Integrable Systems. I

B. A. Dubrovin

I. M. Krichever

S. P. Novikov

Translated from the Russian by G. Wasserman.

Contents

Introduction

Integrable systems which do not have an "obvious" group symmetry, beginning with the results of Poincaré and Bruns at the end of the last century, have been perceived as something exotic. The very insignificant list of such examples practically did not change until the 1960's. Although a number of fundamental methods of mathematical physics were based essentially on the perturbation-theory analysis of the simplest integrable examples, ideas about the structure of nontrivial integrable systems did not exert any real influence on the development of physics.

The situation changed radically with the discovery of the *inverse scattering method*. The ever-growing interest in this method is connected with the fact that it has proved to be applicable to a number of nonlinear equations of mathematical physics which, as became clear in the mid-sixties, possess a remarkable universality property. They arise in the description (in the simplest approximation after the linear one) of the most diverse phenomena in plasma physics, the theory of elementary particles, the theory of superconductivity, in nonlinear optics and in a number of other problems which are reducible to spatially one-dimensional ones. Among the equations referred to are the Korteweg–de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation and many others.

The inverse scattering method allowed people for the first time to discover and to understand a number of principally new effects which had not become apparent in any way in the theory of perturbations. The most striking and important of them are connected with the concept of solitons and their periodic analogues (which will be the topic of discussion to a significant degree later on). The concept of solitons has become one of the fundamental ones in contemporary nonlinear physics.

Although after the papers [54], [59] it subsequently became clear that the equations to which the inverse scattering method is applicable are Hamiltonian and, what is more, are the field analogues of completely integrable Hamiltonian systems, the integration of these equations within the framework of the inverse scattering method does not make use of the Hamiltonian theory. The Hamiltonicity of these equations, the construction for them of variables of the action-angle type turn out to be essential during the following stages—in the construction of a theory of perturbations and of diverse versions of the averaging methods, in the construction of the quantum analogue of the inverse scattering method. These sections remain outside the scope of the present article.

The goal of the present survey is the presentation of the modern theory of integrable systems as a constituent part of the inverse scattering method. Just as in classical analytical mechanics, special emphasis is laid on finite-dimensional systems.

The finite-dimensional dynamical systems to which the inverse scattering method is applicable and to which, basically, this article is devoted (and among them

are contained all the known classical completely integrable systems) are finite-dimensional in their original physical formulation or they arise during the construction of particular classes of exact solutions of the field-theoretic equations as restrictions of the latter to finite-dimensional invariant submanifolds.

One should especially stress the significantly greater effectiveness of the inverse scattering method as compared with the classical methods of integrating Hamiltonian systems. For completely integrable systems, in contrast to the ineffective integration procedure given by Liouville's theorem, the inverse scattering method allows one to explicitly produce solutions of the equations of motion, as well as canonical action-angle variables, in terms of special classes of functions.

In the first chapter of the survey the modern views of the Hamiltonian formalism of both finite-dimensional and field-theoretic systems are presented. Also set forth are the methods, going back to the classical ones, of integrating Hamiltonian systems which have an explicit symmetry or which admit of separation of the variables.

The second chapter is the nucleus of the present survey. In it the paramount concept of the *commutation representation* of evolution systems is introduced, which is the starting point of all the integration schemes which are unified by the ideas of the inverse scattering method. A scheme based on the application of the methods of classical algebraic geometry has proven to be the most fruitful one in the theory of integrable finite-dimensional systems. This scheme, and also its numerous applications, are presented in the second chapter.

It needs to be noted that naturally abutting on the present survey there will be a survey "Integrable systems II" by A.M. Perelomov, M.A. Ol'shanetskij, and M.A. Semenov-Tyan-Shanskij, which will be published in one of the following volumes of the present series. Its first chapter is devoted to group-theoretic methods of integration of some special finite-dimensional systems. The second chapter is devoted to geometric quantization of the open Toda lattice and its generalizations.

CHAPTER 1

Hamiltonian Systems. Classical Methods of Integration

1. The General Concept of the Poisson Bracket. The Principal Examples

From the modern point of view, the concept of the *Poisson bracket* (S. D. Poisson) lies at the basis of the Hamiltonian formalism. Let y^i, $i = 1, \dots, N$ be local coordinates on a manifold Y—the *phase space*. The Poisson bracket of two functions $f(y)$ and $g(y)$ is given by a tensor field $h^{ij}(y)$,

$$\{f, g\} = h^{ij}(y) \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j} \tag{1.1}$$

(here and further on the summation over repeated indices is implied). Here it is required that the following properties be fulfilled:

a) bilinearity

$$\{\lambda f + \mu g, h\} = \lambda\{f, h\} + \mu\{g, h\}, \quad \lambda, \mu = \text{const}, \tag{1.2}$$

and skew-symmetry

$$\{g, f\} = -\{f, g\}; \tag{1.3}$$

b) the *Leibniz identity* (G. W. Leibniz)

$$\{fg, h\} = g\{f, h\} + f\{g, h\}; \tag{1.4}$$

c) the *Jacobi identity* (C. G. J. Jacobi)

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{1.5}$$

Let us note that

$$h^{ij}(y) = \{y^i, y^j\}, \tag{1.6}$$

and the definition (1.1) can be written in the form

$$\{f, g\} = \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j} \{y^i, y^j\}. \tag{1.7}$$

Hamiltonian systems by definition have the form:

$$\dot{y}^i = \{y^i, H\}, \quad i = 1, \dots, N, \tag{1.8}$$

where $H = H(y)$ is an arbitrary function, called the *Hamiltonian*. The vector field $\xi_H = (\xi_H^i)$ corresponding to the Hamiltonian system (1.8) has the form:

$$\xi_H^i(y) = h^{ij}(y) \frac{\partial H(y)}{\partial y^j} = \{y^i, H(y)\}, \quad i = 1, \dots, N. \tag{1.9}$$

Such *vector fields* are called *Hamiltonian*. The commutator of two Hamiltonian fields is connected with the Poisson bracket by the relation

$$[\xi_H, \xi_F] = -\xi_{\{H,F\}}. \tag{1.10}$$

It is clear that the derivative of an arbitrary function $f = f(y)$ by means of the Hamiltonian system (1.8) has the form

$$\dot{f} = \{f, H\} = \xi^i_H \frac{\partial f}{\partial y^i}. \tag{1.11}$$

The flow of (1.8) preserves the Poisson bracket:

$$\{y^i(t), y^j(t)\} = \{y^i(0), y^j(0)\}. \tag{1.12}$$

(*Transformations* which preserve the Poisson bracket are called *canonical*. Any one-parameter group of canonical transformations for non-degenerate brackets $\det(h^{ij}) \neq 0$ has the form (1.8), where the Hamiltonian is possibly defined locally [42] (see also the second chapter of the article by V. I. Arnol'd and A. B. Givental').

It is possible that there are nontrivial functions $f_q(y)$ (maybe given locally on the manifold) such that

$$\{f_q, g\} = 0 \tag{1.13}$$

for any function $g(y)$. In this case the Poisson bracket is called degenerate: the matrix $h^{ij}(y)$ is degenerate. (For a degenerate matrix $h^{ij}(y)$ of constant rank the functions $f_q(y)$ of (1.13) locally always exist.) If all such quantities $f_l(y)$ have been found, then on their common level surface

$$f_l(y) = \text{const} \quad (l = 1, 2, \dots) \tag{1.14}$$

the Poisson bracket no longer stays degenerate.

Let z^q be coordinates on the level surface (1.14). The restriction of the tensor h^{qr} to this surface is no longer degenerate, and there is an inverse matrix

$$h_{qp}h^{pr} = \delta^r_q. \tag{1.15}$$

The inverse matrix defines a 2-form

$$\Omega = h_{qp}(z)\, dz^q \wedge dz^p. \tag{1.16}$$

From property (1.5) it follows that the form Ω is closed,

$$d\Omega = 0, \quad \text{i. e.} \quad \frac{\partial h_{qp}}{\partial z^r} + \frac{\partial h_{rq}}{\partial z^p} + \frac{\partial h_{pr}}{\partial z^q} = 0. \tag{1.17}$$

If the Poisson bracket was non-degenerate right from the start, then the closedness condition (1.17) turns out to be equivalent to the Jacobi identity (1.5) [42]. Thus phase spaces with a non-degenerate Poisson bracket are *symplectic manifolds*.

Let us examine the basic types of phase spaces.

Type I. Constant brackets and Lagrangian variational problems. Let the matrix h^{ij} be constant and skew-symmetric. The Jacobi identity is automatically fulfilled in this case: on a plane where the matrix h^{qr} becomes non-degenerate the corresponding 2-form $\Omega = h_{qr}\, dy^q \wedge dy^r$ has constant coefficients and is therefore

closed. By a linear transformation constant brackets can be reduced to the form

$$(h^{ij}) = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & 0 & & \\ & 0 & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \tag{1.18}$$

Locally any Poisson brackets of constant rank can be brought into this form (the Darboux theorem (G. Darboux)). In the non-degenerate case coordinates $(y) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ may be introduced such that

$$h^{ij} = -h_{ij} = \begin{pmatrix} & & & & 1 & & 0 \\ & & 0 & & & \ddots & \\ & & & & 0 & & 1 \\ -1 & & & 0 & & & \\ & \ddots & & & & 0 & \\ 0 & & & -1 & & & \end{pmatrix} \tag{1.19}$$

The coordinates (x, p) are called canonical ones.

The equations (1.8) take on the form:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, n. \tag{1.20}$$

These are the canonical Hamilton equations (W. R. Hamilton). They can be obtained from the variational principle $\delta S = 0$, where

$$S = \int L(x, \dot{x})\, dt, \tag{1.21}$$

(x) belongs to the configuration space and $L(x, \dot{x})$ is the Lagrangian, if one performs a *Legendre transformation* (A. M. Legendre)

$$\begin{aligned} &p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad i = 1, \ldots, n \\ &H(x, p) = p_i \dot{x}^i - L(x, \dot{x}) \end{aligned} \tag{1.22}$$

(it is assumed that the equations $p_i = \partial L / \partial \dot{x}^i$ can be solved in the form $\dot{x}^i = \dot{x}^i(x, p)$.

Conversely, if $H(x, p)$ is a Hamiltonian, then we have a Lagrangian $L(x, \dot{x})$ defined from the equations

$$\dot{x}^i = \frac{\partial H(x, p)}{\partial p_i}, \quad L(x, \dot{x}) = p_i \dot{x}^i - H(x, p). \tag{1.23}$$

It is assumed that one can solve the equations $\dot{x}^i = \partial H / \partial p_i$ for the variables p_i.

Similar variational problems with higher derivatives [42]

$$\delta S = 0, \quad S = \int L(x, \dot{x}, \ldots, x^{(k)})\, dt \tag{1.24}$$

can be brought into the form (1.20) with the aid of *M. V. Ostrogradskij's transformation*[1]

$$q^j = x^{(j-1)}, \quad p_j = \sum_{s=0}^{k-j} (-1)^s \frac{d^s}{dt^s} \frac{\partial L}{\partial x^{(j+s)}}, \quad j = 1, \dots, k; \tag{1.25}$$

$$-H(q,p) = L - p_1 q^2 - \cdots - p_{k-1} q^k - p_k x^{(k)}, \tag{1.26}$$

$$\dot{q}^j = \partial H/\partial p_j, \quad \dot{p}_j = -\partial H/\partial q^j, \quad j = 1, \dots, k,$$

if the equations (1.25) can be solved uniquely in the form

$$x = x(q,p), \quad \dot{x} = \dot{x}(q,p), \ \dots, \ x^{(2k-1)} = x^{(2k-1)}(q,p). \tag{1.27}$$

Type II. *Lie–Poisson brackets.* Let us now consider the second case in order of complexity, when the tensor h^{ij} is not constant, but depends linearly[2] on the coordinates (y)

$$h^{ij} = c_k^{ij} y^k, \quad c_k^{ij} = \text{const}. \tag{1.28}$$

Let us consider the set L of all linear functions on the phase space, which we shall denote by L^*. For the basis linear forms—the coordinates y^i—the bracket defines a "commutation" operation

$$[y^i, y^j] = c_k^{ij} y^k = \{y^i, y^j\}. \tag{1.29}$$

The requirements (1.3), (1.5) imply that the operation (1.29) turns the linear space L into a Lie algebra (S. Lie) whose dual space L^* is the phase space for the Poisson bracket (1.28). The bracket of this form was first examined by Lie [93]. It was rediscovered by F. A. Berezin [14] and used by A. A. Kirillov and B. Kostant [67] (in the less convenient language of symplectic manifolds) in the theory of infinite-dimensional representations of Lie groups. The bracket (1.28) is in general degenerate.

EXAMPLE 1. The fundamental example of the Hamiltonian formalism of type 1 is the phase space T^*M—the space of covectors (with lowered indices) on a manifold M (the configuration space). On T^*M there are local coordinates x^i (on M) and conjugate momenta p_j (on the fibre) with Poisson brackets

$$\{x^i, x^j\} = \{p_i, p_j\} = 0, \quad \{x^i, p_j\} = \delta_j^i \tag{1.30}$$

and with the form

$$\Omega = dp_i \wedge dx^i. \tag{1.31}$$

EXAMPLE 2. It is useful to consider also a Poisson bracket of the form (1.30) which in addition is distorted by an "external field" $F_{ij} = -F_{ji}(x)$:

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = F_{ij}(x), \tag{1.32}$$

where the 2-form $F = F_{ij} dx^i \wedge dx^j$ is closed, $dF = 0$. The corresponding 2-form Ω has the form:

$$\Omega = dp_i \wedge dx^i + F_{ij} dx^i \wedge dx^j. \tag{1.33}$$

The equations of motion with a Hamiltonian $H(x,p)$ and the Poisson bracket (1.32) represent (for $n = 2, 3$) the equations of motion of a charged particle in the external

[1]Note that in these equations the q^j and p_j are *vectors*, and j essentially indexes the derivatives of different *orders* of the vector x, not the coordinates of x as in (1.20)–(1.23).

[2]The third case in order of complexity, when the tensor $h^{ij}(y)$ depends quadratically on y, is also very interesting and has recently begun to be studied [129].

magnetic field ($n = 2, 3$) F_{ij} (or electromagnetic field for $n = 4$). In a region where $F = dA$ the bracket (1.32) can be reduced to the standard form (1.30). As a rule one can reduce to the form (1.32) (globally) non-degenerate Poisson brackets on the space T^*M which satisfy the following requirement: any two functions f, g on the base space M (not depending on the variables on the fibre, which consists of all the covectors) have a vanishing Poisson bracket: $\{f, g\} = 0$ (see [112]).

Let us turn now to examples connected with the Lie–Poisson brackets.

EXAMPLE 3. Let L be the Lie algebra of the rotation group SO(3). The Killing metric (W. Killing) on L is Euclidean and it allows us not to distinguish between L and L^* (all indices will be considered to be lower ones). The Poisson bracket of the basis functions M_i on L^* has the form:

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \tag{1.34}$$

where

$$\varepsilon_{ijk} = c_k^{ij} = \begin{cases} \text{the signum of the permutation } (i, j, k), \\ \quad \text{if } i, j, k \text{ are all different;} \\ 0, \text{ if there is a pair of coinciding indices } i, j, k. \end{cases} \tag{1.35}$$

The function $M^2 = \sum M_i^2$ is such that

$$\{M^2, M_i\} = 0, \quad i = 1, 2, 3. \tag{1.36}$$

On the level surfaces $M^2 = \text{const}$ (spheres) the bracket (1.34) becomes non-degenerate. The Hamiltonian systems on L^* have the form:

$$\dot{M} = \{M_i, H(M)\}. \tag{1.37}$$

Let $\omega^i = \partial H / \partial M_i$; the Killing metric allows us not to distinguish between upper and lower indices. The equations (1.37) reduce to the form of the "*Euler equations*" (L. Euler)

$$\dot{M} = [M, \omega], \tag{1.38}$$

where the square brackets denote the commutator in L. (When $H = \frac{1}{2}(a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2)$ the equations (1.37) coincide with the equations of motion of a rigid body fixed at its centre of gravity). The derivation of the equations (1.38) is valid for all compact (and semisimple) Lie groups on which there is a Killing metric—a Euclidean (pseudo-Euclidean) metric on the Lie algebra which is invariant with respect to inner automorphisms

$$L \to gLg^{-1}, \tag{1.39}$$

where g is an element of the Lie group, and L is the Lie algebra. Such systems on the groups SO(N) are called the "many-dimensional analogue of a rigid body", in accordance with V. I. Arnol'd, if the Hamiltonian has the aspect of a quadratic form on the space of skew-symmetric matrices $M = (M_{ij})$, where

$$H(M) = \sum_{i<j} d_{ij} M_{ij}^2, \quad d_{ij} = q_i + q_j, \quad q_i > 0. \tag{1.40}$$

EXAMPLE 4. With the Lie algebra L of the group $E(3)$ of motions of three-dimensional Euclidean space some important systems arising in hydrodynamics are connected. This algebra is no longer semisimple. On the phase space L^* there are 6 coordinates $\{M_1, M_2, M_3, p_1, p_2, p_3\}$ and the Poisson brackets

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0. \tag{1.41}$$

The bracket (1.41) possesses two independent functions $f_1 = \sum p_i^2$, $f_2 = \sum p_i M_i$ such that

$$\{f_q, M_i\} = \{f_q, p_i\} = 0, \quad q = 1, 2, \quad i = 1, 2, 3. \tag{1.42}$$

On the level surfaces $f_1 = p^2$, $f_2 = ps$ the bracket (1.41) is non-degenerate. The substitution $\sigma_i = M_i - (s/p)p_i$ sets up an isomorphism of these level surfaces with the tangent bundle T^*S^2 of the sphere, $\sum \sigma_i p_i = 0$. (We identify the tangent bundle with the cotangent bundle by means of the standard Riemannian metric on the sphere.) On these level surfaces the restriction of the Poisson bracket (1.41) is no longer degenerate (when $p \neq 0$). It turns out that the brackets which arise on T^*S^2 can be reduced globally to the form (1.32). The appropriate substitution (see [112]) has the form:

$$\begin{gathered} p_1 = p\cos\theta\cos\psi, \quad p_2 = p\cos\theta\sin\psi, \quad p_3 = p\sin\theta, \\ \sigma_1 = p_\psi \tan\theta\cos\psi - p_\theta \sin\psi, \quad \sigma_2 = p_\psi \tan\theta \sin\psi - p_\theta\cos\psi, \\ \sigma_3 = -p_\psi, \end{gathered} \tag{1.43}$$

where $-\pi/2 \leq \theta \leq \pi/2$, $0 \leq \psi \leq 2\pi$, $\sigma_i = M_i - sp^{-1}p_i$. It is easy to deduce from formula (1.43) that

$$\begin{gathered} \{\theta, \psi\} = \{p_\theta, \psi\} = \{p_\psi, \theta\} = 0, \quad \{\theta, p_\theta\} = \{\psi, p_\psi\} = 1, \\ \{p_\theta, p_\psi\} = s\cos\theta. \end{gathered} \tag{1.44}$$

The corresponding 2-form Ω takes on the form (1.33),

$$\Omega = dp_\theta \wedge d\theta + dp_\psi \wedge d\psi + s\cos\theta\, d\theta \wedge d\psi = d\xi_i \wedge dy^i + F, \tag{1.45}$$

where $y^1 = \theta$, $y^2 = \psi$, $\xi^1 = p_\theta$, $\xi^2 = p_\psi$, $F = s\cos\theta\, d\theta \wedge d\psi$. The integral of the form F (and Ω) over the basis cycle $[S^2] \in H_2(T^*S^2) = \mathbb{Z}$ has the form

$$\iint_{S^2} F = \iint_{[S^2]} \Omega = 4\pi s = 4\pi f_2 f_1^{-1/2}. \tag{1.46}$$

Thus we obtain the standard Poisson bracket on T^*S^2, supplementarily distorted by an effective magnetic field F. When $s \neq 0$ the effective magnetic field is always different from zero and represents a "Dirac monopole" (non-quantized).

Let $H(M, p)$ be a Hamiltonian. Let us introduce the notation $u^i = \partial H/\partial p_i$, $\omega^i = \partial H/\partial M_i$. The Hamilton equations will assume the form of "*Kirchhoff's equations*" (P. Kirchhoff)

$$\dot{p} = [p, \omega], \quad \dot{M} = [M, \omega] + [p, u] \tag{1.47}$$

(the square brackets denote the vector product). The equations (1.47) coincide (for quadratic Hamiltonians $H(M, p)$) with Kirchhoff's equations for the motion of a rigid body in a fluid—in a fluid which is perfect, incompressible, and at rest at infinity [107]. The motion of the fluid itself is considered to be potential. In this case H is the energy, M and p are the total angular momentum and the momentum of the body-fluid system in a moving coordinate system rigidly connected with the body. The energy $H(M, p)$, quadratic in M, p and positive definite, can be given in the form

$$2H = \sum a_i M_i^2 + \sum b_{ij}(p_i M_j + M_i p_j) + \sum c_{ij} p_i p_j. \tag{1.48}$$

One can reduce to the form (1.47) the equations of motion of a rigid body with a fixed point in an axially symmetric force field with a potential $W(z)$. The corresponding Hamiltonian has the form:

$$H = \frac{1}{2}\sum a_i M_i^2 + W(l^i p_i), \tag{1.49}$$

where l^i is the constant vector giving the position of the centre of mass relative to the principal axes and the point of attachment. The quantities p_i here are dimensionless and do not have the physical meaning of momenta. They are the direction cosines of a unit vector, i. e. one always has $f_i = p^2 = 1$. The equations of the dynamics of the spin in the A-phase of superfluid ^{3}He can also be reduced to the form (1.47) (see [112]).

On the surface $f_1 = p^2$, $f^2 = ps$ the Hamiltonians H of the form (1.48) or (1.49) can be written as follows in the variables (y, ξ):

$$H = \frac{1}{2}g^{ab}(y)\xi_a\xi_b + A^a(y)\xi_a + V(y). \tag{1.50}$$

Here for the Hamiltonian (1.48) we shall have

$$\sum a_i\sigma_i^2 = g^{ab}\xi_a\xi_b, \quad \sigma_i = M_i - sp^{-1}p_i, \tag{1.51}$$

$$A^a\xi_a = s(\sum a_i p_i p^{-1}\sigma_i) + p\sum b_{ij}(\sigma_i p_j p^{-1} + \sigma_j p_i p^{-1}), \tag{1.52}$$

$$2V = s^2\sum a_i p_i^2 p^{-2} + 2ps\sum b_{ij}p_i p_j p^{-2} + p^2\sum c_{ij}p_i p_j p^{-2}. \tag{1.53}$$

In view of homogeneity, the Hamiltonian H depends only on sp^{-1}. For the top (1.49), the Hamiltonian can also be written on the level surface $f_1 = 1$, $f_2 = s$ in the form (1.50), where the metric g^{ab} again has the form (1.51), but

$$A^a\xi_a = s\sum a_i\sigma_i p_i, \tag{1.54}$$

$$2V = s^2\sum a_i p_i^2 + 2W(l^i p_i). \tag{1.55}$$

DEDUCTION ([**112**]). *The equations of the Kirchhoff type reduce to a system which is mathematically equivalent to a classical charged particle moving on the sphere S^2 with the Riemannian metric $g_{ab}(y)$, $g^{ab}g_{bc} = \delta_c^a$, in a potential field $U(y)$,*

$$U(y) = V(y) - \frac{1}{2}g_{ab}A^aA^b, \tag{1.56}$$

and also in an effective magnetic field $\tilde{F}_{ab}(y)$,

$$\tilde{F}_{ab} = s\cos\theta - \partial_1 A_2 + \partial_2 A_1, \quad A_a = g_{ab}A^b, \quad s = f_2 f_1^{-1/2}. \tag{1.57}$$

The form $A_a dy^a$ is globally defined on the sphere S^2, therefore

$$\iint_{S^2}\tilde{F}_{12}d\theta\wedge d\psi = \iint_{S^2}F = 4\pi s \tag{1.58}$$

by virtue of (1.46).

REMARK. It has recently become clear that on SO(4) there arise systems which in certain cases describe the motion of a rigid body with cavities filled with a fluid. The integrable cases here were found by V. A. Steklov [133] and have been rediscovered in a number of modern papers (see, for example, [19], [138]).

A number of other applications of Euler equations on Lie algebras in problems of mathematical physics have been found just lately by O. I. Bogoyavlenskij together with the integrable cases in these problems (see [18]).

Now let us consider infinite-dimensional examples of phase spaces—spaces of fields $u = (u^1(x), \dots, u^n(x))$ of some type, where $x = (x^1, \dots, x^m)$ is one of the indices in the formulas. The Poisson bracket is given by a matrix

$$\{u^i(x), u^j(y)\} = h^{ij}(x, y) \tag{1.59}$$

of functions $h^{ij}(x, y)$ (generalized functions), which in general depend on the fields. For two "functions" (functionals) $F[u], G[u]$ the Poisson bracket can be computed by the formula

$$\{F, G\} = \iint \frac{\delta F}{\delta u^i(x)}\, h^{ij}(x, y)\, \frac{\delta G}{\delta u^j(y)}\, d^m x\, d^m y. \tag{1.60}$$

Here $\delta F/\delta u^i(x)$ are variational derivatives, defined by the equalities

$$\delta F = \int \frac{\delta F}{\delta u^i(x)}\, \delta u^i(x)\, d^m x. \tag{1.61}$$

Example 1. The *local field-theoretic brackets* of Lagrangian variational problems. There are two sets of fields $u = (q^1(x), \dots, q^n(x), p_1(x), \dots, p_n(x))$ with pairwise Poisson brackets of the form

$$\begin{aligned} \{q^i(x), q^j(y)\} = \{p_i(x), p_j(y)\} = 0, \\ \{q^i(x), p_j(y)\} = \delta^i_j \delta(x - y), \quad i, j = 1, \dots, n. \end{aligned} \tag{1.62}$$

The Poisson bracket of two functionals F and G has the form:

$$\{F, G\} = \int \left[\frac{\delta F}{\delta q^i(x)}\, \frac{\delta G}{\delta p_i(x)} - \frac{\delta F}{\delta p_i(x)}\, \frac{\delta G}{\delta q^i(x)} \right] d^m x. \tag{1.63}$$

Hamilton's equations can be written in the form

$$\begin{aligned} \dot q^i(x) = \{q^i(x), \mathcal{H}\} = \frac{\delta \mathcal{H}}{\delta p_i(x)}, \\ \dot p_i(x) = \{p_i(x), \mathcal{H}\} = -\frac{\delta \mathcal{H}}{\delta q^i(x)}, \end{aligned} \tag{1.64}$$

where $\mathcal{H} = \mathcal{H}[p, q]$ is the Hamiltonian. They arise, in particular, from the field variational principle

$$\frac{\delta S}{\delta q^i} \equiv \frac{\partial \Lambda}{\partial q^i} - \partial_x \frac{\partial \Lambda}{\partial q^i_x} - \partial_t \frac{\partial \Lambda}{\partial q^i_t} = 0, \quad i = 1, \dots, n, \tag{1.65}$$

$$S = \int dt \int d^m x\, \Lambda(q, q_x, q_t), \tag{1.66}$$

where $\Lambda(q, q_x, q_t)$ is the density of the Lagrangian, with the aid of the field-theoretic version of the Legendre transformation

$$p_i = \frac{\partial \Lambda}{\partial q^i_t}, \quad \mathcal{H} = \int d^m x (p_i(x) q^i_t(x) - \Lambda) \tag{1.67}$$

(it is assumed, just as above in the finite-dimensional case, that the equations $p_i = \partial \Lambda(q, q_x, q_t)/\partial q^i_t$, can be solved for q^i_t).

One can also consider, by analogy with the finite-dimensional case, the distortion of the brackets (1.62) by a "magnetic field"—a closed 2-form on the space of fields $q(x)$. Let us analyze an example connected with the inclusion of "external fields" in the theory of *chiral fields*. As is well-known (see, for example, [115]), the definition of a nonlinear chiral field is as follows: one has arbitrary Riemannian manifolds N^q and M^n; let there be defined a functional $S_0(f)$ on the mappings $f\colon N^q \to M^n$. The functional $S_0(f)$ has the form of a Dirichlet functional, quadratic in the derivatives of the mapping f, possibly with some additional terms. Thus the standard "chiral Lagrangian" for a principal chiral field, where $M^n = G$ is a Lie group with a two-sided invariant metric, has the form:

$$S_0(f) = \frac{1}{2}\int_{N^q} \operatorname{tr}(g^{\mu\nu}A_\mu A_\nu)\sqrt{g}\,d^q y, \tag{1.68}$$

where $g_{\mu\nu}$ is the metric of N^q, $A_\mu = f^{-1}(y)\partial f(y)/\partial y^\mu$.

If the manifold N^q is presented in the form of a product $N^q = P^{q-1}\times\mathbb{R}$, where $\mathbb{R}$ is the axis of the time t and $y = (x,t)$ (for example, N^q is the Minkowski space $N^q = \mathbb{R}^{q-1,1}$), then the Euler–Lagrange equations for the action (1.68) can be brought into the Hamiltonian form by means of the Legendre transformation (1.67), $q(x) = f(x)$.

Now let us define the procedure for including an external field. Let us note beforehand that an arbitrary differential form ω of degree $q+r$ on the manifold M^n defines a differential r-form Ω_r on the space of mappings $\{N^q \xrightarrow{f} M^n\}$ via the formula

$$\Omega_r(\delta_1 f,\dots,\delta_r f)|_f = \int_{N^q} f^*(i_{\xi_1}\dots i_{\xi_r}\omega), \tag{1.69}$$

where

$$\xi_k(y) = \delta_k f(y), \quad k = 1,\dots,r \tag{1.70}$$

are "tangent vectors" to the space of mappings (vector fields on M^n at the points $f(y)$), $i_\xi\omega$ is the inner product of the form $\omega = (\omega_{i_1\dots i_{q+r}})$ with the vector $\xi = (\xi^i)$,

$$(i_\xi\omega)_{i_2\dots i_{q+r}} = \xi^i\omega_{ii_2\dots i_{q+r}}. \tag{1.71}$$

If the form ω is closed, $d\omega = 0$, then the form Ω_r on the infinite-dimensional space of mappings is also closed [112].

On M^n let us fix a closed $(q+1)$-form ω (the "external field"). Then it defines a closed 1-form Ω_1 on the space of mappings $N^q \xrightarrow{f} M^n$ in the way cited above. The closed 1-form

$$\delta S = \delta S_0 + \Omega_1, \tag{1.72}$$

where the functional S_0 is of the type (1.68), defines a so-called "*multi-valued functional*" S of the chiral field f in the external field ω [112]. The extremals of this functional can be determined, as usual, from the Euler–Lagrange equations (L. Euler–J. L. Lagrange)

$$\delta S = 0. \tag{1.73}$$

It turns out that for $N^q_y = P^{q-1}_x\times\mathbb{R}_t$ the inclusion of an external field is equivalent to distortion of the Poisson brackets by a "magnetic field" F—a closed 2-form on the space of fields $\{P^{q-1} \xrightarrow{f} M^n\}$—without changing the Hamiltonian. This 2-form $F = \Omega_2$ can be defined via (1.69) with N^q replaced by P^{q-1}.

EXAMPLE 2. More generally, the field-theoretic brackets (1.59) are called local if the generalized functions $h^{ij}(x,y)$ present themselves as finite sums of the delta function $\delta(x-y)$ and its derivatives with coefficients which depend on the values of the field variables and their derivatives at the points x, y. For these brackets and for local Hamiltonians of the form

$$\mathcal{H} = \int h(u, u_x, \dots, u^{(s)}) d^m x \tag{1.74}$$

the Hamilton equations $\dot{u} = \{u, \mathcal{H}\}$ can be written in the form of partial differential equations.

IMPORTANT EXAMPLE. The case $m = 1$, $n = 1$. Here one has a bracket (the C. Gardner–V. E. Zakharov–L. D. Faddeev bracket) which arises in the theory of the *Korteweg–de Vries equation* (D. J. Korteweg–G. de Vries) (KdV)

$$\{u(x), u(y)\} = \delta'(x-y). \tag{1.75}$$

The Poisson bracket of two functionals has the form:

$$\{F, G\} = \int \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} \, dx \tag{1.76}$$

The skew-symmetry of the brackets (1.75), (1.76) is obvious; the correctness of the Jacobi identity follows from the fact that the "tensor" h^{ij} is constant here (it does not depend on the field variables). The bracket (1.75) is degenerate; the functional $I_{-1} = \int u \, dx$ has vanishing bracket with any other functional F:

$$\{F, I_{-1}\} = 0. \tag{1.77}$$

On a subspace $I_{-1} = \int u \, dx = c$ (for example, $c = 0$) the bracket (1.75), (1.76) is no longer degenerate. The KdV equation itself is given by the Hamiltonian

$$I_1 = \mathcal{H} = \int \left(\frac{u_x^2}{2} + u^3 \right) dx, \tag{1.78}$$

$$u_t = \frac{\partial}{\partial x} \frac{\delta \mathcal{H}}{\delta u(x)} = 6uu_x - u_{xxx}. \tag{1.79}$$

The quantity

$$I_0 = \int \frac{u^2}{2} \, dx, \quad \{u(x), I_0\} = u_x(x), \tag{1.80}$$

plays the rôle of the momentum (the generator of the translations in x). It is curious that one of the manifestations of the integrability of the KdV equation (by the inverse scattering method, see Chap. 2 below) is the presence of another local bracket [94] of the form

$$\begin{aligned} \{F, G\} &= \int \frac{\delta F}{\delta u(x)} A \frac{\delta G}{\delta u(x)} \, dx, \\ A &= -\frac{d^3}{dx^3} + 2\left(u \frac{d}{dx} + \frac{d}{dx} u \right). \end{aligned} \tag{1.81}$$

There is even a family of brackets: one can replace the operator A by $A + \lambda(d/dx)$ (λ is an arbitrary constant). In the new Hamiltonian structure the KdV itself has the form

$$u_t = A \frac{\delta I_0}{\delta u(x)}. \tag{1.82}$$

EXAMPLE 3. Now let us consider continuous examples of the Lie–Poisson brackets, connected with infinite-dimensional Lie algebras L. The starting point for the subsequent constructions will be the Lie algebra L of vector fields on an m-dimensional space. The commutator of two fields $v^i(x), w^i(x)$ has the form:

$$[v,w]^i(x) = v^j(x)\,\frac{\partial w^i(x)}{\partial x^j} - w^j(x)\,\frac{\partial v^i(x)}{\partial x^j}. \tag{1.83}$$

The rôle of the index is played here by pairs (x,i)—a point x and an index i. The operation (1.83) should be written in terms of "structure constants" in the form

$$[v,w]^i(x) = \int d^m y\, d^m z c^i_{jk}(x,y,x) v^j(y) w^k(x). \tag{1.84}$$

By comparing (1.83) with (1.84) we obtain

$$c^i_{jk}(x,y,x) = \delta^i_j \delta(z-x)\partial^{(y)}_k \delta(y-z) - \delta^i_k \delta(y-x)\partial^{(z)}_j \delta(z-y), \tag{1.85}$$

$$\partial^{(x)}_j = \frac{\partial}{\partial x^j}, \quad \int f(z)\partial^{(z)}_j \delta(z-x)\, d^m(z) = -\frac{\partial f(x)}{\partial x^j}. \tag{1.86}$$

The variables $p_i(x)$ conjugate to the velocity components, on the dual space L^* to the vector fields $v^i(x)$, must be such that the quantity

$$\int p_i(x) v^i(x)\, d^m x \tag{1.87}$$

is scalar with respect to change of variables. This means that the variables $p_i(x)$ are covector densities, which under changes of variables are additionally multiplied by the Jacobian determinant (we shall call them momentum densities). By (1.85) the Poisson bracket has the form:

$$\begin{aligned}\{p_j(y), p_k(z)\} &= \int c^i_{jk}(x,y,z) p_i(x)\, d^m x \\ &= p_k(y)\partial^{(y)}_j \delta(y-z) - p_j(z)\partial^{(z)}_k \delta(z-y).\end{aligned} \tag{1.88}$$

In the important special case $m=1$ we get

$$\{p(y), p(z)\} = p(y)\delta'(y-z) - p(z)\delta'(z-y). \tag{1.89}$$

The substitution $p = u^2$ reduces this bracket to the bracket (1.75).

In the algebra L of vector fields on Euclidean space (where there is a distinguished Euclidean metric and the volume element is the mass density, which is considered to be constant) a subalgebra L_0 of divergence-free fields

$$\partial_i v^i = 0 \tag{1.90}$$

is given. The dual space L_0^* can be obtained by factoring by the gradients

$$L_0^* = L^*/(\partial_i \phi). \tag{1.91}$$

In other words, momentum densities $p_i(x)$ give trivial linear forms on L_0 if $p_i(x) = \partial_i \phi(x)$:

$$\int p_i v^i\, d^m x = \int v^i \partial_i \phi\, d^m x = -\int \phi \partial_i v^i\, d^m x = 0. \tag{1.92}$$

The Euler equations for the hydrodynamics of a perfect incompressible fluid can be written as a Hamiltonian system [7], [112] on the space L_0^* with the Hamiltonian

$$H = \int \frac{\rho v^2}{2}\, d^m x, \quad \rho = \text{const}, \quad \partial_i v^i = 0, \quad p_i = \rho v^i \tag{1.93}$$

and the Poisson brackets (1.88). One always writes these equations on the full space L^*, which is equivalent to the space of velocities in the given case

$$\begin{cases} \rho v_t^i = \{p_i, H\} + \partial_i p, \\ \partial_i v^i = 0. \end{cases} \tag{1.94}$$

The terms $\partial_i p$ have arisen because of the transition from L_0^* to the space L^*, where quantities of the form $\partial_i p$ are equivalent to zero. The pressure p is only defined up to a constant here. The Poisson bracket on the space L_0^* may be written in the form

$$\begin{gathered} \{v_i(x), v_j(y)\} = \frac{1}{\rho}(\partial_i v_j - \partial_j v_i)\delta(x-y), \\ p_i = \rho v_i, \quad \rho = \text{const}. \end{gathered} \tag{1.95}$$

The Hamiltonian formalism for a perfect compressible fluid cannot be realized on the algebra L; it represents a special case of the Hamiltonian formalism for fluids with internal degrees of freedom. Even the ordinary compressible fluid has such internal degrees of freedom-the mass density ρ and the entropy density s, whose inclusion requires the extension of the Lie algebra L of vector fields. Besides the vector fields v^i, we shall add another pair of fields v^ρ and v^s with commutators of the form

$$[(v, v^\rho, v^s), (w, w^\rho, w^s)] = ([u, w], v^i\partial_i w^\rho - w^i\partial_i v^\rho, v^i\partial_i w^s - w^i\partial_i v^s). \tag{1.96}$$

We shall denote the algebra (1.96) by $L_{\rho,s}$. The corresponding variables in the dual space $L_{\rho,s}^*$ we shall denote by ρ (the mass density) and s (the entropy density). The Poisson brackets in $L_{\rho,s}^*$ have the form:

$$\begin{gathered} \{p_i(x), \rho(y)\} = \rho(x)\partial_i\delta(y-x), \\ \{p_i(x), s(y)\} = s(x)\partial_i\delta(y-x), \\ \{\rho(x), \rho(y)\} = \{s(x), s(y)\} = \{\rho(x), s(y)\} = 0, \\ \{v_i(x), v_j(x)\} = \frac{1}{\rho}(\partial_i v_j - \partial_j v_i)\delta(x-y) \end{gathered} \tag{1.97}$$

(the velocities are here the covectors $v_i = p_i\rho^{-1}$). The Hamiltonian $H = \int [p^2/2\rho + \varepsilon(\rho, s)]\, d^m x$ is just the energy. The quantities $M = \int \rho\, d^m x$ and $S = \int s\, d^m x$ have vanishing Poisson brackets (the trivial conservation laws). Essentially the Poisson brackets (1.97) were appropriately chosen so that mass and entropy would be transported together with the particles, in contrast to the energy, which is conserved only as a whole. Other examples of Lie–Poisson brackets which arise in hydrodynamics can be found in [112].

EXAMPLE 4. General *brackets of hydrodynamic type*. The Poisson brackets and Hamiltonians considered in the previous example have the following properties:

1) The Hamiltonians have the form:

$$H = \int h(u)\, d^m x, \tag{1.98}$$

where the densities $h(u)$ depend only on the fields $u = (u^1, \dots, u^n)$ and not on their derivatives.

2) The Hamilton equations

$$u_t^i(x) = \{u^i(x), H\} \tag{1.99}$$

are first-order quasilinear equations

$$u^i_t = v^i_j(u)u^j_x, \quad i = 1, \dots, n. \tag{1.100}$$

The most general form for Poisson brackets which lead to the equations (1.100) for Hamiltonians (1.98) is as follows:

$$\{u^i(x), u^j(y)\} = g^{ij\alpha}(u(x))\,\frac{\partial\delta(x-y)}{\partial y^\alpha} + b^{ij\alpha}_k(u(x))\,\frac{\partial u^k(x)}{\partial x^\alpha}\,\delta(x-y). \tag{1.101}$$

The form of equations (1.100), the Hamiltonians (1.98) and the brackets (1.101) is invariant with respect to local changes of field variables

$$u = u(w). \tag{1.102}$$

Let us consider here the one-dimensional case $m = 1$:

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta_y(x-y) + b^{ij}_k(u(x))u^k_x(x)\delta(x-y) \tag{1.103}$$

In this case the symmetric matrix $g^{ij}(u) = g^{ji}(u)$ behaves under the changes (1.102) like a metric (with upper indices) on the space of fields u. If it is non-degenerate, then the quantities Γ^i_{jk}, defined by the equalities

$$b^{ij}_k = -g^{is}\Gamma^j_{sk}, \tag{1.104}$$

transform under the changes (1.102) like Christoffel symbols (see [48]). It turns out [48] that the expression (1.103) gives a Poisson bracket if and only if the connection Γ^i_{jk} is symmetric, compatible with the metric g^{ij} and has zero curvature. This means that by local coordinate changes (1.102), the metric g^{ij} can be reduced to the Euclidean (or a pseudo-Euclidean) one, the connection to zero, and the bracket (1.103), by the same token, to a constant one:

$$\{w^i(x), w^j(y)\} = \pm\delta^{ij}\delta'(x-y). \tag{1.105}$$

It should be noted that the natural "physical" variables u^i in which the equations (1.100) and the brackets (1.103) arise are essentially "curvilinear", i. e. the metric $g^{ij}(u)$ is nontrivial in the coordinates u^i.

In the multidimensional case $m > 1$ a family of metrics $g^{ij\alpha}$, $\alpha = 1, \dots, m$ actually arises. If they are non-degenerate, then the connections $\Gamma^{i\alpha}_{jk}$, where $b^{ij\alpha}_k = -g^{is\alpha}\Gamma^{j\alpha}_{sk}$, are compatible with these metrics, symmetric, and have zero curvature. However all the metrics $g^{ij\alpha}$ cannot as a rule be reduced to a constant form by a single transformation. The obstruction to such a reduction are the tensors $T^{ijk\alpha\beta} = b^{ij\alpha}_l g^{lk\beta} - b^{kj\beta}_l g^{li\alpha}$. For example, for the brackets (1.88) when $m > 1$ such a reduction is impossible. Let us also note that for $m > 2$ the metrics $g^{ij\alpha} = p_s(\delta^{is}\delta^{j\alpha} - \delta^{js}\delta^{i\alpha})$ corresponding to the brackets (1.88) are always degenerate. For non-degenerate metrics $g^{ij\alpha}$ the conditions under which the expression (1.101) gives a Poisson bracket can be written as a set of relations on the tensors $T^{ijk\alpha\beta}$ that we shall not discuss here (see [49]).

The set of relations in [49] is incomplete and incorrect for $N \le 2$. The complete set of relations can be found in [Mo].

In any case, the following theorem is true: if the metric $g^{ij,1}$ is non-degenerate and is reduced to a constant metric, then all other metrics $g^{ij,\alpha}$ are linearly dependent on the coordinates (u^i) (see [49] for $N > 2$ and [Mo] for $N = 2$). Thus, the classification of Poisson brackets of hydrodynamic type is reduced to certain questions in the theory of special infinite dimensional Lie algebras. These questions are

discussed below. Poisson brackets of hydrodynamic type play a fundamental role in the theory of the Hamiltonian systems of hydrodynamic type and, in particular, in the study of the effective integrability of such systems in the case of one spatial variable (see Appendix below).

As was mentioned at the beginning of §1, if the coefficients of the Poisson brackets are linearly dependent on the "fields-coordinates" $\{u(x)\}$, then the "small" Lie algebras form a background of a "big" algebra of Poisson brackets. Such brackets are called the Lie–Poisson–Beresin–Kirillov–Kostant brackets and the systematic development of their theory was initiated in [GDo] and [BN].

We now restrict our attention to the spatially one-dimensional case ($m = 1$) and, following [BN], consider hydrodynamic type Poisson brackets of the form (1.103)

$$g^{ij} = C_k^{ij} u^k, \quad C_k^{ij} = b_k^{ij} + b_k^{ji}, \quad b_k^{ij} = \text{const}.$$

Let B be an algebra with a basis $e^1, \dots, e^N$ and multiplication

$$e^i e^j = b_k^{ij} e^k.$$

Then the structure of an infinite dimensional Lie algebra $\mathcal{L}_B$ on the space of B-valued vector-functions of a single variable x is defined as follows:

$$[p(x), q(x)] = p'q - q'p, \quad p(x), q(x) \in \mathcal{L}_B, \quad x \in S^1,$$

where $p(x) = e^i p_i(x)$, $q(x) = e^i q_i(x)$, and the multiplication law is induced by the multiplication in B. This is a general form of *spatially one-dimensional translation-invariant Lie algebras of the first order* (i. e. depending only on the first derivatives w. r. t. x). The Jacobi identity for $\mathcal{L}_B$ is equivalent to the Jacobi identity for the Poisson bracket of hydrodynamic type (1.103) with $m = 1$ and $g^{ij} = C_k^{ij} u^k$. In particular, if the metric g^{ij} is non-degenerate at the point $u = u_0$, i. e., $\det(g^{ij}(u_0)) \neq 0$, then this metric is flat. Moreover, the Jacobi identity is equivalent to the following two identities in the algebra B:

$$[L_a, L_b] = 0, \quad [R_b, R_c] = R_{bc-cb},$$

where $L_a(b) = R_b(a) = ab$.

Such finite-dimensional algebras B over $\mathbb{R}$ or $\mathbb{C}$ cannot be simple for $N > 1$ (see [Z]). Lie algebras $\mathcal{L}_B$ have large families of central extensions

$$O \to \mathbb{R} \to \hat{\mathcal{L}}_B \to \mathcal{L}_B \to 0$$

defined by cocycles of the form

$$\langle p, q \rangle = \oint_{S^1} \gamma_{(k)}^{ij} p_i^{(k)}(x) q_j(x)\, dx,$$

where (k) is the number of differentiations w. r. t. x and $\gamma_{(k)}^{ij} = \text{const}$.

In the case $B = \mathbb{R}$ there is only one non-trivial "Gelfand–Fuks" cocycle for $k = 3$ (see [GF]). In the general case, there are many non-trivial cocycles for $k = 0, 1, 2, 3$ ([BN]), but their general theory has not been developed. In the important case $b_k^{ij} = b_k^{ji}$ we obtain commutative associative algebras B. If $\det g^{ij} \neq 0$, we obtain Frobenius algebras ([BN]). In this case, it seems that the only non-trivial cocycles are cocycles of order 3 at some point (u_0), i. e., $\gamma_{(3)}^{ij} = C_k^{ij} u_0^k$. This is the only case where the flat coordinates for the metric $g^{ij} = C_k^{ij} u^k$ have been effectively constructed. In general, this problem is solved only in a number of special cases connected with topological quantum field theories [D1], where the theory of Poisson

brackets and systems of hydrodynamic type recently began to play an interesting role (see [D1] and Appendix to [N1]).

Non-local Poisson brackets of hydrodynamic type have the form ([MF], [F]):

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + b_k^{ij}(u)u_x^k\delta(x-y) + \sum_{\alpha=1}^{L}(w^\alpha)_k^i u_x^k \partial^{-1}[(w^\alpha)_s^j(u)u_x^s\delta(x-y)]$$

Here the metric g^{ij} is no longer required to be flat. If $L=1$ and $(w^1)_j^i = c\delta_j^i$, then the curvature is constant. Such brackets are important for the theory of integrable systems of hydrodynamic type (see Appendix). Also see [F] for general theorems.

There is also a higher order analogue of Poisson brackets of hydrodynamic type. These brackets are called "homogeneous differential-geometric brackets of order k" and they have the following form (see [N2]):

$$\begin{aligned}\{u^i(x), u^j(y)\} = {} & g^{ij}(u(x))\delta^{(k)}(x-y) + b_k^{ij}(u)u_x^k\delta^{(k-1)}(x-y) \\ & + [c^{ij}u_{xx}^k + d_{kl}^{ij}u_x^k u_x^l]\delta^{(k-2)}(x-y) + \dots \\ & + [e_s^{ij}u_{\dots x}^k + \dots]\delta(x-y),\end{aligned}$$

where the functions g, b, c, d depend only on u. The theory of such brackets for $k=2$ was constructed in [P1]. The connection $\Gamma_{jk}^i = g_{js}e_k^{is}$ has zero curvature and zero torsion (see the result of Potemin in [N2], also see [D]). For $k=2$, we have the following result: if $\det(g^{ij}(u)) \neq 0$, then the form $g_{ij}du^i \wedge du^j$ is closed if and only if the Poisson bracket can be reduced to a constant bracket.

We would also like to make the following remark about the **Important Example** (above). The theory of integrable systems led to the generalizations of the Gardner–Zakharov—Faddeev bracket (1.75) and the second "Magri bracket" (1.81). Recall that these brackets appear in the study of the KdV hierarchy.

The higher order analogue of the bracket (1.75) has the form [34]:[3]

$$\{u^i(x), u^j(y)\}_1 = M^{ij} \circ \delta(x-y).$$

where M^{ij} is defined by the relation

$$\sum_{i,j=0}^{n-1} M^{ij}\eta_j\partial_s^i = ([X_\eta, L])_+$$

for

$$X_\eta = \sum_{s=0}^{n-1}\partial_x^{-s-1}\circ\eta_s, \qquad L = \partial_x^n + \sum_{k=0}^{n-1}u^k\partial_x^k,$$

and is equal to

$$M^{ij} = \sum_{k-i+j-1}^{n}\left[\binom{k-j-1}{i}u^k\partial_x^{k-i-j-1} - \binom{k-i-1}{j}(-\partial_x^{k-i-j-1})\circ u^k\right].$$

[3]The notation $(\ )_+$ will be explained below (Insertion 2).

The higher order analogue of the bracket (1.81) has the form [2]:

$$\{u^i(x), u^j(y)\}_2 = K^{ij} \circ \delta(x-y), \quad \text{where}$$
$$\sum_{i,j=0}^{n-1} (K^{ij}\eta_j)\partial_x^i = L(X_\eta L)_+ - (LX_\eta)_+ L,$$

and this must be satisfied for any $\eta = (\eta_0, \dots, \eta_{n-1})$.

It is remarkable that the higher order analogues of the second bracket turned out to play an important role in the conformal field theory. They can be interpreted as classical limits of "Zamolodchikov W-algebras" [FL]. Because of this connection with two-dimensional conformal quantum field theories, these Poisson algebras are often called "classical W-algebras".

EXAMPLE 5. Consider one more fundamental class of Poisson brackets related to the theory of integrable systems—The Yang–Baxter brackets. Let the tensor $r\colon V \otimes V \to V \otimes V$ satisfy the "linearized Yang–Baxter equation":

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where $r_{12} = r\colon V \otimes V \otimes 1 \to V \otimes V \otimes 1$ and r_{13}, r_{23} are defined similarly. Then on a space with coordinates t^i_j one can define a Poisson bracket by the following formula:

$$\{t^i_k, t^j_l\} = r^{ij}_{ab} t^a_k t^b_l = t^i_a t^j_b r^{ab}_{kl}.$$

These brackets are very useful for the integration of certain integrable systems in the soliton theory. See [FT] for the detailed discussion on the connection between the theory of the linearized Yang–Baxter equation and the soliton theory and for a large list of references on the subject. Some useful information and references can be found in [DN1].

The connection between the linearized Yang–Baxter equation and another class of Poisson brackets—difference analogues of Poisson brackets of hydrodynamic type—was studied in [D2]. A summary of the results of [D2] can also be found in [DN1].

2. Integrals and Reduction of the Order of Hamiltonian Systems. Systems with Symmetry

A function $F(y)$ is called an *integral of the Hamiltonian system* (1.8) if its bracket with the Hamiltonian $H(y)$ is equal to zero:

$$\{F, H\} = 0. \tag{1.106}$$

Taking (1.11) into account, we get: the quantity F is conserved along the trajectories of the Hamiltonian system (1.8). In particular, the Hamiltonian H itself (if it does not depend on time) is always a conserved quantity. The trajectories of the system (1.8) lie entirely on a level surface $F = \text{const}$. If the Poisson bracket is degenerate, then there are always "trivial" integrals (1.13), which commute with any Hamiltonian. We have looked at an example of the reduction of the Hamiltonian formalism with the help of trivial integrals in §1 in connection with equations of the Kirchhoff type. Hamiltonian systems with one degree of freedom ($N = 2$) with a time-independent Hamiltonian can always be integrated by quadratures. The presence of non-trivial integrals when $N > 2$ which do not depend on the "energy"

H [4] allows one to reduce the order of the Hamiltonian system (1.8) by two all at once. Let us give the appropriate construction (see also vol. 3 of the current publication and Chap. 3, §3 of the article by V. I. Arnol'd and A. B. Givental'). Let $F(y)$ be an integral of a Hamiltonian system with Hamiltonian H, where the vector $(\xi_F^i(y)) = (h^{ij}(y)(\partial F(y)/\partial y^j))$ is independent of ξ_H. Let us consider a level surface M_c:

$$F(y) = c, \tag{1.107}$$

and on it the Hamiltonian flow defined by the Hamiltonian $F(y)$,

$$y_\tau^i = \{y^i, F(y)\}, \quad i = 1, \dots, N. \tag{1.108}$$

The flow (1.8) with Hamiltonian H permutes the trajectories of the flow (1.108) by virtue of the commutation (1.106), and therefore defines a dynamical system on the set of trajectories of the flow (1.108). The trajectories of the flow (1.108) lying on the level M_c are "indexed" by the points of a surface M_c^0 (in general defined locally) transversal to these trajectories. Let us define the "reduction operation" of the original Poisson bracket $h^{ij}(y)$ onto M_c^0. Let us consider the subalgebra of all functions $z(y)$ which commute with $F(y)$,

$$\{F(y), z(y)\} = 0. \tag{1.109}$$

Let the independent functions $z^1(y), \dots, z^{N-2}(y)$ satisfy (1.109) and not functionally depend on $F(y)$. They are constant along the trajectories of the flow (1.108) and together with $F(y)$ and τ they define local coordinates in a neighbourhood of the transverse surface M_c^0. By the same token, the quantities $z^1, \dots, z^{N-2}$ give local coordinates on the transverse surface M_c^0. We obviously have

$$\{\tau, F\} = 1, \quad \{\tau, z^q\} = f^q(z, F), \quad \{z^p, z^q\} = \tilde{h}^{pq}(z, F). \tag{1.110}$$

Therefore we may impose on the choice of coordinates $z^1, \dots, z^{N-2}$ the useful additional conditions

$$\{\tau, z^q\} = 0, \quad q = 1, \dots, N-2. \tag{1.111}$$

The reduced Poisson bracket on M_c^0 has by definition the form

$$\{z^p, z^q\}_{\text{red}} = \{z^p(y), z^q(y)\} = \tilde{h}^{pq}(z, c), \quad p, q = 1, \dots, N-2. \tag{1.112}$$

Obviously, the right-hand side depends only on the coordinates on M_c^0 (and on c), and does not depend on the choice of the surface M_c^0. By virtue of (1.106) the Hamiltonian has the form

$$H(y) = \tilde{H}(z^1, \dots, z^{N-2}, F), \tag{1.113}$$

and therefore the original Hamiltonian system has a well-defined restriction to M_c^0:

$$\dot{z}^q = \{z^q, \tilde{H}(z, c)\}_{\text{red}}, \quad q = 1, \dots, N-2. \tag{1.114}$$

Thus, integration of the original system (1.8) is reduced to the integration of the Hamiltonian system (1.114), whose order has been lowered by two. After this the dependence of the coordinate τ on time can be determined from the equation (taking (1.110), (1.111) into account)

$$\dot{\tau} = \{\tau, \tilde{H}(z, F)\} = \frac{\partial \tilde{H}(z, F)}{\partial F} \tag{1.115}$$

(by one quadrature).

[4] With the aid of the energy integral H the order of the system can also be reduced by two, but the Hamiltonian of the reduced system will depend explicitly on time [7].

Whether it is possible to carry out the reduction procedure globally requires a supplementary investigation. It is sufficient, for example, to suppose that c is a regular value of the function $F(y)$ and the one-parameter group G_t of translations along the trajectories of the system (1.108) is compact and has no fixed points. In the practical realization of the procedure described above the main difficulty lies in the construction of the "transversal" coordinates $z^1, \dots, z^{N-2}$.

EXAMPLE 1. Let $H = H(x, p)$ be a Hamiltonian on the phase space $\mathbb{R}^{2n}$ with canonical coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ of the form (1.19). Let us suppose that $H(x, p)$ is invariant with respect to "spatial translations"

$$x^i \mapsto x^i + a, \quad p_i \mapsto p_i, \tag{1.116}$$

i. e.

$$\sum_{i=1}^{n} \frac{\partial H}{\partial x^i} = 0. \tag{1.117}$$

For this it is sufficient that the Hamiltonian have the form

$$H(x^1 \dots, x^n, p_1, \dots, p^n) = \hat{H}(x_1 - x^n, \dots, x^{n-1} - x^n, p_1, \dots, p_n). \tag{1.118}$$

Then, obviously, the quantity ("total momentum")

$$F = \sum_{i=1}^{n} p_i \tag{1.119}$$

commutes with H, $\{H, F\} = 0$. The coordinates $z = (z^1, \dots, z^{2n-2}) = (\tilde{x}^q, \tilde{p}_q)$ on the reduced phase space have the form

$$\begin{gathered} \tilde{x}^q = x^q - x^n, \quad q = 1, \dots, n-1, \\ \tilde{p}_q = p_q, \quad q = 1, \dots, n-1. \end{gathered} \tag{1.120}$$

The reduced Hamiltonian $\tilde{H}(\tilde{x}, \tilde{p}; c)$ on the surface

$$p_1 + \dots + p_n = c \tag{1.121}$$

has the form

$$\tilde{H}(\tilde{x}, \tilde{p}; c) = \hat{H}\left(\tilde{x}^1, \dots, \tilde{x}^{n-1}, \tilde{p}_1, \dots, \tilde{p}_{n-1}, c - \sum_{q=1}^{n-1} \tilde{p}_q\right). \tag{1.122}$$

The reduced brackets are the canonical ones:

$$\begin{gathered} \{\tilde{x}^q, \tilde{x}^r\}_{\text{red}} = \{\tilde{p}_q, \tilde{p}_r\}_{\text{red}} = 0, \quad \{\tilde{x}^q, \tilde{p}_r\} = \delta^q_r, \\ q, r = 1, \dots, n-1, \end{gathered} \tag{1.123}$$

and the original Hamiltonian system reduces to the system on $\mathbb{R}^{2n-2}$

$$\dot{\tilde{x}}^q = \frac{\partial \tilde{H}}{\partial \tilde{p}_q}, \quad \dot{\tilde{p}}_q = -\frac{\partial \tilde{H}}{\partial \tilde{x}^q}, \quad q = 1, \dots, n-1. \tag{1.124}$$

The dependence of the quantity $\tau = x^n$ on the time t can be found from the equation

$$\dot{\tau} = \tilde{H}_c. \tag{1.125}$$

Now let us suppose that the Hamiltonian system (1.8) with the Hamiltonian H possesses several integrals. Let us note, first of all, a simple but important assertion: the integrals of the system (1.8) form a subalgebra with respect to the Poisson bracket. The proof is obvious from the Jacobi identity (1.5).

The presence of nontrivial pairwise commuting integrals $F_1(y), \dots, F_k(y)$,

$$\{F_i, H\} = 0, \quad \{F_i, F_j\} = 0, \quad i, j = 1, \dots, k, \tag{1.126}$$

allows one, according to the scheme described above, to reduce the order of the Hamiltonian system by $2k$. In particular, if the initial Poisson bracket was non-degenerate, $N = 2n$, then the presence of n pairwise commuting integrals for the Hamiltonian system allows one, in principle, to integrate this system by quadratures. We shall discuss the properties and examples of such systems in §3.

A set of non-commuting integrals also allows one to reduce the order of the original Hamiltonian system; however, here the reduction algorithm is more complicated. Let us first analyze a simple example.

EXAMPLE 2. Let $H(x,p) = (|p|^2/2m) + U(|x|)$ be a spherically symmetric Hamiltonian on $\mathbb{R}^6$, $x = (x_1, x_2, x_3)$, $p = (p_1, p_2, p_3)$. Here one has the three "angular momentum integrals"

$$M_1 = x_2p_3 - x_3p_2, \quad M_2 = x_3p_1 - x_1p_3, \quad M_3 = x_1p_2 - x_2p_1 \tag{1.127}$$

with pairwise brackets

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k. \tag{1.128}$$

(In fact, the whole angular momentum vector

$$M = [x, p] \tag{1.129}$$

is conserved.) Here there are three integrals, but because of their non-commutativity the reduced phase space will have dimension 2. Let us fix a value of the angular momentum

$$M = m \neq 0. \tag{1.130}$$

Without loss of generality we may assume that $m = (\mu, 0, 0)$, and the conditions (1.130) can be written in the form

$$x_2p_3 - x_3p_2 = \mu, \quad x_3p_1 - x_1p_3 = 0, \quad x_1p_2 - x_2p_1 = 0. \tag{1.131}$$

From the last two equations it follows when $\mu \neq 0$ that $x_1 = p_1 = 0$, i. e. the motion takes place in the (x_2, x_3) plane. The flow with the Hamiltonian M_1, which represents a rotation in the (x_2, x_3) and (p_2, p_3) planes by the same angle, thus acts on the three-dimensional surface $x_1 = p_1 = 0$, $x_2p_3 - x_3p_2 = \mu$. If we factor by this flow, we obtain the desired reduced phase space. For the factorization it is most convenient to use polar coordinates r, ϕ in the (x_2, x_3) plane, putting

$$x_2 = r\cos\phi, \quad x_3 = r\sin\phi \tag{1.132}$$

and introducing the conjugate momenta

$$\begin{aligned} p_2 &= p_r\cos\phi - \frac{p_\phi}{r}\sin\phi, \\ p_3 &= p_r\sin\phi + \frac{p_\phi}{r}\cos\phi. \end{aligned} \tag{1.133}$$

Then r, p_r serve as canonical coordinates on the reduced phase space; the reduced Hamiltonian has the form

$$\tilde{H}(r, p_r) = \frac{p_r^2}{2m} + \frac{\mu^2}{2mr^2} + U(r). \tag{1.134}$$

The dependence of ϕ on time is obtained separately from the equation $p_\phi = \mu$, from which we get

$$\dot{\phi} = \frac{\mu}{mr^2} = \frac{\partial H}{\partial p_\phi}. \tag{1.135}$$

The construction of example 2 admits of obvious generalizations, which go back to Jacobi and H. Poincaré and have been formulated in the language of symplectic manifolds in a number of works in the last decades (see also vol. 3 of the present publication and Chap. 3, § 3 of the article by V. I. Arnol'd and A. B. Givental').

Suppose the Hamiltonian system (1.8) possesses integrals $F_1, \ldots, F_r$ whose pairwise brackets can be expressed as linear combinations of these same functions,

$$\{F_i, F_j\} = c_{ij}^k F_k. \tag{1.136}$$

the coefficients c_{ij}^k being constants (this is the next case in order of complexity after commuting integrals; compare the formulas (1.128) of example 2). Thus the space of linear combinations

$$L = \{a^i F_i(y)\}, \quad a^1, \ldots, a^r \text{ are constants}, \tag{1.137}$$

is closed with respect to the Poisson bracket and for this reason forms a finite-dimensional Lie algebra. The functions $F_1(y), \ldots, F_r(y)$ form a basis of L, and the c_{ij}^k are the structure constants. Let G be the corresponding Lie group. Then G acts locally on the phase space by canonical transformations (ones which preserve Poisson brackets): the one-parameter subgroups of G which correspond to the basis vectors F_i of the Lie algebra L are the Hamiltonian flows

$$y_\tau^j = \{y^j, F_i\}, \quad j = 1, \ldots, N. \tag{1.138}$$

For a fixed y the collection of numbers $(F_1(y), \ldots, F_r(y)) = F(y)$ may be considered as the coordinates of a linear form on the Lie algebra L: if $(a^1, \ldots, a^r)$ is a vector in L, then

$$F(y)(a) = a^i F_i(y). \tag{1.139}$$

Thereby we have defined the *momentum mapping*

$$y \mapsto F(y) \in L^*. \tag{1.140}$$

Let us fix some element $c = (c_1, \ldots, c_r) \in L^*$ and let us consider the momentum level surface (the simultaneous level surface of the integrals $F_1, \ldots, F_r$)

$$F(y) = c \leftrightarrow (F_1(y) = c_1, \ldots, F_r(y) = c_r). \tag{1.141}$$

Let us suppose that this surface M_c is a manifold. The Hamiltonian flow corresponding to the Hamiltonian $f_a(y) = a^i F_i(y)$ preserves the level surface M_c if the vector $a = (a^1, \ldots, a^r)$ satisfies the linear relations

$$\{f_a, F_j\}|_{M_c} = a^i c_{ij}^k c_k = 0, \quad j = 1, \ldots, r. \tag{1.142}$$

Such vectors a form a Lie subalgebra $L_c \subset L$. Let l be the dimension of this subalgebra; a basis of it is constituted by the functions

$$f_s(y) = a_s^i F_i(y), \quad s = 1, \ldots, l, \tag{1.143}$$

where (a_s^i) is a fundamental system of solutions of the equations (1.142). (The subgroup $G_c \subset G$ with the Lie algebra $L_c \subset L$ is just the isotropy subgroup of the element $c \in L^*$ in the coadjoint representation Ad^*. In example 2 (above) the subgroup G_c coincided with the rotations about the $c = (\mu, 0, 0)$ axis.)

If we factor M_c by the action of the flows with the Hamiltonians f_a out of the subalgebra L_c, we obtain the reduced phase space M_c^0 (of codimension $r + l$). (All M_c is fibred (locally) over M_c^0 with fibre G_c). As coordinates on M_c^0 one may take functions $z^q = z^q(y)$ such that

$$\{z^q, F_j\}|_{M_c} = 0, \quad j = 1, \dots, r, \tag{1.144}$$

which do not depend on the functions $f_1, \dots, f_l$ (1.143) (the gradients of the functions z^q and f_s must generate the whole tangent space to M_c). As above, we define the reduced brackets on M_c^0 by the equality

$$\{z^q, z^p\}_{\mathrm{red}} = \{z^q(y), z^p(y)\}. \tag{1.145}$$

The Hamiltonian also restricts to M_c^0 in a well-defined way. We obtain the reduced Hamiltonian system

$$\dot{z}^q = \{z^q, \tilde{H}(z, c)\}_{\mathrm{red}}, \quad q = 1, \dots, N - l - r. \tag{1.146}$$

It is clear that in the commutative case $c_{ij}^k = 0$ the subalgebra L_c coincides with the whole Lie algebra L, i. e. $l = r$ and the order of the system is reduced by $2r$.

We have not yet discussed the mechanisms by which integrals of Hamiltonian systems arise. The best known of these mechanisms is a *symmetry of the Hamiltonian system*, i. e. the presence of a continuous group G of canonical transformations of the phase space, preserving the Hamiltonian:

$$H(gy) = H(y), \quad g \in G. \tag{1.147}$$

Let L be the Lie algebra of the group G, let $e_1, \dots, e_r$ be a basis of L, and let the commutators in L have the form:

$$[e_i, e_j] = c_{ij}^k e_k \tag{1.148}$$

Each one-parameter subgroup $\exp(te_i)$ of transformations of the phase space has a Hamiltonian $F_i(y)$ (defined perhaps locally), i. e. the transformations $y \mapsto \exp(\tau e_i)y$ are translations along the trajectories of the system

$$y_\tau^j = \{y^j, F_i(y)\}, \quad j = 1, \dots, N. \tag{1.149}$$

The functions $F_i(y)$ are integrals of the Hamiltonian system (1.147). Indeed,

$$\{H(y), F_i(y)\} = \frac{d}{d\tau} H(\exp(\tau e_i)y)_{\tau=0} = 0.$$

The functions $F_i(y)$ are called *generators of the canonical action of* G.

A canonical *Lie group action* is called *Poisson* if the functions $F_i(y)$ are defined globally and their Poisson brackets have the form (1.136), where the c_{ij}^k are the structure constants (1.148) of the Lie algebra L.

Example. Let the phase space have the form of the cotangent bundle T^*M of a smooth n-dimensional manifold M with the standard brackets (1.30) and let G act on M as a group of diffeomorphisms. The corresponding action of G on T^*M is

canonical. Let us construct the functions $F_i(y)$, where $y = (x, p)$ are the canonical coordinates on T^*M (locally). Let

$$X_i^k(x) = \frac{d}{d\tau}\left(\exp(\tau e_i)x\right)_{\tau=0}^k, \quad k = 1, \dots, n. \tag{1.150}$$

Let us put

$$F_i(x, p) = p_k X_i^k(x), \quad i = 1, \dots, r. \tag{1.151}$$

The functions $F_i(x, p)$ are defined globally on T^*M and are generators of the action of G. Their Poisson brackets, as is easy to see, have the form (1.136). Thus the action of G on T^*M is Poisson.

For an arbitrary phase space a canonical action of G might not be Poisson. In the first place, even if the Poisson bracket is non-degenerate, the integrals $F_i(y)$ might not be globally defined (and single-valued); only their differentials dF_i are well-defined. Let us suppose further that the functions F_i are defined globally (up to a constant). It is not difficult to show that then their Poisson brackets have the form:

$$\{F_i, F_j\} = c_{ij}^k F_k + b_{ij}, \tag{1.152}$$

where the c_{ij}^k are the structure constants of the Lie algebra L, and the $b_{ij} = -b_{ji}$ are certain constants. The skew-symmetric matrix b_{ij} defines a bilinear form on the Lie algebra L, $B(\xi, \eta) = b_{ij}\xi^i\eta^j$, which is a (two-dimensional) cocycle:

$$B([\xi, \eta)], \zeta) + B([\zeta, \xi], \eta) + B([\eta, \zeta], \xi) = 0 \tag{1.153}$$

(a consequence of the Jacobi identity (1.5)). For the action of the group G to be Poisson it is necessary that the matrix b_{ij} should have the form (β_k are certain constants):

$$b_{ij} = \beta_k c_{ij}^k \tag{1.153$'$}$$

(the cocycle b_{ij} is cohomologous to zero). In this case, by substituting $F_j \mapsto F_j + \beta_j$ we obtain a Poisson action.

If the action of the group G on the phase space is Poisson, then the reduction procedure described above for the Hamiltonian formalism can be carried out globally under certain additional restrictions. It is sufficient, for example, to suppose that c is a regular value for the momentum mapping (1.140) (i. e., M_c is a manifold), the isotropy subgroup G_c of the element $c \in L^*$ with respect to the coadjoint representation Ad^* is compact, and its elements act on M_c without fixed points. Thus, for the case T^*M, where the group G acts on M, the reduced phase space has the form $T^*(M/G)$, if of course the quotient manifold M/G is defined.

Example 3 ([112]). *A. J. Leggett's equations* for the dynamics of the "order parameters" in the B-phase of superfluid ^{3}He. In the state of hydrodynamical rest and with non-zero spin a state in the B-phase is defined by a pair—a rotation matrix $R = (R_{ij}) \in \mathrm{SO}(3)$ and $s = (s_i)$, $i = 1, 2, 3$,—the "magnetic moment".

The variables s_i represent coordinates on the dual space to the Lie algebra of the group $\mathrm{SO}(3)$, analogously to the angular momenta M_i. In the variables (s_i, R_{jk}) the standard Poisson brackets on $T^*\mathrm{SO}(3)$ are written thus:

$$\{s_i, s_j\} = \varepsilon_{ijk}s_k, \quad \{R_{ij}, R_{kl}\} = 0, \quad \{s_i, R_{jl}\} = \varepsilon_{ijk}R_{kl}. \tag{1.154}$$

The Hamiltonian of the Leggett system in the B-phase and in an external magnetic field has the form:

$$H = \tfrac{1}{2}as^2 + b\sum s_i F_i + V(\cos\theta), \tag{1.155}$$

where a, b are constants, $F = (F_i)$ is the external field,

$$V(\cos\theta) = \text{const}(\tfrac{1}{2} + 2\cos\theta)^2; \tag{1.156}$$

here R_{ij} is the rotation by the angle θ about the axis of the vector (n_i), $\sum n_i^2 = 1$:

$$R_{ij} = \cos\theta\delta_{ij} + (1-\cos\theta)n_i n_j + \sin\theta\varepsilon_{ijk}n_k, \tag{1.157}$$

$$1 + 2\cos\theta = R_{ij} = \operatorname{tr} R. \tag{1.158}$$

After the substitution

$$as_i = \omega_i, \quad \Omega_{jk} = \varepsilon_{jki}\omega_i = (\dot{R}R^{-1})_{jk} \tag{1.159}$$

we will obtain a Lagrangian system in the variables $(R_{ij}, \dot{R}_{ij})$ on $T^*\mathrm{SO}(3)$, where the kinetic energy is defined by a two-sided invariant Killing metric, and the potential $V(\cos\theta)$ is invariant with respect to inner automorphisms

$$R \mapsto gRg^{-1}, \quad s \mapsto gs, \quad g \in \mathrm{SO}(3). \tag{1.160}$$

If the field $F = (F_i)$ is constant, then the whole Hamiltonian is invariant with respect to the one-parameter group of transformations (1.160), where g belongs to the group of rotations around the axis of the field F. Suppose, further, $F = (F, 0, 0)$.

With zero flux $F = 0$ the system admits the group $\mathrm{SO}(3)$ of transformations (1.160) and is completely integrated in [95]. The transformations (1.160) generate the vector, conserved when $F = 0$:

$$A = (A_j) = (1-\cos\theta)\left[n, \cot\frac{\theta}{2}S + [n, S]\right], \tag{1.161}$$

where the Poisson brackets are the same as for the ordinary angular momentum

$$\begin{gathered}\{A_i, A_j\} = \varepsilon_{ijk}A_k,\\ \{A_i, \tfrac{1}{2}as^2 + V(\cos\theta)\} = 0.\end{gathered} \tag{1.162}$$

The variables s^2 and θ which enter into the Hamiltonian when $F = 0$ generate the closed Poisson bracket algebra $\{s^2, s_\parallel, \theta\}$ [112], where

$$s_\parallel = \sum s_i n_i, \tag{1.163}$$

$$\{s^2, \theta\} = 2s_\parallel, \quad \{s_\parallel, \theta\} = 1, \quad \{s^2, s_\parallel\} = \frac{1+\cos\theta}{\sin\theta}(s^2 - s_\parallel^2). \tag{1.164}$$

The quantity $A^2 = \sum A_i^2 = (1-\cos\theta)(s^2 - s_\parallel^2)$ has zero Poisson bracket with everything in this subalgebra

$$\{A^2, s^2\} = \{A^2, s_\parallel\} = \{A^2, \theta\} = 0. \tag{1.165}$$

In a non-zero magnetic field $(F, 0, 0)$ there remains only one integral (besides the energy)

$$\{A_1, H\} = 0. \tag{1.166}$$

In the present case it proves to be possible to carry through the reduction procedure for the Hamiltonian formalism to the end (globally) and to reduce the system to two degrees of freedom.

The integral A_1 generates a group (1.160), where g is a rotation around the first axis, that of $n = (1, 0, 0)$. Variables which are invariant with respect to this subgroup are the following:

$$s^2, s_\parallel, \theta, n_1, s_1, \tau = s_2 n_3 - n_2 s_3 \tag{1.167}$$

with a constraint of purely geometric origin

$$s^2\tau^2 = (s^2 - s_1^2)(s^2 - s_\parallel^2) - (s^2 n_1 - s_1 s_\parallel)^2. \tag{1.168}$$

It is not hard to work out that the variables (1.167) form a closed Poisson bracket algebra which contains the Hamiltonian H (1.155) and has functional dimension 5. The quantity A_1, which lies in this algebra, has vanishing bracket with all the variables

$$0 = \{A_1, s^2\} = \{A_1, s_\parallel\} = \{A_1, \theta\} = \{A_1, n_1\} = \{A_1, s_1\} = \{A_1, \tau\}. \tag{1.169}$$

Therefore, if we impose the condition $A_1 = \text{const}$ it is possible as before to make use of the formulas for the Poisson brackets of the quantities (1.167) which follow from (1.164). With the condition $A_1 = \text{const}$, we shall choose the following as basis variables:

$$A^2, s_\parallel, \theta, n_1 = n. \tag{1.170}$$

Their brackets have the form:

$$\begin{gathered} \{s_\parallel, \theta\} = 1, \quad \{\theta, n\} = 0, \\ \{A^2, s^2\} = \{A^2, \theta\} = \{A^2, s_\parallel\} = 0, \\ \{A^2, n\} = \sqrt{\tfrac{1}{2}(1-n^2)A^2 - \tfrac{1}{4}A_1^2}. \end{gathered} \tag{1.171}$$

Thus, the canonical variables can be chosen in the form

$$\begin{gathered} x^1 = \theta, \quad p_1 = p_\theta = s_\parallel, \\ x^2 = n, \quad p_2 = p_n = \sqrt{\frac{2A^2}{1-n^2} - \frac{A_1^2}{(1-n^2)^2}}. \end{gathered} \tag{1.172}$$

The Hamiltonian takes on the form:

$$\begin{aligned} H = \frac{1}{2}a\left[p_\theta^2 + \frac{1-n^2}{2(1-\cos\theta)}\left(p_n^2 + \frac{A_1^2}{(1-n^2)^2}\right)\right] \\ + bF\left(np_\theta + \frac{1-n^2}{2}\sin\theta p_n + A_1^2\frac{2-\sin^2\theta}{2(1-\cos\theta)}\right) + V(\cos\theta). \end{aligned} \tag{1.173}$$

Now let us introduce spherical coordinates

$$\theta = 2\chi, \quad n = n_1 = \sin\phi \tag{1.174}$$

and let us pass over to the Lagrangian formalism. We will obtain

$$L = 2a(\dot\chi^2 + \sin^2\chi\dot\phi^2) - \tilde A_1\dot y^1 - \tilde A_2\dot y^2 - U(y), \tag{1.175}$$

where $y^1 = \chi$, $y^2 = \phi$,

$$\begin{gathered} \tilde A_1 = 2b\sin\phi, \quad \tilde A_2 = 8bF\cos\phi\sin^3\chi\cos\chi, \\ U = V(\cos\theta) + \frac{aA_1^2}{4\sin^2\chi\cos^2\phi} + bF\frac{A_1(1-\sin^2\chi\cos^2\chi)}{2\sin^2\chi} \\ - \frac{1}{2}b^2F^2(\sin^2\phi + 4\cos^2\phi\sin^2\chi\cos\chi). \end{gathered}$$

Thus we have obtained a system on a region of the sphere S^2 with the usual metric, where there is an effective magnetic field and a scalar potential. When $A_1 \neq 0$ this system cannot be extended onto the whole sphere, since it has a singularity for $\phi = 0, \pi$.

Now let us consider examples of continuous systems with spare integrals. Let us first consider the case of Lagrangian field-theoretic systems (1.65), described in the Hamiltonian form with the aid of the transformation (1.67). In the case when the density of the Lagrangian $\Lambda(q, q_x, q_t)$ does not depend explicitly on the space-time coordinates x^α, t, $\alpha = 1, \ldots, m$, there hold laws of conservation of the total energy

$$\dot{E} = 0, \quad E = \mathcal{H} = \int d^m x (p_i q_t^i - \Lambda) \tag{1.176}$$

and of the total momentum vector

$$\dot{P}_\alpha = 0, \quad P_\alpha = \int p_i \frac{\partial q^i}{\partial x^\alpha} \, d^m x. \tag{1.177}$$

The functionals P_α are generators of the translations in the spatial variables, i. e.

$$\{p_i(x), P_\alpha\} = \frac{\partial p_i(x)}{\partial x^\alpha}, \quad \{q^i(x), P_\alpha\} = \frac{\partial q^i(x)}{\partial x^\alpha}, \tag{1.178}$$

$$\{P_\alpha, P_\beta\} = 0, \quad \alpha, \beta = 1, \ldots, m. \tag{1.179}$$

Let us stress that the generators of the spatial translations are local field integrals. The conservation laws (1.176), (1.177) are often written in infinitesimal form, by introducing the energy-momentum tensor

$$T_b^a = q_{x^b}^i \frac{\partial \Lambda}{\partial q_{x^a}^i} - \delta_b^a \Lambda, \tag{1.180}$$

where $a, b = 0, 1, \ldots, m$, $x^0 = t$. We have [142]

$$E = \int T_0^0 \, d^m x, \quad P_\alpha = \int T_\alpha^0 \, d^m x, \tag{1.181}$$

$$\frac{\partial T_b^a}{\partial x^a} = 0, \quad b = 0, 1, \ldots, m. \tag{1.182}$$

More generally, one can consider variational problems of the form

$$\delta S = 0, \quad S = \int \Lambda(x, q, q_x) \, d^{m+1} x \tag{1.183}$$

(here, as above, we put $x = (x^a)$, $a = 0, 1, \ldots, m$, $x^0 = t$) which are invariant with respect to more general one-parameter groups of transformations $G_\tau(x, q)$ of the form

$$\begin{aligned} x_\tau^a &= X^a(x), \quad a = 0, 1, \ldots, m, \\ q_\tau^i &= Q^i(x, q), \quad i = 1, \ldots, n. \end{aligned} \tag{1.184}$$

To each such group there corresponds a "conserved current"

$$J^a = \Lambda X^a + \frac{\partial \Lambda}{\partial q_a^i} (Q^i - q_{x^b}^i X^b), \tag{1.185}$$

$$\frac{\partial J_a}{\partial x^a} = 0 \tag{1.186}$$

(*Noether's theorem* (E. Noether) [17]). The quantity

$$\int_{x^0 = \text{const}} J^0 \, d^m x \tag{1.187}$$

is conserved. If the transformations (1.184) do not affect time, i. e. $X^0 = \text{const}$, then they define a family of canonical transformations on the space of fields $(p(x), q(x))$, whose generator is a local field integral with the density

$$J^0 = p_i(Q^i - q^i_{x^\alpha} X^\alpha) \tag{1.188}$$

The second Noether theorem concerns variational problems which admit symmetries with functional parameters (as, for example, in the theory of gauge fields [53]). The equations for the extremals (1.65) are not independent in this case, but satisfy some system of differential relations. We shall not discuss this theorem here.

3. Liouville's Theorem. Action-Angle Variables

In this section we shall restrict ourselves to the consideration of phase spaces with a non-degenerate Poisson bracket. The important *theorem of Liouville* (J. Liouville) studies the case of Hamiltonian systems with n degrees of freedom (i. e. on a $2n$-dimensional phase space) where there are exactly n functionally independent integrals $F_1 = H,\ F_2, \ldots, F_n$ whose pairwise Poisson brackets are equal to zero, $\{F_i, F_j\} = 0$, $j = 1, \ldots, n$. People often call such systems *completely integrable*. In this case the level surfaces of the integrals

$$F_1 = c_1,\ \ldots,\ F_n = c_n \tag{1.189}$$

are quotient groups of $\mathbb{R}^n$ by lattices of finite rank $\leq n$; in particular, compact nonsingular level surfaces are n-dimensional tori. If the level surface (1.189) is compact, then in a neighbourhood of it one can introduce coordinates $s_1, \ldots, s_n, \phi_1, \ldots, \phi_n$ $(0 \leq \phi_i < 2\pi)$ ("*action-angle*" *variables*) such that:

a)

$$\{s_i, s_j\} = \{\phi_i, \phi_j\} = 0, \quad \{\phi_i, s_j\} = \delta_{ij}; \tag{1.190}$$

b) $s_i = s_i(F_1, \ldots, F_n)$, ϕ_j are coordinates on the level surfaces (1.189);

c) in the coordinates (s_i, ϕ_j) the initial Hamiltonian system has the form:

$$\left.\begin{aligned} \dot{s}_i &= 0, \\ \dot{\phi}_i &= \omega_i(s_1, \ldots, s_n) \end{aligned}\right\} \quad i = 1, \ldots, n. \tag{1.191}$$

Let us give the idea of the proof of this theorem (see, for example, [42]). A level surface M_c of the form (1.189) is a smooth manifold by virtue of the independence of the integrals $F_1, \ldots, F_n$ (i. e. the independence of their "gradients" $(\xi_i)^j = h^{jk}(\partial F_i/\partial y^k)$. The group $\mathbb{R}^n$ of the flows with the Hamiltonians $F_1, \ldots, F_n$ acts on this manifold. Let us choose an initial point $x_0 = x_o(c) \in M_c$ and let us pick out a lattice in $\mathbb{R}^n$: a vector $d \in \mathbb{R}^n$ belongs to the lattice if d, acting on x_0, yields x_0 again. A subgroup $\{d\} \subset \mathbb{R}^n$ arises. This subgroup is discrete and is therefore isomorphic to a lattice spanned by k vectors of $\mathbb{R}^n$, where $k < n$. Obviously, only for $k = n$ will we obtain a compact manifold (a torus T^n).

Now let us construct the action-angle variables. On the given level surface M_c one may put together linear combinations of the fields ξ_i:

$$\eta_i = b^j_i \xi_j, i = 1, \ldots, n, \tag{1.192}$$

such that the coordinates introduced with their help on the group $\mathbb{R}^n$ acting on the torus $T^n = M_c$ coincide with the angles $0 \leq \phi_j < 2n$ ($\phi_i = 0$ is just the point x_0).

The coefficients b_i^j will depend on the collection $c_1, \dots, c_n$ in a neighbourhood of the chosen level surface. Thus we have

$$\eta_i = b_i^j(F_1, \dots, F_n)\xi_j. \tag{1.193}$$

This introduces coordinates $\tilde{\phi}_1, \dots, \tilde{\phi}_n$ on a whole region around the given M_c. In this region we have coordinates $(F_1, \dots, F_n, \tilde{\phi}_1, \dots, \tilde{\phi}_n)$ and a non-degenerate matrix of Poisson brackets

$$\begin{pmatrix} \{F_i, F_j\} = 0 & \{F_i, \tilde{\phi}_j\} \\ \{\tilde{\phi}_i, F_j\} & \{\tilde{\phi}_i, \tilde{\phi}_j\} \end{pmatrix}, \tag{1.194}$$

where $\det\{F_i, \tilde{\phi}_j\} \neq 0$. Now let us introduce the action variables. For the phase space $\mathbb{R}^{2n}$ with the canonical coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ the action variables have the form:

$$s_i = \frac{1}{2\pi} \oint_{\gamma_i} p_k \, dx^k, \quad i = 1, \dots, n. \tag{1.195}$$

Here γ_i is the i-th basis cycle of the torus T^n,

$$\gamma_i \colon 0 \leq \tilde{\phi}_i \leq 2\pi, \quad \tilde{\phi}_j = \text{const} \quad \text{for } j \neq i. \tag{1.196}$$

We get:

$$\{\tilde{\phi}_j, s_j\} = \delta_{ij}, \quad i, j = 1, \dots, n. \tag{1.197}$$

On an arbitrary phase space with a form $\Omega = h_{ij}\, dy^i \wedge dy^j$ of the type (1.17) one must do the following: the form Ω vanishes on the tori $T^n = M_c$. Therefore on some neighbourhood of the given torus $T^n = M_c$ this form is exact:

$$\Omega = d\omega.$$

The action variables have the form, analogous to (1.195):

$$s_i = \frac{1}{2\pi} \oint_{\gamma_i} \omega, \quad i = 1, \dots, n. \tag{1.198}$$

Now let us set

$$\phi_i = \tilde{\phi}_i + b_i(s_1, \dots, s_n), \quad i = 1, \dots, n. \tag{1.199}$$

Let us select the b_i according to the condition $\{\phi_i, \phi_j\} = 0$. This can always be done by virtue of (1.197). On each level surface M_c the coordinates $\phi_1, \dots, \phi_n$ coincide up to a translation with the angles $\tilde{\phi}_1, \dots, \tilde{\phi}_n$ chosen earlier. The matrix of Poisson brackets takes on the form (1.190), and the Hamiltonian $H = F_1$ can be written in the form

$$H = \tilde{H}(s_1, \dots, s_n). \tag{1.200}$$

The equations of motion will have the form (1.191). This is conditionally periodic motion along an n-dimensional torus with the frequencies

$$\omega_i(s_1, \dots, s_n) = \frac{\partial \tilde{H}(s_1, \dots, s_n)}{\partial s_i}. \tag{1.201}$$

EXAMPLE 1. Let the level surface $H(x, p) = E$ of a system with one degree of freedom be compact. Then we have the canonical action-angle coordinates

$$s(E) = \int_{H=E} p \, dx, \quad \{s, \phi\} = 1. \tag{1.202}$$

Now let us consider some examples of completely integrable systems with two degrees of freedom. Here, according to Liouville's theorem, it is sufficient for "complete integrability" to know one integral not dependent on the energy H.

EXAMPLE 2. The equations of the rotation of a heavy rigid body with a fixed point can be represented, in accordance with § 1, in the form of a Hamiltonian system on $E(3)$ with the Hamiltonian

$$H(M,p)=\frac{M_1^2}{2I_1}+\frac{M_2^2}{2I_2}+\frac{M_3^2}{2I_3}+\gamma_1p_1+\gamma_2p_2+\gamma_3p_3. \tag{1.203}$$

Here the axes of the coordinate system coincide with the principal axes of the body, the origin is at the point of attachment, I_1,I_2,I_3 are the principal moments of inertia of the body, $\gamma_1,\gamma_2,\gamma_3$ are the coordinates of the centre of mass. The Poisson brackets have the form (1.41). The phase space is six-dimensional here, but the rank of the matrix of Poisson brackets is equal to 4. Therefore for integrability according to Liouville it is enough to know one integral (besides the energy integral). Well known are the following cases of integrability.

a) *The Euler case*: $\gamma_1=\gamma_2=\gamma_3=0$. The extra integral is the square of the total angular momentum $M^2=M_1^2+M_2^2+M_3^2$.

b) *The Lagrange case*: $I_1=I_2$, $\gamma_1=\gamma_2=0$. Here there is an axial symmetry (with respect to the third axis). This gives the extra integral $M_3=\text{const}$.

c) *S. V. Kovalevskaya's case* $I_1=I_2=I_3/2$, $\gamma_3=0$. Here the appearance of the spare integral

$$F=|I_1(M_1+iM_2)^2-2(\gamma_1+i\gamma_2)(p_1+ip_2)|^2 \tag{1.204}$$

is not connected with a symmetry of the system (see Chap. 2 below).

EXAMPLE 3. The problem of the motion of a rigid body in a perfect fluid (see above § 1) is far richer in integrable cases. The simplest of them is the *Kirchhoff case*, where the Hamiltonian has the form (1.48), with $a_1=a_2$, $b_{11}=b_{22}$, $b_{ij}=0$ for $i\neq j$, $c_{11}=c_{22}$, $c_{ij}=0$ for $i\neq j$. Here, just as in the Lagrange case, there is an axial symmetry, and the extra integral is M_3. More complicated integrable cases (with a "hidden symmetry") have the following form.

a) *The Clebsch case* (R. Clebsch). Here the coefficients of the Hamiltonian (1.48) are like this:

$$b_{ij}=0,\quad c_{ij}=c_i\delta_{ij}, \tag{1.205}$$

where the coefficients a_i and c_j satisfy the relation

$$\frac{c_2-c_3}{a_1}+\frac{c_3-c_1}{a_2}+\frac{c_1-c_2}{a_3}=0. \tag{1.206}$$

The supplementary integral has the form

$$M_1^2+M_2^2+M_3^2-(a_1p_1^2+a_2p_2^2+a_3p_3^2). \tag{1.207}$$

b) *The Lyapunov–Steklov–Kolosov case*: $b_{ij}=b_i\delta_{ij}$, $c_{ij}=c_i\delta_{ij}$, where

$$b_j=\mu(a_1a_2a_3)a_j^{-1}+\nu,\quad c_j=\mu^2a_1(a_2-a_3)^2+\nu',\dots \tag{1.208}$$

($\mu,\nu,\nu'=\text{const}$). The supplementary integral is

$$\sum_j[M_j^2-\mu(a_j+\nu)M_jp_j]+\mu^2[(a_2-a_3)^2+\nu'']p_1^2+\dots \tag{1.209}$$

(the parameters ν,ν',ν'' are unessential).

4. The Hamilton–Jacobi Equation. The Method of Separation of Variables—The Classical Method of Integration and of Finding Action-Angle Variables

The theory of completely integrable Hamiltonian systems set forth in the preceding section arose as a generalization of the *Hamilton–Jacobi method* of integrating the canonical equations.

We shall consider here the phase space $\mathbb{R}^{2n}$ with the canonical coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ (see (1.19)). The Hamiltonian system with the Hamiltonian $H = H(x, p)$ has the form:

$$\left.\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i} \end{aligned}\right\} \quad i = 1, \dots, n. \tag{1.210}$$

Let us consider a canonical transformation (i. e. one preserving the Poisson brackets (1.19)) of the coordinates (x, p) to coordinates (X, P) of the form

$$p_i = \frac{\partial S}{\partial x^i}, \quad X^i = \frac{\partial S}{\partial P_i}, \quad S = S(x, P), \tag{1.211}$$

$$dS = p_i\, dx^i + X^i\, dP_i, \quad dp_i \wedge dx^i = dP_i \wedge dX^i. \tag{1.212}$$

In the new coordinates the system (1.210) can be written in the form

$$\left.\begin{aligned} \dot{X}^i &= \frac{\partial K}{\partial P_i}, \\ \dot{P}_i &= -\frac{\partial K}{\partial X^i} \end{aligned}\right\} \quad i = 1, \dots, n. \tag{1.213}$$

where the Hamiltonian $K = K(X, P)$ has the form

$$K(X, P) = H(x(X, P), p(X, P)). \tag{1.214}$$

The idea of the Hamilton–Jacobi method consists in choosing the transformation (1.211), (1.214) appropriately so that in the new coordinates the Hamiltonian K does not depend on X: $K = K(P)$. In this case the variables $P_1, \dots, P_n$ will obviously be variables of action type, and the conjugate variables $X^1, \dots, X^n$ will be the corresponding angles, i. e. the system (1.213) can be written in the form (1.191)

$$\left.\begin{aligned} \dot{X}^i &= \frac{\partial K(P)}{\partial P_i}, \\ \dot{P}_i &= 0 \end{aligned}\right\} \quad i = 1, \dots, n. \tag{1.215}$$

Thus, the problem of integrating the canonical equations (1.210) reduces to finding a function $S(x, P)$ satisfying the *Hamilton–Jacobi equation*

$$H\left(x, \frac{\partial S}{\partial x}\right) = K, \tag{1.216}$$

depending on n parameters $P_1, \dots, P_n$ (it is necessary that the function $S = S(x^1, \dots, x^n, P_1, \dots, P_n)$ be a general integral of equation (1.216), i. e. that $\det(\partial^2 S/\partial x^i \partial P_j) \neq 0$ [7]).

The only method of integrating the Hamilton-Jacobi equation employed with success in classical analytical mechanics is *the method of separation of variables*. Namely, suppose the Hamilton–Jacobi equation (1.216) can be written in the form

$$h\left(f_1\left(x^1, \frac{\partial S}{\partial x^1}\right), \ldots, f_n\left(x^n, \frac{\partial S}{\partial x^n}\right), K\right) = 0, \tag{1.217}$$

where the $f_i(x^i, p_i)$ are certain functions. In this case its general integral may be sought in the form

$$S = S_1(x^1; c_1) + S_2(x^2; c_2) + \cdots + S_n(x^n; c_n), \tag{1.218}$$

where the equations for the functions $S_1, \ldots, S_n$ will be written in the form

$$f_i\left(x^i, \frac{\partial S}{\partial x^i}\right) = c_i, \quad i = 1, \ldots, n, \tag{1.219}$$

and in an obvious manner can be integrated by quadratures. The dependence of the Hamiltonian on the new variables $H(x, p) = K(c_1, \ldots, c_n)$ is determined by the equation

$$h(c_1, \ldots, c_n, K) = 0. \tag{1.220}$$

The variables

$$c_i = \phi(x^i, p_i), \quad i = 1, \ldots, n, \tag{1.221}$$

if they are globally defined, will be variables of action type. The corresponding variables of angle type can be computed by formulas (1.211).

EXAMPLE 1. *Geodesics on an ellipsoid* (Jacobi, 1839). Let the ellipsoid have the form:

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \frac{x_3^2}{a_3} = 1, \quad a_1 > a_2 > a_3 > 0. \tag{1.222}$$

The elliptical coordinates $\lambda_1, \lambda_2, \lambda_3$ in space are defined as the roots of the equation

$$\frac{x_1^2}{a_1 - \lambda} + \frac{x_2^2}{a_2 - \lambda} + \frac{x_3^2}{a_3 - \lambda} = 1, \tag{1.223}$$

where $\lambda_3 < a_3 < \lambda_2 < a_2 < \lambda_1 < a_1$. The ellipsoid (1.222) is obtained for $\lambda_3 = 0$. The Hamiltonian of the free motion of a unit point mass on the surface of the ellipsoid coincides with the kinetic energy (the metric) and has the form

$$H = \frac{2}{\lambda_1 - \lambda_2}\left[\frac{(a_1 - \lambda_1)(\lambda_1 - a_2)(\lambda_1 - a_3)}{\lambda_1}\, p_1^2 + \frac{(a_1 - \lambda_2)(a_2 - \lambda_2)(\lambda_2 - a_3)}{\lambda_2}\, p_2^2\right], \tag{1.224}$$

where

$$p_j = (-1)^{j+1}(\lambda_1 - \lambda_2)\frac{\lambda_j \dot{\lambda}_j}{4(a_1 - \lambda_j)(a_2 - \lambda_j)(a_3 - \lambda_j)}, \quad j = 1, 2. \tag{1.225}$$

The variables have been separated. It is not hard to show [120] that integration of the equations of motion reduces to the hyperelliptic quadratures (of genus 2)

$$\frac{d\lambda_1}{\sqrt{R(\lambda_1)}} = \frac{dt}{\lambda_1 - \lambda_2}, \quad \frac{d\lambda_2}{\sqrt{R(\lambda_2)}} = \frac{dt}{\lambda_2 - \lambda_1}, \tag{1.226}$$

where

$$R(\lambda) = \frac{(\lambda - \alpha)(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)}{\lambda}, \tag{1.227}$$

and $a_3 < \alpha < a_1$ is an arbitrary constant. These equations were integrated in 1861 by K. Weierstrass in theta functions of two variables: The solution of the Hamilton–Jacobi equation has the form:

$$S(\lambda_1, \lambda_2; \alpha, E) = \frac{\sqrt{E}}{\sqrt{2}} \int \frac{\lambda_1 - \alpha}{\sqrt{R(\lambda_1)}}\, d\lambda_1 + \frac{\sqrt{E}}{\sqrt{2}} \int \frac{\lambda_2 - \alpha}{\sqrt{R(\lambda_2)}}\, d\lambda_2, \tag{1.228}$$

where $E = H$ is the energy. From this the variables of angle type are found by the formulas (1.211)

$$\phi_\alpha = \frac{\partial S}{\partial \alpha}, \quad \phi_E = \frac{\partial S}{\partial E}. \tag{1.229}$$

The change of variables (1.229) $(\lambda_1, \lambda_2) \mapsto (\phi_\alpha, \phi_E)$ is an Abel map, corresponding to the hyperelliptic Riemann surface of the root $\sqrt{\lambda(\lambda - \alpha)(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)}$ (of genus 2; see Chap. 2 below). Therefore the invariant tori here can be extended to the complex domain and are abelian.

The question of separation of variables for Hamiltonian systems was studied intensively in the second half of the last century (see the bibliography in [92]). The following criterion was established (by T. Levi-Civita, [91]): the system with Hamiltonian $H(x, p)$ is integrable by the method of separation of variables in a given coordinate system if and only if the function H satisfies the following system of equations

$$\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial x^j \partial x^k} - \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial x^k}\frac{\partial^2 H}{\partial x^j \partial p_k} - \frac{\partial H}{\partial x^j}\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial p_j \partial x^k} + \frac{\partial H}{\partial x^j}\frac{\partial H}{\partial x^k}\frac{\partial^2 H}{\partial p_j \partial p_k} = 0, \tag{1.230}$$

$1 \le j < k \le n$ (there is no summation over repeated indices). The application of this criterion to the investigation of the integrability (via Hamilton–Jacobi) of Hamiltonian systems is a non-trivial problem; advances in certain special classes of Hamiltonians were obtained in [27], [32].

To conclude this section let us note that the system of S. V. Kovalevskaya mentioned above cannot be integrated by the method of separation of variables, and action-angle variables for it were found very recently (see below Chap. 2).

CHAPTER 2

Modern Ideas on the Integrability of Evolution Systems

1. Commutational Representations of Evolution Systems

The algebraic mechanism lying at the foundation of the procedure for integrating the KdV equation with initial conditions rapidly decreasing in x which was proposed in the famous paper of C. S. Gardner, J. Green, M. Kruskal and R. M. Miura [60] was cleared up in P. D. Lax's paper [89]. It was observed that this equation

$$4u_t = 6uu_x + u_{xxx} \tag{2.1}$$

is equivalent to the commutation condition

$$\left[L, \frac{\partial}{\partial t} - A\right] = 0 \iff \frac{\partial L}{\partial t} = [A, L] \tag{2.2}$$

for the auxiliary linear differential operators

$$L = \frac{\partial^2}{\partial x^2} + u(x,t); \quad A = \frac{\partial^2}{\partial x^3} + \frac{3}{2}\,u\,\frac{\partial}{\partial x} + \frac{3}{4}\,u_x. \tag{2.3}$$

Beginning with this paper, all schemes for producing new equations to which the inverse scattering method is applicable were based on various generalizations of the *commutation equation* (2.2).

The first and most natural step is the generalization of equation (2.2) to the case when L and A are arbitrary differential operators

$$L = \sum_{i=0}^{n} u_i(x,t)\,\frac{\partial^i}{\partial x^i}; \quad A = \sum_{i=0}^{m} v_i(x,t)\,\frac{\partial^i}{\partial x^i} \tag{2.4}$$

with matrix $(l \times l)$ or scalar coefficients.

New physically important equations to which the inverse scattering method is applicable were discovered along just this path in the papers of V. E. Zakharov–A. B. Shabat [125], [126] and G. L. Lamb [88].

Let u_n and v_m be constant non-degenerate diagonal matrices with distinct entries on the diagonals. By conjugating with a suitable matrix function $g(x,t)$: $L \to gLg^{-1}$ and $A \to gAg^{-1}$ one can always achieve that $u_{n-1}^{\alpha\alpha} = 0$, $v_{m-1}^{\alpha\alpha} = 0$, $\alpha = 1, \ldots, l$. The equations (2.2) form a system of $n+m$ matrix equations in the coefficients of the operators L and A. It turns out that from the first m equations, obtained by equating to zero the coefficients of the $\partial^k/\partial x^k$, $k = n, \ldots, n+m-1$, one can successively find the $v_j(x,t)$, the matrix entries of which are differential polynomials in the matrix entries $u_i^{\alpha\beta}(x,t)$ and certain constants h_j^{α}; $\alpha, \beta = 1, \ldots, l$. If we substitute the expressions obtained into the remaining n equations, we obtain a system of evolution equations only in the coefficients of the operator L, and these

are called equations of Lax type. There exist a great number of schemes (see, for example, [1], [34], [75], [106], [115], [126], [127]) which by one method or another realize a reduction of the general equation (2.2) to equations in the coefficients of the operator L.

The system (2.2) represents a family of *Lax equations*, parametrized by the constants h_j^α. For example, if $(l = 1)$

$$L = \partial^2 + u, \quad A = \partial^3 + v_1\partial + v_2, \quad \partial = \partial/\partial x, \tag{2.5}$$

then

$$v_1 = \tfrac{3}{2}u + h_1, \quad v_2 = \tfrac{3}{4}u_x + h_2 \tag{2.6}$$

and equation (2.2) is equivalent to the family of equations

$$4u_t = u_{xxx} + 6uu_x + 4h_1u_x. \tag{2.7}$$

Let us give a few simplest examples of equations of Lax type.

EXAMPLE 1 ([142]). If $(l = 1)$

$$L = \partial^3 + \tfrac{3}{2}u(x,t)\partial + w(x,t), \quad A = \partial^2 + u(x,t), \tag{2.8}$$

then equation (2.2) leads to the system

$$\tfrac{3}{2}u_t = w_x - \tfrac{3}{4}u_{xx}, \tag{2.9}$$

$$w_t = w_{xx} - u_{xxx} - \tfrac{3}{2}uu_x. \tag{2.10}$$

If we eliminate w from these equations, we arrive for $u(x,t)$ at the Boussinesq equation (J. V. Boussinesq)

$$3u_{tt} + (u_{xxx} + 6uu_x)_x = 0. \tag{2.11}$$

Two-dimensional systems which admit a representation of the Lax type (2.2) were first discovered in [37], [126]. An important example of such systems is the "two-dimensionalized" KdV equation—the Kadomtsev–Petviashvili (KP) equation

$$\begin{cases} \tfrac{3}{2}u_y + \tfrac{3}{2}u_{xx} - 2w_x = 0, \\ w_x - u_t + u_{xxx} + \tfrac{3}{2}uu_x - w_{xx} = 0. \end{cases} \tag{2.12}$$

In this case

$$L = -\partial_y + \partial_x^2 + u(x,y,t), \quad A = \partial_x^2 + \tfrac{3}{2}u(x,y,t)\partial_x + w(x,y,t). \tag{2.13}$$

REMARK. A number of authors have shown that the usual Lax representation for two-dimensional systems is possible only for operators which involve differentiation of no higher than first order with respect to one of the variables. For example, for the two-dimensional Schrödinger operator $L = \partial_x^2 + \partial_y^2 + u(x,y)$ there is not a single non-trivial equation of Lax type. The correct (non-trivial) analogue of the equations of Lax type for $2+1$-systems was found later in the works [98], [43], [117].

A review of two-dimensional integrable systems, their algebraic-geometrical solutions, spectral theory of two-dimensional operators and their applications is given in the Appendices.

EXAMPLE 2 ([125]). The non-linear Schrödinger equation ($\mathrm{NLS}_\pm$)

$$ir_t = r_{xx} \pm |r|^2 r. \tag{2.14}$$

The operators L and A in this case are matricial and are equal to

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}, \quad q = \mp\bar{r}, \tag{2.15}$$

$$A = i\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \frac{1}{2}\left[\begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} \frac{\partial}{\partial x} + \frac{\partial}{\partial x}\begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}\right] + \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix}\right]. \tag{2.16}$$

EXAMPLE 3. The equations of three-wave interaction

$$\begin{aligned} &u_{13t} + v_{13}u_{13x} = i\varepsilon u_{12}u_{23}; \quad u_{12t} + v_{12}u_{12x} = i\varepsilon u_{13}\bar{u}_{23}, \\ &u_{23t} + v_{23}u_{23x} = i\varepsilon u_{13}\bar{u}_{12}; \quad u_{ij} = u_{ij}(x,t), \quad v_{ij} = \text{const}. \end{aligned} \tag{2.17}$$

The operators L and A are matricial (3×3) and are equal to

$$\begin{gathered} L = I\frac{\partial}{\partial x} + [I,Q], \quad A = J\frac{\partial}{\partial x} + [J,Q], \\ I_{ij} = a_i\delta_{ij}, \quad a_{i+1} > a_i, \quad J_{ij} = b_i\delta_{ij}, \\ Q_{ij} = \bar{Q}_{ji} = -iu_{ij}\sqrt{a_j - a_i}, \quad j > i, \end{gathered} \tag{2.18}$$

$$v_{ij} = \frac{a_ib_j - b_ia_j}{a_i - a_j}. \tag{2.19}$$

EXAMPLE 4. The Toda lattice ((M. Toda) [96], [55], [56]) and the difference analogue of the KdV equation [96], [109]. The inverse scattering method is also applicable to certain differential-difference systems. If L and A are difference operators of the form

$$L\psi_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1}, \tag{2.20}$$

$$A\psi_n = \frac{c_n}{2}\psi_{n+1} - \frac{c_{n-1}}{2}\psi_{n-1}, \tag{2.21}$$

then equation (2.2) leads to the equations

$$2\dot{c}_n = c_n(v_{n+1} - v_n),, \tag{2.22}$$

$$\dot{v}_n = c_n^2 - c_{n-1}^2, \tag{2.23}$$

which, if one sets $c_n = \exp(\frac{1}{2}(x_{n+1} - x_n))$, $v_n = \dot{x}_n$, coincide with the equations of motion of the so-called Toda lattice—the Hamiltonian system of particles on the line with the Hamiltonian

$$H = \tfrac{1}{2}\sum_n p_n^2 + \sum_n \exp(x_{n+1} - x_n). \tag{2.24}$$

If in (2.20) one sets $v_n \equiv 0$ and for A one chooses

$$A\psi_n = \tfrac{1}{2}[c_nc_{n+1}\psi_{n+1} - c_{n-1}c_{n-2}\psi_{n-2}], \tag{2.25}$$

then (2.20) leads to the difference analogue of the KdV equation

$$\frac{d}{dt}\tilde{c}_n = \tilde{c}_n(\tilde{c}_{n+1} - \tilde{c}_{n-1}), \quad \tilde{c}_n = c_n^2. \tag{2.26}$$

With each operator L there is connected a whole *hierarchy of Lax-type equations*, which are the reduction to equations in the coefficients of L of the equations (2.2) with operators A of different orders. One of the most important facts in the theory of integrable equations is the commutativity of all the equations which enter into the general hierarchy.

For the KdV equation the corresponding equations are called the "higher-order KdV equations". They have the form

$$u_\tau = \sum_{k=1}^{n} h_k Q_k(u, \dots, u^{(2k+1)})$$

and are the commutativity condition for the Sturm–Liouville operator with the operators $\partial/\partial t - A$, where A has order $2n+1$.

In the paper [111] a representation of a different type than (2.2) for the higher-order KdV equations was used for the first time—a representation of Lax type in matrix functions depending on an additional spectral parameter.

For the general equation (2.2) such a λ-representation can be constructed in the following manner.

The equation

$$Ly = \lambda y \tag{2.27}$$

is equivalent to the first-order matrix equation

$$\left[\frac{\partial}{\partial x} - U_L(x,t,\lambda)\right] Y(x,t,\lambda) = 0, \tag{2.28}$$

where U_L is an $nl \times nl$ matrix (n is the order of L, and the matricial coefficients of L are $l \times l$). The vector Y is the column composed of the vectors $(\partial^i/\partial x^i)y(x,t,\lambda)$, $i = 0, \dots, n-1$.

The matrix U_L has the following block structure

$$U_L = u_n^{-1} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \lambda - u_0 & -u_1 & & \dots & -u_{n-2} & -u_{n-1} \end{pmatrix}. \tag{2.29}$$

If we act with the operator A on the coordinates of the vector Y and use (2.27) to express the $(\partial^n/\partial x^n)y$ in terms of lower-order derivatives and the parameter λ, we obtain that on the space of solutions of (2.27) the equation

$$\left(\frac{\partial}{\partial t} - A\right) y(x,t,\lambda) = 0 \tag{2.30}$$

is equivalent to an equation

$$\left(\frac{\partial}{\partial t} - V_A(x,y,\lambda)\right) Y(x,t,\lambda) = 0 \tag{2.31}$$

The matrix entries of V_A depend polynomially on λ, the matrix entries of the $u_i(x,t)$, and their derivatives.

The compatibility of equations (2.27) and (2.30) implies the compatibility of (2.28) and (2.31). Hence

$$\left[\frac{\partial}{\partial x} - U_L, \frac{\partial}{\partial t} - V_A\right] = 0. \tag{2.32}$$

For the KdV equation the matrices U_L, V_A have the form [111]:

$$U_L = \begin{pmatrix} 0 & 1 \\ \lambda - \mu & 0 \end{pmatrix}, \tag{2.33}$$

$$V_A = \begin{pmatrix} -\dfrac{u_x}{4} & \lambda + \dfrac{u}{2} \\ \lambda^2 - \dfrac{u\lambda}{2} - \dfrac{u^2}{2} - \dfrac{u_{xx}}{4} & \dfrac{u_x}{4} \end{pmatrix}. \tag{2.34}$$

Subsequently, equation (2.32), where U and V were now arbitrary rational functions of the parameter λ, was proposed in [127] as a general scheme for the production of one-dimensional integrable equations. The beginning of this program goes back to the paper [1], in which for the integration of the sine-Gordon equation an example of a *rational family* was first introduced.

Let $U(x,t,\lambda)$ and $V(x,t,\lambda)$ be arbitrary matrix functions depending rationally on the parameter λ:

$$\begin{aligned} U(x,t,\lambda) &= u_0(x,t) + \sum_{k=1}^{n} \sum_{s=1}^{h_k} u_{ks}(x,t)(\lambda - \lambda_k)^{-s}, \\ V(x,t,\lambda) &= v_0(x,t) + \sum_{r=1}^{m} \sum_{s=1}^{d_r} v_{rs}(x,t)(\lambda - \mu_r)^{-s}. \end{aligned} \tag{2.35}$$

The condition of compatibility of the linear problems

$$\begin{aligned} \left(\frac{\partial}{\partial x} - U(x,t,\lambda) \right) \Psi(x,t,\lambda) = 0, \\ \left(\frac{\partial}{\partial t} - V(x,t,\lambda) \right) \Psi(x,t,\lambda) = 0 \end{aligned} \tag{2.36}$$

is represented by the *equation of zero curvature*

$$U_t - V_x + [U,V] = 0, \tag{2.37}$$

which must be fulfilled for all λ. This equation is equivalent to a system of $1 + \sum_k h_k + \sum_r d_r$ matrix equations in the unknown functions $u_{ks}(x,t)$, $v_{rs}(x,t)$, $u_0(x,t)$, $v_0(x,t)$. These equations arise when one equates to zero all the singular terms on the left-hand side of (2.37) at the points $\lambda = \lambda_k$, $\lambda = \mu_r$, and also the absolute term, equal to $u_{0t} - u_{0x} + [u_0, v_0]$.

The number of equations is one matrix equation fewer than the number of unknown matrix functions. This underdeterminacy is connected with a "gauge symmetry" of the equations (2.37). If $g(x,t)$ is an arbitrary non-degenerate matrix function, then the transformation

$$\begin{aligned} U &\to g_x g^{-1} + gUg^{-1}, \\ V &\to g_t g^{-1} + gVg^{-1}, \end{aligned} \tag{2.38}$$

called a "gauge transformation", takes the solutions of (2.37) over into solutions of the same equation.

A choice of conditions on the matrices $U(x,t,\lambda), V(x,t,\lambda)$ compatible with the equations (2.37) and destroying the gauge symmetry is called a setting of the gauge. The simplest gauge is the pair of conditions $u_0(x,t) = v_0(x,t) = 0$.

Just as in the above-considered case of commutation equations for differential operators, the equations (2.37) are essentially generating equations for a whole family of integrable systems. If the poles of U and V coincide, then these equations can be reduced to a family of equations, which are parametrized by arbitrary constants, in the coefficients only of $U(x,t,\lambda)$. Here, by changing the multiplicity of the poles of V, we will obtain a hierarchy of commuting flows associated with $U(x,t,\lambda)$.

In singling out some particular equations from (2.37), an important issue is singling out the invariant submanifolds for the equation (2.37). This problem reduces to describing the orbits of the coadjoint representation of the current algebra [2], in the framework of which the Hamiltonian theory of equations of zero curvature can naturally be introduced (see [30], [36], [52], [124]).

Leaving aside the further analysis of the questions of reduction and gauge equivalence of systems, which may be found in the papers [2], [36], [52], [106], [126], let us cite the two simplest examples.

EXAMPLE 5. If U and V have one pole each, which do not coincide,

$$U = \frac{u(\xi,\eta)}{1-\lambda}, \quad V = \frac{v(\xi,\eta)}{\lambda+1}, \tag{2.39}$$

then the equations (2.37) (after the substitution $x \to \xi = x' - t'$; $t \to \eta = x' + t'$) lead to the equations of a principal *chiral field* ([106], [122])[1]

$$u_\eta + \tfrac{1}{2}[u,v] = 0, \quad v_\xi = \tfrac{1}{2}[u,v]. \tag{2.40}$$

Here $u(\xi,\eta), v(\xi,\eta)$ are the currents of the chiral field

$$u = G_\xi G^{-1}; \quad u = G_\eta G^{-1}. \tag{2.41}$$

Equation (2.40) yields

$$2G_{\xi\eta} = G_\xi G^{-1} G_\eta + G_\eta G^{-1} G_\xi. \tag{2.42}$$

The last equations are Lagrangian with Lagrangian (1.68) (the currents A_μ in the notation of (1.68) correspond, as is evident from (2.41), to u and v).

As was remarked in [135], the representation (2.37), where U and V are given by the formulas (2.39), simultaneously gives the solutions of the equations of motion of a principal chiral field with a "multi-valued additional term". If in the definition of the corresponding Lagrangian (1.72) one introduces a coupling constant κ, i. e.

$$\delta S = \delta S_0 + \kappa \Omega_1,$$

then the equations of motion can be written in terms of the currents in the form

$$\partial_\xi v = \frac{1+\kappa}{2}\,[v,u], \quad \partial_\eta u = \frac{1-\kappa}{2}\,[v,u]. \tag{2.43}$$

In the inverse scattering-method, as will be stressed repeatedly in the following, the road to constructing solutions of the equations (2.37) goes via the construction with the aid of various schemes ("dressing", algebraic geometric schemes, etc.) of functions $\Psi(\xi,\eta,\lambda)$ which by their construction satisfy the equations (2.36) with some U, V (which will automatically satisfy (2.37)). In the case under consideration, after the function $\Psi(\xi,\eta,\lambda)$ has been constructed by one means or another, the desired solutions of the equations of motion are defined by the formula

$$G(\xi,\eta) = \Psi(\xi,\eta,\kappa).$$

[1]It is interesting that this example was first brought to the open together with the notion of zero curvature in a remarkable (but forgotten) classical paper by René Garnier [61].

EXAMPLE 6. As another example let us cite the Sine-Gordon equation [134], [1]

$$u_{\xi\eta} = 4\sin u, \tag{2.44}$$

which arose in the theory of surfaces of constant negative curvature. By the number of applications (in the theory of superconductivity, in the theories of quasi-one-dimensional conductors, of film-formation processes on crystalline substrates, in the theory of fields) leading to equation (2.44), this equation belongs to the most important in contemporary mathematical physics.

As in the general case of a principal chiral field, the matrices U and V have one pole each

$$U(\xi,\eta,\lambda) = \begin{pmatrix} \dfrac{iu_\xi}{2} & 1 \\ \lambda^{-1} & -\dfrac{iu_\xi}{2} \end{pmatrix}, \tag{2.45}$$

$$V(\xi,\eta,\lambda) = \begin{pmatrix} 0 & \lambda e^{iu} \\ e^{-iu} & 0 \end{pmatrix}. \tag{2.46}$$

It should be noted that an automatic generalization of equations (2.37) to the case of matrices U and V whose spectral parameter is defined on an algebraic curve Γ of genus greater than zero (the case of rational families corresponds to $g = 0$) is obstructed by the Riemann–Roch theorem (G. F. B. Riemann–G. Roch).

Indeed, let $U(x,t,\lambda)$ and $V(x,t,\lambda)$ be meromorphic functions on Γ having poles of total multiplicity N and M. By the Riemann–Roch theorem [131] the number of independent variables (i. e. the dimension of the space of matrix functions with the same poles as U and V) equals $l^2(N-g+1)$ for U and $l^2(M-g+1)$ for V. The commutator $[U,V]$ has poles of total multiplicity $N+M$. Hence the equations (2.37) are equivalent to $l^2(N+M-g+1)$ equations. With gauge equivalence taken into account the number of equations is always greater than the number of variables.

There are two ways of circumventing the obstacle mentioned. One of them was proposed in the papers [84], [86], where the matrices U and V were allowed to have, besides poles stationary with respect to x and t, gl poles depending on x and t in a definite fashion. It was shown that here the number of equations with gauge equivalence taken into account is equal to the number of the independent variables, which (just as in the rational case) are the singular parts of U and V at the stationary poles.

EXAMPLE 7. An example of such an equation is

$$c_t = \frac{1}{4}c_{xxx} + \frac{3}{8c_x}(1-c_{xx}^2) - \frac{1}{2}Q(c)c_x^2. \tag{2.47}$$

(so-called Krichever–Novikov equation).

Here the quantity $Q = \partial\Phi/\partial c + \Phi^2$ is defined through

$$\Phi(c,y) = \zeta(-2c) + \zeta(c-y) + \zeta(c+y)$$

and does not depend, as is easy to verify, on y (ζ is Weierstrass's ζ-function [11]).

This equation together with the following pair with an elliptic spectral parameter (*an elliptic family*) was obtained in [85], [86]. The matrix U equals

$$U = A_1\zeta(\lambda-\gamma_1) + B_1\zeta(\lambda-\gamma_2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\zeta(\lambda) + A_1\zeta(\gamma_1) + B_1\zeta(\gamma_2) + \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix},$$

where

$$A_1 = \frac{1}{\alpha_1 - \alpha_2}\begin{pmatrix} 0 & 0 \\ \alpha_1 & 1 \end{pmatrix}, \quad B_1 = \frac{1}{\alpha_1 - \alpha_2}\begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix}.$$

For V we have the formulas

$$V = A_2\zeta(\lambda - \gamma_1) + B_2\zeta(\lambda - \gamma_2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\wp(\lambda) + \begin{pmatrix} 0 & 1 \\ -\dfrac{u}{2} & 0 \end{pmatrix}\zeta(\lambda) + D,$$

where

$$A_2 = \begin{pmatrix} \dfrac{\alpha_1\alpha_2}{\alpha_1 - \alpha_2} & \dfrac{\alpha_2}{\alpha_1 - \alpha_2} \\ \dfrac{\alpha_1 u}{2(\alpha_1 - \alpha_2)} & \dfrac{u}{2(\alpha_1 - \alpha_2)} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \dfrac{\alpha_1\alpha_2}{\alpha_2 - \alpha_1} & \dfrac{\alpha_1}{\alpha_2 - \alpha_1} \\ \dfrac{\alpha_2 u}{2(\alpha_2 - \alpha_1)} & \dfrac{u}{2(\alpha_2 - \alpha_1)} \end{pmatrix}$$

and

$$D = A_2\zeta(\gamma_1) + B_2\zeta(\gamma_2) + \begin{pmatrix} w_1 & \dfrac{u}{2} \\ w_2 & -w_1 \end{pmatrix}.$$

The equations (2.37) are equivalent to a system of equations in the functions $\gamma_i, \alpha_i, w_i, u$. By successively eliminating from this system the variables w_i, which equal

$$w_1 = \frac{1}{2(\alpha_1 - \alpha_2)}(\wp(\gamma_1) - \wp(\gamma_2)) - \frac{u_x}{4},$$
$$w_2 = w_{1x} - \frac{u^2}{2} + \wp(\gamma_1) + \wp(\gamma_2),$$

and afterwards the α_i, (the formulas for which we shall omit here) and u (see (2.48)), we shall arrive finally at equation (2.47), where

$$\gamma_1 = c(x,t) + y, \quad \gamma_2 = y - c(x,t) + c_0.$$

Each solution of equation (2.47) defines by the formula

$$8u(x,y,t) = (c_{xx}^2 - 1)c_x^{-2} + 8\Phi c_{xx} + 4c_x^2\left(\frac{\partial\Phi}{\partial c} - \Phi\right) - 2c_{xxx}c_x^{-1} \tag{2.48}$$

a rank 2 genus 1 solution of the Kadomtsev–Petviashvili equation (KP).

Equation (2.47) also describes the deformation of commuting linear differential operators of orders 4 and 6. Such an operator of order 4 has the form:

$$\mathcal{L} = (\partial^2 + u^2) + c_x(\wp(\gamma_2) - \wp(\gamma_1))\frac{d}{dx} + \frac{d}{dx}(c_x(\wp(\gamma_2) - \wp(\gamma_1))) - \wp(\gamma_2) - \wp(\gamma_1). \tag{2.49}$$

Equation (2.47), as is shown in [130], is the only one of the equations of the form

$$c_t = \text{const} \cdot c_{xxx} + f(c, c_x, c_{xx})$$

possessing a "hidden symmetry" which cannot be reduced to the ordinary KdV by transformations of "Miura type" $w = w(c, c_x, \dots)$.

By the substitution $v = \zeta(c)$ equation (2.47) reduces to the algebraic form

$$\begin{aligned} v_t &= \frac{1}{4}v_{xxx} + \frac{3}{8v_x}(v_{xx}^2 - P_3(v)), \\ P_3(v) &= 4v^3 - g_2 v - g_3. \end{aligned} \tag{2.50}$$

EXAMPLE 8. The second way of introducing a non-rational spectral parameter is based on the choice of a special form for the matrices U and V and has been successfully realized only in certain examples on elliptic curves Γ ($g = 1$). The physically most interesting example of such equations is the Landau–Lifshitz equation

$$S_t = S \times S_{xx} + S \times JS, \tag{2.51}$$

where S is a three-dimensional vector of unit length, $|S| = 1$, and $J_{\alpha\beta} = J_\alpha \delta_{\alpha\beta}$ is a diagonal matrix. As has been shown in the papers [22], [128], equation (2.51) is the compatibility condition for the linear equations (2.35), (2.36), where the (2×2) matrices U and V are

$$\begin{gathered} U = -i \sum_{\alpha=1}^{3} w_\alpha(\lambda) S_\alpha(x,t) \sigma_\alpha, \\ V = -i \sum_{\alpha,\beta,\gamma} w_\alpha(\lambda) \sigma_\alpha S_\beta S_{\gamma x} \varepsilon^{\alpha\beta\gamma} - 2i \sum_{\alpha} a_\alpha(\lambda) S_\alpha \sigma_\alpha, \end{gathered} \tag{2.52}$$

where the σ_α are the Pauli matrices (W. Pauli), and

$$w_1 = \frac{\rho}{\operatorname{sn}(\lambda,k)}, \quad w_2 = \rho \frac{\operatorname{dn}(\lambda,k)}{\operatorname{sn}(\lambda,k)}, \quad w_3 = \rho \frac{\operatorname{cn}(\lambda,k)}{\operatorname{sn}(\lambda,k)},$$
$$a_1 = -w_2 w_3, \quad a_2 = -w_3 w_1, \quad a_3 = -w_1 w_2,$$

(where $\operatorname{sn}(\lambda,k)$, $\operatorname{cn}(\lambda,k)$, $\operatorname{dn}(\lambda,k)$ are Jacobi's elliptic functions [11]).

The parameters J_α are given by the relations

$$k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}, \quad \rho = \frac{1}{2}\sqrt{J_3 - J_1}, \quad 0 < k < 1. \tag{2.53}$$

EXAMPLE 9. Another interesting example are the equations of an anisotropic O(3)-field [29]

$$u_\xi = [u, Jv], \quad v_\eta = [v, Ju], \tag{2.54}$$

where u and v are currents: $u = g_\eta g^{-1}$, $v = g_\xi g^{-1}$.

The Lagrangian of this model is equal to

$$\mathcal{L} = -\frac{1}{2} \int \operatorname{tr}(g_\xi g^{-1} J g_\eta g^{-1})\, dx. \tag{2.55}$$

The equations (2.54) are equivalent to (2.37), where U and V are closely connected with the pair for the Landau–Lifshitz equation and, just as this latter pair, contain an elliptic dependence on the spectral parameter.

EXAMPLE 10. For a long time the commutational representation

$$\dot{L} = [M, L] \tag{2.56}$$

of Lax type for the equations of motion of the *Moser–Calogero system*[2] (J. Moser–F. Calogero) appeared to be a special case.

This is a system of particles x_n on the line with the Hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + 2 \sum_{i<j} \wp(x_i - x_j), \tag{2.57}$$

[2]Recently this system and its discrete versions has attracted a special interest due to its deep relations to two- and four-dimensional gauge theories and relations with quantum groups [KBBT], [KZ], [KLWZ].

where $\wp$ is the Weierstrass function. For this system L and M are matrices generally not depending on a spectral parameter, and have the form:

$$L_{ij} = p_i\delta_{ij} + (1-\delta_{ij})\,\phi(x_i - x_j), \tag{2.58}$$

$$M_{ij} = \left(\sum_{l\neq i} z(x_i - x_j)\right)\delta_{ij} - (1-\delta_{ij})\,y(x_i - x_j), \tag{2.59}$$

$$y(\xi) = \phi'(\xi), \quad z(\xi) = -\frac{\phi''}{2\phi}, \quad \phi(\xi) = \frac{1}{\operatorname{sn}(\xi, k)}.$$

The representations (2.56) are sufficient for the construction of integrals of the system (2.57), equal to $J_k = (1/k)\operatorname{tr} L^k$, but they are insufficient for the explicit construction of angle-type variables and the integration of the equations of motion in terms of theta functions.

As will be shown in § 5, the matrices L and M admit the introduction of a spectral parameter on an elliptic curve, but here the matrix entries turn out not to be meromorphic functions, but to have an exponential essential singularity.

The examples which have been brought by far do not exhaust all systems to which the inverse scattering method is applicable. A series of important examples will be cited and analyzed in detail in the following sections. Meanwhile, to conclude this section let us underline once more those basic features which are characteristic for systems to which the inverse method is applicable.

First, all such $(1+1)$ systems are equivalent to the compatibility condition (2.37) for the pair of linear problems (2.36), where $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are meromorphic functions of the "spectral parameter λ" (defined in the basic examples on a rational or elliptic curve). Second, each such equation is included in a whole hierarchy of flows which commute with it. The commutativity of the flows allows one to restrict the original system to the set of stationary points of any other flow which enters into the same hierarchy.

Restriction of the KdV equation to the stationary points of the "n-th higher-order analogue" of the KdV equation served as the starting point in the construction of the theory of *finite gap Sturm–Liouville operators* and in the further development of algebraic-geometric integration methods which are applicable to all systems admitting commutational representations [111], [45].

Surveys of the various stages in the development of the theory of *finite gap* or "algebraic-geometric" *integration* can be found in [40], [41], [76], [83], [86], [113], [115].

The stationary points of the "n-th higher-order analogue of the KdV equation" are described by the ordinary differential equation

$$\sum_{k=1}^{n} h_k Q_{2k+1}(u, \dots, u^{(2k+1)}) = 0, \tag{2.60}$$

which is equivalent to the condition of commutation of the Sturm–Liouville operator L and an operator L_1 of order $2n+1$

$$[L, L_1] = 0. \tag{2.61}$$

These equations, as was shown in [111], are a completely integrable Hamiltonian system. An important corollary of this result is that the generic solution of the

purely algebraic problem (2.61) turns out to be periodic and quasiperiodic in the variable x. It defines "finite-gap" Schrödinger Operators in the sense of Spectral Theory on the line. This discovery made by S. P. Novikov in 1974 became an initial point of the periodic soliton theory.

For general Lax equations (2.2) the condition which picks out the algebraic-geometric solutions of these equations also has the form (2.61), where L is the operator (2.4) which enters into the original Lax pair and L_1 is an auxiliary operator. Equation (2.61) describes an invariant submanifold of the initial equation (2.2). When we increase the order of L_1 we obtain an ever-ascending family of such submanifolds, which in a number of cases (for example, for the KdV equation) are everywhere dense in the space of periodic solutions.

Remark. The problem of classifying commuting linear differential operators with scalar coefficients was considered from a purely algebraic point of view in the papers of J. Burchnall and T. Chaundy [26]. It was shown by them that for such operators there can be found a polynomial $Q(\lambda, \mu)$ in two variables such that

$$Q(L, L_1) = 0. \tag{2.62}$$

In the case of operators of relatively prime orders, to each point of the curve Γ given by the equation $Q(\lambda, \mu) = 0$ there corresponds a joint eigenfunction $\psi(x, P)$, $P = (\lambda, \mu)$, unique up to proportionality, of the operators L and L_1, i. e.

$$L\psi(x, P) = \lambda\psi(x, P), \quad L_1\psi(x, P) = \mu\psi(x, P). \tag{2.63}$$

The logarithmic derivative $\psi_x\psi^{-1}$ is a meromorphic function on Γ which generically has g poles $\gamma_1(x), \ldots, \gamma_g(x)$, where g is the genus of the curve Γ (the remaining poles do not depend on x).

In the paper [26] it was shown that commuting operators of relatively prime orders are uniquely determined by the polynomial Q and the assignment of the generic points $\gamma_1(x_0), \ldots, \gamma_g(x_0)$, although finite formulas were not obtained. A program for effectivization of these results was proposed by H. Baker [9]. Unfortunately, Baker's program was never realized and in the course of a long time these papers were undeservedly forgotten.

As has already been said, the equations (2.61) describe the invariant submanifolds of equations of Lax type. These equations were considered from this point of view in the papers [74], [75], in which the results of the twenties were made significantly more effective and were generalized to the case of operators with matrix coefficients. For the coefficients of commuting scalar operators of relatively prime order, explicit expressions in terms of Riemann theta functions were found in these papers which showed that the general solutions of the equations (2.61) in this case are quasi-periodic functions. In 60s Dixmier wrote explicitly a pair of commuting operators of the orders 4 and 6 with polynomial coefficients connected by the equation of the nonsingular elliptic curve. No theta-functions! Investigation of the higher rank problem was started in [35] as a continuation of [74], [75]. A classification was obtained in [78]. For the effective classification (i. e. calculation of coefficients) new method of "the KP deformations of Tyurin parameters" was developed in [84]–[86]. All problems were solved for genus 1, rank 2 and 3 (see [86], [GR?], [MO?])

In the general case the solutions of (2.63) form an r-dimensional linear space, where r is a divisor of the orders of the operators L and L_1. It has been shown that

a ring of commuting operators $\mathcal{A}$ is determined by a curve Γ and a matrix divisor of rank r. The reconstruction of the coefficients of the operators from these data comes down to a linear Riemann problem.

In individual cases, as was shown in [85], it is possible to eliminate the necessity of solving a Riemann problem and to obtain explicit formulas for the operators L and L_1 (the formula for an operator L of order 4 which commutes with an operator of 6th order was quoted above (2.49)).

Let us give a general definition of finite gap solutions for the equations of zero curvature (2.37), which generalizes condition (2.61) in a natural way.

We shall speak of "*finite gap*" or algebraic-geometric *solutions* of equations admitting a commutational representation to mean solutions for which a matrix-valued function $W(x,t,\lambda)$ can be found, depending meromorphically on the parameter λ (which is defined on the same curve as the parameter in U and V) such that

$$\left[\frac{\partial}{\partial x} - U(x,t,\lambda), W(x,t,\lambda)\right] = 0, \tag{2.64}$$

$$\left[\frac{\partial}{\partial t} - V(x,t,\lambda), W(x,t,\lambda)\right] = 0. \tag{2.65}$$

The equations (2.64), (2.65), which in a definition like this one play only an auxiliary role in the process of integrating the original equation (2.37), are also of independent interest, as will be evident in the sequel. To them one can reduce practically all interesting examples of finite-dimensional Hamiltonian systems which are integrable by the inverse scattering method.

2. Algebraic-Geometric Integrability of Finite-Dimensional λ-Families

The basic goal of this section is the presentation of a procedure for integrating equations (2.64) and (2.65).

Let us denote by $\Psi(x,t,\lambda)$ the fundamental solution matrix of the equation

$$\begin{aligned}\left(\frac{\partial}{\partial x} - U(x,t,\lambda)\right)\Psi(x,t,\lambda) = 0,\\ \left(\frac{\partial}{\partial t} - V(x,t,\lambda)\right)\Psi(x,t,\lambda) = 0,\end{aligned} \tag{2.66}$$

normalized by the condition

$$\Psi(0,0,\lambda) = 1. \tag{2.67}$$

It follows from (2.64) that

$$W(x,t,\lambda)\,\Psi(x,t,\lambda) \tag{2.68}$$

is also a solution of equation (2.66). Any solution of (2.66) is uniquely determined by its initial conditions and has the form $\Psi(x,t,\lambda)\,G(\lambda)$, where G does not depend on (x,t). Taking the normalization condition (2.67) into account, we get that

$$W(x,t,\lambda)\,\Psi(x,t,\lambda) = \Psi(x,t,\lambda)\,W(0,0,\lambda). \tag{2.69}$$

Consequently, the coefficients of the characteristic equation

$$Q(\lambda,\mu) = \det(W(x,t,\lambda) - \mu\cdot 1) = 0 \tag{2.70}$$

are integrals of the equations (2.64), (2.65). They are polynomials in the matrix entries of W, or, as was explained in the preceding section, they are differential polynomials of the basic phase variables—the matrix entries of $U(x,t,\lambda)$.

Generically equation (2.70) defines a nonsingular *algebraic curve* Γ (i. e. a compact Riemann surface), to each point of which (i. e. pair (λ,μ)) there corresponds an eigenvector h, unique up to proportionality, of the matrix W

$$W(x,t,\lambda)\,h(x,t,\gamma)=\mu h(x,t,\gamma);\quad \gamma=(\lambda,\mu)\in\Gamma. \tag{2.71}$$

Remark. The notion of genericity in the question at hand is not completely trivial. Although the levels of the integrals of equation (2.64) for which the eigenvalues of W are r-fold degenerate identically in λ (r is obliged to be a divisor of l, where W is an $l\times l$ matrix) unconditionally have nonzero codimension, in the procedure for reconstructing the matrices U and V from these data additional functional parameters appear, as is shown by the results of the papers [36], [78]. The elaboration of this direction, which has led to the construction of finite gap solutions of rank $r>1$ for two-dimensional equations of the Kadomtsev–Petviashvili type depending on functional parameters (see [84], [85], [86]), can be found here in the example 7 above for the special case $r=2$, $g=1$ where this problem was solved effectively.

Let us normalize the vector h by requiring, for example, that $h_1(x,t,\gamma)=1$ or $\sum h_i(x,t,\gamma)=1$. In both cases all of the coordinates $h_i(x,t,\gamma)$ are rational functions of λ and μ, i. e. meromorphic functions on the curve Γ.

If $\Psi^j(x,t,\lambda)$ are the columns of the matrix $\Psi(x,t,\lambda)$, then it follows from (2.69) and (2.71) that the vector function

$$\psi(x,t,\gamma)=\sum_j h_j(0,0,\gamma)\,\Psi^j(x,t,\lambda);\quad \gamma=(\lambda,\mu), \tag{2.72}$$

simultaneously satisfies the equations

$$\left(\frac{\partial}{\partial x}-U(x,t,\lambda)\right)\psi(x,t,\gamma)=\left(\frac{\partial}{\partial t}-V(x,t,\lambda)\right)\psi(x,t,\gamma)=0, \tag{2.73}$$

$$W(x,t,\lambda)\,\psi(x,t,\gamma)=\mu\psi(x,t,\gamma). \tag{2.74}$$

The curve Γ l-foldly covers the curve $\hat\Gamma$ on which the parameter λ is defined. Outside the poles λ_i of the matrices U and V the matrix $\Psi(x,t,\lambda)$ is a holomorphic function of the parameter λ. Consequently, the vector function $\psi(x,t,\gamma)$ is meromorphic on Γ outside the points P_α—the preimages of the poles λ_i of the matrices U and V. (We observe that above the points λ the curve Γ may branch). The poles of $\psi(x,t,\gamma)$ coincide with the poles of $h(0,0,\gamma)$ and therefore do not depend on x,t.

Let us further restrict ourselves to the case of a *rational family* (i. e. λ is defined on the ordinary complex plane). Let us consider the matrix $H(x,t,\lambda)$ whose columns are the vectors $h(x,t,\gamma_i)$, where the $\gamma_u i=(\lambda,\mu_i)$ are the preimages of the point λ on Γ under the natural projection $\Gamma\to\mathbb{C}$, $(\lambda,\mu)\to\lambda$.

The function

$$r(x,t,\lambda)=(\det H(x,t,\lambda))^2 \tag{2.75}$$

does not depend on the numbering of the γ_i and is therefore a well-defined function of λ. Since the $h_i(x,t,\gamma)$ are meromorphic on Γ, $r(x,t,\lambda)$ is a rational function of λ. It has double poles at the images of the poles of $h(x,t,\gamma)$ and zeroes at the

points above which Γ branches. The number of poles of r is equal to the number of zeroes. Hence

$$2N = \nu, \tag{2.76}$$

where ν is the number of branch points with multiplicities counted. A formula is well-known which connects the genus g of a smooth curve Γ l-foldly covering $\mathbb{C}$ with the number of branch points [131]

$$2g - 2 = \nu - 2l. \tag{2.77}$$

Consequently, the number of poles of $h(x,t,\gamma)$ and hence also of ψ is equal to

$$N = g + l - 1. \tag{2.78}$$

Now let us find the behaviour of $\psi(x,t,\gamma)$ in the neighbourhood of the points P_α—the preimages of the poles of $U(x,t,\lambda), V(x,t,\lambda)$.

It follows from (2.71) and (2.74) that the vectors h and ψ are proportional

$$\psi(x,t,\gamma) = f(x,t,\gamma)\, h(x,t,\gamma). \tag{2.79}$$

Let us denote by $\tilde{\Psi}(x,t,\lambda)$ the matrix whose columns are the vectors $\psi(x,t,\gamma_i)$, $\gamma_i = (\lambda, \mu_i)$, and by $F(x,t,\lambda)$ the diagonal matrix $F_{ij}(x,t,\lambda) = f(x,t,\gamma_i)\delta_{ij}$. Then one can write (2.79) in the form

$$\tilde{\Psi}(x,t,\lambda) = H(x,t,\lambda)\, F(x,t,\lambda). \tag{2.80}$$

We have

$$\begin{aligned} U(x,t,\lambda) &= \tilde{\Psi}_x \tilde{\Psi}^{-1} = H_x H^{-1} + H F_x F^{-1} H^{-1}, \\ V(x,t,\lambda) &= \tilde{\Psi}_t \tilde{\Psi}^{-1} = H_t H^{-1} + H F_t F^{-1} H^{-1}, \end{aligned} \tag{2.81}$$

Without loss of generality we may suppose that H is regular and non-degenerate at the points λ_i. From this we get that $F_x F^{-1}$ coincides modulo $O(1)$ with the eigenvalues of the singular part of $U(x,t,\lambda)$ at λ_i; and similarly for F, F^{-1}.

Thus, in a neighbourhood of P_α

$$f(x,t,\gamma) = \exp(q_\alpha(x,t,k_\alpha)) f_\alpha(x,t,\gamma). \tag{2.82}$$

Here $k_\alpha^{-1}(P)$ is a local parameter in the neighbourhood of P_α, $k_\alpha^{-1}(P_\alpha) = 0$, $q_\alpha(x,t,k)$ is polynomial in k, and f_α is a regular function in the neighbourhood of P_α.

Summing up, we come to the following assertion.

THEOREM 2.1. *The vector function $\psi(x,t,\gamma)$*

1°. is meromorphic on Γ outside the points P_α. Its divisor of poles does not depend on x,t. If W is nondegenerate, then generically the curve Γ is nonsingular. The number of poles of ψ (counting multiplicity) is equal to $g + l - 1$, where g is the genus of the curve Γ.

2°. in a neighbourhood of the points P_α the function $\psi(x,t,\gamma)$ has the form:

$$\psi(x,t,\gamma) = \left(\sum_{x=0}^{\infty} \xi_{s\alpha}(x,t) k_\alpha^{-1} \right) \exp(q_\alpha(x,t,k_\alpha)), \tag{2.83}$$

where the first factor is the expansion with respect to the local parameter $k_\alpha^{-1} = k_\alpha^{-1}(\gamma)$ of a holomorphic vector, and $q_\alpha(x,t,k)$ is a polynomial in k.

The basic idea of the algebraic-geometric version of the inverse problem consists in reconstructing the vector $\psi(x,t,\gamma)$ from the enumerated analytic properties. The specific nature of these properties guarantees the existence of $U(x,t,\lambda)$ and $V(x,t,\lambda), W(x,t,\lambda)$ such that (2.66) and (2.69) hold. A consequence of the compatibility of these systems are the equations (2.37), (2.64), (2.65).

As has already been said, the development of the fundamental stages of the theory of finite gap integration is mirrored in detail in [115] and in the surveys [40], [41], [44], [45], [76], [83], [86], [113].

Before going over to the procedure for reconstructing ψ, let us quote what we need to know from the classical algebraic geometry of Riemann surfaces and the theory of theta functions.

An arbitrary compact *Riemann surface* can be given by an equation

$$R(\lambda,\mu)=\sum a_{ij}\lambda^i\mu^j, \tag{2.84}$$

where i,j run through some finite set of integers. Generically this curve will be nonsingular. The genus of this curve can be found conveniently with the aid of the so-called Newton polygon, which is what one calls the convex hull of the integer points with the coordinates i,j for which $a_{ij}\neq 0$ in (2.84). The genus of the curve is equal to the number of integer points lying within the Newton polygon.

A basis of the holomorphic differentials (of the first kind) on a nonsingular curve has the form

$$\eta_{ij}=\frac{\lambda^i\mu^j}{R_\mu(\lambda,\mu)}\,d\lambda, \tag{2.85}$$

where the i,j belong to the interior of the Newton polygon.

On the curve Γ one can choose a basis of cycles $a_1,\dots,a_g,b_1,\dots,b_g$ with the following intersection numbers

$$a_i\circ a_j=b_i\circ b_j=0,\quad a_i\circ b_j=\delta_{ij}. \tag{2.86}$$

By taking suitable linear combinations, we obtain a canonical basis of the holomorphic differentials

$$\omega_1,\dots,\omega_g \tag{2.87}$$

normalized by the conditions

$$\oint_{a_k}\omega_i=\delta_{ik},\quad i,k=1,\dots,g. \tag{2.88}$$

The matrix

$$B_{ik}=\oint_{b_k}\omega_i \tag{2.89}$$

is called the period matrix of the Riemann surface Γ. It is symmetric and has a positive definite imaginary part. The unit basis vectors in $\mathbb{C}^g$ and the vectors B_i with the coordinates B_{ik} generate a lattice in $\mathbb{C}^g$, the quotient by which is a $2g$-dimensional torus $T^{2g}=J(\Gamma)$, called the *Jacobi variety* (or Jacobian) of the curve Γ.

The Riemann theta function of the surface Γ is constructed in terms of the matrix B

$$\begin{gathered}\theta(z)=\sum_{N\in\mathbb{Z}^g}\exp(\pi i\langle BN,N\rangle+2\pi i\langle N,z\rangle),\\ z=(z_1,\dots,z_g),\quad N=(N_1,\dots,N_g),\\ \langle N,z\rangle=N_1z_1+\dots+N_gz_g,\quad \langle BN,N\rangle=\sum B_{ij}N_iN_j.\end{gathered}\tag{2.90}$$

This function is entire. Under translation of the argument by a vector of the lattice it is transformed according to the law

$$\theta(z+N+BM)=\exp(-\pi i(\langle BM,M\rangle+2\langle z,M\rangle))\,\theta(z),\quad N,M\in\mathbb{Z}^g.\tag{2.91}$$

Also often used are theta functions with characteristics

$$\theta[\alpha,\beta](z)=\exp(\pi i(\langle B\alpha,\alpha\rangle+2\langle z+\beta,\alpha\rangle))\,\theta(z+\beta+B\alpha),\quad \alpha,\beta\in\mathbb{R}^g.\tag{2.92}$$

Characteristics $[\alpha,\beta]$ for which all the coordinates of α,β equal 0 or $1/2$ are called half-periods. A half-period $[\alpha,\beta]$ is even if $4\langle\alpha,\beta\rangle=0 \pmod 2$, and odd otherwise.

The Abel map of a Riemann surface Γ into its Jacobi variety $A(P)=(A_1(P),\dots,A_g(P))$ is given in the following way

$$A_k(P)=\int_Q^P\omega_k,\tag{2.93}$$

where Q is a fixed point on Γ.

A *divisor* on Γ is a formal integer combination of points on Γ,

$$D=\sum n_iP_i,\quad n_i\in\mathbb{Z}.\tag{2.94}$$

For any function f meromorphic on Γ there is defined the divisor (f) of its zeroes $P_1,\dots,P_n$ and poles $Q_1,\dots,Q_m$ (with multiplicities $p_1,\dots,p_n$, $q_1,\dots,q_m$ respectively)

$$(f)=p_1P_1+\dots+p_nP_n-q_1Q_1-\dots-q_mQ_m\tag{2.95}$$

(such divisors are called principal).

The divisors form an abelian group. The degree of a divisor is the number

$$\deg D=\sum n_i,\tag{2.96}$$

The Abel map (N. H. Abel) (2.93) can be extended linearly to the group of all divisors.

A divisor for which all $n_i>0$ is called a positive divisor $D\geq 0$ (or an effective divisor).

For any divisor D the linear space $l(D)$ associated with it is the space of meromorphic functions f on Γ such that

$$(f)+D>0.$$

The dimension of this space is given by the Riemann–Roch theorem [131]. For a divisor of degree greater than or equal to g,

$$\dim l(D)\geq\deg D-g+1.\tag{2.97}$$

For generic divisors (2.97) is an equality. The corresponding divisors are called non-special.

Let us consider the *Abel map*[3] of unordered sets $P_1, \dots, P_g$ of points of Γ, i. e. of the g-th symmetric power of Γ

$$A\colon S^g\Gamma \to J(\Gamma), \quad A(P_1, \dots, P_n) = \sum_i A(P_i). \tag{2.98}$$

The problem of inverting this map is known as the Jacobi inversion problem. Its solution (Riemann) can be given in the language of theta functions. Namely, if for the vector $\zeta = (\zeta_1, \dots, \zeta_g)$ the function $\theta(A(P) - \zeta)$ is not identically equal to zero on Γ, then it has on Γ exactly g zeroes $P_1, \dots, P_g$, giving the solution of the inversion problem

$$A(P_1) + \dots + A(P_g) = \zeta - \mathcal{K}, \tag{2.99}$$

where $\mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_g)$ is the vector of Riemann constants [40], which depend only on the Riemann surface, the choice of the basis of cycles on it, and the initial point of the Abel map.

Now we are ready to pass over to the solution of the inverse problem of reconstructing the "eigen"-vector $\psi(x, t, \gamma)$ of the operators of (2.64), (2.65) from its analytic properties.

The fundamental algebraic-geometric tool in the theory of finite gap linear operators and in the algebraic-geometric version of the inverse scattering method are the so-called Baker–Akhiezer functions. The general definition of these functions, including the multi-point ones, was given in [75] on the basis of a generalization of the analytic properties of Bloch eigenfunctions of operators with periodic and almost periodic coefficients [38], [45], [64]. Multipoint functions are functions which have essential singularities of exponential type at several points. Single-point function of this kind (of the exponential type) appears as a joint eigenfunction of a pair of scalar commuting operators of relatively prime order as it was pointed out by A. Baker in the work [9]. He mentioned that it can be computed through the theta-functions using material of the book [8] but his program never has been carried out. The theta-functional formula for this special case was obtained first by A. Its in 1976 (see Appendix in [45]). N. I. Akhiezer [5] stated examples of interpretation of such functions in the spectral theory of operators on the half-line. The connection with periodic problem was not known until 1974.

Definition. Let $P_1, \dots, P_n$ be points on a Riemann surface Γ of genus g; let $k_\alpha^{-1}(P)$ be local parameters in the neighbourhood of these points, $k_\alpha^{-1}(P_\alpha) = 0$, $\alpha = 1, \dots, n$; let $q_1(k), \dots, q_n(k)$ be a set of polynomials; and let D be a divisor on Γ. An *n-point Baker–Akhiezer function* given by these data is a function: a) meromorphic on Γ outside the points P_α, with the divisor of its poles and zeroes (ψ) satisfying the condition $(\psi) + D \geq 0$; and such that b) for $P \to P_\alpha$ the product $\psi(P)\exp(-q_\alpha(k_\alpha(P)))$ is analytic.

Theorem 2.2. *For a non-special divisor D of degree N the dimension of the linear space of functions with the enumerated properties is equal to $N - g + 1$. In particular, if D is a generic set of g points, then ψ is uniquely determined up to a factor. It has the form*:

$$\psi(P) = c\exp\left(\sum_{\alpha=1}^{n}\int_Q^P \Omega_{q_\alpha}\right)\frac{\theta\left(A(P) + \sum_\alpha U^{(q_\alpha)} - \zeta\right)}{\theta(A(P) - \zeta)}. \tag{2.100}$$

[3]Also often called the *Jacobi map* (translator's note).

Here Ω_{q_α} is a normalized abelian differential of the second kind with a principal part at the point P_α of the form $dq_\alpha(k_\alpha(P))$ (normalization means

$$\oint_{a_1} \Omega_{q_\alpha} = 0; \tag{2.101}$$

with this condition Ω_{q_α} exists and is unique); the vector $2\pi i U^{(q_\alpha)}$ the vector of b-periods of the differential Ω_{q_α}; $\zeta = A(D) + \mathcal{K}$.

The proof of formula (2.100) amounts to checking that it correctly defines a function on Γ. Changing the path of integration from Q to P leads to a translation of the arguments of the theta functions by a vector of the period lattice, $N + BM$. The exponent of the exponential is translated by $2\pi i \left\langle \sum_\alpha U^{(q_\alpha)}, M \right\rangle$.

From (2.91) it follows that the value of $\psi(P)$ does not depend on the choice of the path of integration. From (2.100) it follows that the function possesses all the necessary analytic properties. □

By virtue of theorem 2.1, to each finite gap solution of rank 1 of the equations (2.37), i. e. solution of the system (2.64), (2.65), there is associated a Riemann surface Γ, which generically can be considered to be nonsingular, a set of polynomials $q_\alpha(x, t, k)$ and a non-special divisor of degree $g + l - 1$, where g is the genus of the curve Γ. Let us make use of theorem 2.2 for the construction of the inverse mapping.

So, let there be given the set of data enumerated above. In the linear space of Baker–Akhiezer functions corresponding to these data let us choose an arbitrary basis $\psi_i(x, t, P)$ (the polynomials $q_\alpha(x, t, k)$ depend on x and t as parameters; the ψ_i also will obviously depend on the same parameters).

By theorem 2.2 one may choose for the ψ_i the functions given by formula (2.100), in which ζ has been set equal to

$$\zeta_i = \sum_{s=1}^{g-1} A(P_s) + A(P_{g-1+i}) + \mathcal{K}. \tag{2.102}$$

THEOREM 2.3. *Let $\psi(x, t, P)$ be the vector function whose coordinates are the $\psi_i(x, t, P)$ constructed above. There exist unique matrix functions $U(x, t, \lambda)$, $V(x, t, \lambda)$, $W(x, t, \lambda)$, rational in λ, such that*

$$\partial_x \psi = U\psi, \quad \partial_t \psi = V\psi, \quad W\psi = \mu\psi, \quad P = (\lambda, \mu). \tag{2.103}$$

For the proof of the theorem it is enough to consider the matrix $\tilde{\Psi}(x, t, \lambda)$ whose columns are the vectors $\psi(x, t, P_j)$, $P_j = (\lambda, \mu_j)$. This matrix depends on the numbering of the columns (i. e. of the points P_j); however, the matrices

$$(\partial_x \tilde{\Psi})\tilde{\Psi}^{-1}, \quad (\partial_t \tilde{\Psi})\tilde{\Psi}^{-1}, \quad \tilde{\Psi}\mu\tilde{\Psi}^{-1} \tag{2.104}$$

are already well-defined (i. e. do not depend on this numbering) and by virtue of the analytic properties of ψ they are rational functions of λ. These matrices are designated by U, V, W respectively. Here $\hat{\mu}$ is the diagonal matrix equal to $\hat{\mu}_{ij} = \mu_i \delta_{ij}$.

By using the path of the proof of equation (2.78) in the opposite direction, we get that $\det \tilde{\Psi} \neq 0$ if λ is not a branch point. From this it follows that U and V have poles only at the projections of the distinguished points P_α, and W only at the projections of the points on Γ where μ has poles.

COROLLARY. *The matrices U, V, W constructed by the formulas* (2.104) *satisfy the equations* (2.37), (2.64), (2.65).

REMARK. The formulas (2.104) give the most economical way of proving the theorem in general, relating to arbitrary rational families. However in a majority of cases, especially those corresponding to reductions of equations, the explicit computation of the matrices U, V, W can be carried out from the requirement that in the neighbourhoods of the P_α there should hold the congruences:

$$\partial_x \psi(x,t,P) \equiv U(x,t,\lambda)\,\psi(x,t,P) \pmod{O(1)\exp q_\alpha(x,t,k_\alpha)}$$

(and the analogous congruences for V and W). Here the matrix entries of U, V, W turn out to be differential polynomials in the $\xi_{s\alpha}(x,t)$ the expansion (2.83) of the regular part of ψ at the point P_α. This path will be traced in detail later on in examples of the construction of finite gap solutions of equations of the Lax type (see [40], [74], [75], [76]).

In the construction of the vector $\psi(x,t,P)$ from the set of data given before theorem 2.3 there is an arbitrariness connected with the possibility of choosing different bases ψ_i, in the linear space of Baker–Akhiezer functions corresponding to the polar divisor D.

To this arbitrariness, under which $\psi(x,t,P)$ goes over into $g(x,t)\psi(x,t,P)$, where g is a nondegenerate matrix, there corresponds a gauge symmetry (2.38) of the equations (2.37), (2.64), (2.65) (the matrix W goes over under such a transformation into

$$W \to gWg^{-1}). \tag{2.105}$$

Let us consider two vector functions $\psi(x,t,P), \tilde{\psi}(x,t,P)$, corresponding to two equivalent divisors D and $\tilde{D}$. The equivalence of these divisors means that there exists a meromorphic function $f(P)$ such that its poles coincide with D and its zeroes with $\tilde{D}$. From the definition it follows that the components of $f\tilde{\psi}$ possess the same analytic properties as the components of the vector function $\psi(x,t,P)$. Hence

$$\psi(x,t,P) = g(x,t)\,f(P)\,\tilde{\psi}(x,t,P), \tag{2.106}$$

and the functions ψ and $\tilde{\psi}$ define gauge-equivalent solutions.

We shall consider both the equations (2.37), (2.64), (2.65) and their solutions up to gauge transformations (2.38), (2.105). From (2.105) and the definition of Γ (2.70) it follows that the gauge transformations leave the curves Γ invariant.

THEOREM 2.4. *The set of finite gap solutions* (*considered up to gauge equivalence*) *corresponding to a nonsingular curve Γ is isomorphic to a torus—$J(T)$—the Jacobi variety of this curve.*

The assertion of the theorem follows from the fact that by virtue of the well-known theorem of Abel two divisors are equivalent if and only if

$$\deg D = \deg \tilde{D}, \quad A(D) \equiv A(\tilde{D}).$$

The congruence sign means congruence modulo periods of the Jacobian of the curve Γ.

The coefficients of the polynomial $Q(A,p)$ are integrals of the equations (2.37), (2.64). The theorem just formulated means that the level set of these integrals is generically a torus.

For special values of the integrals, for which the surface Γ has singularities, the level manifold of these integrals is isomorphic to the generalized Jacobian of the curve, which is the product of a torus with a linear space.

To multisoliton and rational solutions of the equations (2.37) correspond rational curves with singularities. To the different singularity types there also correspond different solution types. For example, in the case of singularities of the self-intersection type multisoliton solutions are obtained (see, for example, for the KdV equation § 3, [115]), and in the case of singularities of the "cusp" type rational solutions are gotten.

Let us consider at greater length a number of examples connected with *hyperelliptic curves*. As has already been said, finite gap solutions of the KdV equation are the restriction of this equation to the stationary points of one of the higher-order analogues of the KdV equation. They satisfy an ordinary differential equation equivalent to the operator equation

$$[L, A_n] = 0, \tag{2.107}$$

where A_n is a differential operator of order $2n + 1$. As was shown above, this equation admits the λ-representation (2.32), where U_L has the form (2.33), and

$$W_A = \begin{pmatrix} 0 & \lambda^n \\ \lambda^{n+1} & 0 \end{pmatrix} + O(\lambda^{n-1}). \tag{2.108}$$

Hence the characteristic equation (2.70),

$$\det(\mu \cdot 1 - W_A(x, t, \lambda)) = \mu^2 - R_{2n+1}(\lambda) = 0, \tag{2.109}$$

gives a hyperelliptic curve Γ. The coefficients of the polynomial

$$R_{2n+1} = \lambda^{2n+1} + \sum_{i=1}^{2n+1} r_i \lambda^{2n+1-i}$$

are polynomials in $u, u', \ldots, u^{(2n+1)}$ and, in the proved fashion, integrals of the equation (2.107).

The curve Γ may be represented as being glued together out of two copies of the λ plane along cuts joining the E_i—the zeroes of the polynomial R_{2n+1}—and the point at infinity $E = \infty$.

For real $u(x)$ the E_i are the simple points of the spectrum of the periodic and antiperiodic problems for the *Sturm–Liouville operator* (J. C. F. Sturm–J. Liouville)

$$L = -\frac{d^2}{dx^2} + u(x), \tag{2.110}$$

the segments $[E_{2i-1}, E_{2i}]$ are the allowed bands of the spectrum of L on the entire axis, and the $[E_{2i}, E_{2i+1}]$, $i = 1, \ldots, n$ are the forbidden bands[4]. (For more details about the spectral theory of finite gap operators see § 6.)

As the a-cycles it is convenient to choose the cycles situated above the forbidden bands, and as the b-cycles, cycles encompassing the segment $[E_1, E_{2i}]$ of the real axis.

[4]In English, the allowed and forbidden bands are often called the *stable* and *unstable bands* respectively (translator's note).

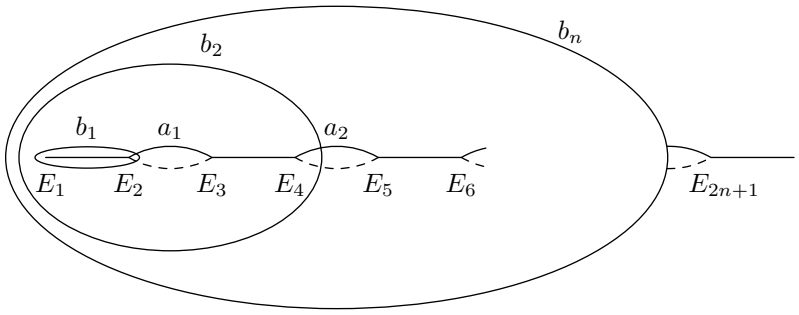

FIGURE 1

Let $\psi(x,t,P)$ be the Baker–Akhiezer function having n poles (n is the genus of Γ) $\gamma_1,\dots,\gamma_n$, and in a neighbourhood of $P_0=\infty$ the form

$$\psi(x,t,P)=\exp(kx+k^3t)\left(1+\sum_{s=1}^{\infty}\xi_s(x,t)k^{-s}\right), \tag{2.111}$$

$k=\sqrt{\lambda}$.

By theorem 2.2 it exists and is unique. Let $u(x,t)=2\xi_1'$. Then a straight substitution of (2.111) gives

$$(-\partial_x^2+u(x,t)+\lambda)\,\psi(x,t,P)=e^{kx+k^3t}O(k^{-1}). \tag{2.112}$$

The function $\tilde\psi(x,t,P)$, equal to the left-hand side of (2.112), satisfies all the requirements defining ψ except one. Its expansion (2.111) in the neighbourhood of P_0 begins with $\tilde\xi_1k^{-1}+\dots$. From the uniqueness of ψ it follows that $\tilde\psi=0$.

Analogously, there exist unique functions $v_1(x,t)$ and $v_2(x,t)$ such that

$$A\psi-\partial_t\psi=O(k^{-1})e^{kx+k^3t}, \tag{2.113}$$

where

$$A=\partial_x^3+v_1\partial_x+v_2. \tag{2.114}$$

From (2.113) we have

$$v_1=-3\xi_1'=\tfrac{3}{2}u.$$

The compatibility of (2.113) and (2.112) is equivalent to the KdV equation.

The normalized *abelian differentials of the second kind* having poles at P_0 of second and fourth order are representable in the form:

$$\Omega^{(2)}=\frac{\lambda^n+\sum_{i=1}^n r_i\lambda^{n-i}}{\sqrt{R_{2n+1}(\lambda)}}\,d\lambda, \tag{2.115}$$

$$\Omega^{(4)}=\frac{\lambda^{n+1}-s_1\lambda^n+\sum_{i=1}^n \tilde r_i\lambda^{n-i}}{\sqrt{R_{2n+1}(\lambda)}}\,d\lambda, \tag{2.116}$$

where $R_{2n+1}(\lambda)=\prod_{i=1}^{2n+1}(\lambda-E_i)$, $s_1=\sum_{i=1}^{2n+1}E_i$.

The coefficients $r_i,\tilde r_i$ are determined by the normalization conditions

$$\int_{E_{2i}}^{E_{2i+1}}\Omega^{(2)}=\int_{E_{2i}}^{E_{2i+1}}\Omega^{(4)}=0,\quad i=1,\dots,n. \tag{2.117}$$

By theorem 2.2

$$\psi(x,t,P)=c\exp\left(x\int_{E_1}^{P}\Omega^{(2)}+t\int_{E_1}^{P}\Omega^{(4)}\right)\frac{\theta(A(P)+Ux+Vt-\zeta)}{\theta(A(P)-\zeta)},\quad c=c(x,t), \tag{2.118}$$

where

$$\pi iU_k=\int_{E_1}^{E_{2k}}\Omega^{(2)}, \tag{2.119}$$

$$\pi iV_k=\int_{E_1}^{E_{2k}}\Omega^{(4)}. \tag{2.120}$$

If we choose E_1 as the initial point for the Abel mapping, then after an explicit calculation of the vector of Riemann constants we get

$$\zeta_k=\sum_{i=1}^{n}\int_{E_{2i}}^{\gamma_i}\omega_k. \tag{2.121}$$

In the neighbourhood of P_0 we have

$$A(P)=-Uk^{-1}+O(k^{-2}) \tag{2.122}$$

and by expanding (2.118) we arrive finally at the Matveev–Its formula [64] for the finite gap solutions of the equation

$$u(x,t)=-2\partial_x^2\ln\theta(Ux+Vt-\zeta)+\text{const}. \tag{2.123}$$

Remark 1. The above construction of finite gap solutions of the KdV equation was carried out with the aid of the original commutational representation without an explicit transition to the λ-representation (2.32) in (2×2) matrices. For a comparison with theorems 2.1, 2.2 let us indicate only that the components of the vector ψ which figures in their formulation are given by $\psi(x,t,\lambda)$ and $\psi_x(x,t,\lambda)$. The divisor of poles of ψ coincides with the divisor of poles of $\psi(x,t,\lambda)$ together with the point P_0.

Let us also indicate that theorem 2.3 associates with the algebraic-geometric spectral data $\Gamma,\gamma_1,\dots,\gamma_g$, besides the operators L and A, a matrix W which is $W_{A_n}=$ the λ-representation of the operator A_n of (2.107). The operator A_n itself can be recovered from these data in an analogous way to the construction of the operator L.

Its coefficients can be determined uniquely from the congruence

$$A_n\psi\equiv\mu\psi\pmod{e^{kx+k^3t}O(k^{-1})},\quad \mu=k^{2n+1}+a_1k^{2n-1}+\dots.$$

From the congruence there follows the exact equality

$$A_n\psi(x,t,P)=\mu(P)\,\psi(x,t,P);\quad(\mu,\lambda)=P.$$

Remark 2. In a number of applications the equations of motion of the zeroes $\gamma_i(x,t)$ of the *Bloch function* $\psi(x,t,P)$, which were first obtained in [38], turn out to be useful.

For this let us consider the functions $\psi_x\psi^{-1}$ and $\psi_t\psi^{-1}$. The function $\psi_x\psi^{-1}$ has poles at the points $\gamma_i(x,t)$ and the form $k+O(k^{-1})$ in the neighbourhood of P_0. Hence it is representable unambiguously in the form

$$\frac{\psi_x}{\psi}=\frac{\sqrt{R(\lambda)}+P(x,t,\lambda)}{\prod_{i=1}^{n}(\lambda-\gamma_i(x,t))},$$

where $P(x,t,\lambda)$ is a polynomial of degree $(n-1)$. It is uniquely determined by the fact that $\psi_x\psi^{-1}$ has a pole over $\lambda=\gamma_i$ only for one sign of $\sqrt{R}$ (for example, for the sign plus). Hence

$$P(x,t,\gamma_i(x,t))=\sqrt{R(\gamma_i(x,t))}.$$

In the neighbourhood of a pole of $\psi_x\psi^{-1}$ we have

$$\frac{\psi_x}{\psi}=\frac{\gamma_i'(x,t)}{\lambda-\gamma_i(x,t)}+O(1),\quad \gamma_i'=\frac{\partial}{\partial x}\,\gamma_i.$$

Comparing the preceding equalities, we finally find that

$$\gamma_i'(x,t)=\frac{2\sqrt{R(\gamma_i(x,t))}}{\prod_{j\neq i}(\gamma_i(x,t)-\gamma_j(x,t))}.$$

Analogously

$$\frac{\psi_t}{\psi}=\frac{\lambda\sqrt{R(\lambda)}+P_1(x,t,\lambda)}{\prod_i(\lambda-\gamma_i(x,t))},$$

and, repeating the derivation of the equations for the γ_i', we get

$$\dot\gamma_i(x,t)=\frac{2\gamma_i\sqrt{R(\gamma_i(x,t))}}{\prod_{j\neq i}(\gamma_i(x,t)-\gamma_j(x,t))},\quad \dot\gamma_i=\frac{\partial}{\partial t}\,\gamma_i.$$

The Abel isomorphism (2.98) linearizes these equations on the Jacobian $J(\Gamma)$.

As a second example let us consider the construction of finite gap solutions of the sine-Gordon equation (2.44), which were first obtained in [68].

It follows from (2.64), (2.65) that $W(\xi,\eta,0)$ commutes with the singular part of U at $\lambda=0$; $W(\xi,\eta,\infty)$ commutes with the singular part of V at the point $\lambda=\infty$. Hence the hyperelliptic curve Γ corresponding to a finite gap solution of the sine-Gordon equation has branching at the points $\lambda=0$, $\lambda=\infty$.

Without retracing word for word the course of the proof of theorem 2.1, let us give the form of the Baker–Akhiezer vector functions for this equation. The components $\psi_i(\xi,\eta,P)$ have n poles γ_i outside the branch points P_+ and P_-, situated above $\lambda=0$, $\lambda=\infty$. In a neighbourhood of these points

$$\psi_1^{\pm}=e^{k(x\pm t)}\left(\sum_{s=0}^{\infty}\chi_{s1}^{\pm}(\xi,\eta)k_{\pm}^{-s}\right), \tag{2.124}$$

$$\psi_2^{\pm}=e^{k(x\pm t)}k^{\pm1}\left(\sum_{s=0}^{\infty}\chi_{s2}^{\pm}(\xi,\eta)k_{\pm}^{-s}\right), \tag{2.125}$$

$$\xi=x+t,\quad \eta=x-t,\quad k_{\pm}=\lambda^{\mp1/2}.$$

The functions ψ_i are determined uniquely by the normalization $\chi_{0i}^{+}\equiv1$. (The divisor D of degree $n+l-1=n+1$ is equal to $\gamma_1+\cdots+\gamma_n+P_+$.)

It follows from the definitions of ψ_1 and ψ_2 that $\partial_\eta\psi_1$ and $\lambda\psi_2$ have the same analytic properties. So they are proportional. For the computation of the constant of proportionality one must compare the coefficients of the term $\lambda^{1/2}$ in the expansions of these functions at P_+. We have

$$\partial_\eta\psi_1=e^{-iu}\lambda\psi_2,\quad e^{-iu}=\frac{\chi_{01}^{-}}{\chi_{02}^{-}}. \tag{2.126}$$

Analogously,

$$\partial_\eta \psi_2 = e^{iu}\psi_1. \tag{2.127}$$

In the same way it can be shown that

$$\partial_\xi \psi_1 = \frac{iu_\xi}{2}\,\psi_1 + \psi_2, \tag{2.128}$$

$$\partial_\xi \psi_2 = \lambda^{-1}\psi_1 + \frac{iu_\xi}{2}\,\psi_2, \tag{2.129}$$

COROLLARY. *The function $u(\xi, t)$ defined out of* (2.126) *is a solution of the sine-Gordon equation.*

Let us find its explicit appearance. It can be shown analogously to theorem 2.2 that

$$\psi_n(x,t,P) = r_n(\xi,\eta)\cdot\exp\left(\xi\int_Q^P \Omega_+^{(2)} + \eta\int_Q^P \Omega_-^{(2)} + n\int_Q^P \Omega_{+-}\right) \times \frac{\theta(A(P)-\zeta+U^+\xi+U^-\eta+Vn)}{\theta(A(P)-\zeta)}. \tag{2.130}$$

Here the $\Omega_\pm^{(2)}$ are normalized abelian differentials with poles of second order at the points $P_\pm, Q_{+-}$ is a differential of the third kind with the residues ± 1 at $P_\pm$, $2\pi i U^\pm$, $2\pi i V$ are the vectors of b-periods of these differentials.

The factor $r_n(\xi,\eta)$ is chosen via the condition that the multiplier in front of the exponential be equal to one at the point P_+. Then χ_{0n}^- equals the value of this multiplier at P_-.

After simple computations we finally arrive at the following expression for the finite gap solutions

$$e^{iu} = \text{const}\frac{\theta^2(U^+\xi+U^-\eta-\zeta)}{\theta(U^+\xi+U^-\eta-\zeta+V)\theta(U^+\xi+U^-\eta-\zeta-V)}. \tag{2.131}$$

Let us note that the vector V is equal to a half-period, since by virtue of the Riemann relations and Abel's theorem

$$2V = 2(A(P_+) - A(P_-)) \equiv 0$$

(the last congruence holds inasmuch as the divisors $2P_+$ and $2P_-$ are equivalent, being the zeroes and the poles of λ on Γ).

Until now we have been talking about the construction of complex solutions of nonlinear equations which admit a commutational representation of one of the enumerated forms. Picking out the real nonsingular solutions among them turns out to be comparatively easy in those cases in which the auxiliary linear problem

$$L\psi = \lambda\psi \tag{2.132}$$

for the Lax representation, or

$$\left(\frac{\partial}{\partial x} + U(x,t,\lambda)\right)\psi = 0, \tag{2.133}$$

in the case of the general representation (2.37), is self-adjoint. However for almost all nonlinear equations (the nonlinear Schrödinger equation, the sine-Gordon equation, the equations of the nonlinear interaction of wave packets etc.) the corresponding linear problems are not self-adjoint.

The typical conditions which select physically interesting real solutions have one of the following types

$$U(x,t,\lambda) = JU^{+}(x,t,\sigma(\lambda))J^{-1}, \tag{2.134}$$

$$U(x,t,\lambda) = J\bar{U}(x,t,\sigma(\lambda))J^{-1}, \tag{2.135}$$

where the cross denotes the Hermitian adjoint, $\sigma(\lambda)$ is an antiholomorphic involution of the λ plane (e. g., $\lambda \to \bar{\lambda}$, $\lambda \to \bar{\lambda}^{-1}$) and J is a diagonal matrix with entries $\varepsilon_k = \pm 1$.

Since one can also subject the matrix $W(x,t,\lambda)$ to the same realness conditions, the curves Γ, given by equation (2.70), which arise in the construction of real finite gap solutions are real, i. e. there is an antiholomorphic involution defined on them

$$\tau\colon \Gamma \to \Gamma,$$

which leaves the distinguished points P_α fixed, or permutes them in a well-defined fashion.

The description of real curve types and, what is the most difficult and interesting part, the distribution on them of the poles γ_i of Baker–Akhiezer functions leading to real solutions, pose problems of real algebraic geometry which until comparatively recently had not been worked out at all. (The first serious progress in the solution of these problems in connection with the nonlinear Schrödinger equation and the sine-Gordon equation was made in [68] and [28], although the results obtained in these papers are far from being effective).

A detailed exposition of recent achievements in real finite gap integration is given in [13], [41], [44], [46]. Here let us describe on the basis of the two examples analyzed above the two basic types of involutions on the set of divisors, whose various combinations give all the realness conditions known at present.

Let Γ be a *real hyperelliptic curve*, i. e. a curve given by equation (2.109) with a real polynomial R_{2n+1}. It the set of points $\gamma_i, \dots, \gamma_n$ is invariant with respect to the antiholomorphic involution

$$\tau\colon (\lambda,\mu) \to (\bar{\lambda},\bar{\mu})$$

and $\psi(x,t,P)$ is the Baker–Akhiezer function corresponding to them, then

$$\bar{\psi}(x,t,\tau(P)) = \psi(x,t,P). \tag{2.136}$$

since both the right and left-hand sides have the same analytic properties and are equal to each other by virtue of the uniqueness of ψ. From this it follows at once that the corresponding finite gap solution $u(x,t)$ of the KdV equation is real.

Now let us suppose that τ has $n+1$ fixed ovals $a_1, \dots, a_{n+1}$ on Γ, on one of which the point $P_0 = \infty$ lies. (In real algebraic geometry the curves of genus g with $g+1$ real ovals are called M-curves[5].) In the case under consideration this means that all the branch points E_i are real.

If the points γ_i are distributed one on each oval, $\gamma_i \in a_i$, then $u(x,t)$ has no singularities. Indeed, as is evident from the construction of ψ, a pole of $u(x,t)$ arises only when one of the n zeroes of ψ hits P_0 (here $\theta(Ux+Vt-\zeta)=0$). But by virtue of (2.136) ψ is real on real ovals. Since ψ has a pole on each a_i it also has at least one zero. Since there are n zeroes in all, they are separate from P_0.

[5]For some authors, M-curves are those with $g+1$ *components*, which need not all be ovals (translator's note).

For the finite gap solutions of the sine-Gordon equation to be real it is necessary that the hyperelliptic curve Γ be real [68]. On it let us consider the anti-involution

$$\tau\colon (\lambda, \mu) \to (\bar{\lambda}, -\bar{\mu}).$$

The action of this anti-involution on the local parameters $k_{\pm}^{-1}$ is such that

$$\tau^*(k_{\pm}) = -\bar{k}_{\pm}.$$

Let the polar divisor of $\psi_n(\xi, \eta, P)$ satisfy the condition

$$D + \tau(D) \equiv K + P_+ + P_-, \tag{2.137}$$

where K is the canonical class, i. e. the zero divisor of a holomorphic differential on Γ.

Condition (2.137) means that D, $\tau(D)$ are the zeroes of a differential of the third kind

$$\omega = \frac{d\lambda}{\lambda} \frac{\lambda^{n+1} + \alpha_1 \lambda^n + \cdots + \alpha_{n+1}}{\sqrt{R_{2n+1}(\lambda)}} \tag{2.138}$$

with poles at the points $P_{\pm}$.

Let us consider the differential

$$\psi_1(\xi, \eta, P)\, \bar{\psi}_1(\xi, \eta, \tau(P))\omega. \tag{2.139}$$

From (2.124) and (2.137) it follows that this is a meromorphic differential with its only poles at the points $P_{\pm}$. The residue of this differential at P_+ is equal to 1. Since the sum of the residues of any meromorphic differential is equal to zero, then

$$\chi_{01}^{-} \cdot \bar{\chi}_{01}^{-} = 1. \tag{2.140}$$

Analogously, if we consider the differential

$$\lambda\psi_2(\xi, \eta, P)\, \bar{\psi}_2(\xi, \eta, \tau(P))\omega,$$

we get

$$\chi_{02}^{-} \cdot \bar{\chi}_{02}^{-} = 1.$$

Hence, by (2.126),

$$|e^{-iu}| = |\chi_{01}^{-}|/|\chi_{02}^{-}| = 1$$

and $u(\xi, \eta)$ is real.

3. The Hamiltonian Theory of Hyperelliptic λ-Families

In this section, following [116], [118], we shall present the Hamiltonian theory of systems which are connected with *hyperelliptic curves* (see the examples of the preceding and the following sections). These systems usually come from systems of the form (2.64), (2.65), where the matrices are (2×2).[6]

The equations (2.64) are ordinary differential equations.

The initial "physical" coordinates on the finite-dimensional space of their solutions are the values of the matrix entries of U and W at some initial point $x = x_0$ (or rather the values of the matrix entries of the singular parts of U and W at their poles).

[6] A generalisation of this theory for general systems which are connected with arbitrary curves and its relations $N = 2$ supersymmetric gauge theories was obtained in [KP].

For example, for the "higher-order KdV equations" the space of solutions of the commutativity equations for the Sturm–Liouville operator L and an operator A_n of order $2n+1$ has dimension $3n+1$. Coordinates on it are given by

$$u(x_0), \dots, u^{(2n+1)}(x_0), h_2, \dots, h_n,$$

where the constants h_j arise in expressing the coefficients of the operator A_n in terms of u and its derivatives.

In the preceding section an isomorphism was set up between this space and the space

$$(\Gamma, P_1, \dots, P_k) = N^{n+k},$$

where Γ is a hyperelliptic curve, given in the form

$$w^2 = \prod_{i=1}^{2g+1} (\lambda - \lambda_i)$$

(as for the KdV equation, the sine-Gordon equation, where $\lambda_1 = 0$), or in the form

$$w^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i)$$

(the nonlinear Schrödinger equation, the Toda lattice etc.). Coordinates in the neighbourhood of P_j are given by $\lambda(P)$—the projections of the points onto the λ plane. In the following (when this does not give rise to misunderstanding) the points P_j will be denoted as $\gamma_j = \lambda(P_j)$ without indicating $\varepsilon_j = \pm$, the number of the sheet of the surface Γ.

The space N^{n+k} is fibred over M^n, the manifold of hyperelliptic curves. Coordinates on M^n are given by the λ_i. The fibre of this fibration

$$N^{n+k} \to M^n$$

is $S^k\Gamma$—the kth symmetric power of the curve Γ.

In the fundamental examples $k = g$ (KdV, sine-Gordon) or $k = g+1$ (NLS). In the first case the fibre, by virtue of Abel's theorem, is birationally isomorphic to a complex torus—the Jacobian of the curve, $J(V)$.

Let us define *analytic Poisson brackets* on the phase space N^{n+k} of our systems.

a) Let A be some set of functions on N^{n+k} which depend only on the point of the base space M^n, i. e. on the hyperelliptic curve. (In the sequel A will play the rôle of the annihilator of the Poisson bracket, which becomes nondegenerate on the manifolds N_A given by the equations $f = \text{const}$ for all $f \in A$; $N_A \to M_A$, $M_A \subset M^n$.)

b) Let a meromorphic 1-form $Q(\Gamma)$ be given on the Riemann surface Γ or on a covering of it $\hat{\Gamma} \to \Gamma$. In local notation

$$Q(\Gamma) = Q(\Gamma, \lambda)\, d\lambda. \tag{2.141}$$

The derivatives of $Q(\Gamma)$ in all directions of the base space tangent to the manifolds M_A are required to be globally defined meromorphic differential forms on the Riemann surface Γ itself (and not on the covering).

c) In all of the major examples the form Q has turned out either to be meromorphic on Γ right from the start, or to be meromorphic on a regular covering $\hat{\Gamma}$ with an abelian monodromy group, where the image of $\pi_1(\hat{\Gamma}) \to \pi_1(\Gamma)$ is generated by a set of cycles with vanishing pairwise intersection numbers.

DEFINITION. If the closed 2-form

$$\Omega_Q = \sum dQ(\Gamma, \gamma_j) \wedge d\gamma_j \tag{2.142}$$

is nondegenerate at a "general" point of a region of the N_A, where the pair (A, Q) has the properties a), b), c), then it will be said that an analytic Poisson bracket with annihilator A is given on an open region of N^{n+k}. The dimension of N_A in this case must be equal to $2k$.

By definition, the Poisson bracket of (2.142) is given by the properties

$$\begin{gathered} \{\gamma_i, \gamma_j\} = 0, \quad \{Q(\gamma_i), Q(\gamma_j)\} = 0, \\ \{Q(\gamma_i), \gamma_j\} = \delta_{ij}, \\ \{f, \gamma_j\} = \{f, Q(\gamma_k)\} = 0, \quad f \in A. \end{gathered} \tag{2.143}$$

If λ_i are any coordinates on the manifold M_A then it follows from (2.142) that Ω_Q contains only terms of the form $d\lambda_i \wedge d\gamma_j$ in its expansion. This at once implies the proposition:

Any two functions g, h which depend only on $\Gamma \in M^n$ are in involution

$$\{g(\Gamma), h(\Gamma)\} = 0. \tag{2.144}$$

Let $\tau_1, \ldots, \tau_k$ be the tangent directions to M_A at a "point in general position". By the definition of an analytic Poisson bracket, $\nabla_{\tau_i} Q$ is a meromorphic differential. Like any other differential, it can be decomposed uniquely (if a basis of a-cycles is fixed (2.86)) as a sum of a holomorphic differential ω_i and normalized (see (2.101)) differentials $\tilde{\omega}_i$ and $\tilde{\tilde{\omega}}_i$ of the second and third kinds respectively:

$$\nabla_{\tau_i} = \omega_i + \tilde{\omega}_i + \sum \tilde{\tilde{\omega}}_{it}, \tag{2.145}$$

where $\tilde{\tilde{\omega}}_{it}$ has a pair of first-order poles at points (P_t', P_t'').

Without loss of generality, when $k \geq g$ one may assume that locally the coordinates τ_i have been chosen so that the ω_i for $i \leq g$ form a normalized

$$\oint_{a_i} \omega_j = \delta_{ij}$$

basis of the holomorphic differentials, and for $j > g$ $\omega_j = 0$.

Let us consider the flows on N^{n+k} generated by Hamiltonians of the form $H(\Gamma)$, which by (2.144) commute with each other.

THEOREM 2.5. *Let the coordinates τ_i be as indicated above. Then at a point*

$$(\Gamma_0, \gamma_1, \ldots, \gamma_k)$$

in general position the complex variables

$$\psi_j = \sum_{i=1}^{k} \int_{P_0}^{\gamma_i} \nabla_{\tau_j} Q \tag{2.146}$$

are independent and have dynamics linear with respect to time.

The proof can be obtained entirely analogously to the standard Liouville procedure.

The definition of the ψ_j (2.146) depends on the choice of the paths between P_0 and the γ_i. Therefore these quantities are determined up to the lattice in $\mathbb{C}^k$ generated by the $2g+l$ periods e_q, e'_q, η_s of the gradient ∇Q:

$$e_q^i = \oint_{a_q} \nabla_{\tau_i} Q = \delta_q^i, \quad e_q'^i = \oint_{b_q} \nabla_{\tau_i} Q, \quad \eta_s^i = \operatorname{res}_{P'_s} \nabla_{\tau_i} Q. \tag{2.147}$$

The transformation (2.146) allows one on the basis of an analytic Poisson bracket (A, Q) satisfying the requirements enumerated above to construct a fibration

$$N_{(Q,A)} \xrightarrow{J_Q(\Gamma)} M_A$$

whose fibre is the quotient of $\mathbb{C}^k$ by the lattice generated, by the vectors (2.147).

Let κ be the number of functionally independent (modulo A) residues of the form Q, $\kappa \leq l$.

In general,

$$2g + \kappa \leq 2k. \tag{2.148}$$

The variables ψ_j form a compact torus T^{2k} only in the case when $2g+\kappa = 2k$. By no means will one always get an abelian torus. For this it is necessary and sufficient that $k = g$ and that all the forms $\nabla_{\tau_i} Q$ be holomorphic.

Comparing (2.146) with the definition of the Abel map (2.93), we get the following theorem.

THEOREM 2.6. *The Abel transformation $S^g\Gamma \to J(\Gamma)$ linearizes the dynamics of all Hamiltonians of the form $H(\Gamma)$ for Poisson brackets given by generic pairs (A, Q), if and only if the derivatives $\nabla_{\tau_i} Q$ in all directions tangent to M_A give a basis of the holomorphic differentials on Γ.*

Only for the real theory does it make sense to discuss a special choice of the vectors τ_i, corresponding to the differentiation of Q with respect to so-called "action variables" canonically conjugate to the "angles" on tori, varying from 0 to 2π.

Let us consider *real hyperelliptic curves* Γ. These are curves with an antiholomorphic involution $\sigma_\Gamma \colon \Gamma \to \Gamma$, which is induced by an antiholomorphic involution σ on the space M^n of all hyperelliptic curves.

The form Q and the annihilator A must also be compatible with σ, σ_Γ in a natural way:

$$\begin{aligned} &\text{a)} \quad \sigma_\Gamma^* Q = \bar{Q}, \\ &\text{b)} \quad \sigma^* A = \bar{A}. \end{aligned} \tag{2.149}$$

The simplest example of real structures, which may be called the "elementary" ones for the Hamiltonian systems which interest us, actually already appeared earlier in the description of the real nonsingular finite gap solutions of the KdV equation.

Let σ_Γ have $g+1$ or g fixed ovals on Γ. (Such curves are called M-curves or $(M-1)$-curves). In the first case $k = g$ or $k = g+1$. In the second case $k = g$.

LEMMA 2.1. *If the Poisson bracket (A, Q) has the properties (2.149) then sets of points γ_i $(i = 1, \ldots, g+1$ or $i = i_1, \ldots, i_g)$ lying on pairwise distinct fixed ovals a_i of the anti-involution σ_Γ are invariant with respect to the dynamics generated by the real Hamiltonians $H(\Gamma)$, $H(\sigma(\Gamma)) = \bar{H}(\Gamma)$.*

For M-curves and $k = g$ the admissible sets of $\gamma_i \in a_i$ form $g+1$ connected components isomorphic to the real torus T^g. For M-curves and $k = g+1$ or for

$(M-1)$-curves and $k=g$ there is only one connected component—a real torus T^{g+1} or T^g.

An example of a non-elementary real structure arose in the description of the real solutions of the sine-Gordon equation (§ 2).

The effective assignment of such a structure is possible only in terms of coordinates on N connected with y by the transformation (2.146).

By a non-elementary real structure will be meant an anti-involution

$$\tau\colon N_{(Q,A)}\to N_{(Q,A)}$$

which is compatible with the fibration

$$N_{(Q,A)}\xrightarrow{J_Q(\Gamma)} M_A.$$

On the fibres of $J_Q(\Gamma)$ there must be a superposition of a translation and an automorphism of $J_Q(\Gamma)$ as a real commutative group.

The real submanifolds in the *phase space* are picked out by the conditions

$$\tau(\eta)=\bar\eta \tag{2.150}$$

or

$$\tau(\eta)=-\bar\eta+\eta_0^\alpha. \tag{2.151}$$

The case (2.151) is realized for the sine-Gordon and the NLS (2.14). For a given curve the vector η_0^α may take on a finite number of values. Their computation for the sine-Gordon equation was done for the first time in [46].

THEOREM 2.7. *For analytic Poisson brackets satisfying the elementary and non-elementary realness conditions, the action variables J_j, canonically conjugate to the coordinates on the tori T^k varying from 0 to 2π, are given by the formula*

$$J_j=\frac{1}{2\pi}\oint_{a_j} Q(\Gamma,\lambda)\,d\lambda. \tag{2.152}$$

The proof of the theorem follows in essence from the course of the proof of Liouville's theorem and from the fact that (2.152) represents the quantity

$$J_j=\frac{1}{2\pi}\oint_{\tilde a_j} p\,dq, \tag{2.153}$$

where the $\tilde a_j$ are the basis 1-cycles on the tori T^k.

For the class of Poisson brackets under study the action variables J_j (2.153) acquire an important interpretation as integrals over the elements a_j of the group $H_1(\Gamma\backslash P,\mathbb{Z})$, where P is the set of poles of the form Q. This results in a significantly greater effectiveness in the construction of action variables than in Liouville's theorem. In particular, for example, until [116] no explicit construction of action variables for the Kovalevskaya case (see below) was known, since the only method for constructing them known to the classical workers was the method of separation of the variables in the Hamilton–Jacobi equation.

Theorem 2.6 gives necessary and sufficient conditions on the bracket (Q,A) which guarantee the linearization by the Abel mapping of the Hamiltonian flows generated by Hamiltonians $H(\Gamma)$.

As is well known, the Abel mapping linearizes all the higher-order KdVs. Let us express the Hamiltonians corresponding to these flows in terms of the form Q.

THEOREM 2.8. *The coefficients of the expansion*

$$Q(\Gamma,\lambda)=\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k q_k(\Gamma),\quad z=\lambda^{-1/2}, \tag{2.154}$$

are such that the $h_l(\Gamma)=q_{2l+3}(\Gamma)$ *are the Hamiltonians of the higher-order KdVs with the number* $l\geq 0$. *The remaining coefficients* q_k *belong to the annihilator* A.

In conclusion let us enumerate a number of major examples.

EXAMPLE 1. The *Gardner–Zakharov–Faddeev bracket*. From [56] one can extract

$$Q=2ip(\lambda)\,d\lambda,\quad A=\left\{T_1,\dots,T_g,\bar{u}=\lim_{T\to\infty}\frac{1}{T}\int_0^T u\,dx\right\} \tag{2.155}$$

$p(\lambda)$ is the quasimomentum, where $dp(\lambda)$ is a differential of the second kind with a unique pole at $\lambda=0$,

$$\oint_{a_j} dp(\lambda)=0,\quad j=1,\dots,g. \tag{2.156}$$

The periods T_j of the quasiperiodic potential $u(x)$ are defined as

$$\frac{1}{2\pi i}\oint_{b_j} dp=T_j. \tag{2.157}$$

EXAMPLE 2. The *Magri bracket* (F. Magri) [94]. In this bracket the higher-order KdV equations have the form

$$\dot{u}=\left(al+b\,\frac{\partial}{\partial x}\right)\frac{\delta H}{\delta u},\quad l=\frac{1}{2}\,\frac{\partial^3}{\partial x^3}+u\,\frac{\partial}{\partial x}+\frac{\partial}{\partial x}\,u. \tag{2.158}$$

Here we have

$$Q=2ip(\lambda)(a\lambda+b)^{-1}d\lambda. \tag{2.159}$$

For $b=0$ the annihilator is

$$A=\{T_1,\dots,T_g,J\},$$

where

$$J=\sum_{k=0}^{g+1}c_kI_{g-k+1} \tag{2.160}$$

is a linear combination of Kruskal integrals, and its extremals are given by the finite gap solutions constructed over Γ.

EXAMPLE 3. The Hamiltonian formalism of the stationary problem for the higher-order KdV.

The commutativity equation (2.61) may be presented in the form

$$\delta J=0, \tag{2.161}$$

where J is the same as in (2.160). This representation naturally gives rise to the Hamiltonian formalism of the system (2.61) (see [18], [20]). From [6] one can extract

$$Q=\sqrt{-R(\lambda)\,d\lambda},\quad R(\lambda)=\prod(\lambda-\lambda_i).$$

The annihilator of the bracket is generated by the first $(g+1)$ symmetric polynomials in the λ_i.

EXAMPLE 4. The Hamiltonian structure generated by the "hidden isomorphism of Moser and Trubowitz" [136], [110] (for more details on which see the examples of the next section) between the dynamics of the KdV on the space of finite gap potentials and the Neumann systems (2.183), (2.184):

$$\begin{aligned} Q &= \sqrt{-R(\lambda)}\prod_j (\lambda-\lambda_{2j})^{-1}d\lambda, \\ A &= \{\lambda_0,\lambda_2,\dots,\lambda_{2g}\} \end{aligned} \tag{2.162}$$

EXAMPLE 5. The integrable case of Goryachev–Chaplygin in the dynamics of a rigid body with a fixed point [71].

Here

$$Q(\Gamma,\lambda)=\arcsin\frac{1}{\mu}\left(\frac{\lambda^2}{2}-\frac{1}{2}H-\frac{2G}{\lambda}\right),$$

where H is the energy of the top, G is the Goryachev–Chaplygin integral, μ is a parameter. The curve Γ is given by the equation

$$y^2=4\mu^2\lambda^2-(\lambda^3-H\lambda-4G)^2.$$

EXAMPLE 6. In the well-known *Kovalevskaya case* the action variables formerly could not be calculated. In the notation of [71] (and of the next section, see (2.174), (2.181)) we have:

$$Q(\Gamma,\lambda)=\frac{1}{2\sqrt{-\lambda}}\ln(\sqrt{-\lambda}(\lambda-6h)^2-k^4)+\frac{v^2}{2\sqrt{-\lambda}}(\lambda-8l^2)+\sqrt{-R_5(\lambda)},$$

where R_5 is given by (2.181).

The curve Γ is given by the equation $y^2=R_5(\lambda)$. By integrating Q over the "real" cycles a_j, on which the $\gamma_j=s_j$ lie—the Kovalevskaya variables, we obtain the action variables J_j.

4. The Most Important Examples of Systems Integrable by Two-Dimensional Theta Functions

By the example analyzed in § 2, the hyperelliptic curve Γ of genus 2

$$y^2=R_5(\lambda)=\prod_{i=1}^{5}(\lambda-\lambda_i) \tag{2.163}$$

generates a pair of commuting operators

$$L=\frac{\partial^2}{\partial x^2}+u(x), \tag{2.164}$$

$$\begin{aligned} A_5 = 16\frac{\partial^5}{\partial x^5}+20\left(u\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial x^3}u\right)+30u\frac{\partial}{\partial x}u-5\left(u''\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u''\right) \\ +h_1\left[4\frac{\partial^3}{\partial x^3}+3\left(u\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u\right)\right]+h_2\frac{\partial}{\partial x}. \end{aligned} \tag{2.165}$$

The commutativity equation (2.107) for the operators L and A_5 on the function u may be written in the Lagrangian form

$$\delta\int\Lambda\,dx=0 \tag{2.166}$$

with the Lagrangian

$$\Lambda = \frac{u''^2}{2} - \frac{5}{2}\,u''u^2 + \frac{5}{2}\,u'' + h_1\left(\frac{u'^2}{2} + u^3\right) + h_2u^2 + h_3u. \tag{2.167}$$

According to [115], the equation (2.166) is equivalent to a Hamiltonian system with two degrees of freedom and with the Hamiltonian

$$H = p_1p_2 + V(q_1, q_2), \tag{2.168}$$

$$\begin{gathered} q_1 = u, \quad q_2 = u'' - 5u^2, \quad p_1 = q_2', \quad p_2 = u', \\ V = -\frac{q_2^2}{2} - \frac{5}{2}\,q_2q_1^2 - \frac{5}{8}\,q_1^4 + \frac{h_2}{2}\,q_1^2 + h_3q_1 \end{gathered} \tag{2.169}$$

(by a substitution $u \to u + \text{const}$ the constant h_1 has been made zero in equations (2.168), (2.169)).

The integrals of the system (2.168) in involution have the form $J_1 = H$,

$$\begin{aligned} J_2 &= p_1^2 + 2q_1p_1p_2 + (2q_2 - h_2)p_2^2 + D(q_1, q_2), \\ D &= q_1^5 + h_2q_1^3 - 4q_1q_2^2 + 2h_2q_1q_2 + 2h_3q_2. \end{aligned}$$

The integrals J_i define a curve (2.163). The corresponding polynomial R_5 is equal to

$$R_5 = \lambda^5 + \frac{h_2}{2}\,\lambda^3 + \frac{h_3}{16}\,\lambda^2 + \left(\frac{J_1}{32} + \frac{h_2^2}{16}\right)\lambda + \frac{J_2 - h_2h_3}{256}. \tag{2.170}$$

The results of § 2 indicate that coordinates on the level manifold $J_1 = \text{const}$, $J_2 = \text{const}$ are given by γ_1, γ_2—the locations of the poles of the corresponding Baker–Akhiezer function. Their connection with the initial variables is given by means of the so-called trace formulas

$$\gamma_1 + \gamma_2 = \frac{u}{2}, \quad \gamma_1\gamma_2 = \frac{1}{8}(3u^2 + u'') + \frac{1}{2}\sum_{i\neq j}\lambda_i\lambda_j, \tag{2.171}$$

where the λ_j are the zeroes of the polynomial (2.170).

The equations on the γ_i which are equivalent to the original system have, in the given case, the form (see § 2)

$$\gamma_1' = \frac{2i\sqrt{R_5(\gamma_1)}}{\gamma_1 - \gamma_2}, \quad \gamma_2' = \frac{2i\sqrt{R_5(\gamma_2)}}{\gamma_2 - \gamma_1}. \tag{2.172}$$

These equations, as was already noted in § 2, are linearized by the *Abel transformation*. The two-gap potential $u(x)$ equals

$$u(x) = 2\,\frac{\partial^2}{\partial x^2}\ln\theta(Ux - \zeta) + \text{const}. \tag{2.173}$$

Later a number of examples will be cited of systems leading to two-gap potentials and integrable, as a consequence, by two-dimensional theta functions.

S. V. Kovalevskaya's Problem. The equations of motion of a heavy rigid body with a fixed point in Kovalevskaya's case have the form:

$$\begin{cases} 2\dot p = qr, \\ 2\dot q = -pr - \mu\gamma_3, \\ \dot r = \mu\gamma_2, \end{cases} \quad \begin{cases} \dot\gamma_1 = r\gamma_2 - q\gamma_3, \\ \dot\gamma_2 = p\gamma_3 - r\gamma_1, \\ \dot\gamma_3 = q\gamma_1 - p\gamma_2, \quad \mu = \text{const}. \end{cases} \tag{2.174}$$

(A representation of the Lax type for this system was found in [121].)

The equations (2.170) have the following integrals

$$\begin{aligned} H &= 2(p^2+q^2)+r^2-2\mu\gamma_1 \quad \text{(the energy)},\\ L &= 2(p\gamma_1+q\gamma_2)+r\gamma_3 \quad \text{(the angular momentum)},\\ K &= (p^2-q^2+\mu\gamma_1)^2+(2pq+\mu\gamma_2)^2 \quad \text{(Kovalevskaya's integral)}. \end{aligned} \tag{2.175}$$

In addition there is fulfilled the constraint condition

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1. \tag{2.176}$$

Let us consider the combined level surface of these integrals

$$H=6h^2, \quad L=2l, \quad K=k^2. \tag{2.177}$$

Under fulfillment of the constraint (2.176) these equations give a two-dimensional invariant submanifold of the original system (2.174).

The Kovalevskaya variables—coordinates on this surface—are defined in the following manner

$$s_{1,2}=3h+\frac{R(x_1,x_2)\mp\sqrt{R(x_1)R(x_2)}}{(x_1-x_2)^2}, \tag{2.178}$$

where

$$\begin{aligned} &x_{1,2}=p\pm iq, \quad R(z)=-z^4+6hz^2+4\mu lz+\mu^2-k^2,\\ &R(x_1,x_2)=-x_1^2x_2^2+6x_1x_2h+2\mu l(x_1+x_2)+\mu^2-k^2. \end{aligned} \tag{2.179}$$

An easy computation shows that in the variables s_i the equations (2.174) have the form:

$$\dot s_1=\frac{i}{2}\,\frac{\sqrt{R_5(s_1)}}{s_1-s_2}, \quad \dot s_2=\frac{i}{2}\,\frac{\sqrt{R_5(s_2)}}{s_2-s_1}, \tag{2.180}$$

$$R_5=(\lambda[(\lambda-3h)^2+\mu^2-k^2]-2\mu^2l^2)((\lambda-3h)^2-k^2). \tag{2.181}$$

These equations coincide up to a factor with the equations (2.172). Consequently, they will be linearized by the Abel substitution.

The expressions for the original variables $p,q,r,\gamma_1,\gamma_2,\gamma_3$ in terms of the Kovalevskaya variables are cited in [63]. As for the variables s_i, they, by the results of § 2, can be defined as solutions of the equations

$$\theta(A(s_i)+Ut-\zeta)=0. \tag{2.182}$$

Here $A\colon \Gamma\to J(\Gamma)$ and Γ is given by equation (2.163) with R_5 equal to (2.181).

The Neumann and Jacobi Problems. The General Garnier System. The equations of motion of a particle on the $(n-1)$-dimensional sphere

$$x^2=\sum_{i=1}^n x_i^2=1 \tag{2.183}$$

under the action of a quadratic potential

$$U(x)=\frac{1}{2}\sum_{i=1}^n a_ix_i^2, \quad a_i=\text{const}, \tag{2.184}$$

have the form:

$$\ddot x_i=-a_ix_i+u(t)x_i, \tag{2.185}$$

where $u(t)$ is a Lagrange multiplier arising because of the imposition of the constraints (2.183). When $n = 3$ this system bears the name Neumann system.

The *Neumann system* may be obtained from the Hamiltonian flow on $\mathbb{R}^6$ with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{3} a_i x_i^2 + \frac{1}{2}(x^2y^2 - (xy)^2) \tag{2.186}$$

by restriction to the surface $x^2 = 1$ (here $xy = \sum_i x_i y_i$). The functions

$$F_k(x,y) = x_k^2 + \sum_{i\neq k} \frac{x_k y_i - y_k x_i}{a_i - a_k}, \quad k = 1,2,3, \tag{2.187}$$

are a system of integrals in involution for (2.186). The Hamiltonian H itself has the form:

$$H = \frac{1}{2}\sum_{i=1}^{3} a_i F_i. \tag{2.188}$$

The transformation

$$\tilde{x} = y, \quad \tilde{y} = -x, \quad \tilde{H} = \frac{1}{2}\sum_{i=1}^{3} \frac{F_i}{a_i} \tag{2.189}$$

takes the constructed Hamiltonian flow over into a geodesic flow on the triaxial ellipsoid (when $a_i > 0$)

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i} = 1$$

The problem of geodesies on a triaxial ellipsoid is called the Jacobi problem.

In the work [110] a trajectory isomorphism was established between the equations in x for the periodic n-gap potentials of the Sturm–Liouville operator and the equations in $t \to x$ for the periodic trajectories of the system (2.183), (2.184). The full phase isomorphism of the systems for n-gap potentials and the system (2.183), (2.184) was proved in [136]. By the same token the general solutions of the latter system can be expressed in n-dimensional theta functions, and those of the Neumann system (and the solutions of the Jacobi problem) in two-dimensional theta functions.

Let us remark that although these systems are trajectorially isomorphic, the corresponding Hamiltonian structures (as was shown in the preceding section) are different.

Let us consider a Baker–Akhiezer function $\psi(x, P)$ associated with a hyperelliptic curve Γ with real branch points $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n+1}$. In § 2 it was shown that it satisfies the equation

$$\psi''(x,P) = -\lambda\psi(x,P) + u(x)\,\psi(x,P), \tag{2.190}$$

where $P = (\lambda, \sqrt{R})$ is a point of Γ. Let us denote $\psi^+(x,P) = \psi(x,\sigma(P))$, where σ is the involution which exchanges the sheets of Γ. Its operation on the local parameter is $\sigma^*(k) = -k$. Hence the function $\psi(x,P)\,\psi^+(x,P)$ is regular at the point at infinity P_0. Besides, it does not depend on the choice of the sheet of Γ , and hence is a rational function of λ

$$\psi(x,P)\,\psi^+(x,P) = \frac{\prod_{i=1}^{n}(\lambda - \gamma_i(x))}{\prod_{i=1}^{n}(\lambda - \gamma_i)}. \tag{2.191}$$

For an arbitrary polynomial $P(\lambda)$ of degree n (whose coefficients may depend on parameters) and for arbitrary points μ_i, $i = 1, \dots, n+1$, there holds the simple identity

$$\sum_{i=1}^{n+1} \frac{P(\mu_i)}{\prod_{j\neq i}(\mu_i - \mu_j)} \equiv 1. \tag{2.192}$$

Since at the branch points $\psi(x, \lambda_i) = \psi^+(x, \lambda_i)$, it follows from (2.191) and (2.192) that the functions

$$\phi_i(x) = \psi(x, \lambda_{2i+1}) \prod_j \left(\frac{\lambda_{2i+1} - \gamma_j}{\lambda_{2i+1} - \lambda_{2j+1}} \right)^{1/2} \tag{2.193}$$

satisfy the identity

$$\sum_{i=1}^{n} \phi_i^2(x) \equiv 1. \tag{2.194}$$

The equalities (2.190) and (2.194) coincide (after renaming x to t) with the equations (2.185) and (2.183). The expressions in terms of theta functions for $\psi(x, P)$ which were obtained in § 2 thereby give the solutions of the system (2.183), (2.184).

For the Neumann system we get, in particular,

$$x_i(t) = \alpha_i \frac{\theta[v_i](tU + \zeta)}{\theta[v_0](tU + \zeta)}, \tag{2.195}$$

where

$$\alpha_1 = -\frac{\theta[v_1](0)}{\theta[v_0](0)}, \quad \alpha_2 = -i\,\frac{\theta[v_2](0)}{\theta[v_0](0)}, \quad \alpha_3 = \frac{\theta[v_3](0)}{\theta[v_0](0)}, \tag{2.196}$$

and the characteristics $[v_i]$ of the theta functions equal

$$\begin{aligned} &v_0 = [(1/2, 1/2), (1/2, 1/2)], \quad && v_1 = [(1/2, 0), (0, 1/2)], \\ &v_2 = [(0, 0), (0, 1/2)], && v_3 = [(0, 0), (1/2, 1/2)]. \end{aligned} \tag{2.197}$$

The system (2.183), (2.184) can be obtained from a more general system, discovered by Garnier [61]

$$\begin{aligned} x_i'' &= x_i(\sum x_i y_i + a_i), \\ y_i'' &= y_i(\sum x_i y_i + a_i), \end{aligned} \tag{2.198}$$

On the invariant plane $x_i = a_i y_i$ we exactly get the Neumann system on the sphere. Another interesting case is the system of anharmonic oscillators, which is obtained from (2.198) by restricting to the plane $x_i = y_i$, [62].

The *Garnier system* is equivalent (under a suitable choice of the parameter τ) to the commutation conditions

$$\frac{dA(\lambda)}{d\tau} = [A(\alpha), A(\lambda)]/(\lambda - \alpha), \tag{2.199}$$

where the matrix $A = (A_{ij})$ has the form:

$$\begin{gathered} A_{11} = \lambda^2 - \sum x_i y_i, \\ A_{1i} = x_{i-1}\lambda + x_{i-1}', \quad A_{i1} = y_{i-1}\lambda - y_{i-1}', \\ A_{ij} = x_{i-1}y_{i-1} - a_{i-1}\delta_{ij}, \quad i, j \geq 2. \end{gathered} \tag{2.200}$$

The Motion of a Body in an Ideal Fluid. Integration of the Clebsch Case. As was already said in Chap. 1, the equations of motion of a rigid body in an ideal fluid have the form:

$$\dot{p} = p \times \frac{\partial H}{\partial M}, \quad \dot{M} = p \times \frac{\partial H}{\partial p} + M \times \frac{\partial H}{\partial M}, \tag{2.201}$$

where $H = H(M, p)$ is the Hamiltonian (1.48),

$$\begin{gathered} M = \{M_1, M_2, M_3\}; \quad p = \{p_1, p_2, p_3\}; \\ \frac{\partial H}{\partial M} = \left\{ \frac{\partial H}{\partial M_i} \right\}; \quad \frac{\partial H}{\partial p} = \left\{ \frac{\partial H}{\partial p_i} \right\}. \end{gathered} \tag{2.202}$$

Below we shall give the commutational representation of equations (2.201) in the integrable cases of Clebsch and of Lyapunov–Steklov–Kolosov (see (1.205)–(1.209)).

The commutational representation for the Clebsch system was found in [119]. The matrix L has the form:

$$L = \lambda A + L_0 - \lambda^{-1} P, \tag{2.203}$$

$$L_0 = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}, \quad P_{ij} = p_i p_j; \quad A_{ij} = a_i \delta_{ij}. \tag{2.204}$$

The matrix M equals

$$M = \lambda C + \begin{pmatrix} 0 & a_3 M_3 & -a_2 M_2 \\ -a_3 M_3 & 0 & a_1 M_1 \\ a_2 M_2 & -a_1 M_1 & 0 \end{pmatrix}, \tag{2.205}$$

$C_{ij} = c_i \delta_{ij}$.

The Clebsch case is the limit under contraction of the group SO(4) to E(3) of the integrable tops obtained in [97] as it was observed first in [112]. If one fixes a basis of the algebra so(4) with the commutation relations

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k; \quad \{M_i, p_i\} = \varepsilon_{ijk} p_k; \quad \{p_i, p_j\} = \varepsilon_{ijk} M_k, \tag{2.206}$$

then this contraction corresponds to a passage to a limit, under which

$$M_i = M_i', \quad p_i \to N p_i', \quad N \to \infty.$$

The Lax pair (2.212) for tops on so(4) diverges under the contraction, although its integrals hold out under this passage and coincide after it with the integrals of the Clebsch case. On the other hand, the pair (2.203)–(2.205) will not endure the deformation of so(4) to e(3). It is of interest to remark that not only the tie indicated above exists between these systems. As was found in [10], [15], the Kirchhoff equations for the Clebsch case go over into the Manakov equations for the algebra so(4) (see below) after a suitable linear change of variables. An analogous linear change takes the integrable case of Lyapunov–Steklov–Kolosov (see below) over into the integrable case of Steklov [133] for the rotation of a rigid body with an ellipsoidal cavity filled with a fluid [15]. The Clebsch case was integrated in [69], [132], [140].

The Lyapunov–Steklov–Kolosov Case. In this case the Hamiltonian, with (1.208) taken into account, has the form:

$$2H = \sum_{\alpha=1}^{3} b_\alpha (M_\alpha - (b_1+b_2+b_3-b_\alpha)\sigma p_\alpha)^2 + A\sum_{\alpha=1}^{3} p_\alpha^2 + B\sum_{\alpha=1}^{3} p_\alpha M_\alpha. \quad (2.207)$$

Let us set

$$2z_\alpha = M_\alpha - \sigma(b_1+b_2+b_3-b_\alpha)p_\alpha. \quad (2.208)$$

The fourth integral has the form

$$2I = \sum_{\alpha=1}^{3} b_\alpha p_\alpha^2. \quad (2.209)$$

The *Lyapunov–Steklov–Kolosov case* was integrated in the paper [70]. It is curious that in this paper for this system there was practically used, between the lines, a commutational representation with an elliptic spectral parameter.

F. Kötter's representation for the equations of motion had the form:

$$\frac{d}{dt}(z_1+\sigma s p_1) = 2(s-b_2)(z_2+s\sigma p_2)\frac{b_3 z_3}{s} - 2(s-b_3)(z_3+s\sigma p_3)\frac{b_2 z_2}{s}$$

(the remaining equations have the same appearance up to a cyclic permutation of the indices).

Here s is the "spectral parameter". These equations, as is simple to verify, are equivalent to the Lax equation $\dot L = [L, M]$, where

$$\begin{aligned} L &= \begin{pmatrix} 0 & \sqrt{c_3}v_3 & -\sqrt{c_2}v_2 \\ -\sqrt{c_3}v_3 & 0 & \sqrt{c_1}v_1 \\ \sqrt{c_2}v_2 & \sqrt{c_1}v_1 0 & \end{pmatrix}, \\ M &= \frac{2}{s}\begin{pmatrix} 0 & \sqrt{c_1c_2}b_3z_3 & -\sqrt{c_1c_3}b_2z_2 \\ \sqrt{c_1c_2}b_3z_3 & 0 & \sqrt{c_2c_3}b_1z_1 \\ \sqrt{c_1c_3}b_2z_2 & -\sqrt{c_2c_3}b_1z_1 & 0 \end{pmatrix}. \end{aligned} \quad (2.210)$$

Here $v_i = z_i + \sigma s p_i$, $c_i = s - b_i$.

Let us set $e_i = b_i - \frac{1}{3}(b_1+b_2+b_3)$, $s = \wp(\lambda) + (b_1+b_2+b_3)/3$, where $\wp$ is the Weierstrass function corresponding to the elliptic curve Γ with the branch points e_i. Then L, M are elliptic functions of λ, defined on Γ.

The Clebsch and Lyapunov–Steklov–Kolosov cases exhaust all the possibilities when the system (2.201) with the Hamiltonian (1.48) has a fourth integral quadratic in (M, p) [119]. Let us note that for general diagonal metrics, as was shown in [73], for the equations of motion (with the exception of the Clebsch case) a splitting of the separatrices occurs, i. e. they are non-integrable.

A Multidimensional Free Rigid Body. The equations of a multidimensional rigid body have the form [7]:

$$\dot M = [\Omega, M], \quad M = J\Omega + \Omega J \quad (2.211)$$

and $J_{ij} = J_i\delta_{ij}$ is the inertia operator[7] of the rigid body. The complete integrability of this system for all n was proved in [97][8]. As was remarked in this paper, the

[7]Often called the inertia *tensor* (translator's note).

[8]For $n = 4$ in [108].

system (2.211) is equivalent to the system

$$[A, \dot{V}] = [[A, V], [B, V]], \tag{2.212}$$

$$[B, V] = \Omega, \quad A = J^2, \quad B = J. \tag{2.213}$$

The commutational representation for (2.211) has the form:

$$\left[\frac{d}{dt} - [B, V] + \lambda B, \lambda A - [A, V]\right] = 0.$$

By the results of § 2, the general solutions can be expressed via theta functions of the Riemann surface Γ of the form

$$\det(\lambda A - [A, V] - \mu \cdot 1) = 0. \tag{2.214}$$

The rigorous and abstract exposition of the Manakov theory and the direct verification of the independence of the Manakov integrals constructed (the coefficients of the characteristic polynomial (2.214)), without relying on the spectral theory of operators, were realized in the paper [57]. A series of subsequent papers, of which a survey is given in [58], were devoted to the transfer of this technique to some other Lie algebras. An investigation of the dynamics of these systems not only on the Lie algebra, but also on the whole Lie group was given in the paper [105].

The equations (2.212) were integrated (for arbitrary A and B) in the paper [39]. As was remarked in [97], for general diagonal matrices A and B the equations (2.212) coincide with the equations of a motion on $\mathrm{SO}(N)$ with the diagonal metric

$$\omega_{ij} = \frac{a_i - a_j}{b_i - b_j},$$

which for $N = 4$ go over under the contraction of $\mathrm{SO}(4)$ to $\mathrm{E}(3)$ into the integrable case of Clebsch.

The solutions of the general equations (2.212):

$$v_{ij} = \frac{\lambda_i}{\lambda_j} \frac{\theta(A(P_i) - A(P_j) + tU + \zeta)}{\theta(tU + \zeta)\, \varepsilon(P_i, P_j)}, \quad i \neq j, \tag{2.215}$$

$$\varepsilon(P, Q)^{-1} = \frac{\sqrt{\partial_{U(P)}\theta[v](0)\, \partial_{U(Q)}\theta[v](0)}}{\theta[v](A(P) - A(Q))},$$

$$\lambda_i = \lambda_i^0 \exp\left(t \sum_{k \neq i} c_i^k b_k\right),$$

$$c_i^k = -\frac{\partial}{\partial P} \ln \varepsilon(P, P_i)|_{P = P_i}.$$

Here the λ_i^0 are arbitrary nonzero constants, the θ-function is constructed with respect to a curve of the form (2.214); P_i are the points at infinity of this curve, where $\mu/\lambda \to a_i$ when $P \to P_i$; the vector U has the form:

$$U = \sum_j b_j U(P_j),$$

$U(P)$ is the period vector of the differential $\Omega_P^{(2)}$ with a double pole at P, $[v]$ is an arbitrary nondegenerate ($\operatorname{grad}\theta[v](0) \neq 0$) odd half-period.

Waves in the Landau–Lifshitz Equation. Following [137], let us look at solutions of the travelling-wave type

$$S(x,t) = q(x - at)$$

for the Landau–Lifshitz equation (2.51).

We have

$$-a\dot{q} = q \times (\ddot{q} + Jq). \tag{2.216}$$

Taking the vector product of this equality with q and using the condition $q^2 = 1$, we get

$$\ddot{q} + Jq = \lambda q + a\dot{q} \times q, \quad \lambda = (q, Jq) - \dot{q}^2.$$

Let us introduce the variable

$$M = \dot{q} \times q + aq.$$

Then equation (2.216) turns out to be equivalent to the already analyzed Clebsch system (2.204)

$$\dot{M} = M \times Jq,$$
$$\dot{q} = q \times M.$$

In [16] finite gap solutions are constructed in terms of Prym theta functions, starting from the Lax pair (2.52) for the Landau–Lifshitz equations (2.51). As a special case they also contain solutions of the travelling-wave type.

A generalization of the Landau–Lifshitz equations is given by the equations

$$\begin{aligned} u_t &= u \times (u_{xx} + Jv), \\ v_t &= v \times (v_{xx} + Ju), \end{aligned} \tag{2.217}$$

which were considered in [139] and which describe a two-sublattice system. In [139] it is shown that the equations describing the travelling wave

$$u = u(x - at), \quad v = v(x - at),$$

can be integrated. They correspond to the Hamiltonian system on E(3)+E(3) with the Hamiltonian

$$H = \tfrac{1}{2}(M^2 + N^2 + 2(Jp, q)). \tag{2.218}$$

Here $p(\xi) = u(\xi)$, $q(\xi) = v(\xi)$ and $M = u \times u_\xi + au$, $N = v \times v_\xi + av$. The pairs p, M and q, N satisfy the commutation relations of E(3).

The matrices L and A which enter into the commutational representation for the equations of motion of the system of (2.218) have the block form

$$L = \begin{pmatrix} \lambda\hat{M} & \lambda^2 p_i q_j + J_i \delta_{ij} \\ \lambda^2 q_i p_j + J_i \delta_{ij} & \lambda\hat{N} \end{pmatrix},$$
$$A = \begin{pmatrix} 0 & \lambda p_i q_j \\ \lambda q_i p_j & 0 \end{pmatrix}.$$

Here $\hat{M}$ and $\hat{N}$ are skew-symmetric 3×3 matrices corresponding in a canonical way to the vectors $\{M_i\}, \{N_i\}$.

A Top in a Gravitational Field. Let us consider, following [19], the problem of the rotation of a top, fixed at its centre of gravity, in the gravitational field being created by an arbitrary body V. Let $\rho(x)$ be the mass density of the body V at its point x; let $R(x)$ be the distance from the point x to the fixation point of the top. Let us write the equations in a coordinate system S rigidly connected with the top, where we shall orient the axes of this system along the principal axes of the inertia operator, i. e. the inertia operator I of the top is diagonal, $I = (I_i\delta_{ij})$. We shall suppose the dimensions of the top to be small in comparison with the distances $R(x)$ to the body V. In this approximation the equations of the rotation of the top can be written in the form

$$\dot{M} = M \times \omega + \int_V 3G\rho(x)R^{-3}(x)\gamma(x) \times I\gamma(x)\, d^3x, \tag{2.219}$$

where $\gamma(x)$ is the unit vector of the direction going from the point x of the body to the top (written in the system S!), M and ω are the angular momentum and angular velocity vectors, G is the gravitational constant. Let us supplement the equations (2.219) with the obvious relation

$$\dot{\gamma}(x) = \gamma(x) \times \omega. \tag{2.220}$$

We shall show that the equations (2.219), (2.220) are integrable, and the integration procedure does not depend on the body V.

Let us associate to the vectors $\gamma(x) = (\gamma_i(x))$, $M = (M_i)$, $\omega = (\omega_i)$ skew-symmetric matrices $\hat{\gamma}(x) = (\hat{\gamma}_i(x))$, $\hat{M} = (\hat{M}_{ij})$, $\hat{\omega} = (\hat{\omega}_{ij})$, setting $\hat{\gamma}_{ij}(x) = \varepsilon_{ijk}\gamma_k(x)$, and so on. Let us introduce, further, the matrix

$$u(x) = \int_V 3G\rho(x)R^{-3}(x)\hat{\gamma}^2(x)\, d^3x.$$

The equations (2.219), (2.220) can be written in the form of the system

$$\begin{cases} \dot{\hat{M}} = [\hat{M}, \hat{\omega}] + [u, C], \\ \dot{u} = [u, \hat{\omega}], \end{cases} \qquad \hat{M} = I\hat{\omega} + \hat{\omega}I. \tag{2.221}$$

Here $C = \operatorname{diag}(C_1, C_2, C_3)$. The system (2.221) is Hamiltonian on the Lie algebra whose elements are pairs of 3×3 matrices (ω, u), where ω is a skew-symmetric matrix and u is a symmetric matrix, and the commutators have the form:

$$[\omega_1, \omega_2] = \omega_1\omega_2 - \omega_2\omega_1, \quad [\omega, u] = \omega u - u\omega, \quad [u_1, u_2] = 0.$$

The Hamiltonian has the form $H = \operatorname{tr}(\frac{1}{2}\hat{M}\hat{\omega} + uC)$. The Lax representation for the system (2.221), obtained in [19], has the form $\dot{L} = [L, A]$, where

$$\begin{gathered} L(\lambda) = \hat{M} + \lambda B + \lambda^{-1}u, \quad A(\lambda) = \hat{\omega} + \lambda C, \\ B = \operatorname{diag}(B_1, B_2, B_3), \quad B_i = I_1I_2I_3I_i^{-1}, \end{gathered} \tag{2.222}$$

where to simplify the formulas we assume that $I_1 + I_2 + I_3 = 0$. From this it follows that the system (2.221) can be integrated in theta functions of the Riemann surface Γ given by the equation $\det(L(\lambda) - \mu \cdot 1) = 0$. On the surface Γ of genus 4 an obvious involution of the form $(\lambda, \mu) \mapsto (-\lambda, -\mu)$ acts with six fixed points, corresponding to $\lambda = 0$ and $\lambda = \infty$. Therefore this surface doubly covers an elliptic curve, and the phase variables of the system (2.221) can be expressed via the Prym theta functions (of three variables) of this covering.

Another application of systems of the form (2.221) is the proof of the integrability of the problem of the rotation of a rigid body about a fixed point in a Newtonian field with an arbitrary quadratic potential $U = 2^{-1}a_{ij}x^ix^j$ [19] (the possibility of applying L-A pairs of type (2.222) to a top in the field of a quadratic potential was noted in [123]). Here the equations of motion can be written in the form (2.221), where the matrix u is constructed as follows. Let Q be the transition matrix from the S-system to the fixed system. Then $u = Q^TaQ$, where $a = (a_{ij})$.

5. Pole Systems

The program for research on the dynamics of poles of solutions of equations to which the inverse scattering method is applicable goes back to the article [87]. In two-dimensional hydrodynamics the poles of the solutions correspond to the dynamics of vortices. In the case of a finite number of vortices the corresponding system turns out to be a finite-dimensional Hamiltonian system.

The connection of the dynamics of poles of rational and elliptic solutions of the KdV equation to the equations of motion of the system (2.57) was first discovered in the paper [4].

Let us remark that elliptic solutions of the KdV equation of the form

$$u(x,t) = 2\wp(x - x_1(t)) + 2\wp(x - x_2(t)) + 2\wp(x - x_3(t))$$

were first constructed without any connection to finite-dimensional systems in the paper [47].

Originally the theory of the *Moser–Calogero systems* (2.57), integrable by the method of L, A pairs, and of their generalizations, which will be discussed in detail in the second part of this work, was developed without the use of a direct connection with solutions of partial-differential wave equations of the KdV type to which the inverse scattering method is applicable. The construction of solutions of the equations of motion of these systems was based on the theory of Lie algebras. For the system (2.57) in degenerate cases of the Weierstrass $\wp$-function—potentials x^{-2} or $\sinh^{-2} x$—it was shown that the coordinates of the particles $x_j(t)$ are the eigenvalues of a matrix depending linearly on t, i. e.

$$\text{const} \times \prod_j (x - x_j(t)) = \det(At + B - x \cdot 1) \tag{2.223}$$

(the matrix entries of A and B can be expressed explicitly in terms of the initial coordinates and momenta of the particles).

But in the elliptic case only the involutivity and independence of the integrals

$$J_k = \frac{1}{k} \operatorname{tr} L^k \tag{2.224}$$

was known, where L is given in (2.58), $J_2 = H$.

In the paper [4] already mentioned above it was shown that the dynamics of the poles $x_j(t)$ of solutions of the KdV equation rational in x, which are obliged to have the form

$$u(x,t) = 2\sum_{j=1}^{N} (x - x_j(t))^{-2}, \tag{2.225}$$

coincides with the Hamiltonian flow generated by the integral J_3, restricted to the fixed points of the original system, $\operatorname{grad} H = 0$. The necessity of restricting the

flow to the stationary points $\operatorname{grad} H = 0$ leads also to a restriction on the number of particles, which may have only the form $N = d(d+1)/2$.

The connection between the rational Moser–Calogero system and rational solutions of nonlinear equations proves to be more natural in the case of two-dimensional systems.

As was remarked in [77], all solutions rational in x of the KP equation

$$\frac{3}{4}\,u_{yy} = \frac{\partial}{\partial x}\left(u_t + \frac{1}{4}(u_{xxx} - 6uu_x)\right) \tag{2.226}$$

which subside as $|x| \to \infty$ have the form $u = 2\sum_{j=1}(x - x_j(y,t))^{-2}$. Here the dynamics of the poles of $x_j(y,t)$ in y and t correspond to the two commuting flows $J_2 = H$, J_3 (2.224). The number N is arbitrary. Using this connection in [77], it was shown that the construction of [77] gives *all* rational solutions of the KP equation.

Rational multisoliton solutions for the KP equation were constructed within the framework of the inverse scattering method in [99].

In the paper [31] the isomorphism of the two problems indicated in [77] was carried over to the elliptic case as well. However till [80] both problems—the construction of angle-type variables for the system (2.57) and the integration of its equations of motion in terms of theta functions, but also the problem of constructing elliptic solutions of the KP equation—remained completely unsolved (except for the simplest, two-particle case).

At the basis of the paper [80], where these problems were solved, lay the commutational representation found for the equations of motion

$$\ddot{x}_i = 4\sum_{j\neq i} \wp(x_i - x_j) \tag{2.227}$$

of the system (2.57). This commutational representation (in contrast to (2.58), (2.59)) involves a spectral parameter defined on the elliptic curve Γ. Moreover with respect to this parameter the matrix entries of U and V are Baker–Akhiezer functions.

Let us define matrices

$$U_{ij} = \dot{x}_i\delta_{ij} + 2(1-\delta_{ij})\Phi(x_{ij},\lambda), \tag{2.228}$$

$$V_{ij} = \delta_{ij}\left(2\sum_{k\neq i}\wp(x_{ik}) - \wp(\lambda)\right) + 2(1-\delta_{ij})\Phi'(x_{ij},\lambda), \tag{2.229}$$

where

$$\Phi(z,\lambda) = \frac{\sigma(z-\lambda)}{\sigma(\lambda)\sigma(z)}\,e^{\zeta(\lambda)z}, \tag{2.230}$$

$$\Phi'(z,\lambda) = \frac{\partial}{\partial z}\Phi(z,\lambda) \tag{2.231}$$

and $x_{ij}(t) = x_i(t) - x_j(t)$.

Proposition. *The equations* (2.227) *are equivalent to the commutational equation*

$$U_t = [V,U]. \tag{2.232}$$

It follows from (2.232) that the function

$$R(k,\lambda)=\det(2k+U(\lambda,t)) \tag{2.233}$$

does not depend on t. The matrix U, which has essential singularities for $\lambda=0$, can be represented in the form

$$U(\lambda,t)=g(\lambda,t)\,\tilde{U}(\lambda,t)g^{-1}(\lambda,t), \tag{2.234}$$

where $\tilde{U}$ does not have an essential singularity at $\lambda=0$, and $g_{ij}=\delta_{ij}\exp(\zeta(\lambda)x_i)$. Consequently, $r_i(\lambda)$—the coefficients of the expression

$$R(k,\lambda)=\sum_{i=0}^{n} r_i(\lambda)k^i, \tag{2.235}$$

are elliptic functions with poles at the point $\lambda=0$. The functions $r_i(\lambda)$ are representable as a linear combination of the $\wp$-function and its derivatives. The coefficients of such an expansion are integrals of the system (2.57). Each set of fixed values for these integrals gives by means of the equation $R(k,\lambda)=0$ an algebraic curve Γ_n which n-foldly covers the original elliptic curve Γ.

Generically the genus of the curve which arises is equal to n. The Jacobian of the curve Γ_n is isomorphic to the level manifold of the integrals r_n, and the variables on it are variables of the angle type.

A further putting to good effect of the solution-of equations (2.227) uses the connection of equation (2.232) with the existence of solutions of a special form for the non-stationary Schrödinger equation with an elliptic potential.

THEOREM 2.9. *The equation*

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}+2\sum_{i=1}^{n}\wp(x-x_i(t))\right)\psi=0 \tag{2.236}$$

has a solution ψ of the form

$$\psi=\sum_{i=1}^{n} a_i(t,k,\lambda)\Phi(x-x_i,\lambda)e^{kx+k^2t} \tag{2.237}$$

if and only if the $x_i(t)$ satisfy equations (2.227).

Here $\Phi(z,\lambda)$ is given by formula (2.230).

A function ψ of the form (2.237), as a function of the variable x, has simple poles at the points $x_i(t)$. Substituting it into (2.236) and equating the coefficients of $(x-x_i)^{-2}$ and $(x-x_i)^{-1}$ to zero, we get that ψ satisfies (2.236) if and only if the vector $a=(a_1,\dots,a_n)$ satisfies the equations

$$U(\lambda,t)a=-2ka, \tag{2.238}$$

$$\left(\frac{\partial}{\partial t}+V(\lambda,t)\right)a=0, \tag{2.239}$$

where U and V are the same as in (2.228), (2.229).

The analytic properties of a on the Riemann surface Γ_n can be clarified analogously to § 2. Let us formulate the final assertion.

THEOREM 2.10. *The eigenfunction $\psi(x,t,\gamma)$ of the non-stationary Schrödinger equation* (2.236) *is defined on the n-fold covering Γ_n of the original elliptic curve.*

The function ψ is a Baker–Akhiezer function with a unique essential singularity of the form

$$\exp(n\lambda^{-1}(x - x_1(0)) + n^2\lambda^{-2}t)$$

at an isolated preimage P_0 on Γ_n of the point $\lambda = 0$.

The explicit expressions for $\psi(x, t, \gamma)$ which were obtained in § 2 give that the poles of ψ in x coincide with the zeroes of the function $\theta(U^{(2)}x + U^{(3)}t + \zeta)$. Comparing with theorem 2.9, we finally get

THEOREM 2.11. *Let Γ_n be given by the equation $R(k, \lambda) = 0$, where R is defined in* (2.233). *Then the equation in x*

$$\theta(U^{(2)}x + U^{(3)}t + \zeta) = 0 \tag{2.240}$$

has n roots in the fundamental cell with periods $2\omega, 2\omega'$—the $x_i(t)$ which satisfy equation (2.227).

Here θ is a Riemann theta function corresponding to the surface Γ_n, and $U^{(2)}$, $U^{(3)}$ are the periods of differentials of the second kind with poles of the second and third order at the distinguished point P_0. These quantifies can he expressed in terms of the integrals r_i of the equations (2.227). *The vector ζ in* (2.240) *is arbitrary and corresponds to variables of the angle type.*

All of the parameters in (2.240) can be expressed by quadratures in terms of $x_i(0)$ and $\dot{x}_i(0)$.

In order to explain the ideas of [60] and show the effectiveness of the algebraic Lax representation (2.5), we consider first the rapidly decreasing case. In this case $u(x) \to 0$ fast enough as $|x| \to \infty$. We introduce the following monodromy matrix $\hat{T}(\lambda)$. Let $\psi_\pm(x, \lambda), \varphi_\pm(x, \lambda)$ be solutions of the equations

$$L\psi = \lambda\psi, \quad L\varphi = \lambda\varphi,$$

such that

$$\psi_\pm \sim \exp\{\pm ikx\}, \quad k^2 = \lambda, \quad x \to -\infty,$$
$$\varphi_\pm \sim \exp\{\pm ikx\}, \quad k^2 = \lambda, \quad x \to +\infty.$$

Then, by definition, the monodromy matrix $\hat{T}(\lambda) = (t_{ij}(\lambda))$ is the transition matrix from the basis $\psi_\pm$ to the basis $\varphi_\pm$:

$$\varphi_+ = t_{11}(\lambda)\psi_+ + t_{12}(\lambda)\psi_-,$$
$$\varphi_- = t_{21}(\lambda)\psi_+ + t_{22}(\lambda)\psi_-.$$

Note that $\det(\hat{T}(\lambda)) = 1$.

For real-valued functions $u(x)$ and $\lambda \in \mathbb{R}$, we have $\varphi_- = \overline{\varphi_+}$, $\psi_- = \overline{\psi_+}$. Thus, in this case $\hat{T}(\lambda) \in \mathrm{SU}(1, 1)$:

$$\hat{T}(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ \bar{b}(\lambda) & \bar{a}(\lambda) \end{pmatrix}, \quad |a|^2 - |b|^2 = 1.$$

The matrix $\hat{T}(\lambda)$ should not be confused with the unitary scattering matrix $S(\lambda)$ from quantum mechanics. The entries (T, R) of the latter are defined by the solution $f(x, \lambda)$:

$$f(x, \lambda) \sim e^{ikx}, \qquad x \to -\infty,$$
$$f(x, \lambda) \sim T(\lambda)e^{ikx} + R(\lambda)e^{-ikx}, \quad x \to -\infty.$$

Here $T = 1/a$, $R = b/a$ are "transition" and "reflection" coefficients respectively. Potentials for which $b(\lambda) \equiv 0$ for $\lambda \in \mathbb{R}$ are called "reflectionless potentials". For example, the famous multi-soliton potentials [60] are of this type. Their evolution with t generates *multi-soliton solutions* of the KdV equation. Such solutions have the property that they can be written as a rational combination of exponential functions, the simplest example being moving wave type solutions ("solitons") decreasing as $x \to \infty$:

$$u(x,t) = -\frac{4a^2}{\operatorname{ch}^2(2a(x-a^2t))}.$$

In the general case $b(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$. The Lax equation $L_t = [A, L]$ (see (2.6)) combined with the KdV equation implies

$$\frac{\partial a}{\partial t} \equiv 0, \quad \frac{\partial b}{\partial t} \equiv 8ik^3 b(\lambda)$$

for all $\lambda \in \mathbb{C}$. From this, one can obtain (using the Gelfand–Levitan–Marchenko integral equation for inverse scattering problems as in [115]) a procedure for solving rapidly decreasing Cauchy problems for the KdV equation, construct exact multi-soliton solutions, investigate asymptotic behavior of solutions, and so on. However, these aspects of the soliton theory are beyond the scope of this paper.

Integrals of motion (Kruskal integrals) can be obtained in the following way [60]. Consider a solution χ of the Ricatti equation

$$\chi'(x,\lambda) = \chi^2 + u - \lambda$$

as a series in k^{-1} (recall that $k^2 = \lambda$):

$$\chi = k + O\left(\frac{1}{k}\right) = k + \sum_{i \geq 1} \chi_i(x) k^{-i}.$$

The quantities χ_{2i} are purely imaginary and are total derivatives. By definition, put

$$\text{const } I_k = \int \chi_{2k+3}(x)\, dx, \quad k = -1, 0, 1, 2, \ldots.$$

Here χ_j are polynomials in u, u', u''. These formulae are not reduced, since one can omit parts of $\chi_j(u, u', u'')$ that are total derivatives. A more compact definition can be given with the help of the resolvent [34]. The right hand sides of the KdV equation and its higher-order analogues can be written in the following form (the Gardner form):

$$u_{t_k} = \partial_x \frac{\delta I_k}{\delta u(x)}, \quad I_{-1} = \int u\, dx, \quad I_0 = \int u^2\, dx, \quad \ldots.$$

As it was noted in [54, 59], this form leads to the Hamiltonian structure described in the **Important Example** of the Poisson brackets (1.75). A general higher-order KdV equation has the form

$$u_{t_k} = \partial_x \frac{\delta}{\delta u(x)} \big(I_k + c_1 I_{k-1} + \cdots + c_k I_0 + c_{k+1} I_{-1}\big).$$

The right hand sides of the higher-order KdV equations can be recursively obtained from each other with the help of the following relation ([59, 89]):

$$A \frac{\delta I_k}{\delta u(x)} = \partial_x \frac{\delta I_{k+1}}{\delta u(x)}, \quad A = \partial_x^3 + 2(u\partial_x + \partial_x u).$$

As it was shown in [89], all higher-order KdV equations admit the Lax type representations (2.6):

$$\frac{\partial u_k}{\partial t_k} = \partial_x \frac{\delta I_k}{\delta u(x)} \iff \frac{\partial L}{\partial t_k} = [A_k, L]$$

with the same operator $L = -\partial_x^2 + u(x)$, but different operators A_k. We have (see (2.6)):

$$A_0 = \partial_x, \quad A_1 = -4\partial_x^3 + 3(u\partial_x + \partial_x u).$$

Following [34], it is convenient to write down the general form of the operator A_k using fractional powers of operators on the real line. To any formal operator given by a series

$$Q = \partial_x^m + a_1(x)\partial_x^{m-1} + \cdots + a_m(x) + \sum_{j=1}^{\infty} a_j(x)\partial_x^{-j} = (Q)_+ + \sum_{j=1}^{\infty} a_j(x)\partial_x^{-j}$$

one can uniquely put in correspondence its m-th root Q_1:

$$Q = Q_1^m, \quad Q_1 = \partial_x + b_0(x) + \sum_{j=1}^{\infty} b_j(x)\partial_x^{-j}.$$

Therefore, one can define fractional powers

$$Q^{n/m} = Q_1^n, \quad (n, m) = 1$$

and their "positive" parts $(Q_1^n)_+ = (Q^{n/m})_+$. In [34, GD], it was shown that for $L = -\partial_x^2 + u(x)$ the operators A_k have the form

$$A_k = ((-L)^{(2k+1)/2})_+, \quad k = 0, 1, 2, \ldots.$$

This gives a higher-order KdV equation for each value of k. In [34], this approach was generalized by changing the second order operator L to an arbitrary scalar differential operator

$$L_n = \partial_x^n + a_2(x)\partial_x^{n-2} + \cdots + a_n(x).$$

Then one obtains analogues of the KdV hierarchy connected with the scalar operators of order greater than 2,

$$\frac{\partial L_n}{\partial t_k} = [A_{k,n}, L_n], \quad (k, n) = 1, \quad A_{k,n} = (L_n^{k/n})_+.$$

The Hamiltonian property of such systems was proved from two different points of view in [2] and [34]. See the end of § 1 for the discussion on two different Poisson brackets generalizing the **Important Example** of § 1.

There is one more interesting feature of the theory of integrable systems induced by scalar operators L. Since it is essentially the same for operators of different order, we consider here the case of the second order operator $L = -\partial_x^2 + u$. This operator can be factored

$$L + \alpha_0 = -(\partial_x + v)(\partial_x - v), \quad v_x + v^2 = u + \alpha_0.$$

The transformation

$$B_{\alpha_0} \colon L \to \tilde{L} = -(\partial_x - v)(\partial_x + v)$$

is called the Backlund–Darboux transformation depending on the parameter α_0. In the soliton theory, the change of variables from $u(x)$ to $v(x)$ is called the Miura

transformation. If we put $\alpha_0 = 0$, it transforms the KdV equation into the MKdV equation

$$v_t = 6v^2 v_x - v_{xxx}$$

which drastically simplifies the second Poisson bracket (1.81).

Consider the following 2×2 matrix operator

$$M = \partial_x + \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}.$$

It is easy to see that the operator L, when written in the matrix form

$$L = \partial_x + \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix},$$

is gauge equivalent to M, i. e., there exists a matrix $V(x)$ such that

$$\begin{pmatrix} v & 1 \\ 0 & -v \end{pmatrix} = V^{-1} \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} V - V^{-1} V_x.$$

Therefore, from the algebraic point of view, the Miura transformation is a transformation from the scalar operator L to M in the class of 2×2 matrix operators of the first order (see [SS] for the case of $n \times n$ matrix systems and [DS] for the generalizations from the Lie algebras point of view). In the set of matrix operators of the first order, this class is called "Shabat–Drinfeld–Sokolov" reduction (SDS-reduction).

Backlund–Darboux transformations $B_{\alpha_0} \colon L \to \tilde{L}$ allow us to construct new solutions of the KdV equation from the known ones. For example, all multi-soliton potentials $u(x)$ and solutions $u(x,t)$ of the KdV equations are obtained from the zero potential (solution) by iteration of such transformations. It is also possible to obtain multi-soliton solutions (potentials) on the background of the finite-gap or algebro-geometric solutions (potentials) [Kri], but the problem of constructing finite-gap solutions (whose theory will be considered below) cannot be reduced to anything simpler. Also, in [W] it was suggested to consider cyclic chains of Backlund–Darboux transformations. Main results were obtained in [SV]. All finite-gap potentials can be obtained from the following cyclic condition on the odd-length chains (proof of the conjecture of Weiss):

$$B_{\alpha_N} \circ B_{\alpha_{N-1}} \circ \cdots \circ B_{\alpha_0}(L) = L, \quad \alpha = \sum_{j=0}^{N} \alpha_j = 0, \quad N = 2K.$$

If $\sum_{j=0}^{N} \alpha_j \neq 0$ then solutions of the above cyclic conditions are oscillator-type potentials $u(x)$ with asymptotic behavior

$$u(x) \sim \left(\frac{\alpha x}{2}\right) + O(x), \quad |x| \to \infty,$$

and with discrete spectrum which consists of the union of $N+1$ arithmetic progressions with the same difference. For $N = 2$, such operators have the form $L = -\partial_x^2 + u$ and satisfy the algebraic relation

$$[L, A] = \alpha A,$$

$$L = \partial_x^2 + u, \quad A = \partial_x^3 + \frac{\alpha x}{2}\,\partial_x^2 + a\partial_x + b.$$

The two dimensional analogues of these results will be discussed in the Appendix.

Now let us consider the periodic case which focuses on the theory of finite-gap (algebro-geometric) operators and corresponding solutions of the KdV equations.

In the periodic case, the spectral theory is completely different and bears no resemblance to the scattering theory. Before the appearance of the KdV theory, inverse problems were not systematically considered. There were some examples of completely integrable potentials (for example, Lame potentials), but since at that time there was no relationship with Bloch or Hill spectral theory (in $\mathcal{L}_2(\mathbb{R})$), these results remained unknown to the specialists.

We begin with general remarks.

Consider a linear ordinary differential operator with the periodic coefficients

$$L = \partial_x^n + \sum_{j=1}^{n} a_j(x)\partial_x^{n-j}, \quad a_j(x+T) = a_j(x).$$

The spectral theory of such operators in the Hilbert space $\mathcal{L}_2(\mathbb{R})$ is based on the notion of Bloch–Floquet functions (or "Bloch waves"). Denote by T the monodromy operator, $T\psi(x) = \psi(x+T)$. This operator commutes with L, $LT = TL$. By definition, Bloch–Floquet functions are common eigenvectors of the operators L and T

$$T\psi(x) = \psi(x+T) = \mu\psi(x),$$
$$L\psi(x) = \lambda\psi(x).$$

The eigenvalues λ, μ must satisfy the relation $\Phi(\lambda, \mu) = 0$. In particular, we can consider λ as a multivalued function on μ. This function is called a "dispersion relation". Below we show that Bloch–Floquet functions always appear in the spectral theory of periodic linear differential operators. This is also true for difference and multidimensional linear periodic operators.

To begin with, let us define the monodromy matrix $\hat{T}(\lambda)$ w. r. t. some basis in the space of solutions of the equation $L\psi = \lambda\psi$. For example, we can take $\psi = (\psi_1, \dots, \psi_n)$ to be the standard basis satisfying

$$L\psi_j = \lambda\psi_j, \quad \psi_j^{k-1}(x_0) = \delta_j^k, \quad j,k = 1,\dots,n.$$

The monodromy operator T, $T\psi(x) = \psi(x+T)$, commutes with L. Hence it maps solutions to solutions and its matrix $\hat{T}$ in the basis ψ is defined by

$$\hat{T}(x_0,\lambda)\psi_j = \sum_q t_{jq}(x_0,\lambda)\psi_q(x,x_0,\lambda).$$

The matrix $\hat{T}$ satisfies the following equation (in x_0):

$$\frac{\partial \hat{T}}{\partial x_0} = [Q(x_0,\lambda), \hat{T}(x_0,\lambda)],$$

where the matrix $Q(x_0,\lambda)$ can be easily computed. The eigenvalues $\mu_j(\lambda)$ of the matrix $\hat{T}(x_0,\lambda)$ are independent of x_0 and we obtain an n-valued function $\mu = \mu_j(\lambda)$, $j = 1,\dots,n$. Denote by Γ the corresponding Riemann surface. Then on Γ we have a complex-valued meromorphic function $\psi(x,\lambda,\mu)$ satisfying the following relations:

$$L\psi = \lambda\psi, \quad T\psi = \mu\psi, \quad \psi(x_0,x_0,\lambda) \equiv 1.$$

Definition 1. The Riemann surface Γ together with the set of poles of ψ is called *inverse spectral data*.

Definition 2. An operator L is called a *finite-gap* or *algebro-geometric* operator if the genus of the Riemann surface Γ of its Bloch–Floquet function ψ is finite. In this case the number of poles of ψ is equal to the genus of Γ.

Strictly speaking, this definition is valid only for operators with periodic coefficients. In the case of quasi-periodic coefficients $a_j(x)$, ψ is called a Bloch–Floquet solution if $(\log\psi)_x$ is a quasi-periodic function with the same group of quasi-periods as the coefficients of the operator L. If the complete Bloch–Floquet solution exists for all $\lambda \in \mathbb{C}$, then there is a Riemann surface Γ with the same analytic properties as in the periodic case. This is always true for surfaces of finite genus.

An important and well-known class of examples is given by the second order operators. It includes the differential Schrödinger operator of the form

$$L = -\partial_x^2 + u,$$

2×2-systems of first order operators, difference operators, and λ-pencils whose Riemann surfaces Γ are hyperelliptic (i. e., Γ is a ramified double cover of a Riemann sphere $\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \{\infty\}$). In some cases the Riemann surface of a Bloch–Floquet function is a double cover of a more complicated algebraic curve of a spectral parameter $\lambda \in \Gamma$ (the coefficients of L depend on x and λ). One interesting example of such behavior for genus $g = 1$ was considered in [GN1] in connection with the theory of commuting operators of rank 2. In this case Γ was a ramified double cover of an algebraic curve of genus 1 and L was a scalar differential operator of order 4.

For a self-adjoint Schrödinger operator

$$L = -\partial_x^2 + u(x), \quad u(x+T) = u(x), \quad u(x) \in \mathbb{R},$$

we have the following property: away from branching points there is a solution ψ such that

$$L\psi = \varepsilon\psi, \quad T\psi = \mu_\pm\psi,$$

where $\mu_\pm = \exp(\pm ipT)$.

For real ε the quantity p ("quasi-momentum") can be real or purely imaginary. Regions with p real are called *allowed* zones (or spectral zones) and regions with purely imaginary p are called *forbidden* zones (or gaps). This terminology comes from the spectral theory of operators in the Hilbert space $\mathcal{L}_2(\mathbb{R})$, where periodic operators are sometimes called Hill operators, and also from solid state quantum physics where the state of an electron in the lattice is determined by Bloch waves and corresponding zones are called allowed and forbidden energy bands. In mechanics and stability theory (with x being a time variable), these zones are called stability and instability zones.

In general, a typical Schrödinger operator L with $\operatorname{Tr}\hat{T} = 2\cos(pT) = \mu_+ + \mu_-$ will have infinitely many forbidden zones satisfying $|\operatorname{Tr}\hat{T}| > 1$.

But there are some cases (for example, $u = \text{const}$) where the lengths of all forbidden zones are contracted to zero, ($|\operatorname{Tr}\hat{T}| \le 1$). For example, for the Lame potentials, $u(x) = n(n+1)\wp(x)$ [9], all but the first $2n+1$ states of a periodic problem are doubly degenerate. Here $x \in R + i\omega_2$, where $2i\omega_2$ is an imaginary period.

Although it was known for a long time that the Lame potentials can be exactly solved formally, it was not until around 1940 that some interesting features of their spectra were noticed ([11]). This is all that was known before the development of the KdV theory. The periodic KdV theory is based on the following fact. There is a large class of periodic or quasi-periodic finite-gap (algebro-geometric) potentials

[9] $\wp(x)$ is the Weierstrass $\wp$-function. Classics considered only the case $x \in R$ where the spectrum is discrete, and Floquet function is meaningless.

$u(x)$ such that all but finitely many of their forbidden zones have length zero. Then the Riemann surface Γ of the corresponding Bloch–Floquet function has genus g, where g is the number of forbidden zones of finite length (any real potential will also have one forbidden zone of infinite length). This theory was created in ([111, 38, 47, 102, 90, N]).

Finite-gap potentials are everywhere dense in the space of all periodic functions [MO]. It is possible to generalize the theta function formulae for the infinite-genus case [103]. This generalization is essentially a "topological closure" of the corresponding formulae for the finite-gap potentials.

Finite-gap potentials generate finite-gap conditionally periodic solutions of the KdV equation.

Below we sketch the theory in a somewhat simpler case of periodic difference operators L. In this case L is always a finite-gap operator, but with a very large number of zones—generically, the genus of the corresponding Riemann surface is equal to the number N of lattice points (i. e., to the period). This "finite-gap" property is not very interesting. It helps to justify general properties, but it is not very useful in practice—explicit theta function formulae have a reasonable form only for very small N. This motivates the following definition. A difference operator L is called *essentially finite-gap* if the number of forbidden zones is much less than the period N. However, even in cases $g = 2, 3$ the theta function formulae are very complicated and they started being used in applications only after the development of this theory.

Note the following property of finite-gap potentials (this is important for the Peierls–Fröhlich model below). Finite-gap potentials are extrema for the Kruskal functionals (defined above). Let $I = I_n + c_1 I_{n-1} + \cdots + c_n I_0$, where c_j are some constants. Then finite-gap potentials are periodic functions satisfying the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = 0.$$

It is important that the functionals I actually depend only on the spectrum of L, i. e. eigenvalues of L and periodic and anti-periodic boundary conditions on the boundaries of the forbidden zones, and are independent of the remaining spectral data.

It is easy to see that any functional of the potential $u(x)$, depending only on the spectrum of $L = -\partial_x^2 + u(x)$, admits an infinite-dimensional symmetry group generated by the KdV system and its higher analogues. Then it is natural to expect that the extrema of such functionals can be found explicitly. This is indeed the case, and the problem of finding these extrema is closely connected with the theory of finite-gap potentials. This idea was first applied in [12, 24] to the "Peierls jelly model" (a well-known model in solid state physics originating in the 1930's), and it was successfully generalized in [51, 81] for a very important difference analogue of the Peierls model. This analogue was developed by physicists in 1970's for the study of "charge density waves". Such waves were experimentally observed in the quasi-one-dimensional materials. This difference model is discussed below.

We now return to the notation of the previous sections.

6. Integrable Systems and the Algebraic-Geometric Spectral Theory of Linear Periodic Operators

The original approach to the construction of finite gap solutions of the KdV equations, the nonlinear Schrödinger equation and a number of others was based on the spectral theory of linear operators with periodic coefficients (see [38], [45], [64], [90], [100], [102], [111], [115]). The term "finite gap solutions" is connected with just this approach. Let us briefly point out the interconnection between this approach and the algebraic-geometric one which was set forth in § 2.

Let $U(x,t,\lambda)$ and $V(x,t,\lambda)$ be solutions of the equations of zero curvature depending periodically on x. Let us consider the matrix

$$W(x,t,\lambda) = \Psi(x+T,t,\lambda)\Psi^{-1}(x,t,\lambda), \tag{2.241}$$

where T is the period and Ψ is a solution of the equations (2.36). This matrix is called the monodromy matrix (describing the translation by a period of the solutions of the linear equations (2.36)).

From the fact that $\Psi(x+T,t,\lambda)$ is also a solution of equations (2.36) it follows that

$$[\partial_x - U, W] = [\partial_t - V, W] = 0,$$

and we arrive at equations (2.64), (2.65).

The matrix $W(x,t,\lambda)$ is analytic outside the poles of U and V, where it generically has essential singularities.

The vector function $\psi(x,t,\gamma)$ defined by the equations (2.66)–(2.72) is an eigenfunction of the period-translation operator: $\psi[x+T,t,\gamma) = \mu(\gamma)\psi(x,t,\gamma)$. In the theory of operators with periodic coefficients such functions are called *Bloch functions*. The Riemann surface Γ on which a Bloch function becomes single-valued has infinite genus in the general case (its branch points accumulate at the poles of U and V).

The finite gap periodic solutions are singled out by the condition that the genus of the surface Γ is finite, which is equivalent to the existence of a solution $W(x,t,\lambda)$ which is rational in λ for equations (2.64), (2.65).

Thus, the periodic solutions of the equations (2.37), (2.64), (2.65) have the property that their corresponding Bloch function is defined on a Riemann surface of finite genus and coincides with the *Baker–Akhiezer function*.

It is clear that the finite gap notion can be carried over verbatim to an arbitrary linear operator $\partial_x - U(x,\lambda)$ irrespective of nonlinear equations. The corresponding matrices U are called finite gap potentials.

The spectral properties of a Sturm–Liouville operator with finite gap potentials (properties obtained in the work presented in [45], [115]) were briefly cited in § 2.

Below we shall describe these properties in greater detail and shall give sketches of the proofs of the fundamental assertions using the example of the spectral theory of the Schrödinger difference operator (2.20)

$$L\psi_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} \tag{2.242}$$

($c_n = c_{n+N} \neq 0$, $v_n = v_{n+N}$), which enters into the Lax representation for the equations of the Toda lattice and for the KdV difference equation (when $v_n \equiv 0$).

Remark. In recent years there have been discovered new remarkable applications of the algebraic-geometric spectral theory to Peierls–Fröhlich problems, which

are among the most fundamental ones in the theory of quasi-one-dimensional conductors. In the continuous limit this model was investigated in the papers [12], [24], where indeed a connection between the Peierls model and the theory of *finite gap* Sturm–Liouville *operators* was discovered for the first time. This theory (the formulas for the variational derivatives of Kruskal integrals, variation with respect to the period group) was first applied to a full extent in [12]. The latter papers served as a starting point for the subsequent investigations [23], [50], [51], [81], in which these results were carried over to the discrete Peierls model and considerably developed.

The *Peierls-Fröhlich model* (R. E. Peierls, H. Fröhlich) describes the self-consistent behaviour of a lattice of atoms with coordinates $x_n < x_{n+1}$ and electrons. There are two models. In the first the atom at each lattice site also possesses an internal degree of freedom: v_n. In the second model, $v_n \equiv 0$.

The electronic energy levels are defined as the points $E_1 < E_2 \leq \cdots \leq E_N$ of the spectrum of the periodic problem for an operator L which has the form (2.242), where $c_n = \exp(x_n - x_{n+1})$, $c_n = c_{n+N}$, $v_n = v_{n+N}$. The energy of the system consists of the energy of the electrons, which at absolute zero occupy the m lowest levels, and the elastic energy of the lattice:

$$H = \frac{1}{N}\left(\sum_{i=1}^{m} E_i + \sum_{n=0}^{N-1} \Phi(c_n, v_n)\right).$$

Here m is the number of electrons and $\Phi(c_n, v_n)$ is the elastic energy potential.

In [23] the case was considered of

$$\sum_n \Phi(c_n, v_n) = \sum_n [\kappa(v_n^2 + 2c_n^2) - P \ln c_n]$$

and the more general one of

$$\sum_n \Phi(c_n, v_n) = \sum_{k=1}^{l} \kappa_k I_k,$$

where the I_k are integrals of the Toda lattice or the Langmuir lattice (J. Langmuir) ($v_n \equiv 0$).

In the first case it was shown that $H(c_n, v_n)$ has a unique extremal, corresponding to a one-gap operator L.

In the continuous limit this extremal goes over into the extremals obtained in [13], [24], which proves that in these papers the ground state was found.

For the more general models the stability of the extremals was investigated and the ground state was found. In addition, the speed of sound and of a charge density wave were found.

The systematic construction of an algebraic-geometric Bloch–Floquet spectral theory on $\mathcal{L}_2(\mathbb{Z})$ for the Schrödinger difference operator (2.242) was first begun by S. P. Novikov [45, Chap. 3, § 1] and by S. Tanaka–E. Date [33]. With the aid of the trace formulas for the function $\chi_n = \psi_{n+1}/\psi_n$ formulas were obtained for v_n. In [45] the symmetric case $v_n = 0$ was also studied. This theory was carried through to the finish in [45], but only in the elliptic case. In the paper [33] the expressions for v_n were written in the form

$$v_n = \frac{\partial}{\partial t} \ln \frac{\theta(Un + Vt + Z)}{\theta(U(n+1) + Vt + Z)} + \text{const.} \tag{2.243}$$

(In [45] an insignificant error was committed, which was rectified in the book [115].)

In the case of the Toda lattice, by virtue of the condition $\dot{x}_n = v_n$ formula (2.243) determines $x_n(t)$ up to a choice of the numbers $x_n(0)$, $-\infty < n < \infty$. The difference KdV was not considered in [33].

These investigations received their completion in [79], in which explicit expressions were obtained for the x_n and the solutions of the difference KdV. The idea of [79] consists in using explicit expressions for the ψ_n in terms of theta functions and analogues of the "trace identities" for the ψ_n, in contrast to [45], [33], where, as has already been said, trace formulas for the χ_n were used, analogously to the continuous case. In the later paper [82] "local trace identities" $c_n = c_n(\gamma_1, \dots, \gamma_n)$ were explicitly obtained whose existence had been ineffectively proved in [45].

The basic contemporary approach to spectral problems for periodic operators is the analysis of the analytic properties of the solutions of the equation

$$L\psi_n = E\psi_n \tag{2.244}$$

(here L is the operator (2.242) with periodic coefficients) for all values, among them also complex ones, of the parameter E.

For any E the space of solutions of equation (2.244) is two-dimensional. Having given arbitrary values to ψ_0 and ψ_1, one can find the remaining values ψ_n in a recursive manner. The standard basis $\phi_n(E)$ and $\theta_n(E)$ is given by the conditions $\phi_0 = 1$, $\phi_1 = 0$, $\theta_0 = 0$, $\theta_1 = 1$. From the recursive procedure for computing $\phi_n(E)$ and $\theta_n(E)$ it follows that (for $n > 0$) they are polynomials in E

$$\begin{aligned}
\phi_n(E) &= \frac{c_0}{c_1 \dots c_{n-1}} \left(E^{n-2} - \left(\sum_{k=2}^{n-1} v_k \right) E^{n-3} + \dots \right), \\
\theta_n(E) &= \frac{1}{c_1 \dots c_{n-1}} \left(E^{n-1} - \left(\sum_{k=2}^{n-1} v_k \right) E^{n-2} \right. \\
&\qquad \left. + \left(\sum_{0<i<j}^{n-1} v_i v_j - \sum_{k=1}^{n-3} c_k^2 \right) E^{n-3} + \dots \right),
\end{aligned} \tag{2.245}$$

The matrix $W(E)$ of the monodromy operator $\hat{T}\colon y_n \to y_{n+N}$ in the basis ϕ_n and θ_n has the form:

$$W(E) = \begin{pmatrix} \phi_N(E) & \theta_N(E) \\ \phi_{N+1}(E) & \theta_{N+1}(E) \end{pmatrix}. \tag{2.246}$$

It easily follows from (2.244) that for any two solutions of this equation, in particular for ϕ and θ, the expression (analogue of the Wronskian)

$$c_n(\phi_n \theta_{n+1} - \phi_{n+1}\theta_n) \tag{2.247}$$

does not depend on n. Since $c_0 = c_N$, we have

$$\det W = \phi_N \theta_{N+1} - \phi_{N+1}\theta_N = \phi_0\theta_1 - \theta_0\phi_1 = 1. \tag{2.248}$$

The eigenvalues w of the monodromy operator can be determined from the characteristic equation

$$w^2 - 2Q(E)w + 1 = 0, \quad 2Q(E) = \phi_N(E) + \theta_{N+1}(E). \tag{2.249}$$

The polynomial Q has degree N and its highest-order terms have the form

$$2Q(E) = \frac{1}{c_0 \dots c_{N-1}} \left(E^N - \left(\sum_{k=0}^{N-1} v_k \right) E^{N-1} + \left(\sum_{i<j} v_i v_j - \sum_{k=0}^{N-1} c_k^2 \right) E^{N-2} + \dots \right) \quad (2.250)$$

The spectra $E_i^{\pm}$ of the periodic and antiperiodic problems for L can be determined from the equations $Q(E_i^{\pm}) = \pm 1$, since when this holds $w = \pm 1$.

Let us denote by E_i, $i = 1, \dots, 2q+2$, $q \leq N-1$, the simple points of the spectrum of the periodic and antiperiodic problems for L, i. e. the simple roots of the equation

$$Q^2(E) = 1. \quad (2.251)$$

For a point E in general position the equation (2.249) has two roots w and w^{-1}. To each root there corresponds a unique eigenvector normalized by the condition $\psi_0 = 1$

$$L\psi_n = E\psi_n, \quad \psi_{n+N} = w\psi_n \quad (2.252)$$

This solution is called a Bloch solution.

THEOREM 2.12. *The two-valued function $\psi_n^{\pm}(E)$ is a single-valued meromorphic function $\psi_n(P)$ on the hyperelliptic curve Γ, $P \in \Gamma$, corresponding to the Riemann surface of the function $\sqrt{R(E)}$*

$$R(E) = \prod_{i=1}^{2q+2} (E - E_i). \quad (2.253)$$

Outside the points at infinity it has q poles $\gamma_1, \dots, \gamma_q$. In the neighbourhood of the points at infinity

$$\psi_n^{\pm} = e^{\pm x_n} E^{\pm n} \left(1 + \sum_{s=1}^{\infty} \xi_s^{\pm}(n) E^{-s} \right). \quad (2.254)$$

Here the $\pm$ signs correspond to the upper and lower sheets of the surface Γ (by the upper sheet will be meant the one on which at infinity $\sqrt{R} \sim E^{q+1}$).

A Bloch solution, like any other solution of equation (2.244), has the form $\psi_n = \psi_0 \phi_n + \psi_1 \theta_n$. The vector (ψ_0, ψ_1) is an eigenvector for the matrix W. Hence $\psi_0 = 1$, $\psi_1 = \frac{w - \phi_N}{\theta_N}$ or

$$\psi_n = \phi_n(E) + \frac{w - \phi_N(E)}{\theta_N(E)} \theta_N(E). \quad (2.255)$$

Let e_j, $j = 1, \dots, N-q-1$ be the double roots of the equation

$$Q^2(E) = 1, \quad \text{i. e.}$$

$$Q^2(E) - 1 = C^2 r^2(E) R(E), \quad r(E) = \prod_{j=1}^{N-q-1} (E - e_j), \quad (2.256)$$

$$C^{-1} = c_0 \dots c_{N-1}.$$

At the points e_j the matrix of the operator $\hat{T}$ with respect to a Bloch basis is equal to ± 1. So it is equal to ± 1 in any other basis. Hence

$$\theta_N(E) = r(E)\tilde{\theta}_N(E), \quad \phi_{N+1}(E) = r(E)\tilde{\phi}_{N+1}(E),$$
$$\phi_N(e_j) = \theta_{N+1}(e_j) = w(e_j) = \pm 1. \tag{2.257}$$

From (2.257) it follows that $Q(E) - \phi_N(E) = r(E)\tilde{Q}(E)$.

Here $\tilde{\theta}_N, \tilde{\phi}_{N+1}, \tilde{Q}$ are polynomials in E. Substituting $w = Q + Cr\sqrt{R}$ in (2.255) and using the preceding equalities, we get

$$\psi_n^{\pm} = \phi_n(E) + \frac{\tilde{Q}(E) \pm C\sqrt{R(E)}}{\tilde{\theta}_N(E)}\,\theta_n(E). \tag{2.258}$$

And this equality means in fact that the double-valued function $\psi_n^{\pm}$ is a single-valued meromorphic function of the point of Γ. The poles of ψ lie at points $\gamma_1, \dots, \gamma_q$, disposed one above each of the roots of the polynomial $\tilde{\theta}_N(E)$. Indeed, if $\tilde{\theta}_N(E) = 0$ then the two roots $w_{1,2}$ are equal to $\phi_N(E)$ and $\theta_{N+1}(E)$). In addition $\phi_N(E) \neq \theta_{N+1}(E)$. Consequently, for one of the roots w (i. e. on one of the sheets of Γ over the root of $\tilde{\theta}_N(E) = 0$) the numerator of the fraction in (2.258) vanishes. The pole of ψ_n lies on the second sheet.

To complete the proof of the theorem it remains to consider the behaviour of $\psi_n^{\pm}(E)$ when $E \to \infty$. From (2.258) it follows that ψ_1 has a simple pole at P^+. We immediately get from (2.244) that ψ_n has a pole at P^+ of n-th order for all $n > 0$. Similarly, ψ_{-n} has a pole of n-th order at P^-. This, together with the fact that w has a pole of N-th order at P^+ and a zero of multiplicity N at P^-, implies equation (2.254), where the x_n are such that $x_0 = 0$, $c_n = \exp(x_n - x_{n+1})$. □

The parameters γ_i, or rather their projections onto the E plane (which, as earlier, we shall for brevity denote the same) have a natural spectral meaning.

LEMMA 2.2. *The set of points e_i (the double points of the spectrum of the periodic and antiperiodic problems for L) and $\{\gamma_i\}$ are the spectrum for the problem* (2.244) *with zero boundary conditions.*

PROOF. The surface Γ has two sheets above the points e_j, on each of which w takes on the same value 1 or -1.

As $\tilde{\psi}_n$ one may take

$$\tilde{\psi}_n(e_j) = \psi_n^+(e_j) - \psi_n^-(e_j) = \frac{2C\sqrt{R(e_j)}}{\tilde{\theta}_N(e_j)}\,\theta_n(e_j). \tag{2.259}$$

The points γ_i are zeroes of $\theta_N(E)$. As was already said above, when $E = \gamma_i$ then for one of the signs in front of $\sqrt{R}$ in (2.258) the numerator of the second term vanishes. Hence for the second it is different from zero. Let this, for example, be the plus sign. Then

$$\tilde{\psi}_n(\gamma_j) = (\tilde{Q}(\gamma_j) + C\sqrt{R(\gamma_j)})\,\theta_n(\gamma_j) \tag{2.260}$$

is a non-trivial solution of equation (2.244), $E = \gamma_j$, with zero boundary conditions. □

Let us consider the inverse problem. Let arbitrary distinct points E_i be given, $i = 1, \dots, 2q+2$, and points $\gamma_1, \dots, \gamma_q$ on the Riemann surface Γ of the function $\sqrt{R(E)}$, whose projections to the E plane are all different. In difference problems the analogue of theorem 2.2 is the Riemann–Roch theorem [131]. In the given

case it states that there exists a meromorphic function $\psi_n(P)$ on Γ, unique up to proportionality, having poles at the points $\gamma_1, \dots, \gamma_q$, an *n-th* order pole at P^+ and an n-th order zero at P^-. The function $\psi_n(P)$ can be normalized up to sign by requiring that the coefficients of $E^{\pm n}$ on the upper and lower sheets at infinity be reciprocal. Having fixed the signs arbitrarily, we shall denote the corresponding coefficients by $e^{\pm x_n}$. With this ψ_n will have the form (2.254) in a neighbourhood of infinity.

LEMMA 2.3. *The constructed functions $\psi_n(P)$ satisfy equation* (2.244)*, where the coefficients of the operator L equal*

$$c_n = e^{x_n - x_{n+1}}, \quad v_n = \xi_1^+(n) - \xi_1^+(n+1). \tag{2.261}$$

PROOF. Let us consider the function $\tilde{\psi}_n = L\psi_n(P) - E\psi_n(P)$. It has poles at the points $\gamma_1, \dots, \gamma_q$. From (2.254), (2.261) it follows that $\tilde{\psi}_n$ has an $(n-1)$-st order pole at P^+ and an n-th order zero at the point P^-. By the Riemann–Roch theorem $\tilde{\psi}_n = 0$.

The method of obtaining explicit formulas for the ψ_n and the coefficients of L is completely analogous to the continuous case. As before, let us fix a canonical set of cycles on Γ. Let us denote by $i\,dp$ the normalized abelian differential of the third kind with its only singularities at infinity

$$i\,dp = \frac{E^q + \sum_{i=0}^{q-1} \alpha_i E^{q-i-1}}{\sqrt{R(E)}}\,dE = \frac{h(E)\,dE}{\sqrt{R(E)}}. \tag{2.262}$$

The coefficients α_i are determined from the normalization conditions

$$\oint_{a_i} dp = 0, \quad i = 1, \dots, q. \tag{2.263}$$

□

LEMMA 2.4. *The function $\psi_n(P)$ has the form*

$$\psi_n(P) = r_n \exp\left(in \int_{e_1}^{P} dp\right) \frac{\theta(A(P) + nU + \zeta)}{\theta(A(P) + \zeta)}, \tag{2.264}$$

where $U_k = (1/2\pi) \oint_{b_k} dp$, r_n is a constant.

In a neighbourhood of the point at infinity on the upper sheet we have

$$\exp\left(i \int_{e_1}^{P} dp\right) = Ee^{-I_0}(1 - I_1 E^{-1} + \dots), \quad P = (E, \sqrt{R}) \in \Gamma.$$

It follows from (2.254) that $e^{2x_n + 2I_0 n}$ equals the ratio of the values of the multipliers attached to the exponential in (2.264) taken at the images $A(P^\pm) = \pm z_0$. From (2.264) and the fact that by the Riemann bilinear relations $2z_0 = -U$, we get

$$c_n^2 = e^{2I_0} \frac{\theta((n-1)U + \tilde{\zeta})\theta((n+1)U + \tilde{\zeta})}{\theta^2(nU + \tilde{\zeta})} \tag{2.265}$$

where $\tilde{\zeta} = \zeta - z_0$.

In a neighbourhood of P^+ we have

$$A(P) = z_0 + VE^{-1} + \dots,$$

where the coordinates V_k of the vector V are defined by the equality

$$\omega_k = (V_k + O(E^{-1}))\,dE^{-1}.$$

By expanding (2.264) in a series in E^{-1}, we get from (2.261)

$$v_n = \frac{d}{dt} \ln \frac{\theta((n-1)U + \tilde{\zeta} + Vt)}{\theta(nU + \tilde{\zeta} + Vt)}\bigg|_{t=0} + I_1. \tag{2.266}$$

THEOREM 2.13. *The formulas* (2.265), (2.266) *recover the coefficients of* L *from the parameters* E_i *and* γ_j.

It is important to note that in general formulas (2.265), (2.266) define quasiperiodic functions c_n and v_n. For c_n and v_n to be periodic it is necessary and sufficient that for the corresponding differential dp the conditions be fulfilled:

$$U_k = \frac{1}{2\pi} \oint_{b_k} dp = \frac{m_k}{N}, \quad \text{the } m_k \text{ being integers.} \tag{2.267}$$

As follows from the definition of the ψ_n, the parameters E_i, γ_i determine them up to sign.

Changes of the signs of the ψ_n result in a change of the signs of the c_n. Operators differing only in the signs of the c_n need not be distinguished, since their eigenfunctions can be trivially transferred into one another.

So far we have been talking about operators L with arbitrary complex coefficients. Now let c_n and v_n be real; then all of the polynomials $\theta_n(E), \phi_n(E), Q(E)$ introduced above will be real. In addition, the periodic and the antiperiodic problems for L are self-adjoint. Hence there are N real points in the spectrum for each of these problems, i. e. the polynomial $Q^2 - 1$ has $2N$ real roots. Hence at the extrema of the polynomial $Q(E)$, $dQ/dE = 0$, one has that $|Q(E)| \geq 1$. The graph of the polynomial Q has the form:

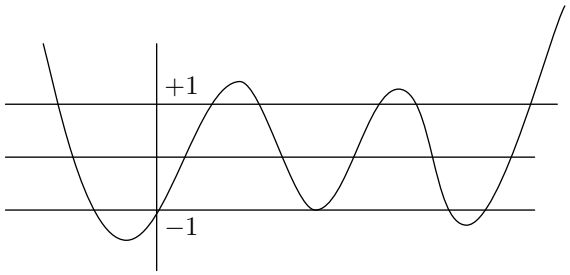

FIGURE 2

The intervals $[E_{2i-1}, E_{2i}]$, on which $|Q(E)| \leq 1$, are called the allowed bands[10] In these intervals $|w| = 1$ and the many-valued function $p(E)$ defined from the equality $w = e^{ipN}$ is real. It is called the quasimomentum. Its differential coincides with (2.262), where in (2.263) the a_i are the cycles situated above the forbidden bands[11] $[E_{2i}, E_{2i+1}]$.

LEMMA 2.5. *The poles* γ_i *of the Bloch function* $\psi_n(P)$ *of a real operator* L *are distributed one in each of the finite forbidden bands,* $E_{2i} \leq \gamma_i \leq E_{2i+1}$.

[10]Or the stable bands (translator's note).

[11]Or unstable bands (translator's note).

PROOF. The poles γ_i are the zeroes of the polynomial $\theta_N(E)$. At these points

$$1 = \det W = \phi_N(\gamma_i)\theta_{N+1}(\gamma_i),$$

Since ϕ_N and θ_{N+1} are real, we have

$$|Q(\gamma_i)| = \tfrac{1}{2}|\phi + N(\gamma_i) + \theta_{N+1}(\gamma_i)| \geq 1$$

and γ_i lies either in a forbidden band or in one of the collapsed bands—the points e_j. At the latter $\psi_n(P)$, as was shown above, has no singularities. □

THEOREM 2.14. *It the points $E_1, \ldots, E_{2q+2}$ are real and the points $\gamma_1, \ldots, \gamma_q$ of the corresponding Riemann surface lie one above each of the forbidden bands $[E_{2i}, E_{2i+1}]$, then the coefficients v_n and c_n of the operator L determined by them by virtue of theorem* 2.13 *are real.*

PROOF. The necessity of the conditions of the theorem within the class of periodic operators is given by lemma 2.5.

Let the E_i be real. Complex conjugation induces an anti-involution τ on the curve Γ, $\tau\colon P = (E, \sqrt{R}) \to \tau(P) = (\bar{E}, \overline{\sqrt{R(E)}})$. The fixed ovals of this anti-involution are the cycles disposed above the intervals $[E_{2i}, E_{2i+1}]$ and above the infinite band which joins through infinity the points E_{2q+2}, E_1.

Let us consider $\bar{\psi}_n(\tau(P))$. This function possesses all the analytic properties of ψ_n. Since ψ_n is determined by these properties up to sign, we get

$$\bar{\psi}_n(\tau(P)) = \pm\psi_n(P). \tag{2.268}$$

From (2.262) it follows that the v_n are real, and the c_n are either real or pure imaginary (i. e. c_n^2 is real).

Let us prove that under the assumptions of the theorem $c_n \neq 0$, $c_n \neq \infty$. The negation of this assertion is equivalent to one or several of the zeroes $\gamma_i(n)$ of the function $\psi_n(P)$ finding themselves at infinity on the upper or the lower sheet of Γ. From (2.268) it follows that on the cycles disposed above $[E_{2i}, E_{2i+1}]\psi_n$ is either real or pure imaginary. On each cycle there is one pole γ_i; therefore there is also at least one zero. Since there are q zeroes in all, the $\gamma_i(n)$ are distributed, like the γ_i, one above each $[E_{2i}, E_{2i+1}]$ and hence are separate from infinity.

By virtue of what has been proved, the sign of c_n^2 does not change under continuous deformations of E_i and γ_i for which the conditions of the theorem are fulfilled. Let us deform them so that all the forbidden bands close up. Here it is easy to check that the operator L is deformed into an operator L_0 which has $v_n = 0$ and $c_n^2 = \text{const} > 0$. The theorem is proved. □

To conclude the section let us examine the conditions which pick out operators L for which $v_n = 0$, i. e.

$$L\psi_n = c_n\psi_{n+1} + c_{n-1}\psi_{n-1}. \tag{2.269}$$

THEOREM 2.15 ([45], Chap. 3, § 1). *Necessary and sufficient conditions for the operator L reconstructed by virtue of theorem* 2.13 *from the data E_i and γ_j to have the form* (2.269)*, i. e. to have $v_n = 0$, are symmetry of the points E_i relative to zero and invariance of the set $\{\gamma_j\}$ with respect to the involution on Γ*

$$(E, \sqrt{R(E)}) \to (-E, \sqrt{R(E)}), \quad R(E) = \prod_{i=1}^{q+1}(E^2 - E_i^2).$$

The necessity of the conditions follows from the fact that if $\psi_n(P)$ is a Bloch solution for the operator (2.269), then for $\tilde{\psi}_n(E) = (-1)\psi_n(E)$ we have

$$L\tilde{\psi}_n = -E\tilde{\psi}_n, \quad \tilde{\psi}_{n+N} = (-1)^N w\tilde{\psi}_n.$$

The sufficiency of the conditions can be proved analogously to the proof of theorem 2.14. □

Let us define a function $\psi_n(t, P)$ which is meromorphic on Γ outside $P^{\pm}$, has poles $\gamma_1, \dots, \gamma_q$, and in a neighbourhood of $P^{\pm}$ has the form:

$$\psi_n^{\pm}(t, E) = e^{\pm x_n} E^{\pm n} \left(1 + \sum_{s=1}^{\infty} \xi_s^{\pm}(n, t) E^{-s}\right) e^{\mp t/2} \tag{2.270}$$

It can be proved in the standard way that such a function satisfies the linear equations

$$L\psi_n = E\psi_n, \quad \frac{d}{dt}\psi_n = A\psi_n, \tag{2.271}$$

where L and A have the form (2.20), (2.21). Consequently, the $x_n = x_n(t)$ satisfy the equations of the periodic Toda lattice.

Analogously to lemma 2.4, it is possible to write out an explicit formula for the $\psi_n(t, P)$ and to find explicit expressions for the $x_n(t)$.

THEOREM 2.16. *The functions*

$$x_n(t) = \ln \frac{\theta(Un + Vt + \zeta)}{\theta(U(n+1) + Vt + \zeta)} + I_1 t - nI_0 \tag{2.272}$$

satisfy the equations of the Toda lattice.

(Here I_1 is the average momentum, $-I_0$ is the mean distance between particles.)

The parameters of the theta function, the vectors U, V, ζ, can be expressed by quadratures in terms of the initial data $-x_n(0), x_n(0)$.

To conclude the chapter, let us cite on the basis of this example one more aspect of the theory of finite gap integration—its connections with variational principles for functionals of the Kruskal type (M. Kruskal).

Let us define the functionals $I_k = I_k\{c_n, v_n\}$ by the formula

$$ip(E) = \ln E - \sum_{k=0}^{\infty} I_k E^{-k}, \tag{2.273}$$

where $p(E)$ is the quasimomentum. These functionals have the form:

$$I_k = \frac{1}{N} \sum_{n=1}^{N} r_k(c_{n+i}, v_{n+i} \mid |i| < k),$$

where the local densities r_k are polynomials.

From (2.250) we have

$$I_0 = \frac{1}{N} \sum_{n=1}^{N} \ln c_n, \quad I_1 = \frac{1}{N} \sum_{n=1}^{N} v_n, \quad I_2 = \frac{1}{N} \sum_{n=1}^{N} \left(c_n^2 + \frac{v_n^2}{2}\right)$$

etc.

THEOREM 2.17. *The operator L is q-gap if and only if its coefficients are extremals of the functional H,*

$$H = I_{q+2} + \sum_{k=0}^{q+1} \alpha_k I_k. \tag{2.274}$$

This assertion follows from the formula

$$\begin{gathered} i\delta p = \frac{l_0 E^{q+1} + \cdots + l_{q+1}}{\sqrt{R(E)}}, \quad l_i = l_i(\delta c_n, \delta v_n), \\ \delta = \sum_n \left(\frac{\partial}{\partial x_c}\, \delta c_n + \frac{\partial}{\partial v_n}\, \delta v_n \right). \end{gathered} \tag{2.275}$$

In fact, by expanding (2.275) in the neighbourhood of P^+ and comparing with the coefficients of (2.273), we get

$$\begin{gathered} l_0 = -\delta I_0, \quad I_1 = -\delta I_1 + \frac{s_1}{2}\delta I_0, \quad s_1 = \sum_i E_i, \\ l_2 = -\delta I_2 + \frac{s_1}{2}\delta I_1 + \left(\frac{s_1^2}{8} - \frac{s_2}{2} \right) \delta I_0, \quad s_2 = \sum_{i<j} E_i E_j, \\ l_k = -\delta I_k + \sum_{i=0}^{k-1} \beta_{ik} \delta I_i. \end{gathered} \tag{2.276}$$

From the first $q+1$ equalities the coefficients l_k will be expressed via the δI_k, $k \le q+1$. Equating the coefficients of E^{-q-2} in the expansion (2.273) and in (2.275), we get that

$$\delta H = 0, \tag{2.277}$$

where the α_k are the symmetric polynomials in the E_i.

The proof of formula (2.274) can be obtained in an entirely analogous way to the proof of its continuous version [38].

Historical Remarks Concerning Algebraic Geometry, Hamiltonian Systems and Spectral Theory

1. First discovery for KdV and periodic 1-dimensional Schrödinger operator, hyperelliptic algebraic curves—Novikov (1974), Dubrovin–Novikov (1974), Its–Matveev (1975), Lax (1975), McKean–van Moerbeke (1975). Toda lattice and periodic difference Schrödinger operators—Henon, Flashka, Manakov (1974), Novikov, Tanaka, Date (1976). Commuting difference operators—Krichever, Mumford (1978). Peierls functionals. Charge Density Waves problem, applications in quantum solid state physics—Belokolos, Brazovsky, Gordunin, Dzyaloshinsky, Krichever (1980–1982).

1a. Periodic Schrödinger operators and approximation of the generic operators by the algebraic-geometrical (finite-gap) operators: Marchenko–Ostrovsky (1977). McKean–Trubowitz (1977). Periodic Darboux chains—Weiss (1988), Shabat–Veselov (1992). Elliptic finite-gap potentials—Dubrovin, Novikov (1974), Treibich–Verdier (1990), Belokolos, Einolsky, Smirnov, Taimanov (1991–1993).

2. Extension to 2-dimensional system, KP hierarchy, purely algebraic construction, all algebraic curves, commuting ordinary differential operators—Krichever (1975–1976). Krichever's solutions and Novikov's conjecture concerning classical Riemann–Shottky problem. Results of Dubrovin (1981), Arbarello, de Conchini (1984) and final solution by Shiota (1986).

2a. KdV and KP theory and poles systems—Airault, Moser, McKean (1977), Krichever (1978), Choodnovski D., Chodnoovski G. (1978). Elliptic case—Krichever (1980), Babelon, Biley, Talon, Krichever (1995), connection with quantum groups—Krichever, Zabrodin (1995), Krichever, Lipan, Zabrodin, Wiegmann (1996–1997).

3. Soliton theory, algebraic geometry and 2-dim Schrödinger operators—Manakov (1976), Dubrovin, Krichever, Novikov (1976). Purely potential Schrödinger operators and Prym Varieties—Novikov, Veselov (1984), Taimanov (1986). Novikov–Veselov hierarchy and deformations of surfaces in 3-dim space preserving Wilmore functional—Taimanov (1995–1996). Modified Novikov–Veselov hierarchy—Bogdanov (1987). Difference 2-dim operators—Krichever (1985), Novikov (1996).

3a. Spectral theory of one energy level and approximation of the generic 2-d potential Schrödinger operators by algebraic-geometrical—Krichever (1989). Compactification of Bloch–Flouque manifolds of eigenfunctions for all energy levels Feldman, Knorrer, Trubowitz (1989). Theory of special solvable operators—Chalikh, Veselov, Styrkas (1993), Novikov, Veselov, Taimanov (1996–1997), Etingof (1997).

4. KP hierarchy, Grassmanians and Representation theory—Sato, Miwa, Jimbo (1979), Segal, Wilson (1980).

5. KP hierarchy and deformations of semi stable vector bundles over algebraic curves. Solutions of rank $l \geq 2$, elliptic cases, commuting ordinary differential operators—Krichever, Novikov (1978–1980), Novikov, Grinevich (1982), Mokhov (1983), Grinevich (1985–1986). Yang–Mills fields and their deformations—Hitchin (1988). Hitchin's hierarchy of deformations of stable holomorphic vector bundles on Riemann surfaces.

6. Hamiltonian theory of the integrable systems (finite-dimensional and field theoretical)—Bogoyavlenski–Novikov (1975), Flashka–McLughlin (1976), Gelfand–Dikii (1977–1979), Adler (1978), Manin–Lebedev (1979), Veselov (1981), Dubrovin, Novikov, Veselov (1982–1984). Connection to Seiberg–Witten solution of $N = 2$ supersymmetric gauge theories—Krichever, Phong (1996).

7. Hamiltonian hydrodynamic type systems, Riemanian Geometry and integrability—Dubrovin, Novikov (1983–1984), Tsarev (1985), Krichever (1988), Potemin (1989), Mokhov (1988–1990), Ferapontov (1990). Topological quantum field theory and geometry—Krichever (1990–1992). Dubrovin (1990–1992). Solvable cases of the associativity equations—Mokhov, Ferapontov, Nutku (1995). n-orthogonal coordinate systems in Euclidean spaces and inverse scattering Zakharov (1996), algebraic-geometrical solutions—Krichever (1996).

8. Soliton theory, bosonic strings and harmonic analysis on the Riemann surfaces. Krichever–Novikov Fourier type basises and almost graded algebras—Krichever, Novikov (1987–1990), Jaffe, Klimek (1988), Bonora (1988), Sadov (1990), Schtihelmeier (1990).

9. Genus 1 surfaces (tori) of constant mean curvature in Euclidean 3-dim space and finite-gap solutions—Wente, Valter (1986), Pinkal, Hitchin, Bobenko (1991–1992).

APPENDIX A

Algebraic-Geometrical Integration of $(2+1)$-Systems

Strange as it may seem, the algebraic-geometrical construction of periodic and quasi-periodic solutions of multi-dimensional $(2+1)$-integrable systems is to some extent "simpler" than the construction of the finite-gap solutions of one-dimensional evolution integrable equations. This slightly exaggerated statement reflects the following inter-relations between spatially two- and one-dimensional integrable systems.

All one-dimensional evolution integrable equations can be considered as a reduction of their two-dimensional analogs. For example, the KdV equation (2.7) is the reduction of the KP equation

$$\frac{3}{4}u_{yy} = \left(u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\right)_x. \tag{A.1}$$

corresponding to the case where there is no dependence on the y variable. (Note that the equation (A.1) is obtained from (2.12) by elimination of w.) The Boussinesq equation (2.11) is the reduction of the KP equation as well. It corresponds to solutions of KP that do not depend on t. Another example is the Toda lattice (2.22), (2.23) which is the reduction of the $2d$-Toda lattice equations

$$\partial^2_{\xi\eta}\varphi_n = e^{\varphi_n - \varphi_{n-1}} = e^{\varphi_{n+1} - \varphi_n}. \tag{A.2}$$

At the same time, the sine-Gordon equation may be considered as another reduction of the same $2d$-Toda lattice equations. Namely, it corresponds to $N = 2$ periodic $\varphi_{n+N} = \varphi_n$ solutions of (A.2).

From the "finite-gap" theory point of view, all these inter-relations are a corollary of the fact that the Baker–Akhiezer functions, that are constructed for the arbitrary Riemann surface with fixed local coordinates at neighborhoods of the punctures, lead to solutions of "unrestricted" two-dimensional integrable systems. The algebraic-geometrical solutions of the equations that are the reductions of these two-dimensional equations correspond to reductions or specifications of the algebraic-geometrical data.

We begin the presentation of the general algebraic-geometrical construction with the definition of the most basic *multi-point* and *multi-variable* Baker–Akhiezer function.

Let Γ be a non-singular algebraic curve of genus g with N punctures P_α and fixed local parameters $k_\alpha^{-1}(Q)$ in neighborhoods of the punctures. For any set of points $\gamma_1, \ldots, \gamma_g$ in general position there exists a unique (up to constant factor $c(t_{\alpha,i})$) function $\psi(t, Q)$, $t = (t_{\alpha,i})$, $\alpha = 1, \ldots, N$; $i \geq 1$, such that:

(i) the function ψ (as a function of the variable Q which is a point of Γ) is meromorphic everywhere except for the points P_α and has at most simple poles at the points $\gamma_1, \dots, \gamma_g$ (it all of them are distinct);

(ii) in a neighborhood of the point P_α the function ψ has the form

$$\psi(t,Q) = \exp\left(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\right)\left(\sum_{s=0}^{\infty} \xi_{s,\alpha}(t) k_\alpha^{-s}\right), \quad k_\alpha = k_\alpha(Q) \tag{A.3}$$

Note that this is the same set of functions that was introduced in Section 2 of Chapter 2 but with a special choice of a set of the external parameters $t = \{t_{1,i}, \dots, t_{N,i}\}$ (that are the coefficients of the polynomials q_α).

In these new variables the theta-functional formula (2.100) becomes

$$\psi(t,Q) = \exp\left(\sum_{i,\alpha} t_{i,\alpha}\Omega_{i,\alpha}(P)\right)\frac{\theta(A(P) + \sum_{i,\alpha} U_{i,\alpha} t_{i,\alpha} + Z)}{\theta(A(P) + Z)}, \tag{A.4}$$

where:

a) $\theta(z) = \theta(z|B)$ is the Riemann theta-function corresponding to the matrix B of b-periods of normalized holomorphic differentials $d\omega_i$, $i = 1, \dots, g$ on Γ;

b) $\Omega_{i,\alpha}(P)$ is an abelian integral

$$\Omega_{i,\alpha}(P) = \int^P d\Omega_{i,\alpha}, \tag{A.5}$$

corresponding to the unique normalized

$$\oint_{a_k} d\Omega_{i,\alpha} = 0, \tag{A.6}$$

meromorphic differential on Γ with the only pole of the form

$$d\Omega_{i,\alpha} = dk_\alpha^i(1 + O(k_\alpha^{-i-1})) \tag{A.7}$$

at the puncture P_α;

c) $2\pi i U_{j,\alpha}$ is the vector of b-periods of the differential $d\Omega_{j,\alpha}$

$$U_{j,\alpha}^k = \frac{1}{2\pi i}\oint_{b_k} d\Omega_{j,\alpha}; \tag{A.8}$$

d) Z is an arbitrary vector (it corresponds to the divisor of poles of the Baker–Akhiezer function).

From the uniqueness of the Baker–Akhiezer function it follows that for each pair (α, n) there exists a unique operator $L_{\alpha,n}$ of the form

$$L_{\alpha,n} = \partial_{\alpha,1}^n + \sum_{j=1}^{n-1} u_j^{(\alpha,n)} \partial_{\alpha,1}^j, \tag{A.9}$$

(where $\partial_{\alpha,i} = \partial/\partial t_{\alpha,i}$) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = 0. \tag{A.10}$$

The idea of the proof of the theorems of this type which was proposed in [74, 75] is universal.

For any formal series of the form (A.3) there exists a unique operator $L_{\alpha,n}$ of the form (A.9) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = O(k^{-1})\exp\left(\sum_{i=1}^{\infty} t_{\alpha,i}k_\alpha^i\right). \tag{A.11}$$

The coefficients of $L_{\alpha,n}$ are differential polynomials with respect to $\xi_{s,\alpha}$. They can be found after substitution of the series (A.3) into (A.9).

It turns out that if the series (A.3) is not formal but is an expansion of the Baker–Akhiezer function in the neighborhood of P_α the congruence (A.11) becomes an equality. Indeed, let us consider the function ψ_1

$$\psi_1 = (\partial_{\alpha,n} - L_{\alpha,n})\psi(t,Q). \tag{A.12}$$

It has the same analytical properties as ψ, with one exception. The expansion of this function in the neighborhood of P_α starts from $O(k^{-1})$. From the uniqueness of the Baker–Akhiezer function it follows that $\psi_1 = 0$ and the equality (A.10) is proved.

Corollary A.1. *The operators $L_{\alpha,n}$ satisfy the compatibility conditions*

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\alpha,m} - L_{\alpha,m}] = 0. \tag{A.13}$$

Remark. The equations (A.13) are gauge invariant. For any function $g(t)$ operators

$$\tilde{L}_{\alpha,n} = gL_{\alpha,n}g^{-1} + (\partial_{\alpha,n}g)g^{-1} \tag{A.14}$$

have the same form (A.9) and satisfy the same operator equations (A.13). The gauge transformation (A.14) corresponds to the gauge transformation of the Baker–Akhiezer function

$$\psi_1(t,Q) = g(t)\psi(t,Q). \tag{A.15}$$

Example. One-puncture Baker–Akhiezer function.

In the one-puncture case the Baker–Akhiezer function has an exponential singularity at a single point P_1 and depends on a single set of variables. Let us choose the normalization of the Baker–Akhiezer function with the help of the condition $\xi_{1,0}$, i. e. an expansion of ψ in the neighborhood of P_1 equals

$$\psi(t_1,t_2,\dots,Q) = \exp\left(\sum_{i=1}^{\infty} t_i k^i\right)\left(1 + \sum_{s=1}^{\infty}\xi_s(t)k^{-s}\right). \tag{A.16}$$

In this case, the operator L_n has the form

$$L_n = \partial_1^n + \sum_{i=0}^{n-2} u_i^{(n)}\partial_1^i. \tag{A.17}$$

For example, for $n = 2, 3$ after redefinition $x = t_1$ we have $L_2 = \sigma L$, $L_3 = A$, where L and A are differential operators (2.12) and

$$u(x,t_2,\dots) = 2\partial_x\xi_1(x,t_2,\dots). \tag{A.18}$$

Therefore, if we define $y = \sigma^{-1}t_2$, $t = t_3$ then $u(x,y,t,t_4,\dots)$ satisfies the KP equation (A.1).

It follows from (A.18) that in order to get the solution of the KP equation, it is enough to take the derivative of the first coefficient of the expansion at the

puncture of the ratio of theta-functions in the formula (A.4). The final formula for the algebraic-geometrical solutions of the KP hierarchy has the form

$$u(t_1, t_2, \dots) = 2\partial_1^2 \ln\theta\left(\sum_{i=1}^{\infty} U_i t_i + Z\right) + \text{const}, \tag{A.19}$$

(see details in [75]).

The formula (A.19) that was derived in [75] has lead to one of the most important pure mathematical applications of the theory of non-linear integrable systems. This is the solution of the famous Riemann–Shottky problem.

According to the Torrelli theorem the matrix of b-periods of normalized holomorphic differentials uniquely defines the corresponding algebraic curve. The Riemann–Shottky problem is to describe symmetric matrices with positive imaginary part that are the matrices of b-periods of normalized holomorphic differentials on algebraic curves. Novikov conjectured that the function

$$u(x, y, t) = 2\partial_1^2 \ln\theta(Ux + Vy + Wt + Z|B) \tag{A.20}$$

is a solution of the KP-equation iff the matrix B that defines the theta-function is the matrix of b-periods of normalized holomorphic differentials on an algebraic curve and U, V, W are vectors of the b-periods of corresponding normalized meromorphic differentials that have one pole at a point of this curve. This conjecture was proved in [Shi].

Let us make a few comments about the multi-puncture case. For each a the equation (A.13) up to gauge transformations, is equivalent to the KP hierarchy corresponding to each set of variables $\{t_{\alpha,i}\}$. One could ask, "What is the interaction between two different KP hierarchies?"

As it was found in [43], for the two-puncture case a full set of equations can be represented in the following form

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\beta,n} - L_{\beta,n}] = D^{\alpha,\beta}_{N,m} H^{\alpha,\beta}. \tag{A.21}$$

where $H^{\alpha,\beta}$ is the two-dimensional Schrödinger operator in a magnetic field

$$H^{\alpha,\beta} = \frac{\partial^2}{\partial_{\alpha,1}\partial_{\beta,1}} + v_1^{\alpha,\beta}\partial_{\alpha,1} + v_2^{\alpha,\beta}\partial_{\alpha,2} + u^{\alpha,\beta} \tag{A.22}$$

and operators $D^{\alpha,\beta}_{N,m}$ are differential operators in the variables $t_{\alpha,1}, t_{\beta,1}$.

The sense of (A.21) is as follows. For the given operator $H^{\alpha,\beta}$ any differential operator D in the variables $t_{\alpha,1}, t_{\beta,1}$ can be uniquely represented in the form

$$D = D_1 H^{\alpha,\beta} + D_2 + D_3, \tag{A.23}$$

where D_2 is a differential operator with respect to the variable $t_{\alpha,1}$ only and D_3 is a differential operator with respect to the variable $t_{\beta,1}$ only. The equation (A.21) implies that the second and the third terms in the corresponding representation for the left hand side of (A.21) are equal to zero. This implies $n + m - 1$ equations on $n + m$ unknown functions (the coefficients of operators $L_{\alpha,n}$ and $L_{\beta,m}$). The equations (A.21) are gauge invariant. That's why the number of equations is equal to the number of unknown functions. Therefore, the operator equation (A.21) is equivalent to the well-defined system of non-linear partial differential equations.

We shall discuss multi-point case at a greater length in the next Appendix. In this section we only consider the $2d$ Toda lattice as a basic two-point example.

Let $\psi_n(t,Q)$ be the Baker–Akhiezer type function that is defined by the following analytical properties: a) ψ, as a function of the variable $Q \in \Gamma$, is meromorphic on Γ outside two punctures P_1, P_2 and has at most simple poles at the points $\gamma_1, \dots, \gamma_g; b)$ at the point P_α the function ψ has the form

$$\psi_n(t,Q) = k^{t_{\alpha,0}} \exp\left(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\right)\left(\sum_{s=0}^{\infty} \xi_{s,\alpha}(n,t) k_\alpha^{-s}\right), \quad k_\alpha = k_\alpha(Q). \tag{A.24}$$

Here $u = 1, 2$ and $t_{1,0} = n$, $t_{2,0} = -n$. This function has the same representation (A.4) through theta functions if we add one term in the arguments of the exponential factor and theta-function. Namely,

$$\psi_n(t,Q) = \exp(n\Omega_\pm + \sum_{i,\alpha} t_{i,\alpha}\Omega_{i,\alpha}(P))\,\frac{\theta(A(P) + nU + \sum_{i,\alpha} U_{i,\alpha} t_{i,\alpha} + Z)}{\theta(A(P) + Z)}, \tag{A.25}$$

where $d\Omega_\pm$ is the normalized differential with simple poles at the points P_1 and P_2 with residues ± 1 and $2\pi i U$ is a vector of b-periods of this differential. Let us normalize this function with the help of the condition $\xi_{0,1} \equiv 1$ and denote $\xi_{0,2}(n,t)$ by $\xi_{0,2} = \exp(\varphi_n(t))$. From (A.25) it follows that

$$\varphi_n = \ln \frac{\theta(A(P_2) + nU + \sum_{i,\alpha} U_{i,\alpha} t_{i,\alpha} + Z)\,\theta(A(P_1) + Z)}{\theta(A(P_1) + nU + \sum_{i,\alpha} U_{i,\alpha} t_{i,\alpha} + Z)\,\theta(A(P_2) + Z)}. \tag{A.26}$$

If we denote the first "times" corresponding to the punctures by $\xi = t_{1,1}$ and $\eta = t_{2,1}$, then the usual arguments prove that

$$\partial_\xi \psi_n(\xi,\eta,Q) = \psi_{n+1}(\xi,\eta,Q) + v_n(\xi,\eta)\psi_n(\xi,\eta,Q), \tag{A.27}$$

$$\partial_\eta \psi_n(\xi,\eta,Q) = c_n(\xi,\eta)\psi_{n-1}(\xi,\eta,Q), \tag{A.28}$$

where

$$c_n = e^{\varphi_n - \varphi_{n-1}}, \quad v_n = \partial_\xi \varphi_n. \tag{A.29}$$

The compatibility conditions of (A.27) and (A.28) are equivalent to (A.2). Hence, the formula (A.26) gives solutions of the $2d$ Toda lattice equations.

Now let us consider the reduction problem. Let us assume that for the algebraic curve Γ and fixed local coordinate k^{-1} in the neighborhood of the puncture P_1 there exists a meromorphic function $E(Q)$ that is holomorphic on Γ outside the puncture and has the form

$$E(Q) = k^n + O(k^{-1}) \tag{A.30}$$

in the vicinity of P_1. Then for the corresponding Baker–Akhiezer function the following identity is fulfilled

$$\psi(t,Q) = e^{E(Q)t_n}\psi(t,Q)|_{t_n=0}. \tag{A.31}$$

For the proof of (A.31) it is enough to note that the right and the left hand sides of it have the same analytical properties. Then the uniqueness of the Baker–Akhiezer function implies that both sides coincide.

The equality (A.31) implies that the corresponding solution of the KP hierarchy does not depend on the variable t_n and the corresponding linear equation (A.10) becomes

$$L_n\psi(t,Q) = E(Q)\psi(t,Q). \tag{A.32}$$

Therefore, if Γ is a hyperelliptic curve that is defined by the equation

$$y^2 = \prod_{i=1}^{2g+1} (E - E_i) \tag{A.33}$$

and the puncture is the "infinity" $E = \infty$, then the formula (A.19) defines the solution of the KdV equation. If we choose the curve given by the equation

$$y^3 + E^n + \sum_{ni+3j \le n-2} a_{ij} y^i E^j = 0$$

then the formula (A.19) gives solutions of the Boussinesq equation.

Now let us consider, as another example, the periodic reductions of the $2d$ Toda lattice. Let $w(Q)$ be a function on the curve Γ such that it has a pole of order N at the point P_1 and a zero of order N at the puncture P_2 and is holomorphic on Γ except at P_1. In that case the corresponding Baker–Akhiezer function satisfies the relation

$$\psi_{n+N}(t, Q) = w(Q)\psi_n(t, Q). \tag{A.34}$$

Again in order to prove (A.34) it is enough to check that the right and the left hand sides of it have the same analytical properties. This equality implies that

$$\varphi_{n+N} = \varphi_n. \tag{A.35}$$

For $N = 2$ we have that the corresponding curve has to be a hyperelliptic curve that can be represented in the form

$$y^2 = w \prod_{i=1}^{2} g(w - w_i). \tag{A.36}$$

Therefore, we conclude that if Γ is defined by the equation (A.36) and two punctures are two branch points, $P_1 = \infty$ and $P_2 = 0$, then the formula (A.26) defines the solution of the Sine-Gordon equation $iu = \varphi_1 - \varphi_0$ (this formula coincides with (2.131)).

APPENDIX B

Two-Dimensional Schrödinger Operators and Integrable Systems

Consider the general two-dimensional Schrödinger operator L for the electric and magnetic fields on the Euclidean plane $\mathbb{R}^2$. After the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$, $z = x + iy$, L can be written in the following complex form

$$2L = (\bar{\partial} + B)(\partial + A) + 2V = (\partial_x + iA_1)^2 + (\partial_y + iA_2)^2 - 2U,$$

where $\partial = \partial_x - i\partial_y$, $\bar{\partial} = \partial_x + i\partial_y$, and A, B, A_1, A_2, V, U are functions of x, y such that

$$2iA_1 = A + B, \quad 2A_2 = A - B6, \quad 2U = A_{\bar{z}} - B_z - 2V.$$

The function $U(x, y)$ is called the *potential* and the function H such that

$$2H = B_z - A_{\bar{z}},$$

is called the *magnetic field*. In the sequel, by the potential we mean the function $V(x, y)$. The operator L is defined up to the gauge transformations

$$L \to e^{-f} L e^{f}, \quad \psi \to e^{-f}\psi.$$

The only invariants of L are the potential V and the magnetic field H. For real V and H we usually choose a gauge in which

$$B = -\bar{A}, \quad A_{1x} + A_{2y} = 0$$

(i.e., A_1, A_2 are real and the Lorenz condition is satisfied).

In the two-dimensional case, there are no non-trivial Lax type equations $L_t = [Q, L]$ for differential operators L. The correct two-dimensional generalization was first developed in [43, 98]. In [98], it was suggested to consider the equation

$$L_t = [Q, L] + PL,$$

where P and Q are differential operators. Then it was shown in [43] that the inverse spectral problem for L can be solved from the data obtained from eigenfunctions of a single energy level,

$$L\psi = 0.$$

Using the methods of algebraic geometry, it is possible to find a large class of exactly solvable systems. This theory was developed in [30, 86, 117] for the periodic case and in [GN2, GM, G] for the rapidly decreasing case. Certain related problems, for example the theory of pole systems, were solved in [KZ, FN*]. Below we outline the main ideas of the theory in the periodic case and consider some interesting two-dimensional systems that can be integrated by these methods.

The simplest and the most interesting example of a non-linear system is the following two-dimensional analogue of the KdV equation from the so called commutative *Novikov–Veselov hierarchy* [NV1]:

$$-\frac{\partial W}{\partial t} = \left(\frac{\partial^3 + \bar\partial^3}{8} + \frac{u_1\partial + u_2\bar\partial}{2}\right) W, \quad u_{1\bar z} = 3W_z, \quad u_{2z} = 3W_{\bar z},$$

where in the real case we put $u_1 = \overline{u_2}$.

This equation is equivalent to the Lax-type representation

$$\frac{\partial L}{\partial t} = [Q, L] + PL$$

with $L = \Delta + W$, and where

$$P = u_{1z} + u_{2\bar z}, \quad \frac{1}{8}(\partial^3 + \bar\partial^3) + \frac{1}{2}(u_1\partial + u_2\bar\partial).$$

Note that this system is more fundamental than the usual KdV or KP equations, since its *y-independent subsystem* coincides with the usual KdV equation, while the KP system can be obtained as a result of a special limiting procedure. There is also an analogue of the *Miura transformation* [Bog]. The substitution

$$\begin{aligned} u_2 &= 3\left(\bar\partial^{-1}\partial(\bar f f) - i\frac{\bar\partial f}{2}\right), \\ u_1 &= 3\left(\bar\partial^{-1}\partial(\bar f f) - i\frac{\partial \bar f}{2}\right), \\ W &= f\bar f - i\frac{\partial f}{2} \end{aligned}$$

transforms this equation into the *modified Novikov–Veselov equation*

$$f_t + \frac{1}{8}(\partial^3 + \bar\partial^3)f + \frac{3}{2}(\partial f)\bar\partial^{-1}\partial(\bar f f) + \frac{3}{2}(\bar\partial f)\partial^{-1}\bar\partial(\bar f f) + \frac{3f}{2}\partial^{-1}\bar\partial(\bar f \partial f) = 0.$$

Interesting classes of solutions of this system can be found by algebro-geometric methods. These include doubly-periodic solutions, solutions which arc quasi-periodic w. r. t. x and y variables, and classes of exactly solvable two-dimensional Schrödinger operators associated with a single energy level $L\psi = 0$. Let all the coefficients of the operator L be periodic in x and y with periods T_1 and T_2 respectively. We call this case *topologically trivial*, since the flux of the magnetic field through the elementary cell is zero:

$$[H] = \int_0^{T_1}\int_0^{T_2} H(x,y)\,dx\,dy = 0, \quad \text{where } 2H = B_z - A_{\bar z}.$$

The Bloch waves ψ such that

$$L\psi = 0, \quad \hat T_1\psi = e^{ip_1T_1}\psi, \quad \hat T_2\psi = e^{ip_2T_2}\psi,$$

where

$$\hat T_1\psi(x,y) = \psi(x+T_1, y), \quad \hat T_2\psi(x,y) = \psi(x, y+T_2)$$

are parameterized by one-dimensional complex manifolds Γ:

$$\psi = \psi(x, y, \mathcal{P}), \quad \mathcal{P} \in \Gamma.$$

Definition. A differential operator L is called *finite-gap* or *algebra-geometric* w. r. t. the level $L\psi = 0$ if Γ is an algebraic curve (i. e., the genus of Γ is finite).

In the general non-singular rank one case we have the following analytic properties. Suppose Γ is an algebraic curve of genus g with two marked "infinity" points $\mathcal{P}_\pm \in \Gamma$. Let $w_\pm = k_\pm^{-1}$ be local parameters in the neighborhood of $\mathcal{P}_\pm$. Then

1) the function $\psi(x, y, \mathcal{P})$ is a meromorphic function on Γ away from $\mathcal{P}_\pm$,
2) ψ has exactly g poles of the first order at the points $\mathcal{P}_1, \dots, \mathcal{P}_g$, $\mathcal{P}_j \neq \mathcal{P}_\pm$,
3) $\mathcal{P}_j$ are independent of x, y,
4) at $\mathcal{P}_\pm$ the function ψ has essential singularities of the special form:

$$\psi(x, y, \mathcal{P}) \sim \exp\{k_+ z\}(1 + O(k_+^{-1})) \qquad \text{as} \quad \mathcal{P} \to \mathcal{P}_+,$$
$$\psi(x, y, \mathcal{P}) \sim c(x, y) \exp\{k_- \bar{z}\}(1 + O(k_-^{-1})) \quad \text{as} \quad \mathcal{P} \to \mathcal{P}_-.$$

Then the operator L, whose coefficients can be expressed with the help of the theta function of the curve Γ, annihilates ψ:

$$L\psi = 0, \quad \text{where} \quad L = \partial\bar{\partial} + (\ln c)_{\bar{z}}\bar{\partial} + V(x, y).$$

In general, the coefficients of L are only quasi-periodic.

The theory of two-dimensional algebro-geometric Schrödinger operators is closely connected with the two-dimensional generalization of commutative rings of differential operators.

Definition. A commutative ring of operators "mod L" is given by the collection of operators

$$Q_0, Q_1, \dots, Q_m, \quad \text{where} \quad Q_0 = L, \quad Q_k = \sum_{i,j \geq 0} a_{ij}^k(z, \bar{z})\partial^i\bar{\partial}^j,$$

satisfying the relations

$$[Q_k, Q_l] = B_{kl}L, \quad k, l = 0, 1, \dots, m,$$

for some differential operators B_{kl}.

Clearly, the operators Q_k are defined up to the additive part

$$Q_k \sim Q_k + S_k L, \quad k > 0.$$

Therefore, any operator Q_k can be reduced to the following form:

$$Q_k \sim \left(\sum_j a_j^k(z, \bar{z})\partial^i\right) + \left(\sum_j b_j^k(z, \bar{z})\bar{\partial}^i\right) + BL.$$

Any algebra of this type, if it is complete enough and has rank one, is isomorphic to the algebra of functions on an algebraic curve Γ, which are meromorphic away from at most two points $\mathcal{P}_\pm \in \Gamma$, and correspond to the Schrödinger operator L [84]. For such algebras of functions and, more generally, of arbitrary order tensors on Γ it is possible ([KN1]) to construct analogues of the Laurent–Fourier bases. These *Krichever–Novikov bases* are necessary for the operator quantization of the bosonic string. In the scalar case (as above), these bases give a structure of an *almost graded algebra* [KN1].

We now explain the conditions necessary to obtain a purely potential self-adjoint operator

$$L = \partial\bar{\partial} + W(x, y), \quad W(x, y) \in \mathbb{R}.$$

Consider the curve Γ whose group of involutions is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Namely, let Γ have a holomorphic involution a with two fixed points

$$\sigma\colon \Gamma \to \Gamma, \quad \sigma^2 = 1, \quad \sigma(\mathcal{P}_\pm) = \mathcal{P}_\pm,$$

and anti-holomorphic involution τ

$$\tau\colon \Gamma \to \Gamma, \quad \tau^2 = 1, \quad \sigma\tau = \tau\sigma, \quad \tau(\mathcal{P}_\mp) = \mathcal{P}_\pm, \quad \tau^* k_+ = \overline{k_-}.$$

The divisor of poles $\mathcal{D} = \mathcal{P}_1 + \cdots + \mathcal{P}_g$ must satisfy

$$\tau(\mathcal{D}) = \mathcal{D}, \quad \sigma(\mathcal{D}) + \mathcal{D} \sim K + \mathcal{P}_+ + \mathcal{P}_-,$$

where K is the canonical divisor of Γ and $\sim$ denotes the linear equivalence. Under these conditions we obtain a self-adjoint purely potential operator $L = \partial\bar\partial + W$ with

$$W = -2\partial\bar\partial \log\theta(V_1 z + V_2 \bar z + V_0) + C(\Gamma, \sigma).$$

Here θ is the standard Prym theta function with zero characteristics, $C(\Gamma)$ is some constant, and V_1, V_2 are vectors of normalized periods of abelian differentials of the second kind corresponding to the points $\mathcal{P}_\pm$. The effective approach for such formulae via non-linear equations (similar to [46] for Sine-Gordon and KP equations) was considered in [Tai]. To study nonlinear systems one must consider ψ with the same analytic properties, but with the different asymptotic behavior near $\mathcal{P}_\pm$:

$$\psi(x, y; t_1^+, t_2^+, \dots; t_1^-, t_2^-, \dots) \sim \exp\left\{ k_+ z + \sum_{i\ge 2} k_+^i t_i^+ \right\}, \quad \mathcal{P} \to \mathcal{P}_+,$$

$$\psi(x, y; t_1^+, t_2^+, \dots; t_1^-, t_2^-, \dots) \sim \exp\left\{ k_- \bar z + \sum_{i\ge 2} k_+^i t_i^- \right\}, \quad \mathcal{P} \to \mathcal{P}_-.$$

In the real case, the following conditions must be satisfied:

$$\tau^* k_+ = \overline{k_-}, \quad \tau t_i^* = \overline{t_i^-}.$$

To obtain a strictly positive purely potential operator L, the following conditions are necessary [Nat1, Nat2] and sufficient [NV1, GN2]. The anti-involution τ must have a maximal number $2g + 1$ fixed cycles

$$C = \{C_{1^+}; C_{1^-}; \dots; C_{g^+}; C_{g^-}; C_{g+1}\},$$

where g is the genus of the curve $\Gamma_0 = \Gamma/\sigma$,

$$\tau|_C \equiv 1,$$

and

$$\mathcal{D} = \mathcal{P}_{1^+} + \mathcal{P}_{1^-} + \cdots + \mathcal{P}_{g^+} + \mathcal{P}_{g^-},$$
$$\mathcal{P}_{j^\pm} \subset C_{j^\pm}, \quad \sigma(\mathcal{P}_{j^+}) = \mathcal{P}_{j^-}, \quad j = 1, 2, \dots, g,$$
$$\sigma\colon C_{j^+} \to C_{j^-}, \quad \sigma\colon C_{g+1} \to C_{g+1}.$$

The correct generalization of this theory for the $g = \infty$ case was developed in [Kr3]. It was shown that the class of *finite-gap operators* is everywhere dense in the space of doubly-periodic potentials $W(x, y)$. Some generalizations of recent results, for example the generalization of the theory of theta functions, were developed in [KT] for exactly the same class of curves as considered in [Kr3].

The rapidly decreasing analogue of this theory was studied in [GN*, GM, GN2, G]. In particular, the case of a positive operator $L = \partial\bar\partial + W$ is considered in [GN2]. In this case, the data is obtained from the level $-L\psi = \varepsilon_0\psi$, where $\varepsilon_0 < 0$ is below

the spectrum, and it is possible to drop the standard assumption, for this theory and the corresponding KP theory, of the small norm of the potential. The example of a simple rational decreasing potential, whose corresponding operator L has a scattering matrix which is trivial on one level, was described in [G].

As was mentioned before, the theory of the dynamical evolution for the pole systems of the rational solutions of the above equation was constructed in [FN*].

Laplace Transformations

The *Laplace transformation* for solutions of the Schrödinger equation is a two-dimensional analogue of the Backlund–Darboux transformation. If $L\psi = 0$, then we put

$$\tilde{\psi} = (\partial + A)\psi, \quad \tilde{L}\tilde{\psi} = 0,$$
$$L = (\tilde{\partial} + B)(\partial + A) + 2V,$$
$$\tilde{L} = V(\partial + A)V^{-1}(\tilde{\partial} + B) + 2V.$$

In other words, we have

$$\tilde{A} = A - (\ln V)_z, \quad \tilde{B} = B, \quad \tilde{V} = V + \tilde{H}, \quad \tilde{H} = H + \frac{\Delta \ln V}{2}.$$

Let $V = \exp(f)$. Consider the infinite chain of Laplace transformations for the potential and the magnetic field:

$$e^{f_{k+1}} = e^{f_k} + H_{k+1},$$
$$H_{k+1} = H_k + \tfrac{1}{2}\Delta f_k.$$

Making a change of variables $f_k = g_k - g_{k-1}$, we obtain a well-known *two-dimensional Toda lattice*

$$\Delta g_k = e^{g_{k+1} - g_k}.$$

This equation and some of its reductions were already known to the classical geometers (Darboux, Tsitseika [T]), whose approach was based on the Laplace transformations (as above). In the soliton theory, this equation and its Lax representation were first obtained in [Mi] as a two-dimensional analogue (via the Zakharov–Shabat approach) of the usual Toda chain. Therefore, it is a difference version of the KP equation w. r. t. the variable x, $x \to n$. Under certain "open w. r. t. n" boundary conditions, this equation is an analogue of the Liouville equation $\Delta = \exp(f)$, corresponding to the simple Lie algebras. In the Toda chain theory, the connection with Lie algebras was first noted and used in [LS]. Periodic chains of Laplace transformations

$$f_{n+k} \equiv f_k$$

were studied by Darboux and Tsitseika [T]. For $n = 2$, the periodicity condition has a solution in the sinh-Gordon form

$$\frac{1}{4}\Delta f_0 = e^{f_1} - e^{f_0} = Ce^{-f_0} - e^{f_0}.$$

If L_0 is a potential operator (i. e., $H_0 = H_n = 0$), then we have:

$$\tfrac{1}{2}\Delta f_0 = Ce^{-2f_0} - e^{f_0}, \quad n = 3,$$
$$\tfrac{1}{2}\Delta f_0 = e^{f_1} - e^{f_0}, \quad \tfrac{1}{2}\Delta(2f_0 + f_1) = Ce^{-3f_0 - 2f_1} - e^{f_0}, \quad n = 4.$$

In general, we can replace constants C by arbitrary harmonic functions. Also, for $n = 4$ there is a reduction $f_0 + f_1 = C_1$ to the sinh-Gordon equation

$$\frac{1}{4}\Delta f_0 = C_2 e^{-f_0} - e^{f_0}.$$

In [NV], it was shown that the algebro-geometric solutions of the two-dimensional Toda lattice, obtained in [Kr2] using the methods of the soliton theory, determine for each value of the discrete parameter k the algebro-geometric (from the spectral theory of the periodic operators point of view) Schrödinger operator L_k. In this case, the dependence of the inverse spectral problem data (i. e., a Riemann surface with two marked points and a divisor of degree g) on the parameter k can be described as follows: the Riemann surface and the marked points are fixed and the divisor of poles is shifting. Moreover, if all f_k are smooth, doubly-periodic, and non-singular, then the corresponding Laplace operator L_k is always algebro-geometric (finite-gap on a single energy level $L_k\psi = 0$). We note that the $n = 2$ case corresponds to the sinh-Gordon equation and, in this case, the above result was already known in connection with the theory of surfaces of the constant mean curvature in $\mathbb{R}^3$, and the topology of T^2 (see the review [Bo, Tai1]).

It is worth mentioning that in the doubly-periodic case any algebro-geometric operator L is topologically trivial, i. e., the magnetic flux is zero:

$$[H] = \iint_K H(x, y)\, dx\, dy = 0.$$

where K is the fundamental domain for the action of the group of periods $\mathbb{Z}^2$ on the plane $\mathbb{R}^2$. Also note that for non-singular smooth periodic f_k, the Laplace transformation preserves the magnetic flux:

$$[H_{k+1}] = [H_k]$$
$$[V_{k+1}] = [V_k] + [H_{k+1}].$$

The case in question is called doubly-periodic topologically non-trivial if H and V are periodic with possibly non-periodic potentials A_1, A_2 (or A, B). In the simplest case $H = \text{const}$ we obtain the Landau case. In this case, the spectrum in $\mathcal{L}_2(\mathbb{R}^2)$ is discrete and forms an arithmetic progression

$$L\psi = \lambda_k\psi, \quad \lambda_k = \lambda_0 + k\varepsilon_0, \quad \varepsilon_0 = H.$$

The levels λ_k are called k-th Landau levels. They are infinitely degenerate. The Landau operator can be considered as a natural two-dimensional analogue of the harmonic oscillator whose spectrum forms an arithmetic progression. In [AC, DN2, DN3, N3], it was shown that, in the rapidly decreasing and periodic cases, the condition $V = H$ implies that the operator L has a strongly degenerate "ground" (basic) level $\lambda_0 = 0$. It is convenient to choose a gauge in which $B = -\bar{A}$ (i. e., A_1, A_2 are real and the Lorenz condition is satisfied). In the periodic case ([DN2, DN3, N3]), if H satisfies the integrality condition

$$[H] = 2\pi m > 0, \quad m \in \mathbb{Z},$$

then the *magnetic Bloch eigenfunctions* of the ground level are given by

$$\psi = e^{\varphi} \prod_{j=1}^{m} \sigma(z - a_j) e^{az}, \quad z = x + iy,$$

where

$$\varphi = -\frac{1}{\pi}\iint_K H(x',y')\ln|\sigma(z-z')|\,dx'dy'.$$

$a_1,\dots,a_m$ are arbitrary constants, and the constant a can be expressed in terms of $a_1,\dots,a_m$. This level (i. e., the whole Hilbert space) is canonically isomorphic to the ground Landau level with the same magnetic flux and $H=\text{const}$.

The *quasi-cyclic chains* $L_0,\dots,L_n$ of Laplace transformations are given by the relations

$$H_0 = V_0, \quad H_n + C_n = V_n, \quad [H_j] > 0.$$

These chains satisfy the strong integrability properties.

Already, for the $n=2$ case, this fact implies the following interesting equation:

$$\frac{1}{2}\Delta f_0 = C_2 - 2e^{f_0}, \quad [H_0] > 0.$$

Let $C_2 > 0$. The operator $L_2 + C_2/2$ has two strongly degenerate energy levels. Since for the operator $L_2 + C_2/2$ we have $H = V$ ([DN2, DN3, N3]), there is a ground level $A_0 = 0$. The second level corresponds to $\lambda_2 = C_2$ and, after the choice of real gauge, the eigenfunctions have the form

$$\tilde{\tilde{\psi}} = e^{f_1/2}(\partial + A_1)e^{f_0/2}(\partial + A_0)\psi,$$

where

$$\psi = e^{\varphi}\prod_{j=1}^{m}\sigma(z-a_j)e^{az}, \quad \Delta\varphi = -\frac{1}{\pi}\iint_K H_0(z')\ln|\sigma(z-z')|\,d^2z,$$

$$H_0 = e^{f_0}, \quad [H_0] = 2\pi m > 0.$$

Both levels 0 and C_2 are strongly degenerate as Landau levels and there must be a nontrivial spectrum in the interval $(0, C_2)$. It seems that in the doubly-periodic case with non-zero magnetic flux (excluding the Landau case), there are at most two strongly degenerate levels.

The non-linear equation $\Delta g = 1 - e^g$ appears in many different problems [dV]. It is not integrable from the soliton theory point of view, but it has large families of non-singular doubly-periodic solutions. Even in the one-dimensional case, the solutions of this equation correspond to interesting one-dimensional "oscillatory-type" potentials whose spectral properties are very different from the spectral properties of the cyclic chains [NV]. Such are the two-dimensional analogues of the theory of cyclic Backlund–Darboux chains.

Now we consider *the discrete analogue of the Laplace transformations*. Although in the continuous case hyperbolic and elliptic operators are formally related by the substitution $\partial \to \partial_x$, $\bar{\partial} \to \partial_y$, in the discrete case they are formally different.

1. In the *hyperbolic case* we begin with the equation $L\psi = 0$, where

$$L\psi_n = \psi_n + a_n T_1\psi_n + b_n T_2\psi_n + c_n T_1T_2\psi_n,$$

where T_j are shifts on the basis vectors of the lattice, $T_1 = (1,0)$, $T_2 = (0,1)$, and $n = (n_1, n_2)$, $n_j \in \mathbb{Z}$. The operator L admits the representation

$$L = f_n[(1 + u_nT_1)(1 + v_nT_2) + w_n].$$

Similarly to the continuous case, put

$$\tilde{L}\tilde{\psi} = 0, \quad \tilde{\psi}_n = (1 + v_n T_2)\psi_n,$$
$$\tilde{L} = \frac{w_n}{1 + w_n}[(1 + v_n T_2)w_n^{-1}(1 + u_n T_1) + 1].$$

The representation of the form

$$L = g_n[(1 + p_n T_2)(1 + q_n T_1) + s_n]$$

generates the inverse representation

$$\tilde{\tilde{L}}\tilde{\tilde{\psi}} = 0, \quad \tilde{\tilde{\psi}}_n = (1 + q_n T_1)\psi_n,$$
$$\tilde{\tilde{L}} = \frac{s_n}{1 + s_n}[(1 + q_n T_1)s_n^{-1}(1 + p_n T_2) + 1].$$

The potential

$$w_n = x_{n-T_1} a_{n-T_1}^{-1} b_n^{-1} - 1$$

and the *magnetic field*

$$e^{H_n} = \frac{a_n b_{n+T_1}}{a_{n+T_2} b_n}$$

are invariant under the gauge transformations

$$L \to gLg^{-1}.$$

The Laplace transformation is given by the following formulae:

$$e^{\tilde{H}_n} = e^{H_n} w_{n+T_2} w_{n+T_1} (w_n w_{n+T_1+T_2})^{-1},$$
$$1 + \tilde{w}_{n+T_1} = e^{-\tilde{H}_n}(1 + w_{n+T_2}).$$

The infinite chain of Laplace transformations gives a complete discrete analogue of the two-dimensional Toda lattice for the variables $w_n^{(k)}, e^{H_n}$. From the second equation above it follows that the pair $(w_n^{(k)}, H_n^{(k)})$ can be obtained from $(w_n^{(k-1)}, H_n^{(k-1)})$ by the Laplace transformation. It is interesting to compare this system with the system constructed in [Kr7] and [KLWZ] using the theory of the Yang–Baxter equation.

It is also possible to impose the following cyclicity condition on chains,

$$w_n^{(k+N)} = w_n^{(k)}, \quad H_n^{k+N} = h_n^{(k)}, \quad n = (n_1, n_2).$$

Then, for $N = 2$, after the reduction

$$w_n^{(1)} = C(w_n^{(0)})^{-1},$$
$$w_{n+T_1+T_2} = w_n^{-1}(C + w_{n+T_1})(C + w_{n+T_2})(1 + w_{n+T_1})^{-1}(1 + w_{n+T_2})^{-1},$$

we obtain the discrete analogue of the sinh-Gordon equation.

2. The Laplace transformation for the *self-adjoint real difference operators* L is considered only for the operators of the form

$$L = a_n + b_n T_1 + c_n T_2 + d_{n+T_1} T_1 T_2^{-1} + b_{n-T_1} T_1^{-1} + c_{n-T_2} T_2^{-1} + d_{n+T_2} T_1^{-1} T_2,$$
$$L^+ = L, \quad T_j^+ = T_j^- 1,$$

where the lattice consists of equilateral triangles,

$$|T_1| = |T_2| = |T_1^{-1} T_2|.$$

Therefore, every vertex has six neighbors. The factorization

$$L = (x_n + y_n T_1 + z_n T_2)(x_n + y_{n-T_1} T_1^{-1} + z_{n-T_2} T_2^{-1}) + w_n$$

implies that for any solution $L\psi = 0$ we can put in correspondence the solution $\tilde{L}\tilde{\psi} = 0$, where

$$\tilde{\psi}_n = (x_n + y_{n-T_1} T_1^{-1} + z_{n-T_2} T_2^{-1})\psi_n,$$
$$\tilde{L} = (x_n + y_{n-T_1} T_1^{-1} + z_{n-T_2} T_2^{-1}) w_n^{-1} (x_n + y_n T_1 + z_n T_2) + 1.$$

As usual, the alternative factorization

$$L = (x'_n + y'_n T_1^{-1} + z'_n T_2^{-1})(x'_n + y'_{n+T_1} T_1 + z'_{n+T_2} T_2) + w'_n$$

corresponds to the inverse Laplace transformation.

As before, we can consider infinite chains of Laplace transformations and the cyclicity conditions, and it leads to new discrete analogues of the two-dimensional Toda lattice and its reductions. The conditions $(w_n = 0)$ or $(w'_n = 0)$ correspond to the theory of zero modes, see [AC, DN2, DN3, N3].

Notice that for both classes of operators—hyperbolic in the square lattice, and real self-adjoint in the equilateral triangle lattice—we have the following result. The factorization of the operator L can be described by local algebraic formulae in the coefficients of L. This is different from the one-dimensional case, where one has to solve the Ricatti-type equations or their discrete analogues.

APPENDIX C

Integrability of Systems of Hydrodynamic Type. The Non-Linear WKB Method

The circle of ideas, discussed below, was developed in [48, Ts, AN, Kr4, P2] and summarized in [DN1]. The new applications of these ideas to the two-dimensional quantum field theories are discussed in Lecture Notes [D1] and the Appendix to [N1J. This is an active area of current research, and it recently attracted a lot of attention after the interesting work of N. Seiberg and E. Witten ([SW1, SW2]) on the supersymmetric Yang–Mills theory.

As was discussed at the end of § 2, the systems of hydrodynamic type have the form

$$u^p_t = v^{p,\alpha}_q(u(x))u^q_\alpha, \quad \alpha = 1, 2, \dots, n; \quad p, q = 1, 2, \dots, m, \quad u^q_\alpha = \frac{\partial u^q}{\partial x^\alpha}.$$

In this equation, $u(x)$ is a map $\mathbb{R}^n \to M^n$, where M^m is some "manifold of components". In the spatially one-dimensional case $n = 1$, there is an important notion of *Riemann invariants*. Riemann invariants $(u^1, \dots, u^m)$ are special local coordinates on the manifold M^m which diagonalize the matrix of velocities:

$$V^p_q(u(x)) = v^p(u)\delta^p_q.$$

A system written down in Riemann invariants is called *diagonal*. In the case $m = 2$, Riemann showed that if the eigenvalues of the matrix V^p_q are real and distinct in the given domain, then it is always possible to find such a coordinate system. The exact solvability of the two-component spatially one-dimensional systems of hydrodynamic type is described by the following theorem (attributed to Riemann): if we do the "hodograph transformation", i. e., rewrite the equation in terms of $x(u^1, u^2)$, $t(u^1, u^2)$, then the new equation would be linear.

In general, for $m > 2$, we cannot always find the set of Riemann invariants. For example, in the theory of the dynamics of gases ($m = 3$), Riemann invariants exist only for a special (and non-interesting from the physical point of view) value of the Poisson adiabatic exponent. However, it is possible to find Riemann invariants for some important systems of hydrodynamic type. These systems arise in the theory of asymptotic methods for the soliton equations of the KdV type and they are a nonlinear analogue of the WKB approximation method or the method of the slow modulation of the parameter (this method is also called the Bogolubov–Whitham method). The Whitham method is described in [Wh1] and the existence of Riemann invariants in this case is discussed in [Wh2, 56].

As was mentioned at the end of § 1 of Chapter 1, the Hamiltonian approach for the systems of hydrodynamic type was developed in [48, 49, Mo] (see also Insert for § 1). According to the *Novikov hypothesis*, the existence of Riemann invariants together with the Hamiltonian properties of the systems of hydrodynamic type

imply their strong integrability. This hypothesis was proved in [Ts, Ts1, DN1], where the beautiful theory of integration for the diagonal Hamiltonian systems of hydrodynamic type was constructed. This theory is based on the Riemannian geometry lying in the foundation of the Hamiltonian approach. In fact, this theory can be extended slightly to the class of *semi-Hamiltonian* diagonal systems. These are systems which are Hamiltonian w. r. t. non-local Poisson brackets of the type considered in [MF, F].

We now consider the Tsarev method for integrating diagonal Hamiltonian systems of the form

$$u^p_t = v^p(u)u^p_x$$

This method is based on the fact that in this case the pseudoriemannian metric, which is used to define the Poisson bracket of hydrodynamic type, would also be diagonal:

$$g^{pq}(u) = g^p(u)\delta^p_q.$$

The Christoffel symbols Γ^k_{kp} for the standard (Levi-Civita) connection are obtained by the formula

$$\Gamma^k_{kp} = \frac{1}{2}\frac{\partial}{\partial u^p}\ln|g_k(u)| \quad \text{(warning: no summation)},$$

and they are connected with the velocities by the relation

$$\Gamma^k_{kp} = \frac{\partial_p v^k}{v^p - v^k}, \quad k \neq p \quad \text{(warning: no summation)}.$$

Therefore, we have the following important Tsarev relation:

$$\partial_q\left(\frac{\partial_p v^k}{v^p - v^k}\right) = \partial_p\left(\frac{\partial_q v^k}{v^q - v^k}\right)$$

This relation is taken as a definition of the *semi-Hamiltonian* diagonal systems of hydrodynamic type. We can always recover the metric from the above formulae, but this (diagonal) metric can have a non-zero curvature tensor. Note that the metrics considered in the theory of the local Poisson brackets of hydrodynamic type are always flat (see § 1). In certain cases, semi-Hamiltonian systems are actually Hamiltonian w. r. t. the generalized non-local Poisson brackets of hydrodynamic type (see Insert for § 1), but this fact is not proved in general. The method for integrating Hamiltonian and semi-Hamiltonian systems of hydrodynamic type is called the *generalized hodograph method* or the Tsarev procedure. It can be shown (though the proof is not effective) that there exists a large family of "symmetries", i. e., systems of hydrodynamic type, which commute with each other and with the given system. Consider the two systems:

$$u^p_t = v^p(u)u^p_x \quad \text{(the original system)},$$
$$u^p_t = w^p(u)u^p_x \quad \text{(symmetry)}.$$

Then a theorem says that the functions $u^p(x,t)$, determined from the equations

$$w^p(u(x,t)) = v^p(u(x,t))t + x, \quad p = 1, 2, \ldots, m$$

form a solution to the original system.

Diagonal Hamiltonian and semi-Hamiltonian systems have ample families of integrals of hydrodynamic type. Although expressed in different terms, this result for the two component systems has been known for a long time [La, YaR]. In the Hamiltonian case, the integrals of motion and symmetries are in one-to-one

correspondence; each integral generates a symmetry, each symmetry corresponds to an integral.

Unfortunately, it is hard to apply these beautiful general differential-geometric theorems to concrete problems. One of the examples in which it is possible is provided by the so called *weakly-linear* systems. Weakly linear systems are the systems of the form

$$\frac{\partial v^p}{\partial u^p}, \quad p = 1, 2, \dots, m \quad \text{(no summation)},$$

such that v^p is independent of u^p.

The systems of hydrodynamic type which arise in the soliton theory (for example, in the KdV theory) from the non-linear WKB method are strongly non-linear. The implementation of the Tsarev method requires the use of algebraic geometry. This technique was developed in [Kr4, PM] and is discussed below. We conclude this review with the following general theorem. If functions $u^1(x), \dots, u^m(x)$ are convex, then the symmetries of hydrodynamic type generate the complete set of flows, commuting with the given diagonal Hamiltonian system. This means that the symmetries form a basis for the tangent space to the generic level surface of the pairwise commuting integrals of hydrodynamic type. In other words, their linear combinations are everywhere dense. Thus, in a neighborhood of convex functions $u^1(x), \dots, u^m(x)$, the system is completely Liouville integrable ([Ts, DN1]).

The systems of hydrodynamic type appear in the following context. For simplicity, assume that

$$\psi^p_t = K^p(\psi, \psi_x, \dots, \psi_{(n)})$$

is an evolution system such that all $\psi = \text{const}$ are solutions. Let $\psi^p(x,t)$ be "slow" functions, i. e.,

(a)

$$\psi^p_t \ll \psi^p, \quad \psi_x \ll \psi.$$

(b) Each next derivative is of a smaller order.

The evolution equation has the form

$$\psi^p_t = G^p(\psi) + V^p_q(\psi) + \dots,$$

where $G^p = 0$ and all omitted terms are small by assumption. Then there is a *dispersionless limit* of this system:

$$\psi^p_t = V^p_q(\psi)\psi^p_x,$$

and it is a system of hydrodynamic type.

Another, more sophisticated method for obtaining systems of hydrodynamic type is a non-linear analogue of the WKB method ([Wh1, Wh2, La, Ma, AB, FFM]). This method requires the existence of a family of exact solutions of the form

$$\psi(x,t;u) = \Phi(Ux + Vt; u^1, \dots, u^m),$$

where the functions $\Phi(\eta_1, \dots, \eta_n)$ are periodic with period 2π w. r. t. the variables $\eta_1, \dots, \eta_k$ and K-vectors U, V can be expressed in terms of m parameters $u^1, \dots, u^m$. We are looking for a solution ψ, which is asymptotic w. r. t. ε, and is

of the form

$$\psi = \Phi(S(X,T); u^1, \dots, u^m) + O(\varepsilon), \quad \text{where}$$
$$u = (u^p(X,T)), \quad X = \varepsilon x, \quad T = \varepsilon t, \quad \varepsilon \to 0, \quad \text{and}$$
$$\frac{\partial S}{\partial X} = U(u^1, \dots, u^m), \quad \frac{\partial S}{\partial T} = V(u^1, \dots, u^m).$$

In certain cases (see the review [DN1]), it is possible to show that the necessary condition for the existence of such solutions is the following condition on the function $u^p(X,T)$. This function must satisfy the so-called "averaged system"

$$\frac{\partial u^p}{\partial T} = V_q^p \left(u(X,T) \frac{\partial u^q}{\partial X} \right),$$

which is a system of hydrodynamic type. The dispersionless limit corresponds to the case $k = 0$. The sufficiency condition is known only for certain cases with $k = 1$. In particular, for the KdV equation such equations of hydrodynamic type (Whitham systems) were obtained in [Wh2] for $k = 1$ and in [FFM] for $k > 1$ from the family of the finite-gap quasi-periodic solutions with $m = 2k + 1$. For these systems, the Riemann invariants exist for all $k \geq 1$. Namely, in the KdV case, the Riemann invariants $r_1, \dots, r_m$ are the boundaries of the Bloch spectral zones in $\mathcal{L}_2(\mathbb{R})$ of the Schrödinger operator with the finite-gap quasi-periodic potential $U = \Phi$ ([FFM, DN1]).

These systems are Hamiltonian [48]. The pseudo-Riemannian metric g^{pq} is flat and has signature $(k, k+1)$.

For Hamiltonian evolutionary systems with a local translation-invariant Poisson bracket $\{\ ,\ \}_0$, can use the method developed in [48]. This method requires the existence of a family of local integrals

$$I_q = \int P_Q(\psi, \psi_x, \dots)\, dx, \quad q = 1, \dots, m.$$

We also require that the parameters u^q for the family of the exact quasi-periodic solutions

$$\psi = \Phi(Ux + Vt; u^1, \dots, u^m)$$

can be taken in the form of the average densities:

$$u^q = \overline{P_q(\psi, \psi_x, \dots)}.$$

Then u^q are said to be *physical variables*. Here we average over the above solutions, and the value of an integral on the space of almost periodic functions is defined as the average of its density in the sense of H. Bohr. Finally, we require the integrals I_q to be in involution:

$$\{I_p, I_q\}_0 = 0, \quad p, q = 1, 2, \dots, m.$$

Let $I_1 = H$ be the Hamiltonian of our original system, $I_2 = P$ be the momentum, and $I_3, \dots, I_s$ be annihilators, i. e., $\{I_q, \cdot\} = 0$ for all $q = 3, \dots, s$. The fundamental principle of the conservation of the Hamiltonicity under averaging claims that the averaged system is also Hamiltonian with the Poisson bracket of hydrodynamic type. Its Hamiltonian coincides with the usual energy:

$$\int u^1(X)\, dX, \quad u^1 = \overline{P_1}, \quad H = \int P_1(\psi, \dots)\, dx.$$

A similar statement holds for the momentum. The Poisson bracket can be written in the "Liouville form" w. r. t. the physical variables $u^1, \dots, u^m$:

$$g^{pq}(u) = \gamma^{pq}(u) + \gamma^{qp}(u),$$
$$b_s^{pq}(u) = \frac{\partial \gamma^{pq}}{\partial u^s}.$$

Since $\{I_p, I_q\} = 0$, the original Poisson bracket has the form

$$\{I_p, P_q(\psi, \psi_x, \dots)\}_0 = \frac{d}{dx}(Q^{pq}(\psi, \psi_x, \dots))$$

Then we have ([DN1]):

$$\gamma^{pq} = \overline{Q^{pq}} + \text{const}.$$

This result was proved in complete generality only recently by A. Ya. Maltsev (to appear in Izvestia, ser. math) (the review [DN1] contains a mistake, see [N4]), its proof for some special cases can be found in [NM]. For the KdV equation this result was obtained earlier in [48].

EXAMPLE 1. Consider the KdV equation and its deformations via the Gardner–Zakharov–Faddeev bracket

$$\{\psi(x), \psi(y)\}_0 = \delta'(x - y),$$
$$H = \int \left(\frac{\psi_x^2}{2} + V(\psi) \right) dx.$$

This system can be written in the Gardner form (see § 1):

$$\psi_t = \frac{d}{dx} \frac{\delta H}{\delta \psi(x)}.$$

The family of exact solutions of the form $\psi(x - vt)$,

$$\psi = \Phi(Ux + Vt; u^1, u^2, u^3),$$

depends on three parameters u^1, u^2, u^3.

These parameters are the average values of the following integrals:

$$u^1 = \overline{\frac{\psi_x^2}{2} + V(\psi)} \quad \text{energy density},$$
$$u^2 = \overline{\psi^2} \quad \text{momentum density},$$
$$u^3 = \overline{\psi} \quad \text{annihilator (or "Casimir") density}.$$

The formulae for the Poisson bracket of hydrodynamic type for these cases can be found in [48], [DN1]. For the usual KdV we have $V(\psi) = \psi^3$, and the averaged densities of the Kruskal integrals I_q, $q = -1, 0, 1, 2, \dots$ define an infinite family of integrals of of hydrodynamic type. We denote their densities by u^q.

The existence of this family is not sufficient to have complete integrability. For $k = 1$, one has to add two more families of integrals; their description can be found in [Ti]. For $k \geq 1$ we have to add $2k$ families. To obtain these families, one has to integrate a certain form $\Omega = p\, d\lambda$ over basic cycles on a hyperelliptic Riemann surface

$$\Gamma\colon \mu^2 = R_{2k+1}(\lambda) = \prod_{j=0}^{2k} (\lambda - r_j),$$

where $p = p(\lambda; r_0, \dots, r_{2k})$ is a so-called *quasi-momentum* connected with the spectrum of the finite-gap Schrödinger operator L. The main family of the Kruskal integrals and their averages is obtained from the quasi-momentum expansion at the "infinity" point.

Any integral of hydrodynamic type generates an exact solution of the averaged Whitham system via the Tsarev procedure (see above). The averaged Kruskal integrals generate self-similar solutions [Kr4]. In particular, the integral I_4 corresponds to a very interesting solution. This solution is called a *dispersive analogue* of a *shock wave* (or the Gurevich–Pitaevskii solution), and it is very important in physical applications. The existence of this solution and some of its qualitative properties were established in [GP], although it was not clear that it was even C^1-smooth. The actual solution was found in [P2] using the procedure from [Kr4]. This solution is discussed in great detail in [DN1].

APPENDIX D

Spectral Theory of Two-Dimensional Periodic Operators

The basic algebraic-geometrical construction allows one to obtain periodic and quasi-periodic solutions for two-dimensional integrable systems but leaves absolutely open the following basic question: "How many algebraic-geometrical solutions are there? And what is their role in the solution of the periodic Cauchy problem for two-dimensional equations of the KP type?"

For finite dimensional $(0+1)$ systems that have the Lax representation of the form

$$\partial_t U(t,\lambda) = [U(t,\lambda), V(t,\lambda)], \tag{D.1}$$

with $U(t,\lambda)$ and $V(t,\lambda)$ rational (or sometimes elliptic) matrix-valued functions of the spectral parameter λ, the answer to the question is as follows: *all* the general solutions are algebraic-geometrical and can be represented in terms of the Riemann theta-functions.

For spacial one-dimensional evolution equations of the KdV type (1+1-systems) the existence of direct and inverse spectral transforms allow one to prove (though it is not always the rigorous mathematical statement) that algebraic-geometrical solutions are dense in the space of all periodic (in x) solutions.

It turns out that the situation for two-dimensional integrable equations is much more complicated. For one of the real forms of the KP equation that is called the KP-2 the algebraic-geometrical solutions are dense in the space of all periodic (in x and y) solutions [Kr3]. It seems, that the same statement for the KP-1 equation ($\sigma = i$) is false. One of the most important problems in the theory of two-dimensional integrable systems which are still unsolved is "in what sense" the KP-1 equation that has the operator representation (2.13) and for which a wide class of periodic solutions has been constructed, is a "*non-integrable*" system.

The proof of the integrability of the periodic problem for the KP-2 equation is based on the spectral Floquet theory of the parabolic operator

$$M = \partial_y - \partial_x^2 + u(x,y), \tag{D.2}$$

with periodic potential $u(x+l_1,y) = u(x,y+l_2) = u(x,y)$. We would like to mention that despite its application to the theory of non-linear equations and related topics, the structure of the Riemann surface of Bloch solutions of the corresponding linear equation that was found in [Kr3] has been used as a starting point for the abstract definition of the Riemann surfaces of infinite genus [KT].

The characteristic equation

$$R(w,E) = \det(w - \hat{T}(E)) \tag{D.3}$$

where $T(E)$ is the monodromy operator for periodic ordinary differential operator L defines the Riemann surface Γ of the Bloch solutions of this operator as a $N = nl$-sheet covering of the complex plane of the spectral parameter E (here n is the order of L and l is the matrix dimension of its coefficients). In [Kr3] another representation of the Riemann surface of the Bloch solutions was proposed. We would like to emphasize that this is the only possible representation that can be used in two dimensions.

Let us consider as an example a non-stationary Schrödinger operator (D.2). The solutions $\psi(x, y, w_1, w_2)$ of the non-stationary Schrödinger equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\psi(x, y, w_1, w_2) = 0 \tag{D.4}$$

with aperiodic potential $u(x, y) = u(x + a_1, y) = u(x, y + a_2)$ are called the Bloch solutions if they are eigenfunctions of the monodromy operators, i. e.

$$\psi(x + a_1, y, w_1, w_2) = w_1\psi(x, y, w_1, w_2), \tag{D.5}$$

$$\psi(x, y + a_2, w_1, w_2) = w_2\psi(x, y, w_1, w_2), \tag{D.6}$$

The Bloch functions will always be assumed to be normalized so that $\psi(0, 0, w_1, w_2) = 1$. The set of pairs $Q = (w_1, w_2)$, for which there exists such a solution is called the Floquet set and will be denoted by Γ. The multivalued functions $p(Q)$ and $E(Q)$ such that

$$w_1 = e^{ipa_1}, \quad w_2 = e^{iEq_2}$$

are called quasi-momentum and quasi-energy, respectively.

The gauge transformation $\psi \to e^{h(y)}\psi$, where $\partial_y h(y)$ is a periodic function, transfers the solutions of (D.4) into solutions of the same equation but with another potential $\tilde{u} = u - \sigma\partial_y h$. Consequently, the spectral sets corresponding to the potentials u and $\tilde{u}$ are isomorphic. Therefore, in what follows we restrict ourselves to the case of periodic potentials such that

$$\int_0^{a_1} u(x, y)dx = 0. \tag{D.7}$$

To begin with let us consider as a basic example the "free" operator

$$M_0 = \sigma\partial_y - \partial_x^2 \tag{D.8}$$

with zero potential $u(x, y) = 0$. The Floquet set of this operator is parametrized by the points of the complex plane of the variable k

$$w_1^0 = e^{ika_1}, \quad w_2^0 = e^{-\sigma^{-1}k^2a_2} \tag{D.9}$$

and the Bloch solutions have the form

$$\psi(x, y, k) = e^{ikx - \sigma^{-1}k^2y}. \tag{D.10}$$

The functions

$$\psi^+(x, y, k) = e^{-ikx + \sigma^{-1}k^2y}. \tag{D.11}$$

are Bloch solutions of the formal adjoint operator

$$(\sigma\partial_y + \partial_x^2)\psi^+ = 0. \tag{D.12}$$

The formulae (D.9) define the map

$$k \in C \mapsto (w_1^0, w_2^0) \in C^2. \tag{D.13}$$

Its image is the Floquet set for the free operator M_0. It is the Riemann surface with self-intersections. The self-intersections correspond to the pairs $k \neq k'$ such that

$$w_i^0(k) = w_i^0(k'), \quad i = 1, 2. \tag{D.14}$$

From (D.9) it follows that

$$k - k' = \frac{2\pi N}{a_1}, \tag{D.15}$$

$$k^2 - (k')^2 = \frac{\sigma 2\pi i M}{a_2}, \tag{D.16}$$

where N and M are integers. Hence, all the resonant points have the form

$$k = k_{N,M} = \frac{\pi N}{a_1} - \frac{\sigma i M a_1}{N a_2}, \quad N \neq 0, \quad k' = k_{-N,-M}. \tag{D.17}$$

The basic idea of the construction of the Riemann surface of Bloch solutions of the equation (D.4) that was proposed in [Kr3] is to consider (D.4) as a perturbation of the free operator (D.8), assuming that the potential $u(x, y)$ is formally small.

For any $k_0 \neq k_{N,M}$ it is easy to construct a formal Bloch solution $\tilde{\psi}$ of the equation (D.4) as a formal series

$$\tilde{\psi} = \sum_{s=0}^{\infty} \tilde{\varphi}_s(x, y, k_0), \quad \tilde{\varphi}_0(x, y, \ , k_0) = \psi(x, y, \ , k_0) = \psi_0. \tag{D.18}$$

This series describes a "perturbation" of the Bloch solution ψ_0 of the non-perturbed equation.

LEMMA D.1. *If $k_0 \neq k_{N,M}$ then there exists a unique formal series*

$$F(y, k_0) = \sum_{s=1}^{\infty} F_s(y, k_0) \tag{D.19}$$

such that the equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\Psi(x, y, k_0) = F(y, k_0)\Psi(x, y, k_0) \tag{D.20}$$

has a formal solution of the form

$$\Psi(x, y, k_0) = \sum_{s=0}^{\infty} \varphi_s(x, y, k_0), \quad \varphi_0 = \psi_0, \tag{D.21}$$

satisfying the conditions

$$\langle \psi_0^+ \Psi \rangle_x = \langle \psi_0^+ \psi_0 \rangle_x, \quad \psi_0^+ = \psi^+(x, y, k_0), \tag{D.22}$$

(*here and below $\langle f(x) \rangle_x$ stands for the mean value in x of the corresponding periodic function f*)

$$\Psi(x + a_1, y, k_0) = w_{10}\Psi(x, y, k_0), \quad w_{10} = w_1^0(k_0), \tag{D.23}$$

$$\Psi(x, y + a_2, k_0) = w_{20}\Psi(x, y, k_0), \quad w_{20} = w_2^0(k_0). \tag{D.24}$$

The corresponding solution is unique and is given by the recursion formulae (D.25)–(D.29).

$$\varphi_s = \sum_{n\neq 0} c_n^s(y,k_0)\psi_n(x,y), \quad s>1, \tag{D.25}$$

$$\psi_n = \psi_n(x,y) = \psi(x,y,k), \quad \psi_n^+ = \psi^+(x,y,k_n), \quad k_n = k_0 + \frac{2\pi n}{a_1} \tag{D.26}$$

$$c_n^s(y,k_0) = \sigma^{-1}\frac{w_{2n}}{w_{20}-w_{2n}}\int_y^{y+a_2}\left(\sum_{i=1}^{s-1} F_i c_n^{s-i} - \frac{\langle\psi_n^+ u\varphi_{s-1}\rangle_x}{\langle\psi_n^+\psi_n\rangle_x}\right)dy', \tag{D.27}$$

$$w_{2n} = w_2^0(k_n), \tag{D.28}$$

$$F_s(y,k_0) = \frac{\langle\psi_0^+ u\varphi_{s-1}\rangle_x}{\langle\psi_0^+\psi_0\rangle_x}. \tag{D.29}$$

From (D.20), (D.23), (D.24) it follows that the formula

$$\tilde\psi(x,y,k_0) = \frac{\Psi(x,y,k_0)}{\Psi(0,0,k_0)}\, e^{-\sigma^{-1}\int_0^y F(y',k_0)\,dy'} \tag{D.30}$$

defines the formal Bloch solution of the equation (D.4):

$$\tilde\psi(x+a_1,y,k_0) = w_{10}\psi(x,y,k_0), \tag{D.31}$$

$$\tilde\psi(x,y+a_2,k_0) = \tilde w_{20}\psi(x,y,k_0), \tag{D.32}$$

where the corresponding Bloch multiplier is equal to

$$\tilde w_{20} = w_{20}e^{-\sigma^{-1}\int_0^{a_2} F(y',k_0)dy'}. \tag{D.33}$$

For sufficiently small $u(x,y)$ it is not too hard to show that the above constructed series of the perturbation theory converges outside some neighborhood of the resonant points (D.17) and therefore determines a function $\tilde\psi(x,y,k_0)$ which is analytic in k_0). This is true for any σ. The principle distinction between the cases $\operatorname{Re}\sigma = 0$ and $\operatorname{Re}\sigma \neq 0$ is revealed under an attempt to extend $\tilde\psi$ to a "resonant" domain. In the case $\operatorname{Re}\sigma = 0$ the resonant points are dense on the real axis. In the case $\operatorname{Re}\sigma \neq 0$ there is only a finite number of the resonant points (D.17) in any finite domain of the complex plane. The discreteness of the resonant points in the last case is crucial for the extension of $\tilde\psi$ to a "resonant" domain (and for the proof of the approximation theorem).

In the stationary case, when u does not depend on y, the preceding formulae turn out to be the usual formulae of the perturbation theory of eigenfunctions corresponding to simple eigenvalues. The condition

$$w_{2n} \neq w_{20} \leftrightarrow k_0 \neq k_{NM} \tag{D.34}$$

is an analog of simplicity of an eigenvalue of an operator. In cases when it is violated, it is necessary to proceed along the same lines as in the perturbation theory of multiple eigenvalues.

As the set of indices corresponding to the resonances we can take an arbitrary set of integers $I \in \mathbb{Z}$ such that

$$w_{2\alpha} \neq w_{2n}, \quad \alpha \in I, \quad n \notin I. \tag{D.35}$$

LEMMA D.2. *There are unique formal series*

$$F^\alpha_\beta(y,w_1)=\sum_{s=1}^{\infty}F^\alpha_{\beta,s}(y,w_1) \tag{D.36}$$

that the equations

$$(\sigma\partial_y-\partial_x^2+u)\Psi^\alpha(x,y,w_1)=\sum_\beta F^\alpha_\beta(y,w_1)\Psi^\beta(x,y,w_1) \tag{D.37}$$

have unique formal Block solutions of the form

$$\Psi^\alpha(x,y,w_1)=\sum_{s=0}^{\infty}\varphi^\alpha_s(x,y,w_1),\quad \varphi^\alpha_0=\psi_\alpha=\psi(x,y,k_\alpha) \tag{D.38}$$

$$\Psi^\alpha(x+a_1,y,w_1)=w_1\Psi^\alpha(x,y,w_1), \tag{D.39}$$

$$\Psi^\alpha(x,y+a_2,w_1)=w_{2\alpha}\Psi^\alpha(x,y,w_1), \tag{D.40}$$

such that

$$\langle\psi^+_\beta\Psi^\alpha\rangle_x=\delta_{\alpha,\beta}\langle\psi^+_\alpha\psi_\alpha\rangle_x. \tag{D.41}$$

The corresponding formulae for F^α_β and Ψ^α are the matrix generalizations of the formulae (D.26)–(D.28) (see details in [Kr3]).

Let us define the matrix $T=T^\alpha_\beta(v,w_1)$ by the equation

$$\sigma\partial_t T+TF=0,\quad T(0,w_1)=1. \tag{D.42}$$

The functions

$$\hat\Psi^\alpha(x,y,w_1)=\sum_\beta T^\alpha_\beta(y,w_1)\Psi_\beta(x,y,w_1) \tag{D.43}$$

are solutions of(D.4). Under the translation by the period in x they are multiplied by w_1, while under the translation by the period in y they are transformed as follows

$$\hat\Psi^\alpha(x,y+a_2,w_1)=\sum_\beta T^\alpha_\beta(w_1)w_{2\beta}\hat\Psi_\beta(x,y,w_1), \tag{D.44}$$

where

$$\hat T^\alpha_\beta(w_1)=T^\alpha_\beta(a_2,w_1). \tag{D.45}$$

It is natural to call a finite set of the formal solutions $\hat T^\alpha_\beta$ *quasi-Bloch*, since it remains invariant under the translation by the periods in x and y.

The characteristic equation

$$R(w_1,\tilde w_2)=\det(\tilde w_2\delta_{\alpha,\beta}-\hat T^\alpha_\beta(w_1)w_{2,\beta})=0 \tag{D.46}$$

is an analog of the "secular equation" in the ordinary perturbation theory of multiple eigenvalues.

Let $h_\alpha(w_1,\tilde w_2)$ be an eigenvector of the matrix $\hat T^\alpha_\beta(w_1)w_{2,\beta}$ normalized so that

$$\sum_\alpha h_\alpha(\tilde Q)\hat\Psi^\alpha(0,0,w_1)=1,\quad \tilde Q=(w_1,\tilde w_2) \tag{D.47}$$

then

$$\tilde\psi(x,y,\tilde Q)=\sum_\alpha h_\alpha(\tilde Q)\hat\Psi^\alpha(x,y,w_1) \tag{D.48}$$

is the formal Bloch solution of (D.4) with multipliers w_1 and $\tilde w_2$, normalized in the standard way. The last statement means that the Bloch solutions are defined (locally) on the Riemann surface (D.46).

Structure of the "Global" Riemann Surface of Bloch Solutions. To begin with, we shall give here the explanation of the structure of a "global" Riemann surface of Bloch solutions in the case of small u. Let us consider some neighborhoods $R_{N,M}$ and $R_{-N,-M}$ of the resonant pair of the points $k_{N,M}$ and $k_{-N,-M}$, respectively. The function $w_1(k)$ (D.9) identifies them with some neighborhood $\hat{R}_{N,M}$ of the point $w_1(k_{N,M}) = w_1(k_{-N,-M})$ on the complex plane of the variable w_1. The series (D.25)–(D.29) of the non-resonant perturbation theory diverge in $R_{N,M}$ and $R_{-N,-M}$, but it turns out that the series of the lemma 3.2 converges in $\hat{R}_{N,M}$ and in this domain defines quasi-Bloch solutions of (D.4) which are analytic in w_1. The characteristic equation (D.46) in this case has the form

$$\tilde{w}_2^2 - f_1(w_1)\tilde{w}_2 + f_2(w_1) = 0 \tag{D.49}$$

and defines a two-sheet covering $\tilde{R}_{N,M}$ over $\hat{R}_{N,M}$ on which the Bloch solutions of (D.4) are defined. The boundary of $\tilde{R}_{N,M}$ can be naturally identified with the boundaries of $R_{N,M}$ and $R_{-N,-M}$. Hence, the structure (local) of the Riemann surface Γ of the Bloch functions looks as follows. Let us cut out $R_{N,M}$ and $R_{-N,-M}$ from the complex plane and glue instead of them a corresponding piece of the Riemann surface $\tilde{R}_{N,M}$. From the topological point of view this surgery is a gluing of a "handle" between two resonant points.

The remarkable thing is that the perturbation approach works even when $u(x, y)$ is not small. Of course, in that case the estimations of the perturbation theory series are much more complicated. In [Kr3] it was proved that if the potential $u(x, y)$ can be analytically extended into a domain

$$|\operatorname{Im} x| < \tau_1, \quad |\operatorname{Im} y| < \tau_2 \tag{D.50}$$

for some τ_1, τ_2; then the perturbation series for the non-resonant case converges outside some central finite domain R_0 and outside $R_{N,M}$ for $k_{N,M} \notin R_0$. Outside R_0 we again have to perform a surgery of the previous type ("glue" handles between $k_{N,M}$ and $k_{-N,-M}$ for $k_{N,M} \notin R_0$). In the central domain R_0 we have to glue some finite genus piece of the corresponding Riemann surface $\tilde{R}_0$ instead of a disc R_0. As a result we obtain the global Riemann surface Γ of the Bloch solutions of the equation (D.4) with $\operatorname{Re}\sigma \neq 0$.

THEOREM D.1. *If the potential $u(x, y)$ of the equation* (D.17) *can be analytically extended into the domain* (D.50) *then the Riemann surface Γ of the Bloch solutions of this equation is a result of the above-defined glueing of the three types of "pieces"*:

1°. *A complex plane of the variable k without small neighborhoods of the finite or infinite set of points $k_{N,M}, k_{-N,-M}$ and without some central domain $|k| > K_0$;*
2°. *A set of "handles " $\hat{R}_{|N,M|}$ that are defined by the equations of the form* (D.4) *as the two-sheets covering of the small neighborhoods of the pairs $k_{N,M}, k_{-N,-M}$;*
3°. *A Riemann surface $\tilde{R}_0$ (with boundary) of a finite genus g_0.*

The Bloch solutions of (D.4) *$\psi(x, y, Q)$, $Q \in \Gamma$, that are normalized by the condition $\psi(0, 0, Q) = 1$ are meromorphic on Γ. Their poles do not depend on x, y. It has one simple pole in each of the domains $\hat{R}_{|N,M|}$. In the domain $\hat{R}_0$ it has g_0 poles, where g_0 in general position (when $\hat{R}_0$ is smooth) equals the genus of $\hat{R}_0$. Outside these domains the function ψ is holomorphic and has no zeros.*

If there is a finite number of handles that are glued then the corresponding curve is compactifed by one point and the corresponding Bloch function is the Baker–Akhiezer function on the compactified Riemann surface.

In the case of real and smooth $u(x,y)$ for $\sigma = 1$ the final form of the Floquet set can be represented in the following form [Kr3]. Let us fix some finite or infinite subset S of integer pairs $(N > 0, M)$. The set of pairs of complex numbers $\pi = \{p_{s,1}, p_{s,2}\}$ where $s \in S$ would be called "admissible", if

$$\operatorname{Re} p_{s,i} = \frac{\pi N}{a_1}, \quad |p_{s,i} - k_s| = o(|k_s|^{-1}), \quad i = 1,2, \tag{D.51}$$

and the intervals $[p_{s,1}, p_{s,2}]$ do not intersect. (Here k_s are resonant points (D.17), $s = (N,M)$.)

Let us define the Riemann surface $\Gamma(\pi)$ for any admissible set π. It is obtained from the complex plane of the variable k by cutting it along the intervals $[p_{s,1}, p_{s,2}]$ and $[-\bar{p}_{s,1}, -\bar{p}_{s,2}]$ and by sewing after that the left side of the first cut with the right side of the second cut and vice versa. (After this surgery for any cut $[p_{s,1}, p_{s,2}]$ there corresponds a nontrivial cycle a_s on $\Gamma(\pi)$.)

THEOREM D.2. *For any real periodic potential $u(x,y)$ which can be analytically extended into some neighborhood of the real values x, y, the Bloch solutions of the equation* (D.4) *with $\sigma = 1$ are parametrized by points Q of the Riemann surface $\Gamma(\pi)$ corresponding to some admissible set π. The function $\psi(x,y,Q)$ which is normalized by the condition $\psi(0,0,Q) = 1$ is meromorphic on Γ and has a simple pole γ_s on each cycle a_s. If the admissible set π contains only a finite number of pairs, then $\Gamma(\pi)$ has finite genus and is compactified by only one point P_1 $(k = \infty)$, in the neighborhood of which the Bloch function ψ has the form*:

$$\psi = e^{kx + k^2 y} = \left(1 + \sum_{s=1}^{\infty} \xi_s(x,y) k^{-s}\right).$$

The potentials u for which $\Gamma(\pi)$ has finite genus are called finite-gap and as it follows from the last statement of the theorem they coincide with the algebraic-geometrical potentials. The following theorem states that the finite-gap potentials are dense in the space of all periodic smooth functions in two variables ([Kr3]).

THEOREM D.3. *Each smooth periodic potential $u(x,y)$ of the equation* (D.4) *with $\operatorname{Re}\sigma \neq 0$ analytically extendable to a neighbourhood of real x, y can he approximated by finite-gap potentials uniformly with any number of derivatives.*

List of Additional References

[AC] Aharonov A., Casher A.: Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field. Phys Rev A (3) *19*, 2461–2462 (1979).

[AB] Ablowitz M., Benney D.: The evolution of multiple modes for nonlinear dispersive waves. Studies AppI Math *49*, No. 3, 225–238 (1970).

[AN] Avilov V.V., Novikov S.P.: Evolution of Whitham zone in the KdV theory. Dokl. AN SSSR (English translation: Sov. Phys. Dokl. *32*, 366–368 (1987)).

[ANK] Avilov V.V., Krichever I.M., Novikov S.P.: Evolution of Whitham zone in the Korteweg-de Vries Theory. Dokl. AN SSSR *295*, 345–349 (1987). (English translation: Sov. Phys. Dokl. *32*, 564–566, (1987)). Zbl.655.65132

[BN] Balinskij A.A., Novikov S.P., Poisson brackets of hydrodynamic type, Frobenious algebras and Lie algebras. Dokl. AN SSSR *283*, 1036–1039 (1985) (English translation: Sov. Math. Dokl. *32*, 228–231 (1985)). Zbl.606.58018

[Bo] Bobenko A.I.: Surfaces of constant mean curvature and integrable equations. Usp. Mat. Nauk *46*, No. 4. 3–42 (1991) (English translation: Russian Math Surveys *46*, No. 4, 1–45 (1991)).

[Bog] Bogdanov L.V.: Novikov–Veselov equation as a natural twp-dimensional generalization of the Korteweg–de Vries equation. Teor. Mat. Fiz *70*, No. 2, 309–314 (1987) (English translation: Theor. Math. Phys. *70*, 219–223 (1987)). Zbl.639.35072

[Bogo] Bogoyavlenskij O.I.: On perturbations of periodic Toda lattice. Commentarii Math. Phys. *51*, 201–209 (1976).

[dV] DeVega H.J., Shaposnik F.A.: Classical vortex solution of the Abelian Higgs model. Phys. Rev. D *14*, 1100–1106 (1976).

[DS] Drinfeld V.G., Sokolov V.V.: Lie Algebras and equations of the Korteweg–de Vries type. Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. (Contemporary problems in Mathematics) 24. VINITI, Moscow 1984 English translation: J. Sov. Math. *30*, No. 2, 1975–2036 (1985)).

[D] Doyle P.W.: Differential geometric Poisson bivectors in one space variable. J. Math. Phys. *34*, No. 4, 1314–1338 (1993). Zbl.778.58022

[D1] Dubrovin B.A.: Lectures on 2D Topological Field Theory, Lecture Notes in Math, *1620*, 120–348 (1996), Springer-Verlag.

[D2] Dubrovin B.A.: On the differentially geometric Poisson brackets on the lattice. Funkts. Anal. Prilozh *23*, No. 2, 57–59 (1989).

[DN1] Dubrovin B.A., Novikov S.P.: Hydrodynamics of weakly deformed soliton lattices, Poisson brackets of Hydrodynamic type and Differential Geometry, Usp. Mat. Nauk *44*, No. 6, 29–98 (1989) (English translation: Russian Math Surveys *44*, No. 6, 35–124 (1989)). Zbl.712.58032

[DN2] Dubrovin B.A., Novikov S.P.: Ground states in a periodic field. Magnetoc Bloch functions and vector bundles. Dokl. AN SSSR *253*, 1293–1297 (1980) (English translation: Sov. Math. Dokl. *22*, 240–244 (1980)).

[DN3] Dubrovin B.A., Novikov S.P.: Ground states of two-dimensional electron in a periodic magnetic field. Zh. Eksp. Teor. Fiz. *79*, 1006–1016 (1980) (English translation: Sov. Phys. JETP *52*, 511–516 (1980)).

[FN*] Françoise J.-P., Novikov R.G.: Rational solutions of KdV type equations in dimension $2+1$ and m-body problems on the line. C. R. Acad. Sci. Paris *314*, Série I, 109–113, 1992.

[FT] Faddeev L.D., Takhtadzhyan A.I.: Hamiltonian Methods in the Theory of Solitons. Nauka, Moscow, 1986 (English translation: Springer-Verlag, Berlin, Heidelberg, New York, 1987).

[FL] Fateev V.A., Lukyanov S.L.; Poisson-Lie groups and classical W-algebras. Internat. J. Modern Phys. A *7*, No. 5, 853–876 (1992).

[FKT] Feldman J , Knorrer H., Trubowitz E.: Riemann surfaces of the infinite genus, preprint ETH Zurich.

[F] Ferapontov E.V.: Nonlocal hamiltonian operators of hydrodynamic type. Differential geometry and applications, Amer. Math. Soc. Translations, Ser. 2, *170*, 33–58 (1996). (ed. by S.Novikov)

[FFM] Flaschka H., Forrest M.G., and McLaughlin D.: Multiphase averaging and the inverse spectral solution of the Korteweg–de Vries equation. Comm. Pure Appl. Math. *33*, 739–784 (1980).

[GD] Gelfand I.M., Dikii L.A.: Fractional powers of operators and Hamiltonian systems. Funkts. Anal. Prilozh. *10*, No. 4, 13–28 (1976) (English translation: Funct. Anal. Appl. *10*, 259–274 (1976)).

[GDo] Gelfand I.M., Dorfman I.Ya.: Hamiltonian operators and infinite dimensional Lie algebras. Funkts. Anal. Prilozh. *15*, No. 3, 23–40 (1981) (English translation: Fund. Anal. Appl. *15*, No. 3. 173–187 (1981).

[GF] Gelfand I.M., Fuks D.B.: Cohomology of Lie algebra of vector fields on the circle. Funkts. Anal. Prilozh *2*, No. 4, 92–93 (1968).

[G] Grinevich P.G.: Rational solutions of the Veselov–Novikov equations are reflectinless two-dimensional potentials at fixed energy. Teor. Mat. Fiz. *69*, No. 2, 307–310 (1986) (English translation: Theor. Math. Phys. *69*, 1170–1172 (1986)). Zbl.617.35121

[GM] Grinevich P.G., Manakov S.V.: Inverse scattering for the two-dimensional Schrödinger operators, $\bar{\partial}$-method and non-linear equations, Funkts. Anal. Prilzh. *20*, No. 1, 14–24 (1986). (English translation: Funct. Anal. Appl. *20*, 94–103 (1986)). Zbl.617.35121

[GN*] Grinevich P.G., Novikov R.G.: Analog of multisoliton potentials for the two-dimensional Schrödinger operator. Funkts. Anal. Prilozh. *19*, No. 4, 32-42 (1985) (English translation: Funct. Anal. Appl. *19*, 276–285 (1985)). Zbl.657.35 122

[GN1] Grinevich P.G., Novikov S.P.; Spectral theory of commuting operators of rank two with periodic coeffitients. Funkts. Anal. Prilozh. *16*, No. 1, 25–26 (1982) (English translation: Funct Anal. Appl. *16*, 19–20 (1982)). Zbl.514.47035

[GN2] Grinevich P.G., Novikov S.P : "Inverse scattering problem" for the two dimensional Schrödinger operator at a fixed negative enery and generalized analytic finctions. Funct. Anal. Appl. *22*, No. 1, 19–27 (1988)

[GP] Gurevich A.V., Pitaevskii L.P.: Nonstationary structure of the collisionless shock wave, Zh. Ehksp. Teor. Fiz. *65*, No. 2, 590–604 (1973).

[Kr1] Krichever I.M.: The potentials with zero reflection coefficient on the background of the finite-gap potentials. Funkts. Anal. Prilozh. *9*, No. 2, 77–78 (1975) (English translation: Funct. Anal. Appl. *9*, 161–163 (1975)). Zbl.333.34022

[Kr2] Krichever I.M.: The periodic nonabelian Toda lattice and two-dimensional generalization. Usp. Mat. Nauk *36*, No. 2, 72-77 (1981) (English translation: Russian Math. Surveys 36. No. 2, ?? (1981))

[Kr3] Krichever I.M.: Spectral theory of two-dimensional periodic operators and its applications. Usp. Mat. Nauk *44*, No. 2, 121–184 (1989). (English translation: Russ. Math. Surv. *44*, No. 2. 145–25 (1989)).Zbl.699.35188

[Kr4] Krichever I.M.: Method of averaging for two-dimensional integrable equations. Funkts. Anal. Prilozh. *22*, No. 3, 37–52 (1988) (English translation: Funct. Anal. Appl. *22*, No. 3, 200–213 (1988)). Zbl.688.35088

[Kr5] Krichever I.M.: The dispersionless Lax equations and topological minimal models. Commun. Math. Phys *143*, 415–429 (1992). Zbl.753.35095

[Kr6] Krichever I.M.: The Tau-function of the universal Whitham hierarchy, matrix modles and topological field theories. Commun. Pure Appl. Math. *47*, 437–475 (1994). Zbl.811.58064

[Kr7] Krichever I.M.: Two-dimensional periodic difference operators and algebraic geometry. Dokl. Akad. Nauk SSSR *285*, No. 1, 31–36 (1985).

[KBBT] Krichever I., Babelon O., Billey E., Talon M.: Spin generalization of the Calogero–Moser system and the Matrix KP equation, Amer Math. Soc. Transl. (Ser. 2) *170*, 83–119 (1995). Zbl.843.58069

[KLWZ] Krichever I., Lipan O., Wiegmann P., Zabrodin A.: Quantum integrable systems and discrete classical Hirota equations. Commun. Math. Phys. *188*, 267–304 (1997).

[KN1] Krichever I.M., Novikov S.P.: Virasoro-type algebras, Riemann surfaces and soliton theory. Funkts. Anal. Prilozh. *21*, No. 2, 46–63 (1987).

[KN2] Krichever I.M., Novikov S.P.: Virasoro-type algebras, Riemann surfaces and strings in Minkovsky space. Funkts. Anal. Prilozh. *21*, No. 4, 47–61 (1987) (English translation: Funct. Anal. Appl. *21*, No. 4, 294–307 (1987)). Zbl.659.17012

[KP] Krichever I.M., Phong D.H.: On the integrable geometry of soliton equations and $N = 2$ supersymmetnc gauge theories. J. Diff. Geom. *45*, No. 2, 349–389 (1997). Zbl.970.48601

[KZ] Krichever I., Zabrodin A.: Spin generalization of the Ruijsenaars–Schneider model, nonabelian 2D loda chain and representations of the Sklyanin algebra, Usp. Mat. Nauk *50*, No. 6, 3–56 (1995) (English translation: Russ. Math. Surv. *50*, No. 6, 1101–1150 (1995)). Zbl.960.43357

[KT] Knörrer H., Trubowitz F.: A directional compactification of the complex Bloch variety. Comment. Math. Helvetici *65*, 114–149 (1990).

[L] Laplace P.: Recherches sur le calcul integral aux differences partielles. Histore de l'Academie de Sciences (1977).

[LS] Leznov A., Saveliev M.: Representation of zero curvature for the system of nonlinear partial differential equation $X_{\alpha,z\bar{z}} = \exp\{KX\}_\alpha$. Lett. Math. Phys. J, No. 6, 489-494 (1979).

[La] Lax P , Levermore C.: The small dispersion limit of the Korteweg–de Vries equation, I, II, III Commun. Pure Appl. Math. *36*, 253–290, 571–593, 809–830 (1983)

[MO] Marchenko V.A., Ostrovski I.: see in the book Marchenko V.A.: Sturm–Liouville operator and applications. Kiev, Naukova Dumka (1977) (in Russian)

[Ma] Manakov S.V.: The method of inverse scattering problem and two-dimensional evolution equations. Usp. Mat. Nauk *31*, No. 5, 245–246 (1976).

[Mas] Maslov V.P.: Limit for $h \to 0$ of Heisenberg equation. Teor. Mat. Fiz. *1*, No. 3, 378–383 (1969).

[Mi] Mikhailov A.V : On the integrability of the two-dimensional generalization of Toda lattices. Zh. Ehksp. Teor. Fiz. *30*, 443–448 (1979) (English translation: Sov. Phys. JETP)

[Mo] Mokhov O.I.: On the Poisson brackets of the Dubrovin–Novikov type (DN-brackets), Funkts. Anal. Prilozh. *22*. No. 4, 92–93 (1988) (English translation: Funct. Anal. Appl. *22*, 336–337 (1988)).

[MF] Mokhov O.I., Ferapontov L.V.: Nonlocal hamiltonian operators of hydrodynamic type associated with constant curvature metrics. Usp. Mat. Nauk *45*, No. 3, 191–192 (1990) (English translation: Russian Math Surveys *45*, No. 3, 218–219 (1990))

[MP] Maltsev A.: Averaged Poisson brackets, preprint Landau Intitule, Chernogolovka, 1997.

[Nat1] Natanzon S.M.: Differential equation for Prym faction. A criterion for two-dimensional finite-gap purely potential Schrödinger operator to be real. Funct. Anal. Appl. *26*, 13–20 (1992).

[Nat2] Natanzon S.M.: Real nonsmgular finite-zone solutions of soliton Equations. AMS Transl. (ser. 2) *170*, 153–184 (1996) (ed. S. Novikov).

[N] Novikov S.P.: Periodic and conditionaly periodic analogues of the many-soliton solutions of the Korteweg–de Vries equation. Zh. Ekspr. Teor. Fiz. *67*, 2131–2144 (1974) (English translation: Sov. Phys. JETP *40*, 1055–1063 (1974)).

[N1] Novikov S.P.: Topology. Itogi Nauki Tekh., Sovr. Probl. Mat., Fund. Napr. 12, VINITI, Moscow 1986 (English translation: Encyclopaedia Math. Sciences, Vol. 12, Springer–Verlag, 1996)

[N2] Novikov S.P,: The geometry of conservative systems of hydrodynamic type. The method of averaging method for field-theoretical systems. Usp. Mat. Nauk *40*, No. 4, 79–89 (1985) (English translation: Russ. Math. Surv. *40*, No. 4, 85–98 (1985)) Zbl.654.76004

[N3] Novikov S.P.: Magnetic Bloch functions and vector bundles. Typical dispertion relations and their quantum numbers. Sov. Math. Dokl. *23*, No. 2, 298–303 (1981).

[N4] Novikov S.P.: Solitons and Geometry, Academia dei Lincei, Scuola Normale Superiore, Lezzioni Fermiane, Pisa 1992 (published in 1994)

[NM] Novikov S.P., Maltsev A.: Liouville form of the averaged Poisson brackets. Usp. Mat. Nauk *48*, No. 1, 155–156 (1993).

[NV] Novikov S.P., Veselov A.P.: Laplace transformations and exactly solvable two-dimensional Schrödinger operators. Usp. Mat. Nauk *50*, No. 6, 71–72 (1995) (English translation: Russian Math Surveys—short notes)

[P1] Potemin G.: On Poisson brackets of differential geometric type. Dokl. AN SSSR 2H6, No. I, 39-42 (1986) (English translation: Sov. Math. Dokl. 33, 30-33 (1986)). Zbl.606.53010

[P2] Potemin G.: Algebro-geometrical construction of self-similar solution to Whitham equation, Usp. Mat. Nauk *43*, No. 5, 211–212 (1988) (English translation: Russian Math. Surv. *43*, No. 5. 252–253 (1988))

[Shi] Shiota T.: Characterization of Jacobian varieties in terms of soliton equations. Invent. Math. *83*, 333–382 (1986). Zbl.621.35097

[SS] Shabat A.B., Sokolov V.V.: (L, A) pairs and Ricatti type transformations. Funkts. Anal. Prilozh. *14*, No. 2, 79–80 (1980).

[SV] Snabat A.B., Veselov A.P.: A dressing chain and the spectral theory of the Schrödinger operator. Funkts. Anal. Prilozh. *27*, No. 2, 1–21 (1993).

[SW1] Seiberg N, Witten E.: Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetnc Yang-Mills theory. Nuclear Phys. B *426*, 19–52 (1994).

[SW2] Seiberg N, Witten E.: Monopoles, duality, and chiral symmetry breaking in $N = 1$ supersymmetric QCD. Nuclear Phys. B *431*, 484–550 (1994).

[Ti] Tian F.R.: The initial value problem for the Whitham averaged system. Commun. Math. Phys., 79–115 (1994).

[Tai] Taimanov I.A.: Pryrn theta functions and hierarhies of nonlinear equations. Mat. Zametki, *50* (1991) (English translation: Math. Notes *50*, No. 1, 723–730 (1991).

[Tai1] Taimanov I.A.: Modified Novikov–Veselov equation and differential geometry of surfaces, Amer. Math. Soc. translations, Ser. 2, *179*, 133–151 (1997).

[Ts] Tsarev S.P.: Poisson brackets and one-dimensional systems of hydrodynamic type. Sov. Math. Dokl. *34*, 534–537 (1987).

[Ts1] Tsarev S.P.: Geometry of the Hydrodynamic type systems. Generalized Hodpgraph method. Izvest. AN SSSR (ser. mat.) *54*, No. 5, 1048–1068 (1990).

[T] Tsiseika G.: Geometrie Differentielle projective des reseans. Paris–Budapest, 1924.

[W] Weiss J.; Periodic fixed points of Backlund transformations and the KdV equation. J. Math. Phys. *27*, 2647–2676 (1986).

[Wh1] Whitham G.B.: Non-linear dispersive waves. Proc. Royal Society A *283*, 238–261 (1965).

[Wh2] Whitham G.B.: Linear and nonliear waves, John Wiley Interscience Publ., New York, 1977

[YaR] Yanenko N., Rozhdestvenskij B.: Systems of quasilinear equations and their applications in gas dynamics. Moscow, Nauka, 1986 (second edition)

[Z] Zelmanov E.I.: On a class of local translation invariant Lie algebras. Dokl. AN SSSR *292*, 1294–1297 (1987) (English translation: Sov. Math. Doklady *35*, 216–218 (1987)). Zbl.629.17002

References[1]

1. The basic concepts of the Hamiltonian formalism go back to the classical work in analytical mechanics, to Poisson, Hamilton, Jacobi, Lie. Different versions of the presentation of these classical concepts are to be found in quite a number of textbooks (see, for example, [7], [42]) and surveys (see [112]). Infinite-dimensional analogues of the Hamiltonian formalism until recently were considered only for Lagrangian field systems in connection with the needs of quantum field theory (see, for example, [17]). More complicated examples arose in the hydrodynamics of a perfect incompressible fluid (see [7], appendix 2), and also in the theory of the Korteweg–de Vries equation [54J, [59]. The modern formalism of Poisson brackets in application to infinite-dimensional (field theoretic) systems was systematically developed in [112]. The general concept of a Poisson bracket of hydrodynamic tvpe was introduced and studied in [48].

2. The use of the symmetry of Hamiltonian systems to construct their integrals and to reduce their order also goes hack to the classical works of Jacobi, Poincare and others; for a modern presentation see [101] (see also [7], appendix 5). The construction of integrals for field-theoretic Lagrangian systems with symmetry was given by E. Noether.

3. The concept of a completely integrable Hamiltonian system arose in the works of Bour and Liouville (see the textbooks [7], [42]). We do not discuss here degenerate completely integrable systems with a larger number of integrals than the number of degrees of freedom (see, for example, [57] and V.V. Kozlov's survey article [72]).

4. The fundamental material of section 4 (chap. 1) is contained in the classical works of Hamilton and Jacobi (see, for example, the text [7]).

5. Beginning with the paper [89], in which the mechanism for integrating the KdV equation which had been proposed in the pioneering paper [60] was cleared up, all schemes for producing integrable evolution equations have been based on representing them in the form of a compatibility condition for the auxiliary linear problems.

The scheme based on the equations of "zero curvature for rational families of operators", proposed in [127], included in a natural way all examples known up till then, in particular such key stages in the development of the method as [54], [88], [96], [56], [59], [1]. (These examples and a number of others are presented together with the history of the development of the first stages of the inverse scattering method in the books [115], [25]). A representation of the KdV equation in the form of the zero curvature equation for polynomial families of operators was first proposed in [111], and an example of a rational family was met with in the paper [1].

For the anisotropic Landau–Lifshitz equation, the papers [22], [128] first used a zero-curvature representation for families of operators with a spectral parameter on an elliptic curve. This line received a further development in the papers [30], [29]. In the papers [84], [86] another way was proposed of generalizing the zero-curvature equations for rational families to the case where the spectral parameter is defined on an algebraic curve of non-zero genus. In the article [80] (for greater detail see [83]) a representation for the Moser–Calogero system was proposed in which the dependence of the matrix entries on a parameter defined on an elliptic curve contained essential singularities of a special form.

6. The program for integration of the periodic problem for the KdV equation was initiated by the paper [111] (somewhat later and in a less effective form it was considered in [90]). The employment of the methods of algebraic geometry for the construction of periodic and quasiperiodic solutions of the kdV and nonlinear Schrödinger equations was begun in the articles [38], [39],

[1]For the convenience of the reader, reference to reviews in Zentralblatt fur Mathcmatik (Zbl.). compiled using the MATH database, and Jahrhuch über die Fortschritte der Mathematik (FdM.), have, as far as possible, been included in this bibliography.

[45], [47], [64]. (Later the papers [102], [103] appeared.) For the sine-Gordon equation finite gap solutions were constructed in [68]. The question of whether one can approximate an arbitrary periodic potential by finite gap potentials of a Sturm-Liouville operator with conservation of the period was settled positively in [100], [104].

The first stage of the theory of finite gap integration was presented in [45], [115].

A general scheme for integrating two-dimensional equations of the type of the Kadomtsev Petviashvili equation with the aid of the methods of algebraic geometry was proposed in [74], [75]. It also included in a natural way the constructions of solutions of one-dimensional evolution equations which were proposed in the works cited above. The concept of the Clebsch–Gordan–Baker–Akhiezer function became the central concept of this scheme. The definition of such functions, including the multi-point ones, was given in [75] on the basis of a generalization of the analytic properties of Bloch functions of finite gap periodic and quasipenodic operators. "Single-point" functions of this kind were introduced as a formal generalization of the concept of exponentials in the 19th century by Gordan and Clehsch (see [8]). Their connection with a joint eigenfunction of a pair of commuting operators of relatively prime orders was first noted in [9] by Baker. N.I. Akhiezer indicated examples of the interpretation of such functions in the spectral theory of operators on the half-line.

The isolation of the real non-singular solutions within the framework of the general scheme, for equations for which the auxiliary linear problem is not self-adjoint, was begun in [28] and was earnestly pushed forward in [13], [41], [44], [46].

A general Hamiltonian theory of systems whose integration is connected with hyperelliptic curves was proposed in [116], [118]. This theory made it possible to examine from a single point of view and to unify not only the Hamiltonian structure itself of diverse systems, but also to give a unified construction of variables of the action-angle type. For Kovalevskaya's system a construction of variables of the action type was obtained for the first time in just these articles. A construction of action-angle variables for the Hamiltonian systems connected with finite gap Sturm–Liouville operators was first obtained in [6], [56]. The relation of the stationary and non-stationary Hamiltonian formalisms for these systems was obtained in [18], [20], [34].

8. References to works devoted to the algebraic-geometric integration of a number of classical systems of mechanics and hydrodynamics are cited in sections 3 and 4 of Chapter 2 in the course of the analysis of a series of prime examples.

9. The program of the research on the dynamics of the poles of solutions of equations to which the inverse scattering method is applicable goes back to the paper [87]. The connection of the dynamics of the poles of rational and elliptic solutions of the KdV equation with the rational and elliptic Moser–Calogero systems was first discovered in [4]. Without any connection with finite-dimensionai systems, elliptic solutions of the KdV equation with three poles were constructed in [47]. The isomorphism of the rational Moser–Calogero system and the polar system of rational solutions of the KP equation was established in [77]. In [31] this result was carried over to the elliptic case. The construction of variables of the angle type for the elliptic Moser–Calogero system and the construction of all elliptic solutions of the KP equation were obtained in [80].

10. The algebraic-geometric Floquet spectral theory of linear operators with periodic coefficients was developed in the publications [38], [45], [115], [64], [111], [100], [90]. The starting point for these works was the problem of constructing periodic solutions of equations of the KdV type.

The possibilities for applying the algebraic-geometric spectral theory to the continuous Peierls–Fröhlich model were discovered in [12], [24].

The construction of the algebraic-geometric spectral theory of the Schrödinger difference operator was begun in the articles [45], [33] and received its completion in [79]. These results were used in the papers [23], [50], [51], [81], in which the discrete Peierls model was integrated and its perturbations were investigated.

Translator's Remark. In the literature list which follows, whenever a Russian work has been translated into English a reference to the translation has been included, and the title I have given is then simply the title of the English translation, unless (as is not infrequent!) the title of the translation is incorrect or differs significantly from the Russian title. In these cases I have supplied my own translation of the Russian title and have indicated how the title of the English translation differs.

However, I have not corrected one "mistake" which is nearly universal in translations of the subject matter treated in this article. It is the lazy translation "finite-zone" (a literal translation of the Russian term) for what English writers generally call *finite gap* (operators, potentials, etc.).

Because "finite-zone" is so frequent (although it is found almost exclusively in translations from the Russian) I have left it unchanged in the English titles but wish to draw the reader's attention to it here.

[1] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Method for solving the sine-Gordon equation. Phys. Rev. Lett. *30*, 1262–1264 (1973).

[2] Adier, M.: On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg–de Vries type equations. Invent. Math. *50*, 219–248 (1979). Zbl. 393.35058

[3] Adler, M., van Moerbeke, P.: Completely integrable systems, Euclidean Lie algebras and curves. Adv. Math. *38*, 267–317 (1980). Zbl. 455.58017

[4] Airault, H., McKean, H.P., Moser, J.: Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem. Commun. Pure Appl. Math. *30*, 95–148 (1977). Zbl. 338.35024

[5] Akhiezer, N.I.: A continuous analogue of orthogonal polynomials on a system of intervals. Dokl. Akad. Nauk SSSR *141*, 263–266 (1961) (English translation: Sov. Math. Dokl. *2*, 1409–1412 (1961)). Zbl. 109.296

[6] Al'ber, S.I.: On stationary problems for equations of Korteweg–de Vries type. Commun. Pure Appl. Math. *34*, 259–272 (1981). Zbl. 456.35075

[7] Arnol'd, V.I.: Mathematical Methods of Classical Mechanics. Nauka, Moscow 1974, 431 p. (English translation: Graduate Texts in Mathematics 60. Springer-Verlag, New York–Heidelberg–Berlin 1978, 462 p.). Zbl. 386.70001

[8] Baker, H.F.: Abelian Functions (the full title is: Abel's Theorem and the Allied Theory Including the Theory of the Theta Functions). Cambridge University Press, Cambridge 1897, 684 p. FdM 28,331

[9] Baker, H.F.: Note on the foregoing paper "Commutative ordinary differential operators," by J.L. Burchnall and J.W. Chaundy. Proc. R. Soc. Lond, Ser. A *118*, 584–593 (1928). FdM. 54,439

[10] Bar'yakhtar, V.G., Belokolos, E.D., Golod, P.I.: One-dimensional magnetic structures and the higher-order Landau–Lifshitz equations (in Russian). Prepr. ITF-84-128 R, Kiev 1984, 28 p.

[11] Bateman, H. (Erdélyi, A., editor): Higher transcendental functions (in three volumes). McGraw-Hill Book Company, New York–Toronto–London 1953/1953/1955, 302/396/292 p. Zbl. 51,303 Zbl. 52,295 Zbl. 64,63

[12] Belokolos, E.D.: Peierls–Fröhlich problem and potentials with finite number of gaps. I, II. I: Teor. Mat. Fiz. *45*. No. 2, 268–275 (1980); II: Teor. Mat. Fiz. *48*. No. 1, 60–69 (1981) (English translation: I: Theor. Math. Phys. *45*, 1022–1026 (1980); II: Theor. Math. Phys. *48*, 604–610 (1981))

[13] Belokolos. E.D., Ehnol'skij, V.Z.: Generalized Lamb ansatz. Teor. Mat. Fiz. *53*, No. 2, 271–282 (1982) (English translation: Theor. Math. Phys. *53*, 1120–1127 (1982)). Zbl. 507.35060

[14] Berezin, F.A.: Some remarks on the universal enveloping algebra of a Lie algebra. Funkts. Anal. Prilozh. *1*, No. 2, 1–14 (1967) (English translation under the title: Some remarks about the associated envelope of a Lie algebra. Fund. Anal. Appl. *1*, 91–102 (1967)). Zbl. 227.22020

[15] Bobenko, A.I.: Euler equations in the algebras e(3) and so(4). Isomorphisms of integrable cases. Funkts. Anal. Prilozh. *20*, No. 1, 64–66 (1986) (English translation: Funct. Anal. Appl. *20*, 53–56 (1986)). Zbl. 622.58010

[16] Bobenko, A.I., Bikbaev, R.F.: On finite gap integration of the Landau–Lifshitz equation. The XYZ case (in Russian). Prepr. LOMI, E-8-83, Leningrad 1983, 27 p.

[17] Bogolyubov (Bogoliubov), N.N., Shirkov, D.V.: Introduction to the Theory of Quantized Fields, Third Edition. Nauka, Moscow 1976, 479 p. (English translation: Wiley-lnterscience. John Wiley & Sons, New York–Chichester–Brisbane–Toronto 1980, 620 p.)

[18] Bogoyavlenskij, O.I.: Integrals of higher-order stationary KdV equations and eigenvalues of the Hill operator. Funkts. Anal. Prilozh. *10*, No. 2, 9–12 (1976) (English translation: Fund. Anal. Appl. *10*, 92–95 (1976)). Zbl. 359.34023

[19] Bogoyavlenskij, O.I.: Integrable Ruler equations on Lie algebras arising in problems of mathematical physics. Izv. Akad. Nauk SSSR, Ser. Mat. *48*, 883–938 (1984) (English translation: Math. USSR. Izv. *25*, 207–257 (1985)). Zbl. 583.58012

[20] Bogoyavlenskij, O.I., Novikov, S.P.: The relationship between Hamiltonian formalisms of stationary and non-stationary problems. Funkts. Anal. Prilozh. *10*, No. 1, 9–13 (1976) (English translation: Fund. Anal. Appl. *10*, 8-11 (1976)). Zbl. 345.70011

[21] Bogoyavlensky, O.I.: On perturbations of the periodic Toda lattice. Commun. Math. Phvs. *51*, 201–209 (1976)

[22] Borovik, A.E.: N-soliton solutions of the nonlinear Landau–Lifshitz equation. Pis'ma Zh. Ehksp. Teor. Fiz. *28*, 629–632 (1978) (English translation: JETP Lett. *28*, 581–584 (1978))

[23] Brazovskij. S.A., Dzyaloshinskij, I.E., Krichever, I.M.: Discrete Peierls models with exact solutions. Zh. Ehksp. Teor. Fiz. *83*, 389–415 (1982) (English translation: Sov. Phys. JETP *56*, 212–225 (1982))

[24] Brazovskij, S.A., Gordyunin. S.A., Kirova, N.N.: An exact solution of the Peierls model with an arbitrary number of electrons in the unit cell. Pis'ma Zh. Ehksp. Teor. Fiz. *31*, 486–491 (1980) [English translation: JETP Lett. *31*, 456–461 (1980))

[25] Bullough, R.K., Caudrey, P.J. (editors): Solitons. Topics in Current Physics 17. Springer-Verlag, Berlin–Heidelberg–New York 1980, 389 p. Zbl. 428.00010

[26] Burchnall, J.L., Chaundy. T.W.: Commutative ordinary differential operators. I, II. I: Proc. Lond. Math. Soc., II. Ser. *21*, 420–440 (1923); II: Proc. R. Soc. Lond., Ser. A *118*, 557–583 (1928). FdM 49.311 FdM 54,439

[27] Burgatti, P.: Determinazione dell'equazioni di Hamilton–Jacobi integrabili mediante la separazione delle variabili. Rend. R. Accad. Lincei *20*, 108–111 (1° semestre 1911)

[28] Cherednik, I.V.: Reality conditions in "finite-zone integration". Dokl. Akad. Nauk SSSR *252*, 1104–1108 (1980) (English translation: Sov. Phys. Dokl. *25*, 450–452 (1980)). Zbl. 491.35044

[29] Cherednik, I.V.: Integrability of the equation ofa two-dimensional asymmetric chiral O(3)-field and of its quantum analog. Yad. Fiz. *33*, 278–282 (1981) (English translation: Sov. J. Nucl. Phys. *33*, 144–145 (1981))

[30] Cherednik, I.V., Gei'fand, I.M.: The abstract Hamiltonian formalism for the classical Yang–Baxter families. Usp. Mat. Nauk *38*, No. 3, 3–21 (1983) (English translation: Russ. Math. Surv *38*, No. 3. 1–22 (1983) (The title of the English translation speaks of Yang–Baxter *bundles* translating the corresponding Russian word literally but inappropriately; what is being referred to is in fact a parametrized family of functions. Transl. note.)). Zbl. 536.58006

[31] Choodnovsky, D.V., Choodnovsky, G.V.: Pole expansions of nonlinear partial differential equations. Nuovo Cimento B *40*, 339–353 (1977)

[32] Dall'Acqua, E.A.: Sulla integrazione delle equazioni di Hamilton–Jacobi per separazione di variabili. Math. Ann. *66*, 398–415 (1909)

[33] Date, E., Tanaka, S.: Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice. Prog. Theor. Phys. *55*, 457–465 (1976)

[34] Dikij, L.A., Gel'fand. l.M.: Asymptotic behaviour of the resolvent of Sturm–Liouville equations and the algebra of the Korteweg–de Vries equations. Usp. Mat. Nauk *30*, No. 5. 67–100 (1975) (English translation: Russ. Math. Surv. *30*, No. 5, 77-113 (1975)). Zbl. 334.58007

[35] Drinfel'd, V.G.: On commutative subrings of certain noncommutative rings. Funkts. Anal. Prilozh. *11*, No. 1, 11–14 (1977) (English translation: Funct. Anal. Appl. *11*, 9–12 (1977)). Zbl. 359.14011

[36] Drinfel'd, V.G.: Hamiltonian structures on Lie groups. Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations. Dokl. Akad. Nauk SSSR *268*, 285–287 (1983) (English translation: Sov. Math. Dokl. *27*, 68–71 (1983)). Zbl. 526.58017

[37] Dryuma, V.S.: Analytic solution of the two-dimensional Korteweg–de Vries (KdV) equation. Zh. Ehksp. Theor. Fiz. Pis'ma *19*, 753–755 (1974) (English translation: JETP Lett. *19*, 387–388 (1974))

[38] Dubrovin, B.A.: Periodic problems for the Korteweg–de Vries equation in the class of finite band potenlials. Funkts. Anal. Prilozh. *9*, No. 3, 41–51 (1975) (English translation: Funct. Anal Appl. *9*, 215–223 (1975)). Zbl. 338.35022

[39] Duhrovin, B.A.: Completely integrable Hamillonian systems associated with matrix operators and Abelian varieties. Funkts. Anal. Prilozh. *11*, No. 4, 28–41 (1977) (English translation: Fund. Anal. Appl. *11*, 265–277 (1977)). Zbl. 377.58008

[40] Dubrovin, B.A.: Theta functions and non-linear equations. Usp. Mat. Nauk *36*, No. 2, 11–80 (1981) (English translation: Russ. Math. Surv. *36*, No. 2. 11–92 (1981)). Zbl. 478.58038

[41] Dubrovin, B.A.: Matrix finite-zone operators, in: Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. (Contemporary Problems of Mathematics) 23. VINITI, Moscow 1983, 33–78 (English translation: J. Sov. Math. *28*, 20–50 (1985)). Zbl. 561.58043

[42] Dubrovin, B.A., Fomenko, A.T., Novikov. S.P.: Modern Geometry—Methods and Applications. Nauka, Moscow 1979. 759 p. (English translation in two parts: Part I: The geometry of surfaces, transformation groups, and fields. Graduate Texts in Mathematics 93, Springer-Verlag. New York–Berlin–Heidelberg–Tokyo 1984, 464 p.; Part II. The geometry and topology of manilolds. Graduate Texts in Mathematics *104*, Springer–Verlag, New York–Berlin–Heidelberg–Tokyo 1985, 430 p.). Zbl. 529.53002, Zbl. 565.57001

[43] Dubrovin, B.A., Krichever, I.M., Novikov. S.P.: The Schrödinger equation in a periodic field and Riemann surfaces. Dokl. Akad. Nauk SSSR *229*, 15–18 (1976) (English translation: Sov. Math. Dokl. *17*, 947–951 (1976)). Zbl. 441.35021

[44] Dubrovin, B.A., Krichever, I.M., Novikov, S.P.: Topological and algebraic geometry methods in contemporary mathematical physics. II. Sov. Sci. Rev., Sect. C, Math. Phys. Rev. *3*, 1–150 (1982). Zbl. 534.58002

[45] Dubrovin, B.A., Matveev, V.B., Novikov, S.P.: Non-linear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties. Usp. Mat. Nauk *31*, No. 1, 55–136 (1976) (English translation: Russ. Math. Surv. *31*, No. 1, 59–146 (1976)). Zbl. 326.35011

[46] Dubrovin, B.A., Natanzon, S.M.: Real two-zone solutions of the sine-Gordon equation. Funkts. Anal. Prilozh. *76*. No. 1, 27–13 (1982) (English translation: Fund. Anal. Appl. *16*, 21–33 (1982)). Zbl. 554.35100

[47] Dubrovin, B.A., Novikov, S.P.: Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg–de Vries equation. Zh. Ehksp. Teor. Fiz. *67*, 2131–2144 (1974) (English translation: Sov. Phys.-JETP *40*, 1058–1063 (1974))

[48] Dubrovin, B.A., Novikov, S.P.: Hamiltonian formalism of one-dimensional systems of hydrodynamic type, and the Bogolyubov–Whitham averaging method. Dokl. Akad. Nauk SSSR *270*, 781–785 (1983) (English translation: Sov. Math. Dokl. *27*, 665–669 (1983) (in the translation, the name *Whitham* is given incorrectly as " *Whitman*")). Zbl. 553.35011

[49] Dubrovin, B.A., Novikov, S.P.: On Poisson brackets of hydrodynamic type. Dokl. Akad. Nauk SSSR *279*, 294–297 (1984) (English translation: Sov. Math. Dokl. *30*, 651–654 (1984)), Zbl. 591 58012

[50] Dzyaloshinskij, I.E., Krichever, I.M.: Commensurability effects in the discrete Peierls model. Zh. Ehksp. Teor. Fiz. *83*, 1576–1586 (1982) (English translation: Sov. Phys.-JETP *56*, 908–913 (1982))

[51] Dzyaloshinskij, I.E., Krichever, I.M.: Sound and charge-density wave in the discrete Peierls model. Zh. Ehksp. Teor. Fiz. *85*, 1771–1789 (1983) (English translation: Sov. Phys.-JETP *58*. 1031–1040 (1983))

[52] Faddeev, L.D.: Integrable models in $1 + 1$ dimensional quantum theory. Preprint S. Ph. T. 82/76. CEN Saclay, Saclay 1982, 97 p.

[53] Faddeev, L.D., Slavnov, A.A.: Introduction to the Quantum Theory of Gauge Fields. Nauka. Moscow 1978, 239 p. (English translation: Gauge Fields: Introduction to Quantum Theory. Frontiers in, Physics 50, Benjamin-Cummings, Reading, Mass. 1980, 232 p.). Zbl. 486.53052

[54] Faddeev, L.D., Zakharov, V.E.: Korteweg–de Vries equation: a completely integrable Hamiltonian system. Funkts. Anal. Prilozh. *5*, No. 4, 18-27 (1971) (English translation: Funct. Anal. Appl. *5*, 280–287 (1971)). Zbl. 257.35074

[55] Flaschka, H.: The Toda lattice. II. Existence of integrals. Phys. Rev., B *9*, 1924–1925 (1974), and On the Toda Lattice. II—inverse-scattering solution. Prog.Theor. Phys. *51*, 703–716 (1974)

[56] Flaschka, H., McLaughlin, D.W.: Canonically conjugate variables for the Korteweg–de Vries equation and the Toda lattice with periodic boundary conditions. Prog. Theor. Phys. *55* 438–456 (1976)

[57] Fomenko, A.T., Mishchenko, A.S.: Euler equations on finite-dimensional Lie groups. Izv. Akad. Nauk SSSR. Ser. Mat. *42*, 396–415 (1978) (English translation: Math. USSR, Izv. *12* 371–389 (1978)). Zbl. 383.58006

[58] Fomenko, A.T., Trofimov, V.V.: Liouville integrability of Hamiltonian systems on Lie algebras. Usp Mat. Nauk *39*, No. 2, 3–56 (1984) (English translation: Russ. Math. Surv. *39*. No. 2, 1–67 (1984)). Zbl. 549.58024

[59] Gardner, C.S.: Korteweg-de Vries equation and generalizations. IV. The Korteweg–de Vries equation as a Hamiltonian system. J. Math. Phys. *12*, 1548–1551 (1971). Zbl. 283.35021

[60] Gardner, C.S., Green, J., Kruskal, M., Miura, R.: Method for solving the Korteweg–de Vries equation. Phys. Rev. Lett. *19*, 1095–1097 (1967)

[61] Garnier, R.: Sur une classe de systèmes différentiels abéliens déduits de la théorie des équations lineaires. Rend. Circ. Mat. Palermo *43*. 155–191 (1919). FdM 47.404

[62] Glaser, V., Grosse, H., Martin. A.: Bounds on the number of eigenvalues of the Schrödinger operator. Commun. Math. Phys. *59*, 197–212 (1978). Zbl. 373.35050

[63] Golubev, V.V.: Lectures on Integration of the Equations of Motion of a Rigid Body about a Fixed Point. Gostekhizdat, Moscow 1953, 287 p. (English translation: Coronet Books, Philadelphia 1953. 240 p.). Zbl. 51,151

[64] Its, A.R., Matveev, V.B.: A class of solutions of the Korteweg–de Vries equation (in Russian), in: Probl. Mat. Fiz. (Problems in Mathematical Physics) 8. Leningrad State University, Leningrad 1976, 70–92.

[65] Kirchhoff, G.: Vorlesungen über mathematische Physik. Mechanik. B.G. Teubner, Leipzig 1876, 466 p. FdM 8,542

[66] Kirillov, A.A.: Elements of the Theory of Representations. Nauka, Moscow 1972, 336 p. (English translation: Grundlehren der mathematischen Wissenschaften 220, Springer-Verlag, Berlin–Heidelberg–New York 1976. 315 p.). Zbl. 264.22011

[67] Kostant, B.: Quantization and unitary representations. Part I: Prequantization, in: Lecture Notes in Modern Analysis and Applications III. Lect. Notes Math. 170. Springer-Verlag, Berlin–Heidelberg–New York 1970, 87–208. Zbl. 223.53028

[68] Kotlyarov, V.P., Kozel. V.A.: Almost periodic solutions of the equation $u_{tt} - u_{xx} + \sin u - 0$ (in Russian). Dokl. Akad. Nauk Ukr. SSR, Ser. A no. *10*, 878–881 (1976). Zbl. 337.35003

[69] Kötter, F.: Über die Bewegung eines festen Körpers in einer Flüssigkeit. J. Reine Angew. Math. *109*, 51–81. 89–111 (1892). FdM 23,977

[70] Kötter, F.: Die von Steklow und Liapunow entdeckten integrablen Fälle der Bewegung ernes starren Körpers in einer Flüssigkeit. Sitzungsber., K. Preuss. Akad. Wiss. Berlin 1900, 79–87.

[71] Kozlov, V.V.: Methods of Qualitative Analysis in the Dynamics of a Rigid Body (in Russian). Moscow State University, Moscow 1980, 230 p. Zbl. 557.70009

[72] Kozlov, V.V.: Integrability and non-integrability in Hamiltonian mechanics. Usp. Mat. Nauk *38*. No. 1, 3–67 (1983) (English translation: Russ. Math. Surv. *38*, No. 1, 1–76 (1983)). Zbl. 525.70023

[73] Kozlov, V.V., Onishchenko, D.A.: Nonintegrability of KirchhofT's equations. Dokl. Akad. Nauk SSSR *266*, 1298–1300 (1982) (English translation: Sov. Math. Dokl. *26*, 495-498 (1982)). Zbl. 541.70009

[74] Krichever, I.M.: Algebraic-geometric construction of the Zakharov–Shabat equations and their periodic solutions. Dokl. Akad. Nauk SSSR *227*, 291–294 (1976) (English translation: Sov. Math. Dokl. *17*, 394–397 (1976)). Zbl. 361.35007

[75] Krichever, I.M.: Integration of nonlinear equations by the methods of algebraic geometry. Funkts. Anal. Prilozh. *11*. No. 1, 15–31 (1977) (EngIish translation: Funct. Anal. Appl. *11*, 12–26 (1977)). Zbl. 346.35028

[76] Krichever, I.M.: Methods of algebraic geometry in the theory of non-linear equations. Usp. Mat. Nauk *32*, No. 6, 183–208 (1977) (English translation: Russ. Math. Surv. *32*, No. 6, 185-213 (1977)). Zbl. 372.35002

[77] Krichever, I.M.: Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line. Funkts. Anal. Prilozh. *12*, No. 1, 76–78 (1978) (English translation: Funct. Anal. Appl. *12*, 59–61 (1978)). Zbl. 374.70008

[78] Krichever, I.M.: Commutative rings of ordinary linear differential operators. Funkts. Anal. Prilozh. *12*, No. 3, 20-31 (1978) (English translation: Funct. Anal. Appl. *12*, 175–185(1978)). Zbl. 408.34008

[79] Krichever, I.M.: Algebraic curves and non-linear difference equations. Usp. Mat. Nauk *33*, No. 4. 215–216 (1978) (English translation: Russ. Math. Surv. *33*. No. 4, 255–256 (1978)). Zbl. 382.39003

[80] Krichever, I.M.: Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles. Funkts. Anal. Prilozh. *14*, No. 4, 45–54 (1980) (English translation: Funct. Anal. Appl. *14*, 282–290 (1980)). Zbl. 462.35080

[81] Krichever, I.M.: The Peierls model. Funkts. Anal. Prilozh. *16*, No. 4, 10–26 (1982) (English translation: Funct. Anal. Appl. *16*, 248–263 (1982))

[82] Krichever, I.M.: Algebro-geometric spectral theory of the Schrödinger difference operator and the Peierls model. Dokl. Akad. Nauk SSSR *265*, 1054–1058 (1982) (English translation: Sov. Math. Dokl. *26*, 194–198 (1982)). Zbl. 527.39001

[83] Krichever, I.M.: Nonlinear equations and elliptic curves, in: Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. (Contemporary Problems of Mathematics) 23. VINITI, Moscow 1983, 79–136 (English translation: J. Sov. Math. *28*, 51–90 (1985)). Zbl. 595.35087

[84] Krichever, I.M., Novikov, S.P.: Holomorphic bundles over Riemann surfaces and the Kadomtsev–Petviashvili (KP) equation. I. Funkts. Anal. Prilozh. *12*, No. 4. 41–52 (1978) (English translation: Funct. Anal. Appl. *12*, 276–286 (1978)). Zbl. 393.35061

[85] Krichever, I.M., Novikov, S.P.: Holomorphic fiberings and nonlinear equations. Finite zone solutions of rank 2. Dokl. Akad. Nauk SSSR *247*, 33–37 (1979) (English translation: Sov. Math. Dokl. *20*, 650–654 (1979)). Zbl. 434.35078

[86] Krichever, I.M., Novikov, S.P.: Holomorphic bundles over algebraic curves and non-linear equations. Usp. Mat. Nauk *35*, No. 6, 47–68 (1980) (English translation: Russ. Math. Surv. *35*, No. 6, 53–79 (1980)). Zbl. 501.35071

[87] Kruskal, M.D.: The Korteweg–de Vries equation and related evolution equations, in: Nonlinear Wave Motion (Proc. AMS-SIAM Summer Sem. Clarkson Coll. of Technology, Potsdam. N.Y. 1972). Lect. Appl. Math. 15. American Mathematical Society, Providence 1974. pp. 61-83. Zbl. 292.35017

[88] Lamb, G.L: Analytical descriptions of ultrashort optical pulse propagation in a resonant medium. Rev. Mod. Phys. *43*, 99–124 (1971)

[89] Lax, P.D.: Integrals of nonlinear equations of evolution and solitary waves. Commun. Pure Appl. Math. *21*, 467–490 (1968). Zbl. 162,411

[90] Lax, P.D.: Periodic solutions of the KdV equation. Commun. Pure Appl. Math. *28*, 141–188 (1975). Zbl. 295.35004

[91] Levi-Civita, T.: Sulla integrazione della equazione di Hamilton–Jacobi per separazione di variabili. Math. Ann. *59*, 383–397 (1904). FdM 35,362

[92] Levi-Civita, T., Amaldi, U.: Lezioni di Meccanica Razionale. Nuova edizione riveduta e corretta. I, II Parte 1, II Parte 2. N. Zanichelli, Bologna 1950/1951/1952, 816/510/673 p. Zbl. 45,123, Zbl. 47,173

[93] Lie, S. (unter Mitwirkung von F. Engel): Theorie der Transformationsgruppen. I.–III. Abschnitt. B.G. Teubner, Leipzig 1888/1890/1893, 632/554/830 p. (reprint: B.G. Teubner. Leipzig–Berlin 1930). I: FdM 20,368

[94] Magri, F.: A simple model of the integrable Hamiltonian equation. J. Math. Phys. *19*, 1156–1162 (1978). Zbl. 383.35065

[95] Maki, K., Ebisawa, H.: Exact magnetic ringing solutions in superfluid ^{3}He-B. Phys. Rev., B *13*, 2924–2930 (1976)

[96] Manakov, S.V.: Complete integrability and stochastization of discrete dynamical systems. Zh. Ehksp. Teor. Fiz. *67*, 543-555 (1974) (English translation: Sov. Phys.-JETP *40*, 269–274 (1974))

[97] Manakov, S.V.: Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body. Funkts. Anal. Prilozh. *10*, No. 4, 93–94 (1976) (English translation: Fund. Anal. Appl. *10*, 328–329 (1976)). Zbl. 343.70003

[98] Manakov, S.V.: The method of the inverse scattering problem and two-dimensional evolution equations (in Russian). Usp. Mat. Nauk *31*, No. 5, 245–246 (1976). Zbl. 345.35055

[99] Manakov, S.V., Zakharov, V.E., Bordag, L.A., Its, A.R., Matveev, V.B.: Two-dimensional solitons of the Kadomtsev–Petviashvili equation and their interaction. Phys. Lett., A *63*, 205–206 (1977)

[100] Marchenko, V.A.: Sturm-Liouville Operators and Applications. Naukova Dumka. Kiev 1977. 331 p. (English translation: Operator Theory: Advances and Applications, Vol. 22. Birkhäuser, Basel–Boston–Stuttgart 1986, 367 p.). Zbl. 399.34022

[101] Marsden, J., Weinstein, A.: Reduction of symplectic manifolds with symmetry. Rep. Math. Phys. *5*, 121–130 (1974). Zbl. 327.58005

[102] McKean, H.P., van Moerbeke, P.: The spectrum of Hill's equation. Invent. Math. *30*, 217–274 (1975). Zbl. 319.34024

[103] McKean, H.P., Trubowitz, E.: Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points. Commun. Pure Appl. Math. *29*, 143–226 (1976). Zbl. 339,34024

[104] Meiman, N.N.: The theory of one-dimensional Schrödinger operators with a periodic potential. J. Math. Phys. *18*, 834–848 (1977)

[105] Meshcheryakov, M.V.: The integration of the equations for geodesies of left-invariant metrics on simple Lie groups using special functions. Mat. Sb., Nov. Ser. *117* (159), 481–493 (1982) (English translation: Math. USSR, Sb. *45*, 473–485 (1983)). Zbl. 506.58028

[106] Mikhajlov, A.V., Zakharov, V.E.: Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Zh. Ehksp. Teor. Fiz. *74*, 1953–1973 (1978) (English translation: Sov. Phys.-JETP *47*, 1017–1027 (1978))

[107] Milne-Thompson, L.M.: Theoretical Hydrodynamics (Fourth Edition). Macmillan, London/St. Martin's Press, New York 1960, 660 p. Zbl. 89,426

[108] Mishchenko, A.S.: Integrals of geodesic flows on Lie groups. Funkts. Anal. Prilozh. *4*, No. 3, 73–77 (1970) (English translation under the incorrect title: Integral geodesics of a flow on Lie groups. Funct. Anal. Appl. *4*, 232–235 (1970)). Zbl. 241.22022

[109] Moser, J.: Three integrable Hamiltonian systems connected with isospectral deformations. Adv. Math. *16*, 197–220 (1975). Zbl. 303.34019

[110] Moser, J.: Various aspects of integrable Hamiltonian systems, in: Dynamical Systems, C.I.M.E. Lectures, Bressanone, Italy, June 1978. Prog. Math., Boston 8. Birkhäuser. Boston–Basel–Stuttgart 1980. pp. 233–289. Zbl. 468.58011

[111] Novikov, S.P.: The periodic problem for the Korteweg–de Vries equation. Funkts. Anal. Prilozh. *8*, No. 3, 54–66 (1974) (English translation: Funct. Anal. Appl. *8*, 236–246 (1974)). Zbl. 299.35017

[112] Novikov, S.P.: The Hamiltonian formalism and a many-valued analogue of Morse theory. Usp. Mat Nauk *37*, No. 5, 3–49 (1982) (English translation: Russ. Math. Surv. *37*, No. 5, 1–56 (1982)). Zbl. 571.58011

[113] Novikov, S.P.: Two-dimensional Schrödinger operators in periodic fields, in: Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. (Contemporary Problems of Mathematics) 23. VINITI, Moscow 1983, 3–32 (English translation: J. Sov. Math. *28*, 1–20 (1985)). Zbl. 564.35083

[114] Novikov, S.P., Grinevich, P.G.: Spectral theory of commuting operators of rank two with periodic coefficients. Funkts. Anal. Prilozh. *16*, No. 1, 25–26 (1982) (English translation: Funct. Anal. Appl. *16*, 19–20 (1982)). Zbl. 514.47035

[115] Novikov, S.P., Manakov, S.V., Pitaevskij (Pitaevskii), L.P., Zakharov, V.E.: Theory of Solitons. The Inverse Scattering Method. Nauka, Moscow 1980, 319 p. (English translation: Contemporary Soviet Mathematics. Consultants Bureau [Plenum], New York–London 1984, 276 p.). Zbl. 598.35003

[116] Novikov, S.P., Veselov, A.P.: On Poisson brackets compatible with algebraic geometry and Korteweg–de Vries dynamics on the set of finite-zone potentials. Dokl. Akad. Nauk SSSR *266*, 513–537 (1982) (English translation: Sov. Math. Dokl. *26*, 357–362 (1982)). Zbl. 555.35106

[117] Novikov, S.P., Veselov, A.P.: Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations. Dokl. Akad. Nauk SSSR *279*, 20–24 (1984) (English translalion: Sov. Math. Dokl. *31*), 588–591 (1984)). Zbl. 613.35020

[118] Novikov, S.P., Veselov, A.P.: Poisson brackets and complex tori. Tr. Mat. Inst. Steklova *165*, 49–61 (1984) (English translation: Proc. Steklov Inst. Math. *165*, 53–65 (1985:3)). Zbl. 565.58022

[119] Perelomov, A.M.: Some remarks on the integrability of the equations of motion of a rigid body in an ideal fluid. Funkts. Anal. Prilozh. *15*, No. 2, 83–85 (1981) (English translation: Funct. Anal. Appl. *15*, 144–146 (1981)). Zbl. 495.70016

[120] Perelomov, A.M.: Integrable systems of classical mechanics and Lie algebras. Systems with constraints (in Russian). Prepr., Inst. Teor. Ehksp. Fiz., Moskva, 1983. No. 116, Moscow 1983. 63 p.

[121] Perelomov, A.M.: Integrable systems of classical mechanics and Lie algebras. The motion of a rigid bodv about a fixed point (in Russian). Prepr., Inst. Teor. Ehksp. Fiz., Moskva, 1983, No. 147. Moscow 1983, 64 p.

[122] Pohlmeyer, K.: Integrable Hamiltonian systems and interactions through quadratic constraints. Commun. Math. Phys. *46*, 207–221 (1976)

[123] Rejman, A.G.; Integrable Hamiltonian systems connected with graded Lie algebras. Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova *95*, 3–54 (1980) (English translation: J. Sov. Math. *19*, 1507–1545 (1982)). Zbl. 488.70013

[124] Semenov-Tyan-Shanskij, M.A.: What is a classical r-matrix? Funkts. Anal. Prilozh. *17*, No. 4. 17–33 (1983) (English translation: Funct. Anal. Appl. *17*, 259–272 (1983)). Zbl. 535.58031

[125] Shabat, A.B., Zakharov. V.E.: Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Zh. Ehksp. Teor. Fiz. *61*, 118-134 (1971) (English translation: Sov. Phys.-JETP *34*, 62–69 (1972))

[126] Shahat, A.B., Zakharov, V.E.: A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I. Funkts. Anal. Prilozh. *8*. No. 3, 43–53 (1974) (English translation: Funct. Anal. Appl. *8*, 226–235 (1974)). Zbl. 303.35024

[127] Shahat, A.B., Zakharov, V.E.; Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. Funkts. Anal. Prilozh. *13*, No. 3, 13–22 (1979) (English translation: Funct. Anal. Appl. *13*, 166–174 (1979)). Zbl. 448.35090

[128] Sklyanin, E.K.: On complete integrability of the Landau–Lifshitz equation (in Russian). Prepr. LOMI. E-3-79, Leningrad 1979, 32 p. Zbl. 449.35089

[129] Sklyanin, E.K.: Some algebraic structures connected with the Yang–Baxter equation. Funkts. Anal. Prilozh. *16*. No. 4. 27–34 (1982) (English translation: Funct. Anal. Appl. *16*, 263–270 (1982)). Zbl. 513.58028

[130] Sokolov, V.V., Svinolupov, S.I.: Evolution equations with nontrivial conservative laws. Funkts. Anal. Prilozh. *16*, No. 4, 86–87 (1982) (English translation: Funct. Anal. Appl. *16*, 317–319 (1982)). Zbl. 508.35072

[131] Springer, G.: Introduction to Riemann Surfaces. Addison-Wesley, Reading, Mass. 1957. 307 p. Zbl. 78.66

[132] Steklov, V.A.: On the motion of a rigid body in a fluid (in Russian). Khar'kov 1893, 98 p.

[133] Steklov, V.A. (Stekloff, W.): Sur Ie mouvement d'un corps solide ayant une cavité de forme ellipsïdale remplie par un liquide incompressible et sur les variations des latitudes. Ann. Fac. Sci. Toulouse, III. Ser., Math.-Phys. *1*, 145–256 (1909)

[134] Takhtadzhyan, L.A.: Exact theory of propagation of ultrashort optical pulses in two-level media. Zh. Ehksp. Teor. Fiz. *66*, 476–489 (1974) (English translation: Sov. Phys.-JETP *39*, 228–233 (1974))

[135] Takhtadzhyan, L.A., Veselov, A.P.: Integrability of the Novikov equations for principal chiral fields with a multivalued Lagrangian. Dokl. Akad. Nauk SSSR *279*, 1097–1100 (1984) (English translation: Sov. Phys. Dokl. *29*, 994–996 (1984))

[136] Veselov, A.P.: Finite-zone potentials and integrable systems on a sphere with quadratic potential. Funkts. Anal. Prilozh. *14*, No. 1, 48–50 (1980) (English translalion: Funct. Anal. Appl. *14*, 37–39 (1980)). Zbl. 574.34001

[137] Veselov, A.P.: The Landau-Lifshits equation and integrable systems of classical mechanics. Dokl. Akad. Nauk SSSR *270*, 1094–1097 (1983) (English translation: Sov. Phys. Dokl. *28*, 458–459 (1983))

[138] Veselov, A.P.: On integrability conditions for the Euler equations on so(4). Dokl. Akad. Nauk SSSR *270*. 1298–1300 (1983) (English translation: On integrability conditions for the Euler equations on SO(4). Sov. Math. Dokl. 27, 740–742 (1983)). Zbl. 539,58013

[139] Veselov, A.P.: Cnoidal solutions of the Landau–Lifshitz equation for two-sublattice magnetic materials. Dokl. Akad. Nauk SSSR *276*, 590–593 (1984) (English translation: Sov. Phys. Dokl. *29*, 393–395 (1984))

[140] Weber, H.: Anwendung der Thetafunctionen zweier Veränderlicher auf die Theorie der Bewegung ernes festen Körpers in einer Flüssigkeit. Math. Ann. *14*, 173–206 (1879)

[141] Whittaker, E.T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Fourth Edition). Cambridge University Press, Cambridge 1937, 456 p. FdM 63,1286

[142] Zakharov, V.E.: On stochastization of one-dimensional chains of nonlinear oscillators. Zh. Ehksp.Teor. Fiz. *65*, 219–225 (1973) (English translation: Sov. Phys.-JETP *38*, 108–110 (1974))