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Discrete Triangular Systems

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References: see Novikov's Homepage
www.mi.ras.ru/~snovikov
click Scientific Publications; The items 136,137,138,140,159,163, 173,174,175 present development of this ideas.
Veselov, Taimanov, Dynnikov, Grinevich, R.Novikov also worked here
The item 185 is dedicated to the new results (joint with Grinevich) presented in this talk.

Dedicated to the 100th

Anniversary of I.M.Gelfand

Problem: There are many discretizations of the same system.

What is an Optimal Discretization of the Continuous System?

The same problem we have for Quantization. In both cases the answer is:

It should preserve as many "symmetry" of the original system as possible.

The Group Symmetry leads to Integrals of Motion in the Hamiltonian Structures. There are other symmetries like conservation of diff forms. Good discretizations preserve some of these properties as much as possible.

Huge Phys and Math Literature is dedicated to the study of Completely Integrable Classical and Quantum Systems based on the various more complicated types of Symmetry. Our case today is following.

Consider the KdV Type Nonlinear PDE Systems. A lot of such systems are known now in the dimensions 1+1 and 2+1. They are based on the "Isospectral Symmetries" of the 1D and 2D linear Schrodinger Operators known since XVIII Century:

They are based on the Elementary "Factorization":

We have 2-parametric family of Strong Factorization for D=1

$$L = -\partial_x^2 + u(x) = Q^+ Q + const,$$

$$Q = \partial_x + a(x), Q^+ = -\partial_x - a(x)$$

Here $u = a_x + a^2 + const.$

For every eigenfunction $L\psi = \lambda\psi$

and every factorization the function $\psi' = Q\psi$ satisfies to the equation

$$L'\psi' = \lambda\psi', L' = QQ^{+} + const$$

We consider Darboux transformations $L \to L'$ as a symmetry of the whole class of such operators. It generates the theory of "isospectral" KdV type systems deforming operators within this class.

The optimal discretization of this class is

$$L = a_n T + a_{n-1} T^{-1} + u_n$$

where $T : n \to n + 1$ is a shift. We have family of Right and Left Factorizations

$$L = Q^+ Q + const = P^+ P + const$$

$$Q = x_n + y_n T, P = u_n + v_n T^{-1}$$

There is an integrable system here in the variables n, t (The whole

1D Toda Lattice Hierarhy), deforming operators within this class.

The standard discretization of second derivative $T + T^{-1} + u$ used before is not invariant under Darboux Transformations. It is not optimal.

For $D = 2$ we have Weak Right and Left Factorization for the hyperbolic and elliptic cases:

Hyperbolic

$$L = \partial_x \partial_y + a\partial_x + b\partial_y + c =$$

$$= Q_1Q_2 + W = Q_2Q_1 + V$$

$Q_1 = a + \partial_y, Q_2 = b + \partial_x$ and equation $L\psi = 0$. The Right Laplace Transformation is

$$\psi' = Q_2\psi, L' = WQ_2W^{-1}Q_1 + W$$

the Left One is similar interchanging Q_1 and Q_2. The gauge equivalence group has a form

$$L \to fLg, \psi \to g^{-1}\psi$$

Elliptic Self-Adjoint

$$L = -\partial\bar{\partial} + a\partial + b\bar{\partial} + c =$$

$$= Q^{+}Q + W = QQ^{+} + V$$

$$Q = \partial + A, Q^{+} = -\bar{\partial} + \bar{A}$$

Here $\partial = \partial_x - i\partial_y, Q_1 = \bar{Q}_2$, Laplace Transformation is the same. The gauge group is $L \to f^{-1}Lf, \psi \to f^{-1}\psi$ for the zero level $L\psi = 0, |f| = 1$.

The sequence of Laplace Transformations (right only)

$$... \to L_n \to L_{n+1} \to ...$$

where $L_{n+1} = L'_n$, is equivalent to the famous 2D Toda Lattice System in the variables x, y, n (Hyperbolic) or $z, \bar{z}, n$ (Elliptic).
The Hyperbolic case was studied since XIX Century and used by the Darboux school in the 3D Geometry.
Sasha Veselov and myself used the elliptic case in 1990s studying spectrum of 2D Schrodinger operators in periodic magnetic field

The Optimal Discretizations are:

Hyperbolic Case: Take Square Lattice with shifts T_1, T_2 and Equation $L\psi = 0$ where

$$L = a + bT_1 + cT_2 + dT_1T_2$$

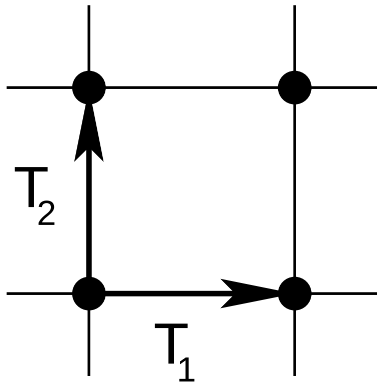
with Right and Left Factorizations

$$L = f[(1 + uT_1)(1 + vT_2) + W] =$$

$$= g[(1 + vT_2)(1 + uT_1) + V]$$

See Fig 1

Fig 1 (Hyperbolic Case)



The operator

$$L = 1 + iT_2 - T_1T_2 - iT_1$$

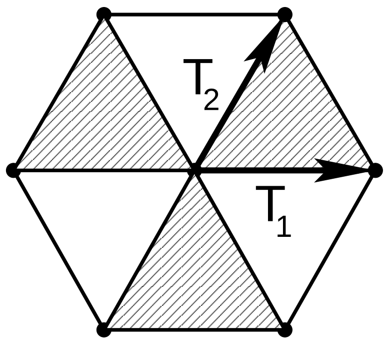
has been used as discrete analog of the Cauchy-Rieman operator $\bar{\partial}$ since 1940s.

Elliptic Case: Take Equilateral Triangle Lattice with basic shifts T_1, T_2. Consider class of real self-adjoint second order operators

$$L = a + bT_1 + cT_2 + dT_1^{-1}T_2 +$$

$$+ T_1^{-1}b + T_2^{-1}c + T_1T_2^{-1}d$$

Fig 2 (Elliptic case)



It can be weakly factorized: $L = Q_b^+ Q_b + W = Q_w^+ Q_w + V$, $Q_b = u + vT_1 + wT_2$ (black triangle operator), $Q_w = x + yT_1^{-1} + zT_2^{-1}$ (white triangle operator)

Laplace transformations–Right and Left– were invented in 1997, $\psi \to Q_b\psi$ and $\psi \to Q_w\psi$ They are inverse to each other up to gauge equivalence. Every nonzero solution $Lf = 0$ allows to choose representative $\tilde{L}$ in the gauge class such that $\tilde{L}(1) = 0$ taking $\tilde{L} = fLf$

Examples:

1.Complex Analysis is the best

known Completely Integrable System. The operators $Q_b = 1 + T_1 + T_2, Q_w = 1 + T_1^{-1} + T_2^{-1}$ were taken by the present author and I.Dynnikov as discrete analogs of Cauchy-Rieman operators constructing New (Triangular) Discretization of Complex Analysis in 2002-2003. Ring structure is missing in our discretization (as well as in the standard one). There is a unique canonical definition of holomorphic polynomials and rational functions in our discretization. Factorization property of the standard 2D Laplace-Beltrami Operator $\Delta = \partial\bar{\partial}$ is preserved here $-\Delta = Q_b Q_w + Const.$

2.A number of discrete 2D operators with remarkable "solvable" spectrum were constructed for this lattice in 1997 including q-analog of the famous Landau Operator in the constant (homogeneous) magnetic field. There are 3D examples among them with solvable infinitely degenerate ground state.

3.Every pair $Q_b = u + vT_1 + wT_2$, $Q_w = x + yT_1^{-1} + zT_2^{-1}$ define the operator $Q = Q_b \oplus Q_w$. It maps the space of functions of vertices into functions of triangles because every triangle is either black or white. We call formal equation $Q\psi = 0$ "The Discrete GL_2–Connection." Theory of such connections (see below) was constructed in the series of works of the present author in collaboration with Dynnikov and Grinevich.

Discrete GL_n Connections and Operators:

Triangle Operators, Connections:

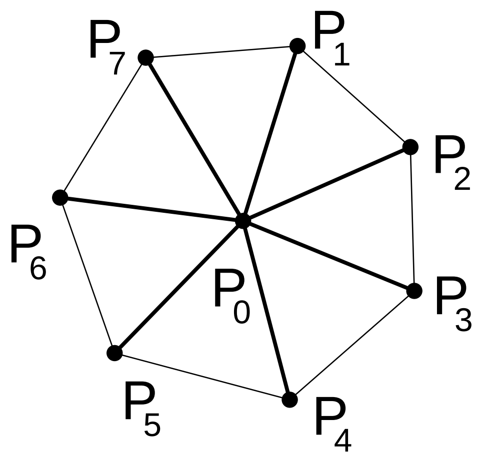
Let K be a n-dimensional simplicial complex and X selected family of n-simplices. The operator Q^X below is called Triangle

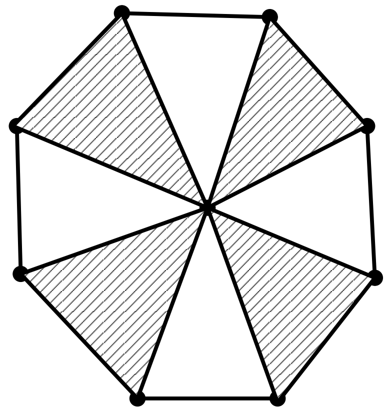
$$Q^X_T = \sum_{P \in T} b_{P:T} \psi_P, T \in X$$

first order difference operator.

We call formal equation $Q\psi = 0$ GL_n Connection if family X consists of all n-simplices in K.

Fig 3 (Curvature, Colorings)





Important examples:

1.X contains all n-simplices. It is GL_n-connection characterized by the set of data $\mu^T_{PP'} = b_{P:T}/b_{P':T}$

2.N-simplices in K are black-white colored, and X consists of all black or white simplices.

We call triangulation K Even if every closed thick path

$$\gamma = [T_0, T_1, T_2, ..., T_m = T_0]$$

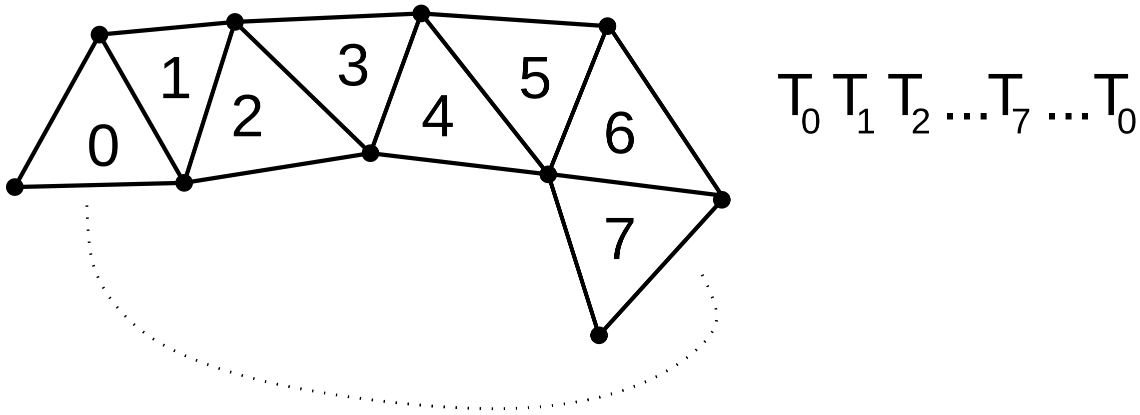
consists of even number of n-simplices $m = 2k$. Here $T_i \cap T_{i+1}$ is an $(n-1)$-face Δ_i.

A natural Holonomy homomorphism is defined from the semigroup of thick paths with fixed initial $n-1$-simplex $\Delta_0 = T_0 \cap T_1$ into the group GL_n

$$\gamma \in \Omega_{\Delta_0}^{thick} \to GL_n$$

The Nonabelian Curvature corresponds to the thick paths belonging to simplicial star of one vertex. For n-manifolds $K = M^n$ Curvature is completely characterized by the thick paths around $n-2$-simplices.

Fig 4 (Thick Paths)



For the case of Trivial Curvature corresponding Holonomy map depends only on fundamental group.

Complex K is Even iff n-simplices can be black-white colored.

Framed Path

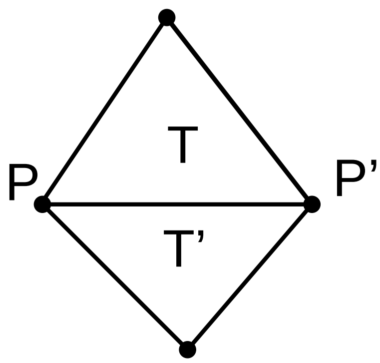
$$\gamma^{fr} =< P_0, P_1, ..., P_m; T_1, T_2, ...T_m >$$

consists of edges $P_{i-1}P_i$ belonging to the n-simplex T_i. Every closed framed path $P_m = P_0$ determines a Framed Abelian Holonomy

$$\mu(\gamma^{fr}) = \prod_i \mu^{T_i}_{P_{i-1}P_i} \in k^*$$

where k^* is multiplicative group of our basic field. We assume now that $k = R$ and all coefficients are positive. Existence of such invariants is specific for the discrete case. Nothing like that appears in the continuous case.

Fig 5 (Local Framed Invariants)



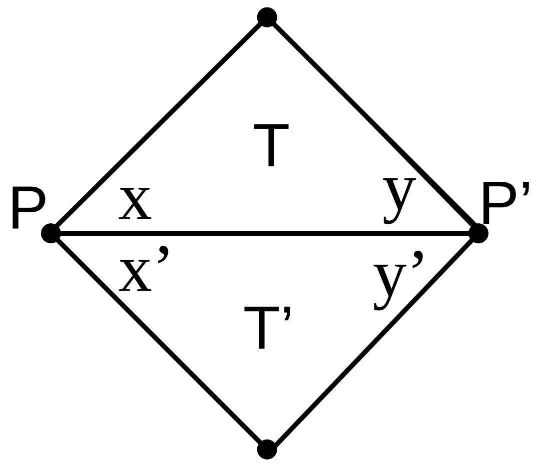
The Inverse Problem is: How to reconstruct Connection from the Holonomy?

Framed Abelian Holonomy (FAH) determines the Nonabelian Holonomy and (generically) back. Inverse Problem was solved for manifolds by the present author few years ago. No theory of Characteristic classes has been constructed yet. The "curvature type" local gauge invariants are $\rho_{PP'}^{TT'} =$

$\mu^T_{PP'}/\mu^{T'}_{P'P}$ for $PP' \in T \cap T'$ and framed path $< PP'P, T, T', T >$. They determine connection jointly with 1-dimensional homological invariants of FAH.

Operators and SL_2 Reductions. Let K be a 2D simplicial complex with 2nd order real self-adjoint difference operator L acting on vertices as above. We assume that every edge belongs to so some 2-simplex in K, and every vertex belongs to 3 or more 2-simplices.

Fig 6 (SL_2 Reductions)



Theorem 1. 1.There exists a $SL_2^{\pm}$ Connection with Triangle Op-

erator Q such that

$$L = Q^+ Q + W$$

where W is a scalar function (potential).

2.For Even Complex $K = M^2$ with black-white colored triangulation operator L can be uniquely presented in the forms

$$L = Q_b^+ Q_b + U = Q_w^+ Q_w + V =$$

$$= 1/2 Q^+ Q + W$$

Here U, V, W are potentials and the connection $Q = Q_b \oplus Q_w$ is SL_2. Here $SL_2^{\pm} \subset GL_2$ is subgroup with $\det A = \pm 1$

Theorem 2. Let Q be a GL_2-Connection operator in K, and $PP' = T \cap T'$ with following property: $b_{P:T} b_{P':T} = b_{P:T'} b_{P':T'}$. Then Connection is $SL_2^{\pm}$. If $K = M^2$ is black-white colored then this condition is necessary and sufficient to be SL_2 Connection.

Remark: We proved that for $K = M^n, n > 2$ every $SL_n^{\pm}$ Connection is equivalent to the Canonical Connection with all $\mu_{PP'}^T = 1$. So our reductions are non-trivial only for $n = 2$.

We define Right and Left Laplace Transformations for all black-white colored $K = M^2$ as above using factorizations $L = Q_b^+ Q_b + U = Q_w^+ Q_w + V$ mapping $\psi \to Q_b \psi$ and

$\psi \to Q_w \psi$ correspondingly As results we get operators L'_b and L'_w acting on the functions of black (or white) triangles.

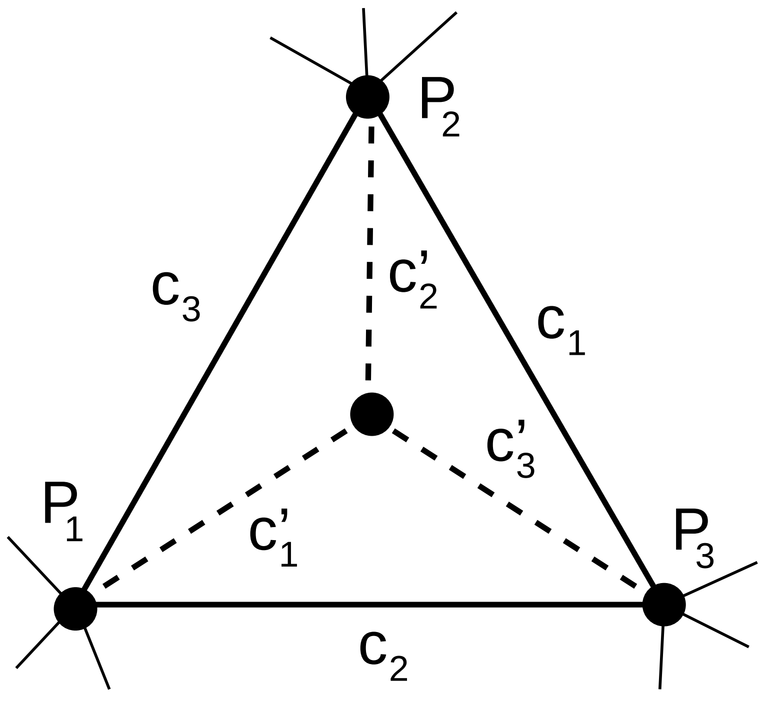
Electric Chains:

Let us compare these transformations with some XIX Century constructions in the Theory of Electric Chains. Electric Chain is realized as a 1-dimensional graph Γ with edges $I = P_1P_2$ having electrical resistance equal to $r_I = 1/c_I$ where c is Conductivity. Applying Voltage function U_P to every vertex $P \in \Gamma$ we obtain currents through the oriented edge $I = P_1P_2$ equal to $J_I = c_I(U_{P_2} - U_{P_1})$. The current through vertex P is equal to the sum of currents entering P. So we have for the remaining currents in the vertices

$$J_P = \partial C \partial^*(U)$$

in the standard topological notations. Here C is diagonal Conductivity operator acting on edges $I \to c_I I$. All c_I are positive. As we can see, this is our standard real selfadjoint difference operator written in the gauge form such that constant belongs to the Kernel.

Fig 7 (Electric Chains, Star-Triangle)



The Star-Triangle Transformation:

Take any triangle of edges $P_1P_2P_3$ with conductivities c_3, c_1, c_2 for the edges $12, 13, 23$. Introduce one new vertex T. Join it by new edges I_1, I_2, I_3 with vertices P_1, P_2, P_3 with conductivities $c_i' = (c_1c_2 + c_1c_3 + c_2c_3)/c_i = \sigma_2(c)/c_i$. Let $U_i = U_{P_i}$. Take $U_0 = U_T = (c_1c_2U_3 + c_1c_3U_2 + c_2c_3U_1)/\sigma_2(c)$ or $U_T^{ext} = QU$.

Lemma 1. With this choice of Voltage function U^{ext} and conductivities C^{new} the total current through vertex T is equal to zero. The currents through the new edges I_j are equal to the sum of currents through the old edges P_iP_k entering the same vertex. In the new graph Γ^{new} with T, I_j added to the set of vertices and edges, and edges P_jP_k removed, we have the same set of currents through vertices

$$L^{ext}U^{ext} = \partial C^{new}\partial^*(U^{ext})|_{\Gamma^{new}} =$$

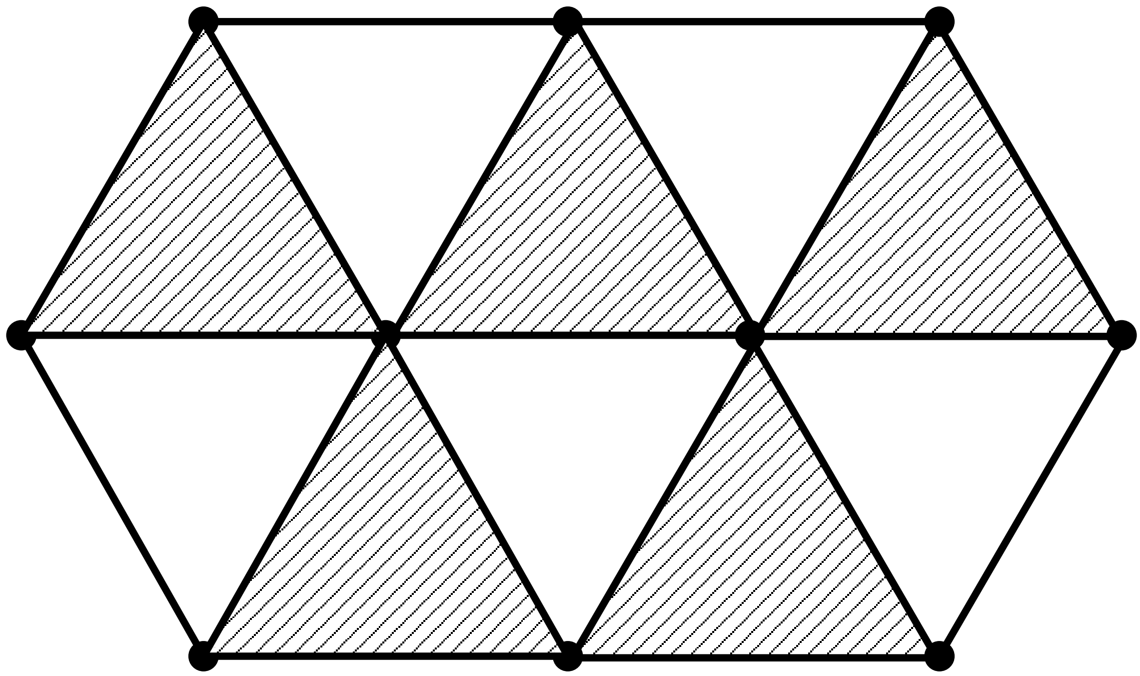
$$= \partial C\partial^*(U)|_{\Gamma}$$

and $U^{ext} = [U(P), QU(T)]$

Now we consider the "2D Lattices" (maybe irregular), which are by definition the graphs $\Gamma =$

$K^1 \subset K$. Here K is a 2D simplicial complex, and graph coincides with 1D skeleton. We assume that every edge belongs to exactly one 2-simplex of K, like black (or white) triangles in the colored 2-manifold $K \subset M^2$. We call 2D simplices in K "black". Let the structure of Electric Chain be given in K, i.e. set of conductivities c_I for all edges I. Consider the operator $L = \partial C \partial^*$ in $\Gamma = K^1$. The Inner Part of K is set of vertices belonging to 3 or more 2D black simplices.

Fig 8 (Star-Triangle and 2D Lattices)



Theorem. The operator L can be factorized in the form:

$$L = Q^{+}(C')^{-1}Q - W$$

with diagonal operator $C' : T \to (\sum_i c'_i(T))T$ and black triangle operator

$$(QU)_T = \sum_i c'_i U_{P_i \in T}$$

We perform the Star-Triangle Transformation along all (black) triangles. It defines Laplace Transformation $U_P \to (QU)_T$ which is set of values of extended Voltage function U^{ext} in the centers of black triangles T. It maps the Kernel $(LU)_P = 0$ into the Kernel of Laplace Transformed Operator L'

$$L' = QW^{-1}Q^{+} - C',$$
$$L'(QU)_T = 0$$

So we conclude that the triangular Laplace Transformation admits physical interpretation in the theory of electric chains. New extended operator L^{ext} acts on the union of old and new vertices P, T. If function $U(P)$ belongs to the Kernel of original operator $LU = 0$, its extension U_T^{ext} restricted to the subset of new vertices T, satisfies to the equation $L'U^{ext}(T) = 0$ on this subset only: $L^{ext}U(T) = L'U(T)$