S.Novikov

Singular Solitons and Indefinite Metric

I.M.Gelfand's 100 Anniversary,

Moscow, December 2013

Collaborators: P.Grinevich

References: Novikov's Homepage
www.mi.ras.ru/˜snovikov
click Publications, items 175,176,182, 184. New Results published recently in the Journal of Brazilian Math Society, 2013, special volume dedicated to the 60th Anniversary of IMPA

Problem:

Construct Spectral Theory on the real line for the special class of real singular 1D Schrodinger Operators $L = -\partial_x^2 + u(x)$

Our Class contains all "singular multisolitons" and "singular finite-gap KdV solutions" (algebro-geometric solutions). They have the property (below) :

All solutions (for all λ) $L\psi = \lambda\psi$ are meromorphic at the real line. This property we take as a definition of the whole class.

Questions:

1. Which singularities we allow?

2. Which spaces of functions and inner products we are going to use?

3. Which case corresponds to the right analog of the Fourier Transform on Riemann Surfaces?

4. For which cases we can prove Completeness Theorem?

A simple lemma is true (missed in the classical literature): such potentials have isolated singularities of the form (only)

$u = \frac{n_k(n_k+1)}{(x-x_k)^2} + \sum_j b_{jk}(x-x_k)^{2j} + o((x-x_k)^{2n_k})$ for $j \geq 0$

The chosen class of functions is C^∞ plus singularities such that $\psi(x) = \sum_{j \leq n_k} q_j (x-x_k)^{-n_k+2j} + o((x-x_k)^{n_k})$ for $j \geq 0$, nearby of every real singularity. We call it $F_{x_1,\ldots,x_M;n_1,\ldots,n_M} = F_{X;N}$

The inner product in the space $F_{X;N}$ is
$< \psi, \phi >= \int \psi(x)\bar{\phi}(\bar{x})dx$

It is well-defined here using complex contours avoiding singularities because all residues of the product are equal to zero.

This inner product is indefinite.

We consider either functions rapidly decreasing at infinity or periodic (quasiperiodic) with condition

$\psi(x+T)=\varkappa\psi(x), \psi \in F_{X,N}(\varkappa)$

for $|\varkappa|=1$. The number of negative squares of inner product in the space $F_{X,N}(\varkappa)$ is equal to

$m_{x_1,\ldots,x_M;n_1,\ldots,n_M} = m_{X;N} =$

$= \Sigma_{period} n_k = ||N||$ (the "Total Singularity").

KdV and Schrodinger Operator:

$$u_t(x,t) = 6uu_x - u_{xxx}$$

$$L\psi = -\psi_{xx} + u\psi = \lambda\psi$$

Spectral Theory of Rapidly Decreasing and Periodic Schrodinger Operators L requires NONSINGULARITY of Potential $u(x)$ as well as physical derivation of KdV in the Theory of Solitons.

A number of other applications of KdV theory was discovered later which do not require nonsingularity.

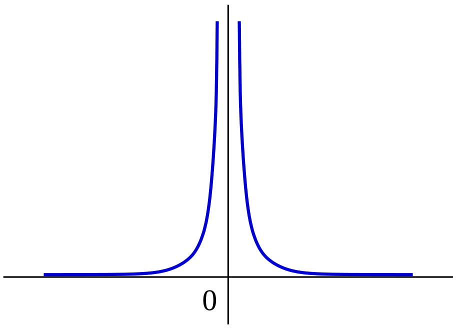
Huge Literature is dedicated to the singular KdV Solutions. A Theory of Rational and Elliptic Solutions is especially popular.

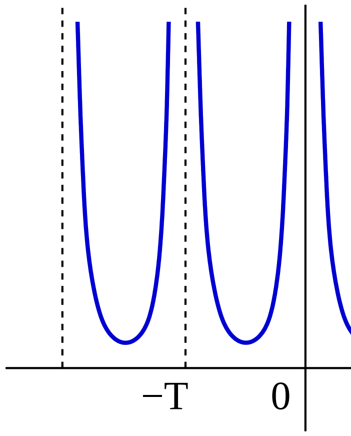
Example: For $j = 1, \ldots, \frac{n(n+1)}{2}$ there are Real Rational and Elliptic Solutions

$$u(x,t) = \Sigma_j \, 2/(x - x_j(t))^2$$

$$u(x,t) = \Sigma_j \, 2\wp(x - x_j(t))$$

let $u(x,0) = n(n+1)/x^2$





and $u(x,0) = n(n+1)\wp(x)$;

(the famous Lame' Potentials.)

Hermit found Spectrum with Dirichlet boundary conditions for $x = 0, T$. Here T is a real period. No spectral theory was constructed on the real line. For $n = 1$ this solution is a SINGULAR TRAVELING WAVE $u = 2\wp(x - at)$ with 2nd order pole in the point $x = at$. Don't Confuse it with NONSINGULAR TRAVELING WAVE $u = 2\wp(x + i\omega' - at)$ where $2i\omega'$ is an imaginary period. This is a first example of periodic finite-gap potentials found in 1950s.

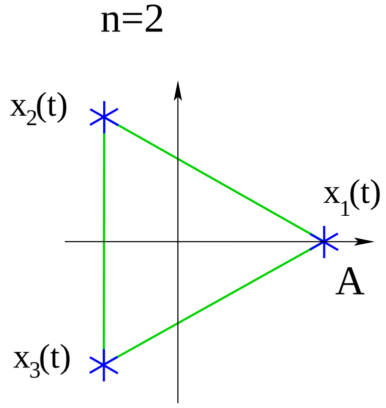
The evolution of Lame' Potentials

$$u(x,0) = n(n+1)\wp(x)$$

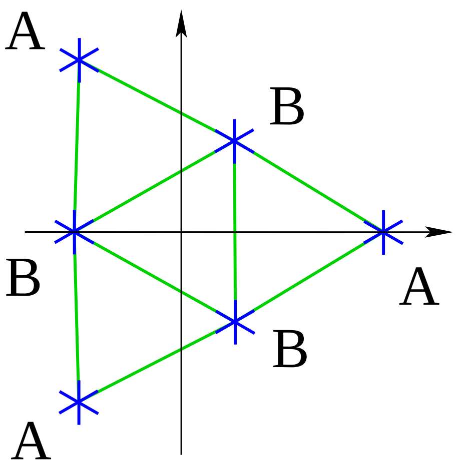
or $u(x,0) = n(n+1)/x^2$ leads to singular solutions

Important Technical Question:

How many real poles these solutions have for $t > 0$?

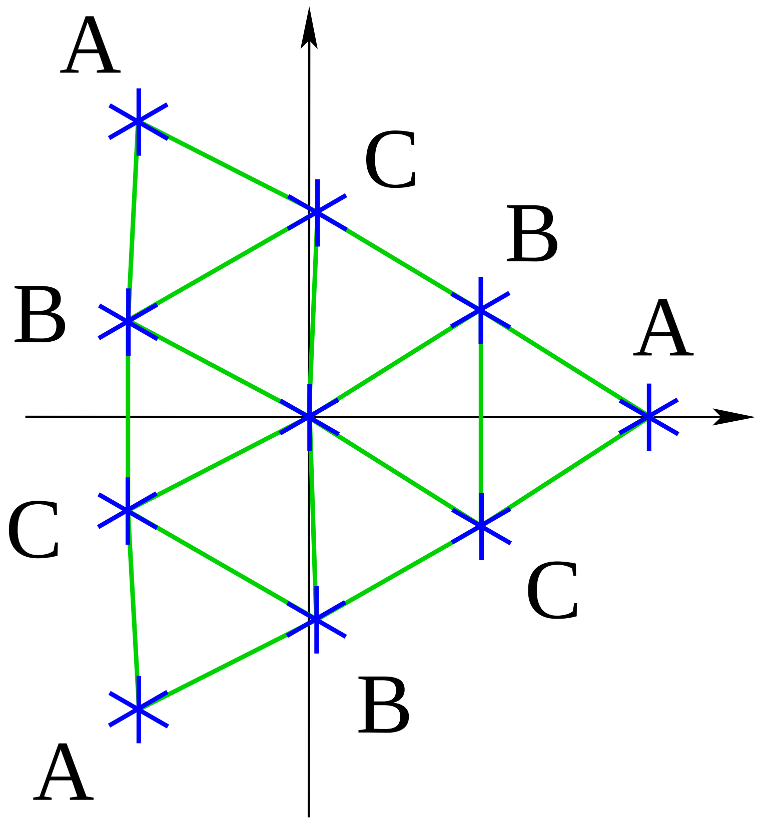


n=3



The orbits of group $Z/3Z$ are marked here. We have $1, 1, 2, 2, 3, \ldots$ real poles for $n = 1, 2, 3, 4, 5 \ldots.$

n=4 $\qquad \frac{n(n+1)}{2} = 10$



$$x_j \sim r_j t^{1/3}$$

The symmetry group $Z/3Z$ acts here

$r_j \to \zeta r_j, \zeta^3 = 1$

Our Result: The number of real poles is equal to $[(n+1)/2]$. This number is equal to the number of negative squares for the Inner Product in the Spaces of functions on the real line where the operator $L = -\partial_x^2 + u(x,t)$ is symmetric. Arkad'ev, Polivanov and Pogrebkov constructed some kind of Scattering Theory for the potentials with singularities like $2/(x - x_k)^2$. No spectral theory was discussed.

Consider All Real Singular "Algebrogeometric" or "Singular finite-gap" Potentials. Every solution $L\psi = \lambda\psi$ is meromorphic in x near the real axis with negative part: $\psi(y) = a_0/y^n + a_1/y^{n-2} + \ldots + O(1))$ where $y = x - x_j$. So we have parameters $a_0, \ldots, a_{[(n+1)/2]}$.

Take the space $F_{X,N}$ of smooth periodic (quasiperiodic) or rapidly decreasing functions f defined by the set X of points x_j and set N of numbers n_j.

How Singular Solitons can be used? We used them to define right analog of Fourier Transform on Riemann Surfaces.

What is Fourier Transform on Riemann Surfaces? Which Problems need it? Why singular Solitons are important?

Example: The Fourier/Laurent Series for the contours which are the time-sections of the world-sheet Riemann Surfaces (the "String Diagrams") was constructed by Krichever-Novikov (1986-1990) realizing The Program of Operator Quantization of the Closed Strings). Some Singular solutions to the 2D Toda System were used.

Continuous Fourier Transform on Riemann Surfaces was invented in our works with Grinevich using singular finite-gap solutions to KdV and KP (2003-2010). It requires Indefinite Inner Product for genus $g > 0$.

The ordinary Fourier Series and Transform are based on the standard exponential base in the Space of functions:

I. On the circles in Riemann Sphere:

$\Psi_n(k) = k^n = e^{in\phi}, x = n \in Z, k = e^{i\phi}, |k| = const$ belongs to the circle

(Fourier Series)

The Laurent series is defined in domains between such circles.

II. On the real line.

$\Psi(x,k) = \exp(ikx), x \in R, k \in R$. The k-circle in $S^2 = \Gamma$ is passing through infinity

(Fourier Transform)

The exceptional property is following

The Graded Multiplication:

$\Psi_n(k)\Psi_m(k) = \Psi_{m+n}(k)$,

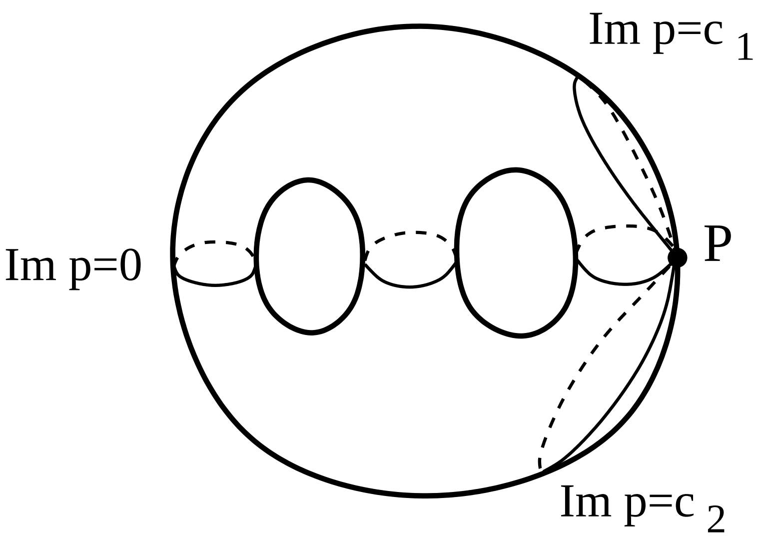
$\Psi(x,k)\Psi(y,k) = \Psi(x+y,k)$

Here genus is equal to zero $g = 0$, and k belongs to the "Canonical Contour" κ_0 on Riemann Surface $\tau = 0$ which is $|k| = 1$ in discrete case and $k \in R$ in continuous case.

There are many orthonormal bases in Mathematics and Applications ("Wavelets", for example) but only Fourier base has such multiplicative property.
It is important for Nonlinear Problems. The notion of Resonances is bases on it.

These bases were constructed for operator quantization of the closed bosonic string. It was critical to have bases with good MULTIPLICATIVE properties.

Continuous analog of the Krichever-Novikov bases.



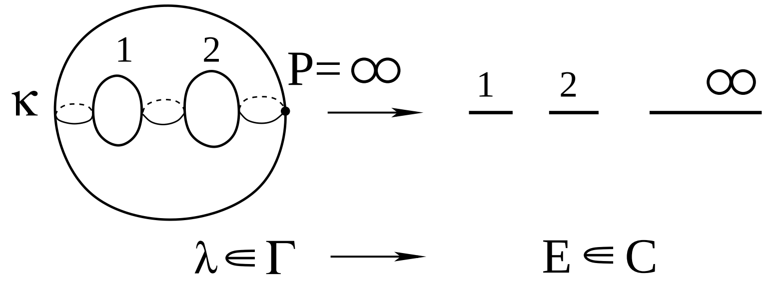
Step 1: Select Contour.

Let $z = 1/k$ be local parameter near P,

dp is a meromorphic differential with a second-order pole at $P = \infty$, $dp = dk + O(1)$, $Im(\oint dp) = 0$ for all closed paths. So $\tau = Im(p)$ is well-defined. Our Canonical Contour is $\kappa_0 : Im(p) = 0$.

Step 2: Take Real Inverse Spectral Data,
and construct Ψ-function:
1) A compact Riemann surface Γ of genus g with an "infinite" point $P = \infty$ and

local parameter $z = 1/k$ near $P, z(P) = 0$. 2) A collection of points ("Divisor") $D = \gamma_1, \ldots, \gamma_g$ (the poles of ψ-function). 3) The "reality conditions" for Γ and poles should be satisfied. Γ is hyperelliptic (2-sheeted over λ-plane) for KdV and 1D Schrodinger Operator.



1)The eigenfunction $\Psi(\lambda, x)$, $\lambda \in \Gamma$, $x \in R$, is meromorphic in $\Gamma \backslash \infty$ with simple poles $\gamma_1, \ldots,$ γ_g, $\Psi(\lambda, x_0) = 1$.

2. $\Psi(\lambda, x) = (1 + o(1)) \exp(ik(x - x_0)), \lambda \to \infty$.

Let $g = 0$ and $\Gamma = C \cup \infty, P = \infty$. Here k is the standard coordinate $k = \lambda$. Then $p = k$, $\Psi(\lambda, x) = \exp(ikx)$

is the standard Fourier base on the real line $Im(k) = 0$

A continuous analog of the Fourier bases is defined by the Special Ψ-function with following Singular Data:

$\gamma_1 = \ldots = \gamma_g = \infty, x_0 = 0$

Ψ-functions form an almost-graded algebra:

$$\Psi(\lambda, x)\Psi(\lambda, y) = l\Psi(\lambda, x+y)$$

$$[l = \partial_z^g + \Sigma_{j>0}\, c_j(x,y)\partial_z^{g-j}]_{z=x+y}$$

$c_0 = \zeta(x+y)$ for $g = 1, j = 1$.

We study functions of λ here; x is a parameter numerating our basic functions.

The functions $\Psi(\lambda, x)$ are Singular in x. Example: The classical periodic Lame' operators $-\partial_x^2 + g(g+1)\wp(x)$.

Do Singular Operators have reasonable spectral theory on the whole real line x?

Classical people like Hermit considered spectrum only at the interval $[0, T = 2\omega]$ with zero boundary conditions. We need to use the whole line in order to construct Fourier Transform with good multiplicative properties.

Ψ-functions for Regular Real Periodic Operators never form an almost-graded multiplicative system for $g > 0$.

Consider real (may be singular) "finite-gap" periodic operator with spectral curve (Riemann Surface) Γ given by the equation

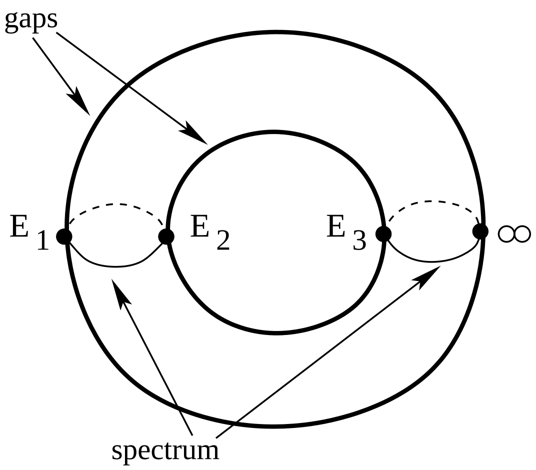
$\mu^2 = (E - E_1) \cdots (E - E_{2g+1})$ with permutation of sheets $\sigma(E, \mu) =$

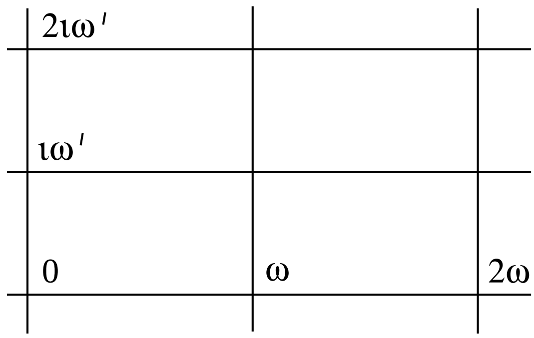
$(E, -\mu), \sigma^2 = 1$ and poles of Ψ-function $D = \gamma_1, ..., \gamma_g$.

Real case corresponds to the data where Γ and divisor of poles are real i.e. collection of branching points is invariant under complex conjugation $\tau(E, \mu) = (\bar{E}, \bar{\mu})$.

Example 1: Let $g = 1$ (Γ is a torus)

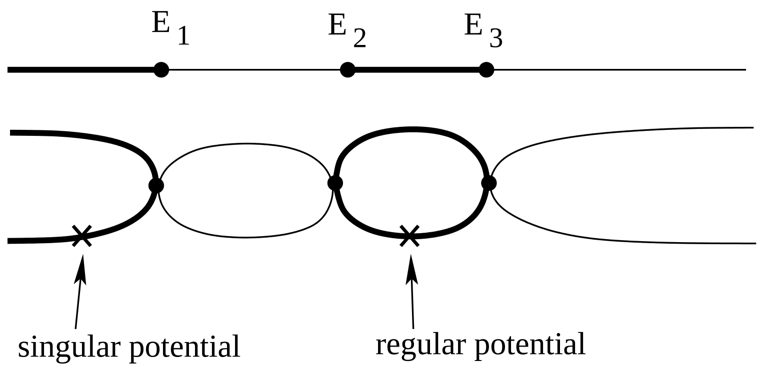
and all E_j are real, $j = 1, 2, 3$:





The lattice of periods of the Weierstrass $\wp$ -function is **rectangular with periods** $2\omega, 2i\omega'$.

The spectrum is real , and spectral gaps are $[-\infty, E_1]$ and $[E_2, E_3]$

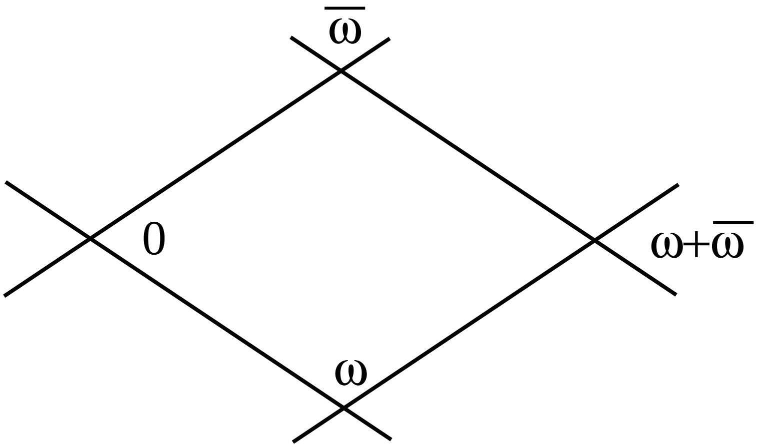


κ_0 is represented by fine lines.

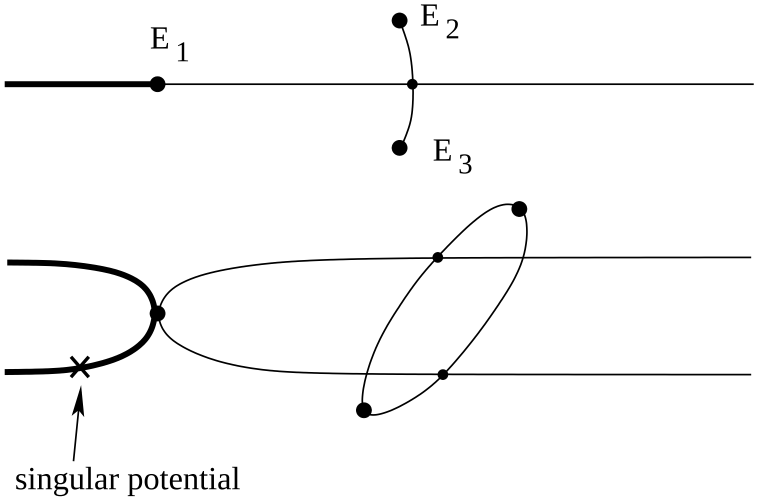
The contour κ_0 has 2 components here: infinite and finite. There is only one pole γ: For Regular Case it belongs to the finite gap, for the Singular Case it belongs to the infinite gap

(They both are the shifted Hermit-Lame Operators but in regular case the shift is imaginary, in singular case the shift is real). The spectrum on the whole line is the same but eigenfunctions and functional spaces on the x-line are different.

Example 2. Let $g = 1, E_1 \in R, E_3 = \overline{E_2}$:



The lattice of periods is **rombic**.



κ_0 given by fine lines.

The spectrum on the whole line coincides with the projection of the contour κ_0 on the $E-line$. It contains complex arc joining $E_2, \bar{E}_2$. Spectral theory of singular operators on the whole line was not discussed before.

Define the "spectral measure" $d\mu$. Let λ_j= projection of poles, $\lambda = \lambda(E)$:

$$d\mu = \frac{(E-\lambda_1)\ldots(E-\lambda_g)dE}{2\sqrt{(E-E_1)\ldots(E-E_{2g+1})}}$$

For every smooth function on the contour κ_0 parametrized by λ, with decay fast at infinity, we define

Direct and Inverse Spectral Transform:

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\kappa_0} \phi(\lambda)\Psi(\sigma\lambda, x)d\mu \quad (1)$$

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_R \tilde{\phi}(x) \Psi(\lambda, x) dx \tag{2}$$

We call it R-Fourier Transform if all $\lambda_j = \infty$; $d\mu^F = dE/2\sqrt{(E - E_1) \ldots (E - E_{2g+1})}$ and our base has good multiplicative properties

In the Regular Case this Spectral Transform is an Isometry between the Hilbert spaces with inner products

$$< \psi_1, \psi_2 >_{\kappa_0} = \int_{\kappa_0} \psi_1(\lambda)\overline{\psi}_2(\lambda)d\mu(\lambda)$$

$$< f_1, f_2 >_R = \int_R f_1(x)\overline{f}_2(x)dx$$

Consider the Singular Potentials

1) Formula for the Spectral Transform remains valid; For the Inverse Transform it remains valid

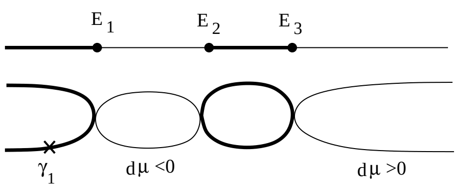
after a natural regularization.

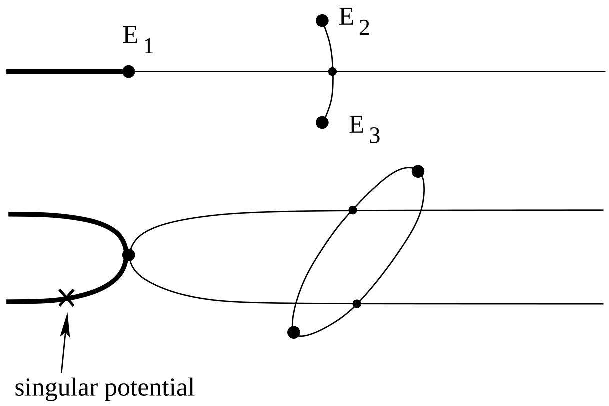
2) Spectral Transform is an isometry between the spaces with indefinite metric described above. All singularities have a form described above

Example 1. All branching points are real: τ acts identically on κ_0, the form $d\mu$ is negative somewhere. For R-Fourier Transform we have:

$d\mu^F/dp > 0$ exactly in every second component starting from the infinite one; So we have $[(g+1)/2]$ "negative" finite components in κ_0. Example 2. Some pair of branching points is complex adjoint: τ is not identity in the nonreal components of κ_0; So the inner product is nonlocal and therefore indefinite.

We proved Completeness Theorem in the spaces $F_{X,N}(\varkappa)$ which are similar to the





"Pontryagin-Sobolev spaces". Every function $f(x) \in \mathcal{L}_2(R)$ can be written for real x as $f(x) = \oint \hat{f}(\varkappa, x) d\phi(\varkappa), \varkappa = e^{i\phi}$

Here $f(\varkappa, x+T) = \varkappa f(\varkappa, x)$. The space $F_{X,N}$ is represented as a direct integral of Bloch-Floquet spaces $f \in F_{X,N}(\varkappa), |\varkappa| = 1$: Our inner product has r negative squares in the space $F_{X,N}(\varkappa), r = [(g+1)/2]$ for the R-Fourier case.

Remark: Singular Bloch-Floquet eigenfunctions are known for the $k+1$-particle

Moser–Calogero operator with Weierstrass elliptic pairwise potential if coupling constant is equal to $n(n+1), n \in Z$. They form a k-dimensional complex algebraic variety. No one function is known for $k > 1$ serving the discrete spectrum in the space $\mathcal{L}_2$ of the bounded domain inside of poles. Our case corresponds to $k = 1$. We believe that for all $k > 1$ this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space R^k.