On Kripke's theory of truth — II

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27 May 2022

Kleene's $\mathcal{O}$

Remember that Kleene's system of notation for C-Ord consists of:

- a special partial function $\nu_{\mathcal{O}}$ from $\mathbb{N}$ onto C-Ord;
- an appropriate ordering relation $<_{\mathcal{O}}$ on $\mathrm{dom}\,(\nu_{\mathcal{O}})$ — which mimics the usual ordering relation on C-Ord.

Call $n \in \mathbb{N}$ a notation for $\alpha \in$ C-Ord iff $\nu_{\mathcal{O}}\,(n) = \alpha$. To simplify the statements I often write $n \in \mathcal{O}$ instead of $n \in \mathrm{dom}\,(\nu_{\mathcal{O}})$.

Folklore

$\mathrm{dom}\,(\nu_{\mathcal{O}})$ *is* Π^1_1*-complete.*

Fix one's favorite universal partial computable (two-place) function U.

Folklore

There exists a computable function f such that for every $n \in \mathcal{O}$,

$$\{k \in \mathbb{N} \mid k <_{\mathcal{O}} n\} \; = \; \mathrm{dom}\left(U_{f(n)}\right).$$

Folklore (Effective Transfinite Recursion)

Suppose f is a computable function such that for any $e \in \mathbb{N}$ and $n \in \mathcal{O}$,

$$\{k \in \mathbb{N} \mid k <_{\mathcal{O}} n\} \; \subseteq \; \mathrm{dom}\left(U_e\right) \quad \Longrightarrow \quad n \; \in \; \mathrm{dom}\left(U_{f(e)}\right).$$

Then there is a $c \in \mathbb{N}$ for which $U_{f(c)} = U_c$, and $\mathrm{dom}\left(\nu_{\mathcal{O}}\right) \subseteq \mathrm{dom}\left(U_c\right)$.

About Least Fixed-Points

Let us call a valuation scheme V ordinary iff for any $\alpha \in \mathrm{Ord}$, $\chi \in \mathit{Sen}$, $\psi \in \mathit{Sen}_T$ and $\varphi(x) \in \mathit{For}_T$ the following conditions hold:

1. $\mathrm{T}_V^{\alpha} \subseteq \mathrm{T}_V^{\alpha+1}$;
2. $\chi \in \mathrm{T}_V^{\alpha}$ iff $\alpha \neq 0$ and $\mathfrak{N} \models \chi$;
3. $\psi \in \mathrm{T}_V^{\alpha}$ iff $T(\ulcorner \psi \urcorner) \in \mathrm{T}_V^{\alpha+1}$;
4. $\forall x\, \varphi(x) \in \mathrm{T}_V^{\alpha+1}$ iff $\{\varphi(\underline{n}) \mid n \in \mathbb{N}\} \subseteq \mathrm{T}_V^{\alpha+1}$;
5. $\chi \wedge \psi \in \mathrm{T}_V^{\alpha}$ iff $\mathfrak{N} \models \chi$ and $\psi \in \mathrm{T}_V^{\alpha}$;
6. if $\chi \vee \psi \in \mathrm{T}_V^{\alpha}$ and $\mathfrak{N} \models \neg\chi$, then $\psi \in \mathrm{T}_V^{\alpha}$;
7. if $\mathfrak{N} \models \chi$ and $\alpha \neq 0$, then $\chi \vee \psi \in \mathrm{T}_V^{\alpha}$.

In effect, except for V_{wK}, all the schemes considered above are ordinary.

Given V, by the rank of $\psi \in Sen_T$ — denoted by $rank_V(\psi)$ — I mean the least ordinal α for which $\psi \in \mathrm{T}_V^{\alpha+1}$.

Proposition

Let V be a valuation scheme satisfying (3–4). Then for every $\psi \in Sen_T$ and every $\varphi(x) \in For_T$,

$$\begin{aligned} rank_V(T(\ulcorner\psi\urcorner)) &= rank_V(\psi) + 1 \quad \textit{and} \\ rank_V(\forall x\, \varphi(x)) &= \sup\{rank_V(\varphi(\underline{n})) \mid n \in \mathbb{N}\}. \end{aligned}$$

Proposition ⋆

For each ordinary scheme V there exists a computable function ρ_V such that for every $n \in \mathcal{O}$, $rank_V(\rho_V(n)) = \nu_{\mathcal{O}}(n) + 1$.

Corollary ∗

Each ordinary scheme V has the following property: for every ordinal α, if $\mathrm{T}_V^\alpha = \mathrm{T}_V^{\alpha+1}$, then $\alpha \geqslant \omega_1^{\mathrm{CK}}$ and T_V^α is Π_1^1-hard.

The technique used in the proofs of these facts can be applied in various other situations as well. Let us see how it works e.g. for V_{wK}. Still, as it was shown by Cain and Damnjanovic, one should be warned:

> *Actually certain complexity results for the weak Kleene scheme depend on the Gödel numbering and the language of the "standard" model of arithmetic we choose.*

I am aiming at a deeper understanding of this intensionality phenomenon.

In their article from 1991, Cain and Damnjanovic suggested expanding σ to avoid the conflict. More precisely, assuming an appropriate coding $\mathrm{M}_0, \mathrm{M}_1, \ldots$ of all Turing machines, they added a new function symbol π of arity 4, whose interpretation is given by

$$\pi(e, i, j, k) := \begin{cases} n & \text{if } \mathrm{M}_e \text{ halts on input } i \text{ at step } j \text{ with output } n, \\ k & \text{if } \mathrm{M}_e \text{ does not halt on input } i \text{ at step } j. \end{cases}$$

Clearly this function is primitive recursive. *So what can we do with* π*?*

Observation ♯

If we include π *in* σ*, then both Proposition* $\star$ *and Corollary* $*$ *generalise to arbitrary valuation schemes satisfying (1–5).*

As an alternative to Cain–Damnjanovic' suggestion, I propose to add a symbol $\dot{-}$ for cut-off subtraction, i.e. $i \dot{-} j := \max\{0, i - j\}$:

Observation ♡

Similar to Observation ♯, but with $\dot{-}$ instead of π.

Another modification, with $\mathcal{L}$ unchanged, deals with the following condition (for any $\alpha \in \mathrm{Ord}$ and $\theta(x) \in For$):

8. $\exists x (\theta(x) \wedge T(x)) \in \mathrm{T}^{\alpha}_{V}$ iff $\mathfrak{N} \models \theta(\underline{n})$ and $T(\underline{n}) \in \mathrm{T}^{\alpha}_{V}$ for some $n \in \mathbb{N}$.

Observation ♠

The analogues of Proposition $\star$ and Corollary $*$ hold for all valuation schemes satisfying (1–5) and (8).

In fact, although (8) fails for the weak Kleene scheme, the customary treatment of $\exists$ in the case of V_{wK} does not seem to be well motivated. Alternatively, we can define V^*_{wK} exactly as V_{wK} except that

$$V^*_{\mathrm{wK}}(S)(\exists x\,\varphi(x)) \;:=\; \begin{cases} 1 & \text{if } V^*_{\mathrm{wK}}(S)(\varphi(t)) = 1 \text{ for some closed } \mathcal{L}\text{-term } t, \\ 0 & \text{if } V^*_{\mathrm{wK}}(S)(\varphi(t)) = 0 \text{ for all closed } \mathcal{L}\text{-terms } t, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

(treating $\exists$ like in the strong Kleene scheme V_{sK}). Then V^*_{wK} satisfies (1–5) and (8), so Observation ♠ applies.

Earlier we took $\rightarrow$ as an abbreviation. However, interpreting $\varphi \rightarrow \psi$ as $\neg\varphi \vee \psi$ is not always the right choice. To avoid confusion, I add a new connective symbol $\twoheadrightarrow$ to the original three (viz. $\neg$, $\wedge$ and $\vee$). Of course *For*, *For*$_T$, *Sen* and *Sen*$_T$ are easily modified to accommodate $\twoheadrightarrow$. Now consider the following variation on (6–7):

- 6' if $\chi \twoheadrightarrow \psi \in \mathrm{T}^{\alpha}_{V}$ and $\mathfrak{N} \models \chi$, then $\psi \in \mathrm{T}^{\alpha}_{V}$;
- 7' if $\mathfrak{N} \models \neg\chi$, then $\chi \twoheadrightarrow \psi \in \mathrm{T}^{\alpha}_{V}$

— where χ and ψ range over the modified versions of *Sen* and *Sen*$_T$ resp. Evidently, even when we treat $\twoheadrightarrow$ as the material conditional on $\{0, 1\}$, the meanings of $\varphi \twoheadrightarrow \psi$ and $\neg\varphi \vee \psi$ may differ on $\{0, \frac{1}{2}, 1\}$.

Observation ♢

If we expand $\mathfrak{L}$ and $\mathfrak{L}_T$ by adding $\twoheadrightarrow$, then the analogues of Proposition $\star$ and Corollary $*$ hold for all val. schemes satisfying (1–5) and (6'–7').

This is closely related to a three-valued scheme from Feferman's article 'Axioms for determinateness and truth'. It can be obtained by extending V_{wK} to formulas containing $\twoheadrightarrow$ by setting

$$V'_{\mathrm{wK}}(S)(\varphi \twoheadrightarrow \psi) := \begin{cases} 1 & \text{if } V'_{\mathrm{wK}}(S)(\varphi) = 0 \text{ or } V'_{\mathrm{wK}}(S)(\psi) = V'_{\mathrm{wK}}(S)(\varphi) = 1, \\ 0 & \text{if } V'_{\mathrm{wK}}(S)(\varphi) = 1 \text{ and } V'_{\mathrm{wK}}(S)(\psi) = 0, \\ \frac{1}{2} & \text{otherwise;} \end{cases}$$

(the other clauses are the same as in the definition of V_{wK}). Now V'_{wK} satisfies (1–5) and (6'–7'), so Observation ♢ applies.

Note that every $\mathcal{L}_T$-formula can be viewed as an arithmetical monadic second-order $\sigma_{\mathbb{N}}$-formula whose only set variable is T, and vice versa. Given an $\mathcal{L}_T$-sentence ψ and an $\mathcal{L}$-formula $\chi(x)$, we construct

$$\psi_\chi \ := \ \text{the result of replacing each } T(t) \text{ in } \psi \text{ by } \chi(t) \wedge T(t).$$

Observation ♣

Let $\chi(x)$ be an $\mathcal{L}$-formula defining an infinite computable subset of $\mathbb{N}$ in $\mathfrak{N}$. Then $\{\psi_\chi \mid \psi \in Sen_T \text{ and } \mathfrak{N} \models \forall T\, \psi_\chi(T)\}$ is Π^1_1-complete.

It gives an alternative and probably the shortest proof for the following.

Theorem (Welch, Hjorth, Meadows)

For any $V \in \{V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}, V_{\mathrm{L}}\}$ and $\alpha \in \mathrm{Ord}^+$, T^α_V is Π^1_1-hard.

Proof.

Assume $V = V_{\mathrm{L}}$. Take A to be $\# \{\mu, T(\ulcorner \mu \urcorner), T(\ulcorner T(\ulcorner \mu \urcorner) \urcorner), \dots\}$ with μ denoting some fixed 'truthteller', and let χ be an $\mathcal{L}$-formula defining A in $\mathfrak{N}$. Since $A \cap \mathrm{G} = \varnothing$, we obtain

$$
\begin{aligned}
\#\psi_\chi \in \mathrm{T}_V^{\beta+1} &\iff \#\psi_\chi \in \mathrm{G}_{\beta+1} \text{ and } \langle \mathfrak{N}, \mathrm{T}_V^\beta \rangle \models \psi_\chi \\
&\iff \mathfrak{N} \models \forall T\, (\psi_\chi\, (T \cap \mathrm{G}_\beta) \leftrightarrow \psi_\chi\, (T)) \wedge \psi_\chi(\mathrm{T}_V^\beta) \\
&\iff \mathfrak{N} \models \forall T\, (\psi_\chi\, (\varnothing) \leftrightarrow \psi_\chi\, (T)) \wedge \psi_\chi(\varnothing) \\
&\iff \mathfrak{N} \models \forall T\, \psi_\chi\, (T).
\end{aligned}
$$

Clearly $\mathrm{T}_V^\alpha = \bigcup_{\beta<\alpha} \mathrm{T}_V^{\beta+1}$, so the Π_1^1-hardness of T_V^α follows by Observation ♣. Perfectly analogous arguments apply to the other schemes. □

- J. P. Burgess (1986). The truth is never simple. *Journal of Symbolic Logic* 51:4, 663–681.
- J. Cain and Z. Damnjanovic (1991). On the weak Kleene scheme in Kripke's theory of truth. *Journal of Symbolic Logic* 56:4, 1452–1468.
- S. Feferman (2008). Axioms for determinateness and truth. *Review of Symbolic Logic* 1:2, 204–217.
- M. Fischer, V. Halbach, J. Kriener and J. Stern (2015). Axiomatizing semantic theories of truth? *Review of Symbolic Logic* 8:2, 257–278.
- S. Kripke (1975). Outline of a theory of truth. *The Journal of Philosophy* 72:19, 690–716.

- H. Leitgeb (2005). What truth depends on. *Journal of Philosophical Logic* 34:2, 155–192.
- T. Schindler (2015). *Type-Free Truth* (Ph.D. Thesis). Ludwig-Maximilians-Universität München.
- S. O. Speranski (2017). Notes on the computational aspects of Kripke's theory of truth. *Studia Logica* 105:2, 407–429. ✓
- P. D. Welch (2014). The complexity of the dependence operator. *Journal of Philosophical Logic* 44:3, 337–340.