FINITE-ZONE, TWO-DIMENSIONAL, POTENTIAL SCHRÖDINGER OPERATORS. EXPLICIT FORMULAS AND EVOLUTION EQUATIONS

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I. According to [1] (see also [2]), to distinguish purely potential and real Schrödinger operators among those two-dimensional operators which are "finite-zone with respect to a single energy level" it suffices to require that the collection of "data of the inverse problem" [3]—a nonsingular Riemann surface Γ of finite genus $g = 2h$, a pair of labelled points P_1 and P_2 on it together with distinguished local parameters w_1, w_2 near them, and also a divisor D consisting of g distinct points of general position—possess the following symmetry. There is given the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by a holomorphic involution σ and an antiholomorphic involution $\tau\colon \Gamma \to \Gamma$. The points P_1 and P_2 must be the only fixed points of σ where $\tau(P_1) = P_2$,

$$\begin{aligned} &\tau(w_1) = \bar{w}_2, \quad \sigma(w_i) = -w_i, \quad i = 1, 2, \\ &\tau(D) = D, \quad D + \sigma(D) \sim K + P_1 + P_2, \end{aligned} \tag{1}$$

and K is the divisor of zeros of a holomorphic form on Γ. On the surface $\Gamma_0 = \Gamma/\sigma$ of genus h the involution τ induces an antiholomorphic involution τ_0. *Condition*: the pair (Γ_0, τ_0) is an M-curve, i.e., to has the maximal number of fixed ovals $h+1$. We assume that these conditions are satisfied everywhere below.

In [1] sufficient conditions are given for the positivity and smoothness of the operator L thus obtained: it is necessary to require that the pair (Γ, τ) be an M-curve; in this case there is exactly one σ-invariant connected oval which is fixed relative to τ, and the points of the divisor D are situated one each on the remaining $2h = g$ ovals. When the σ-invariant oval degenerates into a point we obtain singular curves corresponding to the base state ψ_0 of the operator L where $(L\varphi, \varphi) > 0$ on $\mathcal{L}_2(\mathbb{R}^2)$ and $L\psi_0 = 0$ (see [1]).

II. We consider the multiparameter function $\psi(P, x, y, t_1, \ldots, t_n, \ldots)$ constructed on the basis of the collection of data (1) which, as a function of the point P, is meromorphic everywhere except at the points P_1 and P_2 and has a fixed divisor of poles D; for $P \to P_i$ there are the asymptotic expressions

$$\begin{aligned} \psi &= \exp\left(k_1 z + \sum_{n=1}^{\infty} k_1^{2n+1} t_n\right)\left(1 + \sum_{i=1}^{\infty} \xi_i k_1^{-i}\right), \\ \psi &= \exp\left(k_2 \bar{z} + \sum_{n=1}^{\infty} k_2^{2n+1} t_n\right)\left(1 + \sum_{i=1}^{\infty} \bar{\xi}_i k_2^{-i}\right), \end{aligned} \tag{2}$$

where $k_i = w_i^{-1}$, $i = 1, 2$, $z = x + iy$ and $\bar{z} = x - iy$.

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Theorem 1. *A function ψ with the properties indicated exists, is unique, and satisfies the equations*

$$L\psi = 0, \quad \left(\frac{\partial}{\partial t_n} - (A_n - \bar{A}_n)\right)\psi = 0, \quad n = 1, 2, \dots, \tag{3}$$

where $L = \partial\bar{\partial} + V$, $A_n = \partial^{2n+1} + a_{2n-1}\partial^{2n-1} + \cdots + a_0$,

$$\partial = \partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y), \quad \bar{\partial} = \partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y),$$

and the coefficients V and a_k are uniquely determined on the basis of the coefficients of (2).

Corollary. *Deformations of the operator L with respect to any variable t_n are described by L-A-B-triples* [4] *of the form*

$$\partial L/\partial t_n = [L, A_n + \bar{A}_n] + (B_n + \bar{B}_n)L, \tag{4}$$

where $A_n = P_{2n+1}(\partial)$ and $B_n = Q_{2n-1}(\partial)$ are differential operators. The flows (4) *preserve the class of real potential Schrödinger operators L and their spectral data corresponding to the zero energy level.*

Main Example. $n = 1$. In this case $A_1 = \partial^3 + u\partial$, where u is determined from the condition $\bar{\partial}u = 3\partial V$, and $B_1 + \bar{B}_1$ is the operator of multiplication by the function $f = \partial u + \bar{\partial}\bar{u}$. The corresponding equation has the form

$$V_t = \partial^3 V + \bar{\partial}^3 V + \partial(uV) + \bar{\partial}(\bar{u}V). \tag{5}$$

The coefficient u is found from V up to a function depending analytically on z. This fact is general for the entire hierarchy; it is related to the conformal invariance of the equation $L\psi = 0$. We have

$$u = \frac{1}{\pi i}\iint \frac{dz'\,d\bar{z}'}{z - z'}\frac{\partial V}{\partial z'} + \varphi(z, t), \quad \bar{\partial}\varphi = 0. \tag{6}$$

a) If V decreases faster than any power r^{-n} as $r^2 = x^2 + y^2 \to \infty$, then the function uV has the same property if φ is a polynomial in z. Hence, this is an invariant class of functions for the system (5) which is determined by the z-polynomial $\varphi(z, t)$ as a parameter.

b) If V decreases like $r^{-\alpha}$ while the derivatives of order k of V decay like $r^{-\alpha-k}$, $\alpha > 0$, then (6) correctly determines the system (5) on this class if $\varphi = c(t)$.

Remark. In the absence of dependence on y, (5) reduces to the Korteweg–de Vries (KdV) equation in the unusual representation (4). Thus, (5) is a new integrable two-dimensionalization of KdV which with regard to its physical menaing may be no less important than the familiar Kadomtsev–Petviashvili equation.

III. Before giving formulas for the potentials and Bloch functions in theta functions of Prym varieties, we present the necessary facts regarding the latter in the case we require (see, for example, [5]). Suppose that on the curve Γ of genus g there is an involution σ with two fixed points P_1 and P_2; in this case $g = 2h$, where h is the genus of the curve $\Gamma_0 = \Gamma/a$. On Γ it is possible to choose a basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ such that $\sigma(a_i) = -a_{i+h}$ and $\sigma(b_i) = -b_{i+h}$, $i = 1, \dots, h$, and also a corresponding basis of holomorphic differentials $\omega_1, \dots, \omega_g$ normalized by the conditions

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}, \quad j, k = 1, \dots, g.$$

The columns of the matrix $B_{ij} = \int_{b_j} \omega_i$ together with the vectors $2\pi i e_j$ (e_j is a basis in the space $\mathbb{C}^g$) define a lattice T in $\mathbb{C}^g$, $\mathbb{C}^g/T = J(\Gamma)$.

Let $A(Q_1, \dots, Q_g) \in J(\Gamma)$, $Q_i \in \Gamma$, be the image of the Abel mapping:

$$A(Q_1, \dots, Q_g) = \sum_{i=1}^{g} A(Q_i), \quad A(Q) = \sum_{i=1}^{g} \int_{P_1}^{Q} \omega_i,$$

where one of the fixed points of σ, say P_1, is chosen as the initial point of the integtation (this explains a certain lack of symmetry in the following formulas with respect to P_1 and P_2). The Prym variety (the "Prymian") $\mathrm{Pr}_\sigma(\Gamma)$ is distinguished in $J(\Gamma)$ by the equation $\sigma_*(x) = -x$, where σ_* is the natural action of the involution of $J(\Gamma)$. In our case $\mathrm{Pr}_\sigma(\Gamma)$ is given by points of the form $(z_1, \dots, z_h, z_1, \dots, z_h)$. This is an Abelian variety $\mathbb{C}^h/T_\sigma$ where the lattice T_σ is generated by the vectors $2\pi i f_k$ (f_k is a basis in $\mathbb{C}^h$) and the columns of the matrix Π: $\Pi_{ij} = \int_{b_j} \eta_i$, $\eta_i = \omega_i + \omega_{i+h}$, $i = 1, \dots, h$, are normalized Prym differentials. The natural imbeddings $i\colon \mathrm{Pr}_\sigma(\Gamma) \to J(\Gamma)$ and $\pi^*\colon J(\Gamma_0) \to J(r)$ are defined as is the mapping $\eta\colon \Gamma \to \mathrm{Pr}_\sigma(\Gamma)$, where $\eta(P)^i = \int_{P_1}^{P} \eta_i$.

We introduce the Abelian differentials of second kind Ω_1 and Ω_2 normalized by the condition $\int_{a_j} \Omega_i = 0$ having a single pole at the points P_1 and P_1 respectively of the form $d(w_i^{-1})$. Let

$$U_1^i = \int_{b_i} \Omega_1, \quad U_2^i = \int_{b_i} \Omega_2, \quad i = 1, \dots, h.$$

We note that by virtue of the normalization conditions Ω_1 and Ω_2 are Prym differentials i.e., $\sigma^*\Omega_k = -\Omega_k$, while the vectors of their b-periods have the form $\hat{U}_k = (U_k, U_k)$. We denote by $\theta[\alpha, \beta](z)$ the theta function with characteristics (α, β) (see [5] and [6]) corresponding to the Prym variety, and by $\theta[\mu, \nu](w)$ the theta function on $J(\Gamma)$.

On the basis of the divisor D we define a point $e \in \mathrm{Pr}_\sigma(\Gamma)$ by

$$i(e) = A(D) - A(P_2) + \pi^*(R),$$

where R is the vector of Riemann constants of the curve Γ_0 (see [6] and [6]).

Theorem 2. *For a divisor D of general position the function*

$$\psi(z, \bar{z}, P) = \frac{\theta(\eta(P) + z_1 U + \bar{z} U_2 - e)\theta(e)}{\theta(\eta(P) - e)\theta(z_1 U + \bar{z} U_2 - e)} \exp\left[z\left(\int_{P_0}^{P} \Omega_1 - \alpha\right) + \bar{z}\int_{P_1}^{P} \Omega_2\right] \tag{7}$$

has the required analytic properties on Γ and satisfies the equation $L\psi = \varepsilon_0\psi$, where $L = \partial\bar{\partial} + V$,

$$V = 2\partial\bar{\partial} \ln \theta(zU_1 + \bar{z}U_2 - e), \quad \varepsilon_0 = \sum_{i,j=1}^{j} \hat{U}_1^i \hat{U}_2^j \partial_i \partial_j \ln \hat{\theta}[\nu](z)|_{z=A(P_2)}. \tag{8}$$

Here the constant a is chosen so that $(\int_{P_0}^{P} \Omega_1 - \alpha) \sim 1/w_1 + O(w_1)$ for $P \sim P_1$, and ν is an arbitrary nondegenerate odd half period, i.e., $\nu = (\alpha, \beta)$, $2\nu \in \mathbb{Z}_2^{2g}$, $4\sum_1^g \alpha_i\beta_j \equiv 1 \pmod 2$ and $\operatorname{grad} \hat{\theta}[\nu](0) \neq 0$.

Clarification. The integrals in the definition of $\eta(P)$ and in the argument of the exponential functions are chosen in a consistent manner; this is achieved by fixing some path from P_0 to P_1. It is just this path that is present in the definition of α

whose introduction together with the point P_0 is needed only to give meaning to the expression $\int_{P_1}^{P} \Omega_1$.

We now consider the simplest singular case corresponding to a base state: suppose the singular curve Γ has one "double" point, while the involutions σ and τ of Γ are the same as in Theorem 5 of [1] (see the beginning of the article). In this case $\psi_0 = \psi(Q)$, where Q is the "double" point, corresponds to a base state:

$$(L - \varepsilon_0)\psi_0 = 0, \quad (L\varphi, \varphi) > \varepsilon_0 \|\varphi\|^2, \quad \varphi \in \mathcal{L}_2(\mathbb{R}^2).$$

We present a formula for this eigenfunction. Let $\tilde{\Gamma}$ and $\tilde{\Gamma}_0$ be the curves obtained from Γ and $\Gamma_0 = \Gamma/\sigma$ by resolution of the singularity, and let the points Q_1 and Q_2 of $\tilde{\Gamma}_0$ correspond to the "double" point of Γ_0. The genus of $\tilde{\Gamma}$ is equal to $g = 2h+1$, where h is the genus of $\tilde{\Gamma}_0$. We choose a basis of cycles a_i, b_i on $\tilde{\Gamma}$ so that $\sigma(a_i) = -a_{i+h}$, $\sigma(b_i) = -b_{i+h}$ $(i = 1, \dots, h)$, $\sigma(a_g) = -a_g$ and $\sigma(b_g) = -b_g$; the cycles a_g and b_g here "hang" over paths on $\tilde{\Gamma}_0$ joining the points P_1 and Q_1 and the points Q_1 and Q_2 respectively. The differentials ω_i and Ω_k on $\tilde{\Gamma}$ are defined as in the nonsingular case; $\eta_i = \omega_i + \omega_{i+h}$, $i = 1, \dots, h$, and $\eta_{h+1} = \omega_g$. The matrix Π corresponding to the Prym variety of the singular curve Γ is defined as follows: $\Pi_{ij} = \int_{b_j} \eta_i$, $i = 1, \dots, h+1$, $j = 1, \dots, h$, and $\Pi_{i(n+1)} = \frac{1}{2} \int_{b_g} \eta_i$; the vectors U_1 and U_2 are defined similarly:

$$U_k^i = \int_{b_i} \Omega_k, \quad i = 1, \dots, h, \quad U_k^{h+1} = \frac{1}{2} \int_{b_g} \Omega_k, \quad k = 1, 2.$$

Theorem 3. *The ψ-function of the base stage has the form*

$$\psi_0(z, \bar{z}) = \frac{\theta[0, \beta](zU_1 + \bar{z}U_2 - e)\theta(e)}{\theta[0, \beta](e)(zU_1 + \bar{z}U_2 - e)},$$

where $\beta = (0, 0, \dots, 0, \frac{1}{2})$, and e belongs to the corresponding real component of the Prymian described above. The potential V and the energy of the base state are given by (8)*, where $\hat{\theta}[\nu](z)$ corresponds to the Jacobian of the curve $\tilde{\Gamma}$.*

We note that ψ_0 is real and has the same group of periods as the potential V. In the case $g = 3$, $h = 1$ they are both doubly periodic.

In conclusion we discuss the question of the position of the "finite-zone" operators under study among all potential Schrödinger operators.

Conjecture. Any smooth, real, doubly periodic potential $V(x, y)$ can be approximated by potentials which are finite-zone with respect to one energy level.

At the intuitive level this conjecture is quite clear. Suppose there is given a uniformly convergent sequence $V_n(x, y) \to V(x, y)$ of smooth, real, doubly periodic potentials which are finite-zone with respect to the zero energy level, where the genus $g_n = g(\Gamma_n) \to \infty$. Then outside any fixed neighborhoods of the points P_1 and P_2 (we identify them for all n) the collection of data of the inverse problem converges, including the finite parts of the surfaces Γ_n, the involutions σ and τ and the poles of ψ. With the exception of the finite part, which does not grow with n as $n \to \infty$, the entire collection of fixed ovals of the anti-involutions τ and $\sigma\tau$ and the position of the poles are the same as in Theorem 5 of [1]; the size of the ovals of the involution τ decrease with the index of the oval, while their positions can asymptotically be computed precisely near the points P_1 and P_2.

References

[1] A. P. Veselov and S. P. Novikov, Dokl. Akad. Nauk SSSR **279** (1984), 784–788; English transl. in Soviet Math. Dokl. **30** (1984).

[2] S. P. Novikov, Uspekhi Mat. Nauk **39** (1984), no. 4(238), 97–98. (Russian)

[3] B. A. Dubrovin, I. M. Krichever and S. P. Novikov, Dokl. Akad. Nauk SSSR **229** (1976), 15–18; English transl. in Soviet Math. Dokl. **17** (1976).

[4] S. V. Manakov, Uspekhi Mat. Nauk **31** (1976), no. 5(191), 245–246. (Russian)

[5] John D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Math., vol. 352, Springer-Verlag, 1973.

[6] B. A. Dubrovin, Uspekhi Mat. Nauk **36** (1981), no. 2(218), 11–80; English transl. in Russian Math. Surveys **36** (1981).

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