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Homepage www.mi.ras.ru/$\tilde{}snovikov$ (click publications),

items 177, 178, 179

collaborators: P.Grinevich, A.Mironov

On the 2D Nonrelativistic Purely Magnetic Pauli Operator (spin 1/2) :

The Algebro-Geometric Theory of the Ground Level

In 1979-1980 three groups of authors completely calculated the ground states using following property of the 2D Pauli Operator with zero electric field (Avron-Seiler[AS], Aharonov-Casher[AC], Dubrovin and myself[DN]) using appropriate units and gauge conditions: $L_P = L \oplus \tilde{L} =$

$$= QQ^{+} \oplus Q^{+}Q, \; Q = \partial_z + A$$

Here $\partial_z = \partial = \partial_x + i\partial_y$ and magnetic field $B = 1/2(A_{\bar{z}} - \bar{A}_z)$

The most interesting classes of magnetic fields are

1.AC: Rapidly decreasing fields with flux $|[B]| = |\int_{R^2} B dx dy| < \infty$. The ground states form a finite-dimensional space of dimension $m \in Z, m \leq [B] < m+1$

2.DN: Periodic fields with integer flux through the elementary cell $\int_{cell} B dx dy = [B] = m \in Z$. The ground states form an infinite dimensional subspace in the Hilbert Space $L_2(R^2)$ isomorphic to the Landau level for the same value of the magnetic flux.

In both cases ground states are The Instantons belonging to one spin-sector only:

a. They satisfy to the 1st order equations $Q^+\psi = 0$ for the case $[B] > 0$ and $Q\psi = 0$ for the case $[B] < 0$.

b. They belong to the Hilbert Space $L_2(R^2)$

In the latest literature started in 1980s this operator was associated with the "Super-Symmetry" operator $L \rightarrow \tilde{L}$ (known under the name "Laplace Transformation" in this case since XVIII Century). It implies only that all higher levels are double-degenerate having representatives in the both spin-sectors.

Theory of 2 + 1-Solitons: The Inverse Problem for 2D Schrodinger Operators is based on the Selected Energy Level. The Magnetic Field should be "Topologically trivial in periodic case (Magnetic flux through the elementary cell is equal to zero) Question: Is Theory of Ground Level for the Purely Magnetic Pauli Operators related somehow to the Theory of Solitons and Algebro-Geometric Theory of the scalar 2D Schrodinger Operators based on the Selected Energy Level? Do Corresponding 2D Soliton Hierarchies have Reduction of that kind?

The Algebro-Geometric Spectral Theory of the 2D Second Order Scalar Schrodinger Operators and Corresponding Soliton Hierarchies based on the selected energy level were started in 1976 by Manakov(M) and Dubrovin, Krichever and myself (DKN). The Reduction on Algebro-Geometric Data leading to the zero magnetic field, were found by Veselov and myself (1983). Our Problem: Which Data Leads to the Factorized Operators of the Form $L = QQ^+$? Such Operators are associated with 2D Purely Magnetic Pauli Operators

Make replacement $x, y \to z, \bar{z}$. Consider elliptic operators $L = \partial\bar{\partial} + G\bar{\partial} + S$ gauge equivalent to the self-adjoint operators? The magnetic field $B = 1/2G_{\bar{z}}$ should be real. For $S = 0$ this condition is sufficient for the nonsingular fields. We call corresponding Nonlinear Hierarhy "A $2D$ Burgers Hierarhy". Konopelchenko pointed out on it in 1988. It can be linearized by the substitution $B = 1/2\Delta \log c$. The linearity in the variable c plays fundamental role in our results (see below).

The Algebro-Geometric Data for self-adjoint Operators L: a.Riemann Surface Γ of genus g with 2 selected "infinite points" ∞_1, ∞_2, local parameters k''^{-1}, k'^{-1} and Divisor $D = P_1, ..., P_g$ of degree g. They define the operator L.

b.Let an antiinvolution $\sigma : \Gamma \to \Gamma, \sigma^2 = 1$ be given such that $\sigma(D) + D \sim K + \infty_1 + \infty_2$, $\sigma^*(k') = -\bar{k}''$, $\sigma(\infty_1) = \infty_2$ Such Data (generically) define a self-adjoint operator L.

We construct a "two-point Baker-Akhiezer function" $\Psi(z, \bar{z}, k)$ meromorphic in the variable $k \in \Gamma$ outside of infinities with simple poles in the points of Divisor D and asymptotic at infinities:

$\Psi = \exp\{k'' z\}(1 + O(k_1^{-1}))$

$\Psi = c(z, \bar{z}) \exp\{k' \bar{z}\}(1+O(k_2^{-1}))$ It satisfies to the equations $L\Psi = 0$.

Here $L=\Delta+G\bar{\partial}+S, G=(\log c)_{\bar{z}}$. It is gauge equivalent to self-adjoint operator $L=QQ^{+}+V$. Our main result describes The Inverse Problem Data corresponding to The Purely Magnetic Pauli Operators: Take Riemann Surface splitted into the nonsingular pieces $\Gamma=\Gamma'\cup\Gamma''$ of genus $g'=g''$, intersecting each other in $k+1$ points $Q_0,...,Q_k$(see Fig 1).

Take infinities $\infty_1 \in \Gamma', \infty_2 \in \Gamma''$ with local parameters k'^{-1}, k''^{-1}. Let an antiinvolution $\sigma : \Gamma' \to \Gamma''$ be given such that $\sigma^*(k') = -\bar{k''}, \sigma(Q_s) = Q_{l_s}$. Take divisors D', D'' of degree $g'+k, g''$ correspondingly not crossing infinities and points Q_s. Assume that the total divisor $D = D' + D''$ satisfies to the "Degenerate Cherednik Type Equation" $D + \sigma(D) = K + \infty_1 + \infty_2$. Here $K = K' + K'' + Q_0 + ... + Q_k$ with condition on residues in the crossing points Q_s.

Such Data generate a symmetric scalar operator $c^{-1/2}Lc^{1/2} = = QQ^{+}, Q = \partial + 1/2(\log c)_z$, $c \in R$. Here $S = 0$. Therefore they generates a Purely Magnetic Pauli Operator $L_P = QQ^{+} \oplus Q^{+}Q$. Magnetic Field is real $B = -1/2\Delta \log c$. It is nonsingular if $c \neq 0$, so the operator is self-adjoint. *Otherwise we need to consider operator in the domain whose boundary consists of zeroes of c if they are nondegenerate. Here we have a ground state $c^{1/2}$ in one spin-sector only. Only this State belongs to the Hilbert Space with Dirichlet boundary condition.*

How to find ground states?

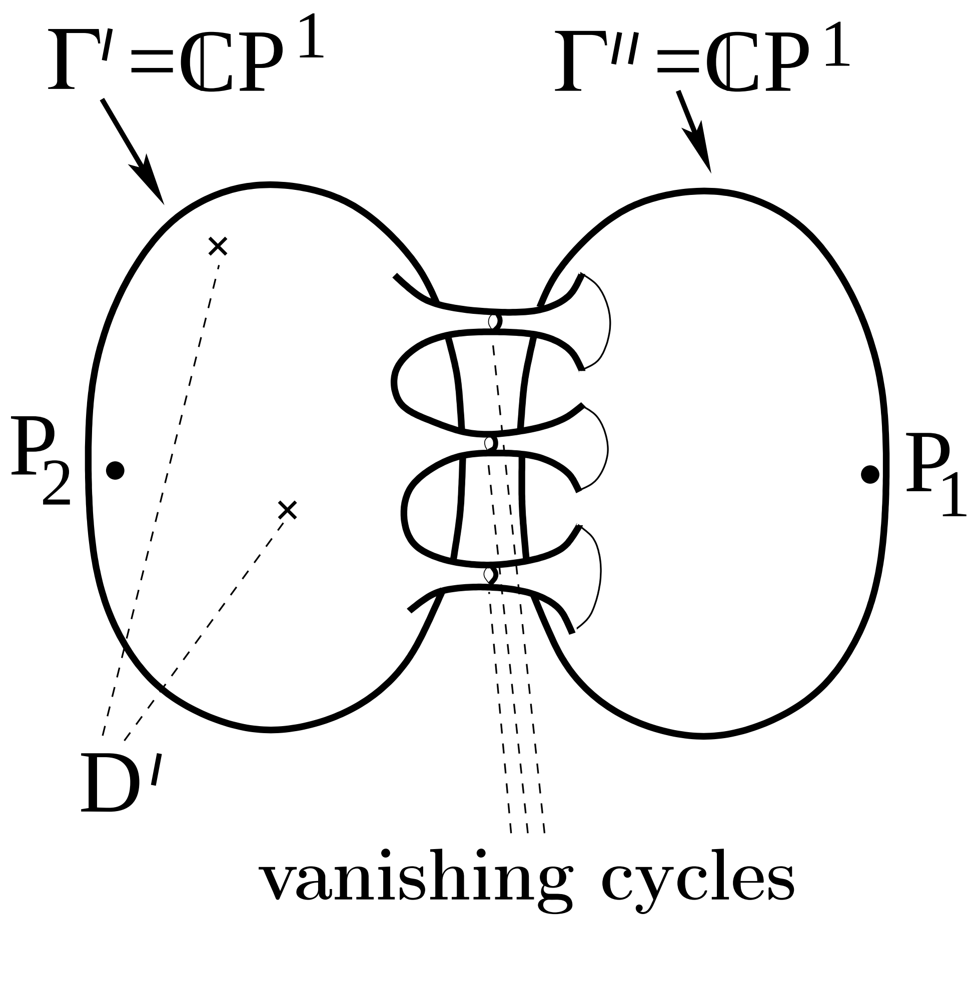
We have to take $\psi_0 = c^{1/2}$ in the first spin-sector because $Q^+\psi_0 = 0$.

We have to take $\phi_0 = c^{-1/2}$ in the second sector because $Q\phi_0 = 0$.'

In the case of periodic $c \neq 0$ we have periodic ground states $c^{\pm 1/2}$ in both sectors. They do not belong to the Hilbert Space $L_2(R^2)$, so they belong to the bottom of continuous spectrum. A full family of Ψ-functions gives a complex family of the Bloch-Floquet functions with unimodular multipliers (equal to one) in isolated points only. The Magnetic Field is Topologically Trivial (i.e. its flux through the Elementary Cell $[B]$ is equal to zero).

The Case of Genus Zero (Fig 1)

Fig 1



We take $l+1$ intersection points presented as $k'=k_s$ and $k''=p_s$ in Γ', Γ'', and divisor $D'=(a_1,...,a_l)$ of degree l in Γ'. We have $\Psi = e^{k'\bar{z}}\frac{w_0 k'^l+...+w_l}{(k'-a_1)...(k'-a_l)}, \Psi|_{k'=k_s}=e^{p_s z}$.

As we can see, $c=w_0$.

So the solution is $c=\sum_{s=0}^{l}\kappa_s e^{W_s(z,\bar{z})}$, Here W_s is a linear form. All complex coefficients are possible.

$W_s=\alpha_s x+\beta_s y, (\alpha_s,\beta_s)\in C^2_W$.

Transformation $c\to c'=ce^{\gamma+\alpha x+\beta y}$ leads to the gauge equivalent operator (with the same magnetic field)

There exist 3 types of Real Solutions:

1.Purely Exponential Positive Case $\kappa_s, (\alpha_s, \beta_s) \in R, \kappa_s > 0$. 2.Purely Trigonometric Real Case. 3. Mixed exponential/trigonometric case.

Consider the case 1. Let "the Tropical Sum" of the forms in the set $\{W\}$ is nonnegative $I'_{\{W\}}(\phi) = \max_s(\alpha_s \cos\phi + \beta_s \sin\phi) \geq 0$.

Then $c^{-1/2}$ is bounded in R^2

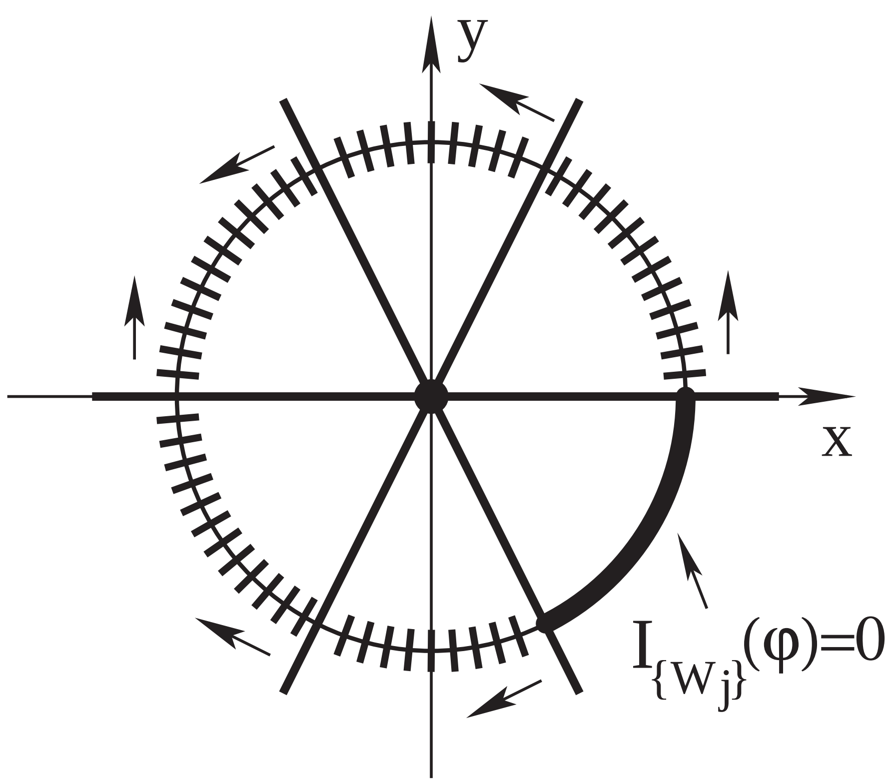
For the angles $I'_{\{W\}}(\phi) > 0$ we have a rapid decay

$c^{-1/2} \to 0, R \to \infty$,

Let $I(\phi) = \max\{I'(\phi), 0\}$

Fig 2a

$$c = e^{y} + e^{y-2x} + e^{-y-2x}$$

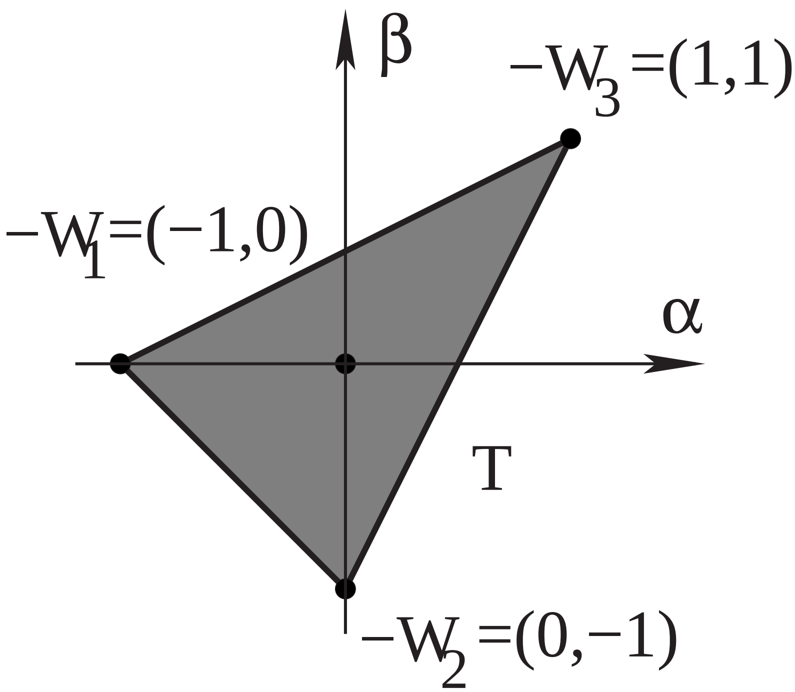


In every class $c' \in ce^W, W' \in R^2_W$, the set of representatives c' with nonnegative $I = I'_{\{W'\}}(\phi) \geq 0$ forms a convex polytop $\bar{T}_c$. Its inner part $T_c \subset \bar{T}_c$ consists of all c' such that $I_{\{W'\}} > 0$. Open part T_c is always nonempty for $l > 2$. $\bar{T}_c$ is nonempty for $l > 1$. (see Fig 2b for $l = 3$)

Fig 2b

b) $\mathbb{R}^2 = (\alpha, \beta)$

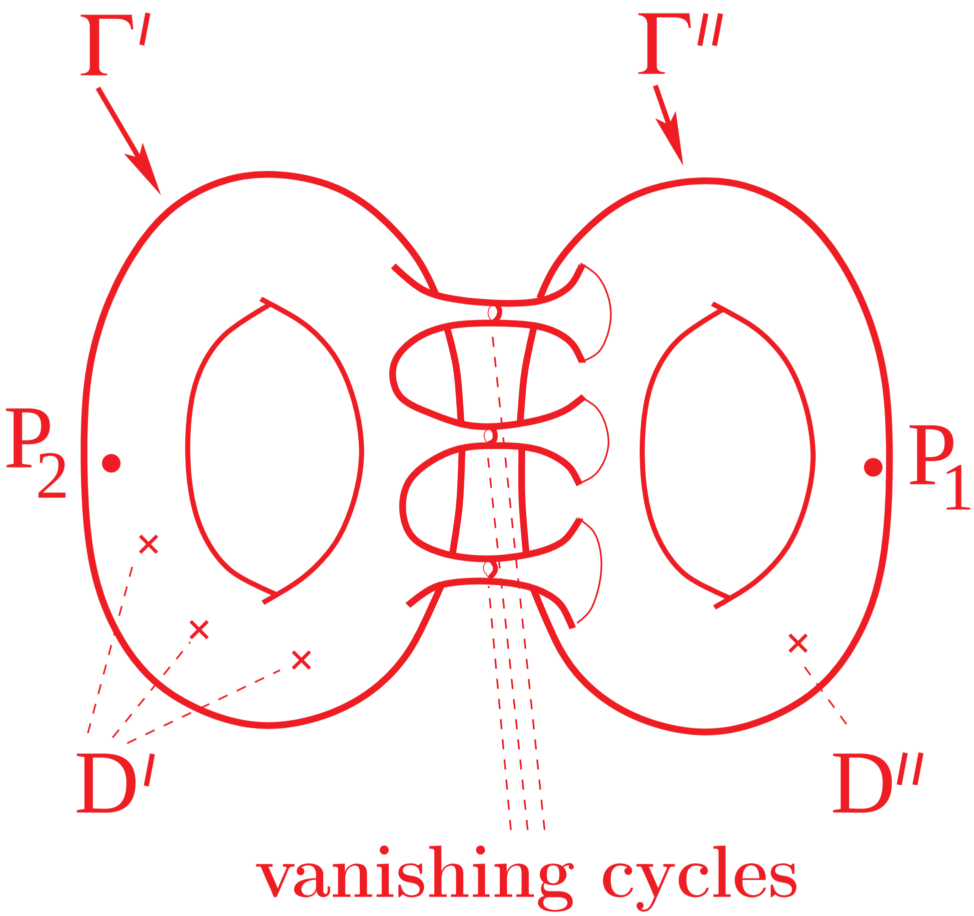
$W_j = \alpha_j x + \beta_j y$



Magnetic field is decaying for $R \to \infty$ except some selected angles, it is a Lump Type Field analogous to the KP "Lump Potentials". A linear sum under the $1/2\Delta \log()$ reflects linearization of the Burgers Hierarchy in the variable c. $[B] = \int\int_{D^2_R} B dx dy =$
$= -1/2 R \oint_{S^1} I_{\{W\}}(\phi) d\phi + O(R^{-1})$
All points in T_c define ground states in the Hilbert Space $L_2(R^2)$. The boundary points define the bottom of continuous spectrum.

The case of genus 1.

Fig 3



Take pair of "real" elliptic curves $\Gamma' = \Gamma'' = C/\Lambda$ with euclidean local parameters k, p and periods $2\omega \in R, 2i\omega' \in iR$ We have $\eta = \zeta(\omega) \in R$ and $i\eta' = \zeta(i\omega') \in iR$. The point 0 presents "infinities" for both of them. They have $n+1$ intersection points $Q_0, Q_1, ..., Q_n \in \Gamma'$ and $R_0, ..., R_n \in \Gamma''$. The divisors $D' = (P_1, ..., P_{n+1})$ and $D'' = P$ have degrees $n+1, 1$ correspondingly.

We have $\psi' = e^{-\bar{z}\zeta(k)} \times \frac{\prod_s \sigma(k-Q_s)}{\prod_l \sigma(k+P_l)} \times$ $\times (\sum_j w_j \frac{\sigma(k+\bar{z}+\tilde{P}+\tilde{Q}-Q_j)}{\sigma(k-Q_j)})$. Here $\tilde{P} = P_1 + ... + P_{n+1}, \tilde{Q} = Q_0 + ... + Q_n$, sum as in C

$\psi'' = e^{-z\zeta(p)} \sigma(p+z+P)/(\sigma(z+P)\sigma(p+P))$, $\psi'(Q_s) = \psi''(R_s)$.

All singularity of the quantity c disappear after multiplication $c \to \tilde{c}$: $\tilde{c} = c\sigma(\bar{z}+\tilde{Q}+\tilde{P})\sigma(z+P)$

Easy to prove that every solution can be presented in the form $\sum_q \alpha_q \exp\{-z\zeta(R_q) + \bar{z}\zeta(Q_q)\}\times$ $\times\sigma(z + R_q)\sigma(\bar{z} - Q_q)$. Real solutions should look like sum of individual terms with either $\alpha_q \in R, R_k = -\bar{Q}_k$ or pairs $q = (j, l)$ where $\alpha_j = \bar{\alpha}_l, R_j = -\bar{Q}_l, R_l = -\bar{Q}_j$

For the nonzero $\tilde{c} \neq 0$ magnetic field is nonsingular. Simplest real periodic nonsingular solution we obtain for $n = 2$ (3 intersection points): $\alpha \exp\{z\eta + \bar{z}\bar{\eta}\}|\sigma(z-\omega)|^2 + [\beta \exp\{\bar{z}\zeta(Q_1) - z\zeta(R_1)\}\sigma(z-Q_1)\sigma(\bar{z} + R_1) + CC]$ Here CC means complex conjugate.

With α, β and lattice fixed, we can choose a countable number of $R_2 = -R_1 = -\bar{Q}_1 = \bar{O}_2, R_0 = -\omega = -Q_0$ such that magnetic field is nonsingular and periodic with same periods as the lattice. We need to solve equations $U - \bar{U} = -i\pi n$,

$V + \bar{V} = -\pi m, \; m, n \in Z$ and $U = \lambda\eta - \omega\zeta(\lambda), V = \lambda\eta' - \omega'\zeta(\lambda)$ Sum of such (distinct) expressions $\tilde{c} = \sum_s \tilde{c}_s$ also leads to the

Algebro-Geometric Periodic Magnetic Fields with flux equal to one quantum unit. We expect that infinite sums of that kind can present every analytic (smooth) magnetic field which is periodic with flux equal to one quantum unit.

So our Conclusion is:

The magnetic field $\tilde{B} = -1/2\Delta\tilde{c}$ is periodic, nonsingular with magnetic flux equal to ONE QUANTUM UNIT. The magnetic field $B = -1/2\Delta c$ has magnetic flux equal to zero through the elementary cell and δ-singularity in the point P. We have the "Bohm-Aharonov" situation.

So extracting the BA δ-shape singularity from the field $\tilde{B}$ we obtain the Algebro-Geometric Topologically Trivial Operators corresponding to data described above.

For $g > 1$ we should take data corresponding to Elliptic Solutions of the KP Equation (studied in particular by Krichever since 1979) in order to get double periodic real nonsingular purely magnetic Pauli operators similar to those obtained here but with magnetic flux equal to some number of Quantum Units. This number is equal to the intersection number of algebraic curve Γ' with the Θ-divisor in the Jacobian Variety $J(\Gamma')$.

The basic elliptic curve Γ_0 responsible for the field of functions is defined by the z-direction in the Jacobian Variety $J(\Gamma')$ if it is compact. In this case Γ' can be realized as a ramified covering over the elliptic curve Γ_0. The number of sheets is equal to the number of quantum units. A number g is the genus of Γ' and Γ''.