operator L.

Although a weakly commuting family L_0, L_1, L_2 of rank $l > 1$, generally speaking, cannot be approximated by families of rank 1, nevertheless, apparently the single operator $L = L_0$ important for us and contained in the family can be approximated by algebraic operators (i.e., finite-zone operators in the sense indicated above for $n = 2$) of rank 1. This conjecture has so far not been rigorously proved.

It is clear, however, that in the case of general position we shall obtain a surface Γ_ε of infinite genus but of rank 1, i.e., the Bloch function of the given "typical" energy level on the complex Fermi surface Γ_ε has multiplicity one.

There is reason to suppose that the "data of the inverse problem," i.e., a Riemann surface (of infinite genus, generally speaking), a collection of poles Q_j $(j = 1, 2, \dots)$, and a pair of "singular" points (infinities) near which the Bloch function ψ has the asymptotics (4′) and where the poles Q_j accumulate, vary with change of the energy ε subject to certain equations (in ε) which it would be interesting to obtain and investigate. All this, however, has so far not been studied in detail.

It is not hard to give conditions on the collection of data $(\Gamma, \mathcal{P}_1, \mathcal{P}_2, w_1, w_2, Q_1, \dots, Q_g)$ so that a purely real operator L of the form (7) is obtained. For this it suffices, for example, that Γ possess an antiinvolution $\sigma \colon \Gamma \to \Gamma$ such that $\sigma(\mathcal{P}_1) =$

$\mathcal{P}_2$, $\sigma(\mathcal{P}_2) = \mathcal{P}_1$, $-\sigma^* w_1 = \bar{w}_2$, $-\sigma^* w_2 = \bar{w}_1$, $\sigma(Q_j) = Q_j$. However, this is of no special interest, since realness of L implies the presence of a purely imaginary vector potential of the magnetic field if it is nonzero (it is hereby not possible to distinguish in a natural way the class of potential operators with zero magnetic field in the language of data of the inverse problem).

Distinguishing data of the inverse problem corresponding to self-adjoint operators is a difficult and unsolved problem. Cherednik [30] first obtained some sufficient conditions which make it possible to prove in an ineffective way the existence of a large number of nontrivial finite-zone operators of rank 1 for any genus $g \geq 2$. These results were carried over to higher ranks in [21]. It is necessary that on the Riemann surface there be given an antiinvolution $\sigma\colon \Gamma \to \Gamma$ permuting the infinitely distant points and $\sigma^* w_1 = -w_2$. On divisors of degree $g-1$ there is defined a standard involution

$$\Delta\colon D \to K - \sigma(D),$$

where K is the canonical class (of degree $2g-2$). Subtracting out the infinitely distant points, we obtain an involution on divisors of degree g

$$\tilde{\Delta}(Q_1 + \cdots + Q_g) = \Delta(Q_1 + \cdots + Q_g - \mathcal{P}_1). \tag{12}$$

The invariance $\tilde{\Delta}(D) \approx D$ is sufficient for the self-adjointness of L. This is an ineffective description of the class of poles $Q_1, \dots, Q_g$. Further, for equations of sine-Gordon type the situation is similar (but simpler; see [30]).

It is appropriate to point out that the results of [15, 17] on the theory of the KP equation and the construction of its "algebraic" or "finite-zone" solutions can essentially be interpreted as the construction of the Bloch function with respect to the zero level of the two-dimensional linear operators of the special form

$$\begin{gathered} L = \alpha\partial/\partial y - L_{0n}, \quad L_{0n} = \sum_{i\geq 0} a_i(x, y, \dots)\frac{\partial^{n-i}}{\partial x^{n-i}}, \\ a_0 = \text{const}, \quad \alpha = \text{const}, \end{gathered} \tag{13}$$

where the coefficients a_i depend on $x, y, \dots$. Namely, the Gordon–Clebsch–Baker–Akhiezer function $\psi(x, y, \mathcal{P})$ on the surface Γ with distinguished "infinitely distant" point $\mathcal{P}_1$, local parameter $w = k^{-1}$, and divisor of poles $Q_1 + \cdots + Q_g$ is actually the Bloch function with respect to all variable solutions of the equation $L\psi = 0$. The KP equation arose as a compatibility condition for two such operators

$$\begin{gathered} L_1 = \partial/\partial t - L_{03}, \quad \psi = \psi(x, y, t, \mathcal{P}), \\ L_2 = \alpha\partial/\partial y - L_{02}, \quad [L_1, L_2] = 0. \end{gathered} \tag{14}$$

This case is degenerate: solutions of the equation $L\psi = \lambda\psi$ can formally be obtained from solutions of the equation for $\lambda = 0$ by the substitution $\psi \to \varepsilon^{\lambda y}\psi$. The finite-zone operators L_1, L_2 themselves commute with ordinary differential operators in x whose coefficients depend also on $y, t, \dots$ as parameters. The weakly commuting families of operators (11) and the two-dimensional operators which are "finite-zone" operators with respect to a single energy level are the correct generalization of this situation to nondegenerate operators of the type of the two-dimensional, stationary Schrödinger operator.

The problem of distinguishing all data leading to real solutions of the KP equation is simpler for the case where in the Lax pair the operator L is of parabolic form (13), $\alpha = 1$,

$$L = \partial/\partial y - L_0.$$

The more complicated case where $\alpha = i$ has so far not been investigated.

CHAPTER 3

TWO-DIMENSIONAL SCHRÖDINGER OPERATOR IN PERIODIC FIELDS WITH NONZERO FLUX

4. General Properties of the Schrödinger Operator in a Magnetic Field and a Lattice. Rational and Irrational Cases. Perturbation Theory and Its Topological Consequences. Discussion of the Inverse Problem

In contrast to the one-dimensional case, for $n = 2$, according to the geometric interpretation of a field as the curvature of a bundle with fiber C^1, the appearance of periodic electromagnetic fields with nonzero "topological charge" (characteristic class) is possible which leads to Schrödinger operators (stationary and time-dependent) with periodic coefficients. In a number of cases this leads to great difficulties which will be discussed below.

The simplest "topologically nontrivial" fields are the following.

I. There is given an electric field in x, t space directed along x of the form

$$E(x,t) = \bar{E} + E_0(x,t),$$

where $\bar{E} = \text{const}$ and the function E_0 is periodic in x, t with periods T_1, T_0 and with zero mean over a period $\bar{E}_0 = 0$. The time-dependent Schrödinger operator has the form for a charged spinless particle

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + e(\bar{E}x + u_0(x,t))\psi = 0, \tag{1}$$

where u_0 is a periodic function.

The topological charge—the flux through an elementary two-cell in the lattice—is equal to

$$\varphi = \int_0^{T_1}\int_0^{T_2} E\,dx\,dt = \bar{E}T_1T_2 \quad (\hbar = m = c = 1). \tag{2}$$

The discrete group of translations, commuting with Eq. (1), acts as follows:

$$\begin{aligned} T_1^*\varphi(x,t) &= \varphi(x+T_1,t)\exp(-ie\bar{E}T_1t), \\ T_0^*\varphi(x,t) &= \varphi(x,t+T_0). \end{aligned} \tag{3}$$

This is a projective representation of the translation group, since it follows from (3) that

$$T_0^*T_1^* = T_1^*T_0^*\exp(-ie\varphi). \tag{4}$$

If the integral condition is satisfied,

$$e\varphi = 2\pi N, \tag{5}$$

where N is an integer, then the group (3) is Abelian. In this case it is possible to define the "electric Bloch solution" of Eq. (1)

$$T_1^*\psi = \exp(ip_1T_1)\psi, \quad T_0^*\psi = \exp(ip_0T_0)\psi, \tag{6}$$

where p_0 and p_1 are the "quasienergy" and "quasimomentum." If E_0 does not depend on t, then the electric Bloch solutions can be sought in the form (9).

Suppose that $\varphi = e^{i\omega t}\varphi_0(x,\omega)$, where

$$\frac{1}{2}\varphi_0'' + e[\bar{E}(x-\omega/e\bar{E}) + u_0]\varphi_0 = 0. \tag{7}$$

Thus, $\varphi_0(x,\omega)$ is a solution of the equation ($\bar{x} = x - \omega/e\bar{E}$):

$$\varphi_0''/2 + e\bar{E}\bar{x} + u_0(\bar{x}_\omega/e\bar{E})]\varphi_0 = 0. \tag{7$'$}$$

The condition that the solution φ_0 decrease as $|x| \to \infty$ distinguishes the "phase" $\alpha(\omega)$: $y = -2e\bar{E}\bar{x}$:

$$\begin{aligned} \varphi_0 &\sim \frac{1}{2y^{1/4}}\exp\left(-\frac{2}{3}y^{3/2}\right), \quad y \to +\infty, \\ \varphi_0 &\sim \frac{1}{|y|^{1/4}}\sin\left(\frac{2}{3}|y|^{2/3} + \frac{\pi}{4} + \alpha(\omega)\right), \quad y \to -\infty. \end{aligned} \tag{8}$$

In this case we can form the sum

$$\psi(x,t) = \sum_n (T_1^*)^n[e^{ipT_1n+1\omega t}\varphi_0(x,\omega)], \quad p = p_1. \tag{9}$$

This is the Bloch solution of Eq. (1). Here the family of Bloch solutions is two-dimensional. There is another simple, but physically meaningful case where E_0, to the contrary, depends only on t and does not depend on x (L. V. Keldysh). Suppose that $v_0(t)$ is a periodic function of t such that $v_0'' = E_0$. The states with particular momentum k (relative to the group of electric translations) for translations along the x axis have the form

$$\varphi = \exp\left\{i\left[\frac{e^2\bar{E}^2\bar{t}^3}{6} + ev_0(t) + e\bar{E}\bar{t}x\right]\right\},$$
$$\bar{t} = t + k/e\bar{E}.$$

The electric Bloch states were not studied previously; they are obtained if we form the sum

$$\psi(x,t,p_0,p_1) = \sum_n (T_0^*)^n e^{-i\omega T_0 n + i\left\{e\bar{E}x\bar{t} + \frac{e^2\bar{E}^2\bar{t}^3}{6} + ev_0(t)\right\}}. \tag{9$'$}$$

The series (9′) converges for all real ω and $k = \operatorname{Re} k + i\varepsilon^2$, $p_1 = k$, $p_0 = \omega \bmod 2\pi T_0^{-1}$. As $\varepsilon^2 \to 0$ we obtain the Bloch eigenfunction ψ for real values of the quasimomenta as a generalized function on the torus $T^2(p_0,p_1)$. As simple examples show, the analytic properties of electric Bloch functions are very interesting, and they have absolutely not been investigated. For zero flux $\bar{E} = 0$ the Bloch function $\psi(x,t,p_0,p_1)$ is concentrated on a curve $\varphi(p_0,p_1)=0$ in the usual way. The case $\bar{E}\neq 0$ is very extraordinary with regard to the analytic properties of ψ.

II. We now consider another case—a purely stationary magnetic field directed orthogonally to the plane (x,y) of the form

$$\begin{gathered} H(x,y)=\bar{H}+H_0(x,y), \quad \bar{H}_0=0, \quad \bar{H}=\text{const}, \\ H(x+T_1,y)=H(x,y+T_2)=H(x,y). \end{gathered} \tag{10}$$

The operators of "magnetic translations" commuting with the Hamiltonian are

$$\begin{aligned} T_1^*\varphi &= \varphi(x+T_1,y)\exp(-ief_1), \\ T_2^*\varphi &= \varphi(x,y+T_2)\exp(-ief_2), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \partial_i f_1 &= \Delta_1 A_i = A_i(x+T_1,y)-A_i(x,y), \\ \partial_i f_2 &= \Delta_2 A_i = A_i(x,y+T_2)-A_i(x,y). \end{aligned} \tag{12}$$

It is easy to verify the formula

$$\begin{gathered} T_1^*T_2^* = T_2^*T_2^*\exp(-ie\varphi), \\ \varphi=\bar{H}T_1T_2, \quad c=\hbar=m=1. \end{gathered} \tag{13}$$

Only in the case of integral magnetic flux $e\varphi=2\pi N$ do the magnetic translations commute: $T_1^*T_2^*=T_2^*T_1^*$. In this case only there arise the "magnetic Bloch" functions

$$T_1^*\psi=\exp(ip_2T_1)\psi, \quad T_2^*\psi=\exp(ip_2T_2)\psi, \quad L\psi=\varepsilon\psi. \tag{14}$$

The spectrum of the Schrödinger operator L in $\mathcal{L}_2(R)$ is attended by real (p_1,p_2); there is defined a collection of "dispersion laws" (infinite in number):

$$\varepsilon_j(p_1,p_2), \quad j=1,2,\ldots$$

If the magnetic flux is rational $e\varphi=2\pi(NM^{-1}+N_1)$, $N<M$ and the fraction NM^{-1} is irreducible, then the group of magnetic translations (13) is noncommucative. However, in this group sublattices of index M are all Abelian. Having fixed any such sublattice $\Gamma'\subset\Gamma$, we can define the Bloch function $\psi_{\Gamma'}(x,y)$ relative to the sublattice. It is true that the choice of sublattice Γ' of index M is not unique. The following procedure is more invariant: we consider the M-fold sublattice (MT_1,MT_2) which has index M^2. It is invariantly defined, and connected with it is the Bloch function $\psi=\psi(x,y,\bar{p}_1,\bar{p}_2)$, where $\bar{p}_1,\bar{p}_2$ are the quasimomenta relative to the translations MT_1^*,MT_2^*. The function $\psi(x,y,\bar{p}_1,\bar{p}_2)$ is a vector, since it is NM-fold degenerate, as follows easily from the algebraic structure of the group of magnetic translations. Expanding to an Abelian subgroup of index M, it is possible to introduce N vectors $\psi^{(s)}(x,y,p_1,\bar{p}_2)$ which are eigenvectors with respect to the translation T_1^*, $s=0,\ldots,M-1$;

$$\psi=(\psi^{(0)},\ldots,\psi^{(M-1)}), \quad T_1^*\psi^{(s)}=\exp(ip_1^{(s)}T_1)\psi^{(s)}, \tag{15}$$

where $\bar{p}_1=Mp_1^{(s)}$, $p_1^{(s)}=p_1^{(0)}+2\pi s/M$.

To the one value $\bar{p}_1$ there correspond the M values $p_1^{(s)}$ (mod 2π). Since $T_2^* T_1^* = T_1^* T_2^* \exp(-ie\varphi)$, the function $T_2^* \psi^{(s)}(p_2^{(s)}, \bar{p}_2)$ is an eigenfunction for T_1^* with eigenvalue $\exp(ip_1 T_1 + ie\varphi)$. Thus, the quasimomentum p_1 is shifted by the quantity $e\varphi T_1^{-1}$: $p_1 \to p_1 + e\Phi T_1^{-1}$. Repeating this shift M times, we move the quasimomentum p_1 into itself (into an equivalent quasimomentum). Thus, the energy operator reduces to a matrix $\hat{\varepsilon}_j$ of order MN depending on $\bar{p}$ which reduces naturally to block-scalar form

$$\hat{\varepsilon}_j = \begin{pmatrix} A_j & & 0 \\ & A_j & \\ 0 & & \ddots \end{pmatrix} = \hat{\varepsilon}_j(\bar{p}_1, \bar{p}_2),$$

where A_j is an $N \times N$-matrix.

The elementary algebraic analysis following from the properties of the group of magnetic translations was discussed in [43], [37] by Brown and Zak who first introduced this group.

In the general case of an irrational flux $e\varphi(2\pi)^{-1}$ we arrive at a situation never studied in mathematics; the Schrödinger operator L in $\mathcal{L}_2(R^2)$, although it corresponds to a physically double periodic problem, is nevertheless an operator covering an elliptic operator on a compact manifold of type T^n in the sections of some vector bundle; the Chern class c_1 here is irrational. It is possible chat objects of the type of "Neumann factors," etc., connected with the group of magnetic translations should be associated with this problem, although their structure is very sensitive to the precise value of the magnetic flux.

However, "averaged" characteristics of states that in a particular sense change well together with the magnetic flux φ, which is also the external parameter in the physics of a solid body, would appear natual physically. Real magnetic fields are "weak," i.e., the flux amounts, for example, to 10^{-6} of a single integral quantum, since the area of a unit cell in a metal is $\sim 10^{-16}$ cm^2. Here it is difficult to distinguish a rational number from an irrational number if there is no "superlattice."

As is known, in a homogeneous magnetic field $H = \bar{H} = \text{const}$ the Schrödinger equation can be solved completely, and there arises a discrete collection of "Landau levels"

$$L\psi = -\frac{1}{2}\left[\frac{\partial^2}{\partial y^2} + \left(\frac{\partial}{\partial x} - ieHy\right)^2\right]\psi = \varepsilon_m \psi, \tag{16}$$

where

$$\psi = e^{ikx}\psi_0(y,k),$$
$$L_0\psi_0 = \left[-\frac{\partial^2}{2\partial y^2} + \frac{1}{2}e^2H^2\left(y - \frac{k}{e\bar{H}}\right)^2\right]\psi_0 = \varepsilon_m\psi_0. \tag{17}$$

An irreducible, unitary, infinite-dimensional representation of the continuous group of magnetic translation isomorphic to the Heisenberg–Weyl group is realized at each level ε_m.

The function $\psi_0^{(m)}(y - k/e\bar{H})$ coincides with the eigenfunction of the quantum oscillator

$$\psi_0^{(m)}(\bar{y}) = \exp\left(-\frac{|e\bar{H}|}{2}\bar{y}\right) H_m(e\bar{H}\bar{y}), \quad \bar{y} = y - \frac{k}{e\bar{H}}, \tag{18}$$

where H_m is the Hermite polynomial. In correspondence with general rules, from the functions (18) it is easy to construct magnetic Bloch eigenfunctions. We prescribe an integer $N = 1, 2, 3, \dots$, and we choose a lattice (T_1, T_2) in the x, y plane such that the flux has the form

$$e\varphi = 2\pi N = e\bar{H}T_1T_2. \tag{19}$$

The (magnetic Bloch) eigenfunctions are conveniently given by summing the y-localized states (16) over the lattice on the y-line $(k = p_1)$

$$\begin{gathered} \psi^{(m)}(x, y, p_1, p_2) = \sum_n (T_2^*)^n (e^{ikx - ip_2T_2n}\psi_0^{(m)}(\bar{y})), \\ T_2^*\psi = \psi(x, y + T_2)\exp(-ieHT_2x), \quad eHT_2 = 2\pi N T_1^{-1}. \end{gathered} \tag{19^1}$$

For $m = 0$, $N = 1$, it is easy to see that the Bloch functions $\psi^{(0)}$ can be computed in terms of the elliptic Jacobi θ function or the Weierstrass σ function, and they coincide with the "Cartier basis" in the infinite-dimensional, irreducible representation of the Heisenberg–Weyl group of magnetic translations. Connecting the formulas in elliptic functions, see the next section. It is not hard to show that the operator B^m takes Bloch functions of level ε_0 into functions of level ε_m with the same quasimomenta p_1, p_2

$$B = -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \frac{e\bar{H}}{2}\left\{\bar{z} + z\left(\frac{T_1\eta_2}{\pi} - \frac{1}{2}\right)\right\} - \frac{e\varphi}{4\pi}(\eta_1 + i\eta_2), \quad \varphi = HT_1T_2. \tag{20}$$

Here the lattice is rectangular, as above, and

$$\eta_1 = \xi\left(\frac{T_1}{2}\right), \quad i\eta_2 = \xi(iT_2/2),$$

where $\xi(z)$ is the Weierstrass function such that

$$-\xi' = \wp(z), \quad \xi(z) = \sigma'/\sigma, \quad z = x + iy.$$

The operator B^m realizes an isomorphism of the families of Bloch functions.

It is essential to note the following: if the number of quanta of the magnetic flux is equal to N, i.e., $e\varphi = 2\pi N$, then the Bloch functions of any Landau level defined by formulas (19) represent an N-dimensional vector bundle $\eta_N^{(m)}$ with fiber C^N and base $T^2(p_1, p_2)$. The first Chern class $c_1(\eta_N^{(m)})$ always coincides with the basis class of the cohomology group $H^2(T^2, Z)$.

We now consider doubly periodic perturbations of the Schrödinger operator in a homogeneous magnetic field

$$L = -\frac{\partial^2}{\partial y^2} - \left(\frac{\partial}{\partial x} - ie\bar{H}y\right)^2 + gV(x, y), \tag{21}$$

[1]Numbering as in Russian original — Publisher.

where g is small. A more general class of perturbations also contains a periodic component of the magnetic field with zero mean. For the time being, we consider purely potential perturbations of the form (21) where

$$V(x+T_1,y)=V(x,y+T_2)=V(x,y).$$

As is known, in first order of perturbation theory in the parameter g the eigenfunctions of the operator are the same as for $g=0$, while the energy acquires a nontrivial increment of order g. Thus, to first order in g we use either a basis of solutions of the form (18) or a basis of Bloch solutions of the form (19) for a fixed Landau level ε_m. An elementary computation shows the following.

In the basis (18) of solutions local in y of the form $e^{ikx}\psi_0^{(m)}(y,k)$ the perturbation can be written (for periodic V) as the difference equation (see [35])

$$\sum_n b^m_{n,k} f(k+n\Delta)=\varepsilon_m^{(1)} f(k),\quad \varepsilon=\varepsilon_m+g\varepsilon_m^{(1)}+O(g^2), \tag{22}$$

where the coefficients $b^{(m)}_{n,k}$ can easily be computed proceeding from the expansion $V=\sum_n \varphi_n(y)\exp(2\pi i n x T_1^{-1})$ in a Fourier series: the following formulas hold ($\Delta=2\pi T_1^{-1}$):

$$b^{(m)}_{j,k}=\int dy\,\varphi_j(y)\psi_0^{(m)}\left(y-\frac{k+j\Delta}{e\bar H}\right)\psi_0^{(m)}\left(y-\frac{k}{e\bar H}\right). \tag{23}$$

If V is a trigonometric polynomial in x, then the operator (22) is a difference operator of finite order. For example, if only $\varphi_1,\varphi_0,\varphi_{-1}$ are nonzero, then we obtain a difference equation of second order:

$$\begin{gathered} b^{(m)}_{-1,k} f(k-\Delta)+b^{(m)}_{+1,k} f(k+\Delta)+b^{(m)}_{0,k} f(k)=\varepsilon_m^{(1)} f,\\ c_k^{1/2}=b_{-1,k}=b_{1,k-1},\quad v_k=b_{0,k}. \end{gathered} \tag{24}$$

An equation of this type arises in the theory of the Toda lattice [6]. Equation (22) is analogous to a one-dimensional differential operator with quasiperiodic coefficients: Here one period lies on the step Δ and the other in the period of the coefficients $b^{(m)}_{n,k}$. Setting $\Delta=1$ and varying the magnetic field, we see that it would be natural to study those characteristics of the spectrum of the operators (22), (24) which vary "well" together with the magnetic flux is integral with $e\varphi=2\pi N$, then the coefficients $b^{(m)}_{n,k}$ of the operators (22) and (24) are periodic in k with period $N\Delta=e\bar H T_2$ which corresponds to the shift $y\to y+T_2$ in the variable $\bar y=y-k(e\bar H)^{-1}$. To the Bloch functions of the original operator $L+gV$ there correspond (to first order in g for perturbations of the level ε_m) the Bloch solution

$$\begin{gathered} \hat V f_0(k)=\varepsilon_m^{(1)} f_p(k),\\ f_p(k+N\Delta)=\exp(ipT_2) f_p(k), \end{gathered} \tag{25}$$

where $p=p_2$, $k \pmod{2\pi T_1^{-1}}=p_1$. Obviously, for fixed (p_1,p_2) we have exactly N solutions of the unperturbed operator which split in the first order of perturbation theory. The operator (22) is a direct integral of operators on lattices with the "initial" value

$$k_0 \leq k \leq k_0 + \Delta.$$

We thus have the dispersion laws

$$\begin{aligned} &\varepsilon_m^{(1)}(k,p,j), \quad j = 1,2,\dots,N, \\ &0 \leq k < \Delta, \quad 0 \leq p < 2\pi T_2^{-1}. \end{aligned} \tag{26}$$

Returning to the basis of Bloch solutions of the unperturbed operator at the level ε_m, we see that the operator of multiplication by $V(x,y)$ in this basis is a collection of Hermitian forms $\hat{\varepsilon}_m(p_1,p_2)$ on the N-dimensional fibers of the bundles $\eta_M^{(m)}$ with base T^2, fiber C^N, and Chern class c_1 (above). Suppose the potential $V(x,y)$ is given in the form of a trigonometric polynomial. Any doubly periodic potential can obviously be approximated by such potentials.

A. S. Lyskova proved the theorem that any continuous Hermitian form on the bundle $\eta_N^{(m)}$ can be approximated by forms $\hat{\varepsilon}_m(p_1,p_2)$ corresponding to potentials of the form of trigonometric polynomials for a fixed Landau level ε_m. This implies an elegant proof of the theorem (Novikov [25]) that for $N > 1$ any topological type is possible for the decomposition of Bloch functions of one Landau level into dispersion laws of general position—one-dimensional bundles $\eta_{(s)}^{(m)}$, $\sum_s \eta_{(s)}^{(m)} = \eta_N^{(m)}$ with any Chern classes where $\sum_{s=1}^N c_1(\eta_{(s)}^{(m)}) = c_1(\eta_N^{(m)})$, i.e., for $N > 1$ there exists potentials possessing in the spectrum dispersion laws with any "quantum numbers" $c_1(\eta_{(s)}^{(m)})$. There is probably no dependence between the topological types of different Landau levels, although analytically, according to [23], the single form $\hat{\varepsilon}_m(p_1,p_2)$ already determines the potential uniquely. The behavior of the Chern numbers of "decomposed" (after perturbation) dispersion laws $c_1(\eta_{(s)}^{(m)})$ of the Landau level ε_m is unclear; possibly, it possesses some regularity as $m \to \infty$ or, on the contrary, possibly it is purely random; for sufficiently large indices m any periodic potential has an effect on the spectrum which is small compared with the magnetic field $\bar{H}$ [the effect of the variable component of the field H_0 of (10), where $\bar{H}_0 = 0$, also dies out as $m \to \infty$ if $\bar{H}_0 \neq 0$]. Hence, as $m \to \infty$ we are always in the domain of applicability of perturbation theory discussed above.

We now turn to a discussion of the correct formulation of the inverse problem of spectral theory for the Schrödinger operator in the periodic fields considered with nonzero topological charge. We consider the same cases as above:

$$\begin{aligned} &\text{I.} \quad E(x,y) = \bar{E} + E_0(x,t), \quad H = 0, \quad \bar{E} \neq 0, \\ &\text{II.} \quad H(x,y) = \bar{H} + H_0(x,y), \quad E = 0, \quad \bar{H} \neq 0. \end{aligned}$$

In case I the Schrödinger equation has the form (1), while in case II it has the form (3). In both cases we require that the fluxes be integral, and we pose the question of the formulation of correct spectral data—information regarding the Bloch solution (6), (14) making it possible to uniquely recover the coefficients of the operator L (or the electric and magnetic field). We recall that in the topologically trivial cases this problem has already discussed in Sec. 3 (above). We shall now consider case II in which we have the possibility of advancing interesting conjectures.

First of all, in this problem it is not possible to formulate any natural generalization of the analytic properties "at infinity" in the complex domain of a family of Bloch functions (with all complex quasimomenta) corresponding to a single complexified Fermi surface $\varepsilon(p_1, p_2) = \varepsilon_0$ which are analogous to (4) (Sec. 3) for operators with $\bar{H} = 0$. There arises the natural supposition that a reasonable (for the inverse problem) collection of spectral data can, on the contrary, be connected with all energy levels for $\bar{H} \neq 0$ but without passing into the complex domain in the energy and quasimomenta. We shall consider the case $n = 2$ and the situation of "general position" where all the dispersion laws do not intersect,

$$\varepsilon_j(p) \neq \varepsilon_k(p), \quad j \neq k$$

(for all p, j, k).

The magnetic Bloch function $\psi_j(x, y, p_1, p_2)$ corresponding to the dispersion law ε_j can be considered in two ways: 1) as a vector of Hilbert space with fixed p_1, p_2

$$\psi_{j,p_1,p_2} \in H_{p_1,p_2},$$

2) as a function of the four variables x, y, p_1, p_2. As a vector of Hilbert space the function ψ_{j,p_1,p_2} has an algebraic number of zeros equal to $c_1(\eta_j)$—the first Chern class of the corresponding one-dimensional bundle. These are zeros identically in x, y. Moreover, the function $\psi_j(x, y, p_1, p_2)$ also possibly has other zeros—the two-dimensional manifold of zeros

$$M_j^2 \subset T^4(x, y, p_1, p_2),$$

so that the equation $\psi_j = 0$ actually constitutes two conditions. The position of the zeros can locally be represented in the form

$$p_{\alpha j} = \gamma_{\alpha j}(x, y), \quad \alpha = 1, 2. \tag{27}$$

For large energies $j \to \infty$ the position of the zeros "stabilizes" with respect to energy, since $\varepsilon_j \to \bar{\varepsilon}_m(j)$, where $\bar{\varepsilon}_m$ is the Landau level in a homogeneous magnetic field in the absence of an electric field. Our conjecture is that the collection of dispersion laws $\varepsilon_j(p)$ together with the Chern numbers $c_1(\eta_j)$ and the collection of zeros $\gamma_{\alpha j}(x, y)$ at a single point (x_0, y_0) completely determines the coefficients of the operator—the electric and magnetic fields. In analogy to the one-dimensional case ($n = 1$), collective equations in x and y should be satisfied. Stabilization for large energies $j \to \infty$ gives a natural topological closure of this infinite system.

It is curious that in a magnetic field $H = H_0$ with zero mean $\bar{H} = 0$ such stabilization as $j \to \infty$ does not occur; the entire investigation (Sec. 3) was carried out for one energy, but in the complex domain compactification as $\operatorname{Im} p \to \infty$ was achieved on the basis of asymptotics in the quasimomenta of the form (4) for which there is no analogue for $\bar{H} \neq 0$.

This formulation of the inverse problem for $\bar{H} \neq 0$ is analogous to the theory of the one-dimensional stationary Schrödinger operator $L = -\partial^2 + u$, $L\psi = \varepsilon\psi$; in a periodic potential $u(x + T) = u(x)$.

In the theory of the one-dimensional operator L the zeros of the Bloch function $\psi_j(x,p)$, denoted by $\gamma_j(x)$, were each contained in a forbidden zone (lacuna) of the spectrum where the quasimomentum p is purely imaginary. It is evident, however, that just this consideration is analogous to the behavior of the zeros for $n=2$ in the permitted zones $\varepsilon_j(p_1,p_2)$: indeed, in the one-dimensional case in a permitted zone, where ψ is bounded on the entire line and is complex, this function, generally speaking, has no zeros (the two conditions $\operatorname{Re}\psi=0$, $\operatorname{Im}\psi=0$ on the single variable x, as a rule, are incompatible). On the other hand, in a forbidden zone the function $\psi_\pm(x,p)$ is real and has a "typical" zero $\gamma_j(x)$—one in each zone. Moreover, as $j\to\infty$ the lengths of the lacunae tend to zero; therefore, stabilization of the position of the zero $\gamma_j(x)$ as $j\to\infty$ near particular constants depending only on the period occurs. This is similar to the situation for $n=2$, $\bar H\neq 0$ but for the zeros in permitted zones [more precisely, in the dispersion laws $\varepsilon_j(p_1,p_2)$, since the projection onto the energy axis does not determine all quantum numbers of the spectrum].

The analytic structure of the manifolds of zeros $M_j^2\subset T^4$, however, has so far not been studied.

The situation for a periodic electric field $E(x,t)=\bar E+E_0(x,t)$ (case I above) is also curious.

5. A Particle with Spin 1/2 in a Periodic Magnetic Field. An Exact Solution for the Basic States

We consider the motion of a two-dimensional quantum particle with spin 1/2 in the plane x,y under the influence of a doubly periodic magnetic field $H(x,y)=\bar H+H_0(x,y)$ directed orthogonally to this plane; here $\bar H_0=0$. In the nonrelativistic approximation this motion is described by the Pauli operator which for $n=2$ has the form

$$\hat H\psi=\varepsilon\psi,\quad \hat H=\frac12(i\partial_1+eA_1)^2+\frac12(i\partial_2+eA_i)^2+\sigma_3H\cdot e,\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{1}$$

where $H=\partial_1A_2-\partial_2A_1$, $\hbar=m=c=1$.

Suppose, further, that $\partial_1A_1+\partial_2A_2=0$. It is easy to see that for $n=2$ we have

$$\sigma_3\hat H=\hat H\sigma_3.\tag{2}$$

Therefore, Eq. (1) decomposes into 2 scalar equations on the subspaces $\mathcal{L}_\pm$

$$\sigma_3\psi=\pm\psi,\quad \psi\in\mathcal{L}_\pm.\tag{3}$$

The operators $H_\pm$ are hereby equivalent to scalar Schrödinger operators in the same magnetic field and electric potentials $u=\pm H(x,y)$.

We assume that the magnetic flux is integral

$$e\varphi=e\int_0^{T_1}\int_0^{T_2}H\,dx\,dy=2\pi N.\tag{4}$$

Using the group of magnetic translations [see (11), Sec. 4], we define the Bloch states

$$T^*_\alpha \psi = \exp(ip_\alpha, T_\alpha)\psi$$

on the subspaces $\psi \in \mathcal{L}_\pm$.

We shall here discuss a rather general (as compared with a homogeneous field) integrable case where the Bloch functions of the base state can be found exactly (see [10, 11]). On the subspaces $\mathcal{L}_\pm$ we consider vector-valued functions with a fixed three-component of spin $\sigma_3\psi = \pm\psi$. We next use the trick first found in [32, 34] for other classes of magnetic fields, in particular, for fields localized on the plane in [32]. Namely the operators $H_\pm$ have the form

$$H_+ = AA^*, \quad H_- = A^*A, \quad A = -i(\partial_1 - eA_2) - (\partial_2 - eA_1). \tag{5}$$

From this it follows, in particular, that the spectrum $\varepsilon \geq 0$, i.e., the operator $H = H_+ \oplus H_-$ is nonnegative. Suppose, further, that $\varphi > 0$. For localized fields, for example, if we assume that $\varepsilon = 0$ is a point of the spectrum, then we find the following: $\hat{H}\psi = 0$ implies either $AA^*\psi = 0$ or $A^*A\psi = 0$.

We have further ($\psi \in \mathcal{L}_2(R)$):

$$\begin{gathered} AA^*\psi = 0 \to \langle A^*\psi, A^*\psi\rangle = 0 \to A^*\psi = 0, \\ A^*A\psi = 0 \to A\psi = 0. \end{gathered} \tag{6}$$

For $\varphi > 0$ it is easy to see that the equation $A^*\psi = 0$ is not solvable. Thus, if $\psi \in \mathcal{L}_+$, then $A\psi = 0$.

Using the gauge $\partial_1 A_1 + \partial_2 A_2 = 0$, we make the substitution

$$A_1 = -\partial_2\varphi, \quad A_2 = \partial_1\varphi. \tag{7}$$

This implies

$$\Delta\varphi = H(x, y). \tag{8}$$

The formal substitution

$$\psi = \exp(-e\varphi) f(x, y) \tag{9}$$

after trivial computations leads to the conclusion that

$$(\partial_1 + i\partial_2) f = 0 \quad \text{or} \quad f = f(z)$$

is an analytic function on the plane (an entire function).

This formal side is true, of course, for any boundary conditions. We obtain the general conclusion: if we can pass into an admissible class of functions by choosing the analytic function $f(z)$ and the potential φ, then $\varepsilon = 0$ is the bottom level of the spectrum, and functions of the form (9) are the base states. Both [32, 34] and [10, 11] are based on this consideration which is analogous to the theory of instantons. In the case of localized fields $H \in \mathcal{L}_2(R^2)$, following [38], it is necessary to choose the usual potential

$$\varphi = \frac{1}{2\pi} \iint_{R^2} \ln|r - r'|\, H(r') d^2 r', \quad r = (x, y). \tag{10}$$

From the form of the potential it follows easily that $f(x, iy)$ is a polynomial of degree $\leq [e\varphi(2\pi)^{-1}] - 1$. We shall not discuss the class of functions of [34].

In the periodic fields $H = \bar{H} + H_0(x, y)$ of interest to us under the condition that the magnetic flux (4) be integral it is natural to attempt to find magnetic Bloch functions of the form (9). We recall that the Weierstrass σ function is defined by the infinite product

$$\sigma(z) = z \prod_{n^2+m^2 \neq 0} \left(1 - \frac{z}{z_{m,n}}\right) \exp\left(\frac{z}{z_{m,n}} + \frac{z^2}{2z_{m,n}^2}\right), \qquad z_{m,n} = mT_1 + inT_2. \tag{11}$$

Under shifts by periods the transformations are as follows:

$$\begin{gathered} \sigma(z + T_1) = -\sigma(z) \exp(2\eta_1(z + T_1/2)), \\ \sigma(z + iT_2) = -\sigma(z) \exp(2i\eta_2(z + iT_2/2)), \\ \eta_1 = \xi(T_1/2), \quad i\eta_2 = \xi(iT_2/2), \quad \xi = \sigma'/\sigma. \end{gathered} \tag{12}$$

We shall determine a solution of Eq. (8) of the form

$$\varphi = \frac{1}{2\pi} \iint_K \ln|\sigma(z - z'|\, H(z')\, d^2 z', \quad z = x + iy, \tag{13}$$

where K is an elementary cell $0 \leq x \leq T_1$, $0 \leq y \leq T_2$.

We suppose that the magnetic Bloch eigenfunctions can be sought in the form

$$\psi = \lambda \exp(-e\varphi) \prod_{j=1}^{N} \sigma(z - a_1) \cdot \ldots \cdot \sigma(z - a_N) e^{az}, \tag{14}$$

where $\lambda \neq 0$, $a_1, \ldots, a_N, a$ are constants, and $e\varphi = 2\pi N$.

For magnetic translations we have

$$\begin{gathered} T_1^* \psi = \psi(x + T_1, y) \exp(-ie\varphi y \eta_1/\pi), \\ T_2^* \psi = \psi(x, y + T_2) \exp(-ie\varphi x \eta_2/\pi), \\ \eta_1 T_2 - \eta_2 T_1 = \pi \end{gathered} \tag{15}$$

All functions (14) are eigenfunctions for the magnetic translations (15) with complex quasimomenta p_1, p_2.

The condition that p_1, p_2 be real reduces to the relations

$$\begin{aligned} \operatorname{Re} a &= \operatorname{Re} \left[\eta_1/T_1 \left(2 \sum_{j=1}^{N} a_j - \frac{e}{\pi} \iint_K zH \, dx \, dy \right) \right], \\ \operatorname{Im} a &= \operatorname{Im} \left[\eta_2/T_2 \left(2 \sum_{j=1}^{N} a_j - \frac{e}{\pi} \iint_K zH \, dx \, dy \right) \right]. \end{aligned} \tag{16}$$

The quasimomentum p_1, p_2 is easily computed. In particular, for the quantity $p = p_1 + ip_2$ there is the formula

$$p = p_1 + ip_2 = -\frac{2\pi i}{T_1 T_2} \sum_{j=1}^{N} a_j + \text{const}, \tag{17}$$

where the constant depends on the magnetic field by (16). The entire manifold of Bloch functions constitutes a bundle with base the torus T^2 and fiber C^N. The Chern class c_1 is equal to the basis cocycle in $H^2(T^2, Z)$. We thus see that the entire collection of Bloch functions of the base state for a particle of spin 1/2 in any doubly periodic magnetic field is exactly the same as in a homogeneous field $H = \bar{H}$ where we have

$$\varphi = \varphi/2\pi \left(\frac{\eta_1}{T_1} x^2 - \frac{\eta_2}{T_2} y^2 - \eta_1 x + \eta_2 y \right). \tag{18}$$

A rigorous proof of the completeness of this basis for $\varepsilon = 0$ uses the theory of the index [42], $\mathcal{L}_+ \oplus \mathcal{L}_- = \mathcal{L}_0 \oplus \mathcal{L}_\perp$, $\varepsilon(\mathcal{L}_0) = 0$. The conclusion is that for any purely magnetic doubly periodic perturbation (spin 1/2, $n = 2$) the bottom Landau level remains unchanged, and, according to [5], it is separated from the rest of the spectrum by a finite integer in the energy: $\min \varepsilon(\mathcal{L}_\perp) \geq \Delta > 0$.

An electric potential destroys the base state. If it is small a discussion of perturbation theory can be found in Sec. 4.

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