PERIODIC EXTREMALS OF MANY-VALUED OR NOT-EVERYWHERE-POSITIVE FUNCTIONALS

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I. We consider (possibly many-valued) functionals of the type of length in Finsler metric, which in local coordinates x^j_α have the form

$$l^{(\alpha)}(\gamma) = \oint F(x, \dot{x})\, dt + \oint_\gamma A^{(\alpha)}_j(x)\, dx^j_\alpha. \tag{1}$$

Here it is assumed that $F(x, \lambda\dot{x}) = \lambda F(x, \dot{x})$, for $\lambda > 0$, $F > 0$, $\partial^2 F/\partial\dot{x}^i\,\partial\dot{x}^j$ is a positive form, and the expression $\Omega_{ij} = \partial_i A^{(\alpha)}_j - \partial_j A^{(\alpha)}_j$ (magnetic field) is uniquely defined on the whole manifold M^n as a closed but possibly nonexact 2-form. The first summand is defined as a positive Finsler metric, but the second may make the functional nonpositive or many-valued. This note is a continuation of [1]–[5].

Remark. This note contains, in particular, corrections of certain errors in [2]–[4]. It is necessary to remove the term "nonselfintersecting" from all statements except those in §II of this article. Moreover, in §5 of [3] there is an assertion in which the term "flat section" should be replaced by the term "T-invariant curve". These curves, however, can be considered nonselfintersecting only for functionals invariant under reversal of direction.

II. **Two-dimensional problems.** Functionals of the form (1) occur, for example, as Maupertuis–Fermat functionals for a charged particle in an external magnetic field or (if $M^n = \mathrm{SO}_3$) for the motion of a solid body under the action of gyroscopic forces. These functionals, as was shown in [2], occur on the two-dimensional sphere S^2 as the result of a reduction of the Hamiltonian formalism of the Kirchhoff–Thompson equations for the motion of a solid in a fluid or of a body around a fixed point, with the total flux of the "effective magnetic field" Ω_{12} through the sphere S^2 proportional to the area constant. Let $\hat{\Omega}^+$ denote the space of directed nonselfintersecting (piecewise smooth) curves. On this space a functional of the form (1) is single-valued. We normalize it so that it takes the value zero on one-point curves. In the many-valued case the functional is normalized so that it takes a positive value on the second component of the set of one-point curves, which is "split into two" by the construction of this single-valued branch of the functional (see [3], the beginning of §2).

Theorem 1. *If on S^2 there exists at least one nonselfintersecting closed curve γ such that $l(\gamma) < 0$, then there exists a nonselfintersecting periodic extremal γ_0 such that $l(\gamma_0) < 0$.*

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Remark. This extremal may not be a minimum of the functional among nonselfintersecting curves. This is the point of an error in §1 of [3].

Theorem 1 follows from the following lemmas.

Lemma 1. *If a minimum $\bar{\gamma}$ of the functional l lying in the closure of the set of nonselfinterecting curves is not a periodic extremal, then there exists a closed curve $\bar{\bar{\gamma}}$ consisting of a single nonselfintersecting closed piece of an extremal containing one turning point* (*zero angle*) *with $l(\bar{\bar{\gamma}}) < 0$.*

Lemma 2. *Given a functional of the form* (1) *on the plane and a convex extremal polygon $\bar{\bar{\gamma}}$ with all the angles between the links not greater than π* (*the zero angle is allowed*) *and $l(\bar{\bar{\gamma}}) < 0$, there exists within its interior a smooth nonselfintersecting periodic extremal γ such that $l(\gamma) < 0$.*

Theorem 2 follows from the fact that, due to the convexity, the shortening deformations lead into the interior of $\bar{\bar{\gamma}}$.

Theorem 2. *Suppose that on the torus T^2 there is given a functional of the form* (1)*, where the total flux of the field Ω_{12} through T^2* (*or a chamber in the plane R^2*) *is nonzero. Then in the general case there exist at least* 4 *nondegenerate periodic extremals on R^2* (*up to a translation by a vector in the lattice $\mathbb{Z} + \mathbb{Z}$*) *with index* $1, 2, 2, 3$ *for which the Whitney number is zero and $l > 0$.*

Corollary 1. *All these extremals are geometrically distinct, since the Whitney number of an iterated curve is nonzero.*

The proof follows [3], §4.III (see the footnote), starting from the "principle of throwing out cycles", in this case first formulated in [3]; for its generalization see [5]. However, in the process of deformation "downwards" there appear selfintersecting curves, and only the Whitney number remains invariant.

III. We turn now to arbitrary dimensions.

Lemma 3. *If a functional of the form* (1) *is single-valued and positive on the space of curves nullhomotopic in $\Omega^+_1(M^n)$, then it is semibounded and single-valued on all spaces of curves $\Omega^+_g(M^n)$ of arbitrary homotopy classes g* (*with a fixed starting point*).

If the functional is many-valued on some space $\Omega^+_g(M^n)$, $g \neq 1$, then in Ω^+_g the group π_1 is nontrivial. A map $S^1 \xrightarrow{\varphi} \Omega^+_g$ defines a map $T^2 \xrightarrow{f} M^n$, $f_*\pi_1(T^2) \neq 0$, such that the flux of the magnetic field Ω_{ij} through the cycle $f(T^2)$ is nonzero. We obtain a map of coverings $R^2 \xrightarrow{\hat{f}} \hat{M}^n$. The image $\bar{\gamma}_R = \hat{f}(\gamma_R)$ of a circle of radius R in R^2 as $R \to \infty$ is a curve for which $l(\bar{\gamma}_R) < 0$. From this Lemma 3 follows easily. Hence we obtain

Theorem 3. *On an arbitrary non-simply-connected manifold* (*closed*) *the functional* (1) *possesses at least one periodic extremal.*

For the simply-connected case the theorem was obtained in [1].

Condition. a) $H_{2k+1}(M^n; R) \neq 0$ for some k.

b) For some g, $g \neq 1$, the number of generators of $\pi_1(\Omega^+_g(M^n))$ is not less than two and the class g^{2m+1} does not coincide with g^{2n} for any $n, m \geqslant 0$.

Theorem 4. *If the condition holds, then there exist at least two nondegenerate extremals of the functional* (1) *in general position on a closed manifold, which extremals are geometrically distinct and homotopic to each other.*

The existence of at least two generators in $\pi_1(\Omega_g^+)$ gives the existence of two extremals in Ω_g^+ of index 0 and 1, if the functional is single-valued and semibounded on Ω_1^+. Otherwise, the functional is nonpositive on Ω_1^+, where, therefore, there exist geometrically distinct extremals of index 1 and $2k+2$ [5]. The claim that the extremals in Ω_g^+ are geometrically distinct is proved using Bott's theorem [6].

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