On the quantified version of the Belnap–Dunn modal logic

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Abstract

We develop a quantified version of the propositional modal logic BK from an article by S. P. Odintsov and H. Wansing, which is based on the (non-modal) Belnap–Dunn system; denote this version by QBK. First, by using the canonical model method we shall prove that QBK, as well as some important extensions of it, is strongly complete with respect to a suitable possible world semantics. Then we shall define translations (in the spirit of Gödel–McKinsey–Tarski) that faithfully embed the quantified versions of Nelson's constructive logics into suitable extensions of QBK. In conclusion, we shall discuss interpolation properties for QBK-extensions.

Keywords: modal logic, constructive logic, strong negation, possible world semantics, quantification

Contents

1 Introduction

Quantified intuitionistic logic, QInt, plays a key role in constructive mathematics. Among its interpretations, a special place is occupied by *Kleene's realizability semantics* and the informal *Brouwer–Heyting–Kolmogorov interpretation*; see [8] and, say, [20, Chapter 1]. However, QInt has a certain drawback: although from each derivation of Φ in intuitionistic number theory one can extract a way of verifying Φ, a derivation of $\neg\Phi$ does not give a direct way of falsifying Φ, but only reduces the assumption that Φ is verifiable to absurdity; see [10] and [11].[1] In particular, this implies the failure of the 'negative disjunction property': from the derivability of $\neg(\Phi \wedge \Psi)$ we cannot, in general, obtain the derivability of $\neg\Phi$ or $\neg\Psi$.

In order to eliminate the drawback mentioned above, D. Nelson proposed to enrich the language of QInt by adding a 'strong negation', $\sim$, which is directly responsible for falsification, and expand the realizability semantics to the new language; see [11], and also [1]. This is how the logic $\mathsf{QN3}$ arose. Then its useful generalization $\mathsf{QN4}$ was described, which allows one to deal with inconsistent data; see [2]. Here $\mathsf{QN3}$ extends $\mathsf{QN4}$ by adding the scheme

$$\sim\Phi \to (\Phi \to \Psi).$$

In what follows, for every quantified logic $\mathsf{Q}L$ we shall denote its propositional version by L.[2] Notice that if we exclude intuitionistic implication, $\mathsf{N3}$ turns into Kleene's strong three-valued logic, which has an algorithmic interpretation, and $\mathsf{N4}$ turns into the well-known Belnap–Dunn four-valued logic; see [9, § 64] and [3, 4].

A very important role in understanding QInt is played by the Gödel–McKinsey–Tarski translation from QInt into the modal logic $\mathsf{QS4}$, i.e. the reflexive-transitive extension of the modal logic QK. We would like to have a similar understanding of the logics $\mathsf{QN3}$ and $\mathsf{QN4}$. In the propositional case, this problem was solved by S. P. Odintsov and H. Wansing in [18]:

- they defined the *propositional Belnap–Dunn modal logic*, which enriches K and is denoted by BK, and showed the strong completeness of BK and some of its extensions with respect to a suitable possible worlds semantics;
- by enriching the propositional version of the Gödel–McKinsey–Tarski translation they showed that $\mathsf{N3}$ and $\mathsf{N4}$ are faithfully embedded into suitable BK-extensions.[3]

However, so far nothing has been known about the situation in the quantifier case, despite the fact that constructive theories are formulated exactly in a quantified language. Our goal is to develop a quantified version of BK, prove the strong completeness theorems for it and some of its extensions with respect to a suitable possible worlds semantics, and also generalize the result about faithful embeddings to $\mathsf{QN3}$ and $\mathsf{QN4}$. We shall consider semantics with expanding domains as well as with constant domains. Furthermore, we shall discuss interpolation properties for QBK-extensions.

[1] Here and elsewhere $\neg\Phi$ is an abbreviation for $\Phi \to \bot$.

[2] We shall assume that the language of $\mathsf{N4}$ includes that of Int, and therefore contains $\bot$; a lot of information about the lattice of extensions of $\mathsf{N4}$ is contained in [13].

[3] The lattice of extensions of BK has been actively studied in [14, 15, 16].

2 Syntax

For simplicity we shall restrict ourselves to signatures without equality and without function symbols.[4] Let σ be a signature. Denote by Pred_σ and Const_σ the sets of all its predicate and constant symbols respectively. Here and elsewhere we assume that $\mathrm{Pred}_\sigma \neq \varnothing$.

Fix a countable set Var, whose elements will be called *variables*. Denote by Term_σ the set of all σ-terms. Thus $\mathrm{Term}_\sigma = \mathrm{Var} \cup \mathrm{Const}_\sigma$. Our logical vocabulary will consist of:

- the connective symbols $\wedge$, $\vee$, $\to$, $\perp$, $\sim$, $\Box$ and $\Diamond$;
- the quantifier symbols $\forall$ and $\exists$.

Denote by Form_σ the set of all σ-formulas. For each $\Phi \in \mathrm{Form}_\sigma$, take

$$\mathrm{FV}(\Phi) \;:=\; \{x \in \mathrm{Var} \mid x \text{ is free in } \Phi\}.$$

By σ-*substitutions* we mean functions from Var to Term_σ. If $\mathrm{FV}(\Phi) \subseteq \{x_1, \ldots, x_n\}$, then for each σ-substitution λ we denote by $\lambda\Phi$ the result of (simultaneously) replacing all free occurrences of x_1, …, x_n in Φ by $\lambda(x_1)$, …, $\lambda(x_n)$ respectively. In the case when

$$\lambda \;=\; \{(x, t)\} \cup \{(y, y) \mid y \in \mathrm{Var} \text{ and } y \neq x\},$$

where $x \in \mathrm{Var}$ and $t \in \mathrm{Term}_\sigma$, we shall often write $\Phi(x/t)$ instead of $\lambda\Phi$. Finally, a σ-substitution λ is called *ground* if $\lambda(x) \in \mathrm{Const}_\sigma$ for all $x \in \mathrm{Var}$.

For convenience we introduce the following abbreviations:

Abbreviation	**Definition**	**Name**
$\neg\Phi$	$\Phi \to \perp$	weak negation
$\Phi \leftrightarrow \Psi$	$(\Phi \to \Psi) \wedge (\Psi \to \Phi)$	weak equivalence
$\Phi \Leftrightarrow \Psi$	$(\Phi \leftrightarrow \Psi) \wedge ({\sim}\Phi \leftrightarrow {\sim}\Psi)$	strong equivalence

Denote by Sent_σ the set of all σ-sentences, i.e. σ-formulas without free variables. Arbitrary subsets of Sent_σ will be called σ-*theories*.

As usual, by σ-*structures* we mean non-empty sets augmented with interpretations for the symbols of σ over them. Let $\mathfrak{A}$ be an arbitrary σ-structure, A be its domain. For each $\varepsilon \in \sigma$, define

$$\varepsilon^{\mathfrak{A}} \;:=\; \text{the interpretation of } \varepsilon \text{ in } \mathfrak{A}.$$

If we expand σ to the signature

$$\sigma_A \;:=\; \sigma \cup \{\underline{a} \mid a \in A\},$$

where $\underline{a}$ are new constant symbols, then we can pass from $\mathfrak{A}$ to its σ_A-expansion $\mathfrak{A}^*$ such that

$$\underline{a}^{\mathfrak{A}^*} \;:=\; a \quad \text{for every } \; a \in A.$$

In this case σ_A-formulas will be also called A-*formulas*, and σ_A-substitutions will be called A-*substitutions*. If Φ is a A-sentence, we shall often write $\mathfrak{A} \Vdash \Phi$ instead of $\mathfrak{A}^* \Vdash \Phi$.

[4] This is due to the fact that in the context of expanding domain semantics some well-known problems related to equality and function symbols arise; see discussion in [6].

3 Hilbert-style calculus

Our calculus for a quantified version of BK extends the deductive system for BK from [18]. It employs the following axiom schemes.

- Axioms for classical logic in the language $\{\wedge, \vee, \rightarrow, \bot\}$:

  I1. $\Phi \rightarrow (\Psi \rightarrow \Phi)$;

  I2. $(\Phi \rightarrow (\Psi \rightarrow \Theta)) \rightarrow ((\Phi \rightarrow \Psi) \rightarrow (\Phi \rightarrow \Theta))$;

  C1. $\Phi \wedge \Psi \rightarrow \Phi$;

  C2. $\Phi \wedge \Psi \rightarrow \Psi$;

  C3. $\Phi \rightarrow (\Psi \rightarrow \Phi \wedge \Psi)$;

  D1. $\Phi \rightarrow \Phi \vee \Psi$;

  D2. $\Psi \rightarrow \Phi \vee \Psi$;

  D3. $(\Phi \rightarrow \Theta) \rightarrow ((\Psi \rightarrow \Theta) \rightarrow (\Phi \vee \Psi \rightarrow \Theta))$;

  N1. $\bot \rightarrow \Phi$;

  N2. $\Phi \vee (\Phi \rightarrow \bot)$;

  Q1. $\forall x\, \Phi \rightarrow \Phi\,(x/t)$, where t is free for x in Φ;

  Q2. $\Phi\,(x/t) \rightarrow \exists x\, \Phi$, where t is free for x in Φ.

- Axioms for strong negation:

  SN1. $\sim\sim \Phi \leftrightarrow \Phi$;

  SN2. $\sim(\Phi \wedge \Psi) \leftrightarrow (\sim\Phi \vee \sim\Psi)$;

  SN3. $\sim(\Phi \vee \Psi) \leftrightarrow (\sim\Phi \wedge \sim\Psi)$;

  SN4. $\sim(\Phi \rightarrow \Psi) \leftrightarrow (\Phi \wedge \sim\Psi)$;

  SN5. $\sim\bot$;

  SN6. $\sim\forall x\, \Phi \leftrightarrow \exists x \sim \Phi$;

  SN7. $\sim\exists x\, \Phi \leftrightarrow \forall x \sim \Phi$.

- Axioms for $\Box$:

  $\Box$1. $(\Box\Phi \wedge \Box\Psi) \rightarrow \Box\,(\Phi \wedge \Psi)$;

  $\Box$2. $\Box\,(\Phi \rightarrow \Phi)$.

- Modal interaction axioms:

  M1. $\neg\Box\Phi \leftrightarrow \Diamond\neg\Phi$;

  M2. $\neg\Diamond\Phi \leftrightarrow \Box\neg\Phi$;

  M3. $\Box\Phi \Leftrightarrow \sim\Diamond\sim\Phi$;

  M4. $\Diamond\Phi \Leftrightarrow \sim\Box\sim\Phi$.

It also employs the following inference rules.

- The *modus ponens* rule, i.e.

$$\frac{\Phi \qquad \Phi \to \Psi}{\Psi}\ (\mathtt{MP}).$$

- The monotonicity rules for $\Box$ and $\Diamond$:

$$\frac{\Phi \to \Psi}{\Box\Phi \to \Box\Psi}\ (\mathtt{MB}) \qquad \text{and} \qquad \frac{\Phi \to \Psi}{\Diamond\Phi \to \Diamond\Psi}\ (\mathtt{MD}).$$

- The Bernays rules for $\forall$ and $\exists$:

$$\frac{\Phi \to \Psi}{\Phi \to \forall x\, \Psi}\ (\mathtt{BR1}) \qquad \text{and} \qquad \frac{\Psi \to \Phi}{\exists x\, \Psi \to \Phi}\ (\mathtt{BR2}),$$

  where x does not occur free in Φ.

Denote by QBK_σ the least set of σ-formulas containing all axioms of our calculus (in the signature σ) and closed under all its inference rules. In what follows, when the choice of σ is not of significant importance or the whole logic is meant (with no reference to a specific signature), the lower index $\cdot_\sigma$ will often be dropped.

Proposition 3.1. QBK *includes the Kripke scheme*

$\mathtt{K}.\ \Box(\Phi \to \Psi) \to (\Box\Phi \to \Box\Psi).$

Furthermore, QBK *is closed under the normalization rule*

$$\frac{\Phi}{\Box\Phi}\ (\mathtt{RN}).$$

Proof. The Kripke scheme is obtained in a standard way:

1	$(\Phi \to \Psi) \wedge \Phi \to \Psi$	classical logic
2	$\Box((\Phi \to \Psi) \wedge \Phi) \to \Box\Psi$	from 1 by $\mathtt{MB}$
3	$\Box(\Phi \to \Psi) \wedge \Box\Phi \to \Box((\Phi \to \Psi) \wedge \Phi)$	$\Box 1$
4	$\Box(\Phi \to \Psi) \wedge \Box\Phi \to \Box\Psi$	from 3, 2
5	$\Box(\Phi \to \Psi) \to (\Box\Phi \to \Box\Psi)$	from 4.

Now let us check the closedness of QBK under the normalization rule. Suppose that Φ belongs to QBK. Then $\Box\Phi$ will also belong to QBK:

1	Φ	by hypothesis
2	$(\Phi \to \Phi) \to \Phi$	from 1
3	$\Box(\Phi \to \Phi) \to \Box\Phi$	from 2 by $\mathtt{MB}$
4	$\Box(\Phi \to \Phi)$	$\Box 2$
5	$\Box\Phi$	from 4, 3.

□

For each $\Gamma \subseteq \mathrm{Form}_\sigma$, define

$$\mathrm{Disj}\,(\Gamma) \ := \ \{\Phi_1 \vee \cdots \vee \Phi_n \mid n \in \mathbb{N} \text{ and } \{\Phi_1, \ldots, \Phi_n\} \subseteq \Gamma\}.$$

Here the empty disjunction is identified with $\bot$. Given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$, we write $\Gamma \vdash \Delta$ if and only if some element of $\mathrm{Disj}\,(\Delta)$ can be obtained from elements of $\Gamma \cup \mathsf{QBK}_\sigma$ by means of `MP`, `BR1` and `BR2`.[5] Since the modal rules (`MB` and `MD`) are not used here, the derivability relation $\vdash$ has a local character in the sense that it is intended to preserve truth in specific worlds (see the definition of the semantical consequence relation in § 4); at the same time, `MB` and `MD` will apply globally, i.e. preserve truth in models, but not in specific worlds of these models. Of course, when $\Delta = \{\Phi\}$, we usually write $\Gamma \vdash \Phi$ instead of $\Gamma \vdash \{\Phi\}$.

Theorem 3.2 ((on deduction)). *For any* $\Gamma \cup \{\Phi\} \subseteq \mathrm{Sent}_\sigma$ *and* $\Psi \in \mathrm{Form}_\sigma$*,*

$$\Gamma \cup \{\Phi\} \vdash \Psi \quad \Longleftrightarrow \quad \Gamma \vdash \Phi \to \Psi.$$ [6]

The proof is similar to the case of classical first-order logic.

Like other logics with strong negation, BK is not closed under the usual replacement rule; see [18]. Of course, the same will hold for QBK, but to formally justify this fact, we need a suitable possible world semantics, which will appear only in Section 4. Nevertheless, the ‘positive’ and ‘weak’ replacement rules from [18] can be generalized, without much effort, to the quantifier case.

Theorem 3.3 ((positive replacement rule)). *Let* $\{\Phi, \Psi, \Theta\} \subseteq \mathrm{Form}_\sigma$. *Suppose that* Φ' *is obtained from* Φ *by replacing some occurrences of* Ψ *by* Θ*, and none of these occurrences are in the scope of* $\sim$. *Then* $\vdash \Psi \leftrightarrow \Theta$ *implies* $\vdash \Phi \leftrightarrow \Phi'$.

Proof. We shall restrict ourselves to the cases when Φ begins with $\forall$ or $\exists$, because the remaining cases have actually been considered by S. P. Odintsov and H. Wansing. Furthermore, we may assume that $\Psi \neq \Phi$, since otherwise the statement is trivial.

Suppose $\Phi = \forall x\, \Omega$. Clearly, the part Ω corresponds to Ω' such that $\Phi' = \forall x\, \Omega'$. Then $\Phi \leftrightarrow \Phi'$ is derived in a standard way:

1	$\Omega \leftrightarrow \Omega'$	inductive hypothesis
2	$\forall x\, \Omega \to \Omega$	`Q1`
3	$\forall x\, \Omega \to \Omega'$	from 2, 1
4	$\forall x\, \Omega \to \forall x\, \Omega'$	from 3 by `BR1`
5	$\forall x\, \Omega' \to \forall x\, \Omega$	by analogy with 4
6	$\forall x\, \Omega \leftrightarrow \forall x\, \Omega'$	from 4, 5.

Similarly for $\Phi = \exists x\, \Omega$. □

For convenience, in what follows we shall denote applications of Theorem 3.3 by `PR`, from ‘positive replacement’.

Theorem 3.4 ((weak replacement rule)). *Let* $\{\Phi, \Psi, \Theta\} \subseteq \mathrm{Form}_\sigma$. *Suppose that* Φ' *is obtained from* Φ *by replacing some occurrences of* Ψ *by* Θ. *Then* $\vdash \Psi \Leftrightarrow \Theta$ *implies* $\vdash \Phi \Leftrightarrow \Phi'$.

[5] The requirement that Γ consist of sentences is related to standard difficulties in defining Hilbert-style derivations from sets of formulas with free variables.

[6] Since $\vdash$ has been defined as a relation between sets of sentences and formulas, Φ must contain no free variables.

Proof. We shall restrict ourselves to the cases when Φ begins with $\sim$, $\forall$ or $\exists$. Furthermore, we may assume that $\Psi \neq \Phi$.

Suppose $\Phi = \sim\Omega$. Clearly, the part Ω corresponds to Ω' such that $\Phi' = \sim\Omega'$. Then $\Phi \Leftrightarrow \Phi'$ can be easily derived:

1	$\Omega \Leftrightarrow \Omega'$	inductive hypothesis
2	$\Omega \leftrightarrow \Omega'$	from 1
3	$\sim\sim\Omega \leftrightarrow \Omega$	SN1
4	$\Omega' \leftrightarrow \sim\sim\Omega'$	SN1
5	$\sim\sim\Omega \leftrightarrow \sim\sim\Omega'$	from 3, 2, 4
6	$\sim\Omega \leftrightarrow \sim\Omega'$	from 1
7	$\sim\Omega \Leftrightarrow \sim\Omega'$	from 6, 5.

Suppose $\Phi = \forall x\,\Omega$. So the part Ω corresponds to Ω' such that $\Phi' = \forall x\,\Omega'$. Then $\Phi \Leftrightarrow \Phi'$ can be derived in the following way:

1	$\Omega \Leftrightarrow \Omega'$	inductive hypothesis
2	$\Omega \leftrightarrow \Omega'$	from 1
3	$\forall x\,\Omega \leftrightarrow \forall x\,\Omega'$	from 2 by PR
4	$\sim\Omega \leftrightarrow \sim\Omega'$	from 1
5	$\exists x \sim\Omega \leftrightarrow \exists x \sim\Omega'$	from 4 by PR
6	$\sim\forall x\,\Omega \leftrightarrow \exists x \sim\Omega$	SN6
7	$\exists x \sim\Omega' \leftrightarrow \sim\forall x\,\Omega'$	SN6
8	$\sim\forall x\,\Omega \leftrightarrow \sim\forall x\,\Omega'$	from 6, 5, 7
9	$\forall x\,\Omega \Leftrightarrow \forall x\,\Omega'$	from 3, 8.

Similarly for $\Phi = \exists x\,\Omega$. □

For convenience, in what follows we shall denote applications of Theorem 3.4 by WR, from 'weak replacement'.

Say that a σ-formula Φ is a *negation normal form*, or an *n.n.f.* for short, if each occurrence of $\sim$ in Φ immediately precedes some atomic subformula.[7] Our next goal is to prove a strong version of the n.n.f. theorem for QBK. Here 'strong' indicates that we shall use not $\leftrightarrow$ (as in [15], for example), but $\Leftrightarrow$ — cf. [19, Proposition 3.1]. On this path what turns out to be useful is:

Proposition 3.5. *The following schemes are derivable in* QBK*:*

A1. $\sim\neg\Phi \Leftrightarrow \neg\neg\Phi$*;*

A2. $(\Phi \to \Psi) \Leftrightarrow (\neg\Phi \vee \Psi)$.[8]

Proof. $\boxed{\text{A1}}$ First we derive $\sim\neg\Phi \to \neg\neg\Phi$:

1	$\sim(\Phi \to \bot) \leftrightarrow (\Phi \wedge \sim\bot)$	SN3
2	$\sim(\Phi \to \bot) \to (\Phi \wedge \sim\bot)$	from 1
3	$\sim(\Phi \to \bot) \to \Phi$	from 2
4	$\Phi \to \neg\neg\Phi$	classical logic
5	$\sim(\Phi \to \bot) \to \neg\neg\Phi$	from 3, 4

[7] Here $\bot$ is treated as an atomic formula.

[8] A semantical proof of this fact can be extracted from [18, Section 5]; see also [14, Section 3].

(remember that $\neg\Phi$ is an abbreviation for $\Phi \to \bot$). Now we derive $\neg\neg\Phi \to {\sim}\neg\Phi$:

1	$\neg\neg\Phi \to \Phi$	classical logic
2	${\sim}\bot$	SN5
3	$\neg\neg\Phi \to (\Phi \wedge {\sim}\bot)$	from 1, 2
4	$(\Phi \wedge {\sim}\bot) \leftrightarrow {\sim}(\Phi \to \bot)$	SN3
5	$\neg\neg\Phi \to {\sim}(\Phi \to \bot)$	from 3, 4.

Thus $\vdash {\sim}\neg\Phi \leftrightarrow \neg\neg\Phi$ for all $\Phi \in \mathrm{Form}_\sigma$. Finally, note that:

1	${\sim}{\sim}\neg\Phi \leftrightarrow \neg\Phi$	SN4
2	$\neg\Phi \leftrightarrow \neg\neg\neg\Phi$	classical logic
3	$\neg\neg\neg\Phi \leftrightarrow {\sim}\neg\neg\Phi$	by what has already been proved
4	${\sim}{\sim}\neg\Phi \leftrightarrow {\sim}\neg\neg\Phi$	from 1, 2, 3.

Hence $\vdash {\sim}\neg\Phi \Leftrightarrow \neg\neg\Phi$ for every $\Phi \in \mathrm{Form}_\sigma$.

$\boxed{\texttt{A2}}$ Clearly, $(\Phi \to \Psi) \leftrightarrow (\neg\Phi \vee \Psi)$ can be derived as in classical logic. Let us show the derivability of ${\sim}(\Phi \to \Psi) \leftrightarrow {\sim}(\neg\Phi \vee \Psi)$:

1	${\sim}(\Phi \to \Psi) \leftrightarrow (\Phi \wedge {\sim}\Psi)$	SN3
2	$\Phi \leftrightarrow \neg\neg\Phi$	classical logic
3	$\neg\neg\Phi \Leftrightarrow {\sim}\neg\Phi$	A1
4	$\Phi \leftrightarrow {\sim}\neg\Phi$	from 3, 4
5	$(\Phi \wedge {\sim}\Psi) \leftrightarrow ({\sim}\neg\Phi \wedge {\sim}\Psi)$	from 4 by PR
6	$({\sim}\neg\Phi \wedge {\sim}\Psi) \leftrightarrow {\sim}(\neg\Phi \vee \Psi)$	SN3
7	${\sim}(\Phi \to \Psi) \leftrightarrow {\sim}(\neg\Phi \vee \Psi)$	from 1, 5, 6.

$\square$

For each $\mathtt{S} \in \{\mathtt{SN1}, \mathtt{SN2}, \mathtt{SN3}, \mathtt{SN6}, \mathtt{SN7}, \mathtt{M1}, \mathtt{M2}\}$ denote by $\mathtt{S}^*$ the scheme obtained from S by replacing $\leftrightarrow$ by $\Leftrightarrow$.

Lemma 3.6. *$\mathtt{SN1}^*$, $\mathtt{SN2}^*$, $\mathtt{SN3}^*$, $\mathtt{M1}^*$ and $\mathtt{M2}^*$ are derivable in* QBK.[9]

Proof. $\boxed{\texttt{SN1}^*}$ Obviously, ${\sim}{\sim}{\sim}\Phi \leftrightarrow {\sim}\Phi$ belongs to QBK (as a special case of SN1).

$\boxed{\texttt{SN2}^*}$ Let us show that ${\sim}{\sim}(\Phi \wedge \Psi) \leftrightarrow {\sim}({\sim}\Phi \vee {\sim}\Psi)$ belongs to QBK:

1	${\sim}{\sim}(\Phi \wedge \Psi) \leftrightarrow (\Phi \wedge \Psi)$	SN1
2	$\Phi \leftrightarrow {\sim}{\sim}\Phi$	SN1
3	$(\Phi \wedge \Psi) \leftrightarrow ({\sim}{\sim}\Phi \wedge \Psi)$	from 2 by PR
4	$\Psi \leftrightarrow {\sim}{\sim}\Psi$	SN1
5	$({\sim}{\sim}\Phi \wedge \Psi) \leftrightarrow ({\sim}{\sim}\Phi \wedge {\sim}{\sim}\Psi)$	from 4 by PR
6	$({\sim}{\sim}\Phi \wedge {\sim}{\sim}\Psi) \leftrightarrow {\sim}({\sim}\Phi \vee {\sim}\Psi)$	SN3
7	${\sim}{\sim}(\Phi \vee \Psi) \leftrightarrow {\sim}({\sim}\Phi \wedge {\sim}\Psi)$	from 1, 3, 5, 6.

$\boxed{\texttt{SN3}^*}$ This case is similar to that of $\mathtt{SN2}^*$.

$\boxed{\texttt{M1}^*}$ Let us show that ${\sim}\neg\Box\Phi \leftrightarrow {\sim}\Diamond\neg\Phi$ belongs to QBK:

[9] A semantical proof of this fact can be extracted from [18, Section 5]; see also [14, Section 3].

1	$\sim\neg\Box\Phi \leftrightarrow \neg\neg\Box\Phi$	A1
2	$\neg\Box\Phi \leftrightarrow \Diamond\neg\Phi$	M1
3	$\neg\neg\Box\Phi \leftrightarrow \neg\Diamond\neg\Phi$	from 1 by PR
4	$\neg\Diamond\neg\Phi \leftrightarrow \Box\neg\neg\Phi$	M2
5	$\neg\neg\Phi \leftrightarrow \sim\neg\Phi$	A1
6	$\Box\neg\neg\Phi \leftrightarrow \Box\sim\neg\Phi$	from 5 by PR
7	$\Box\sim\neg\Phi \leftrightarrow \sim\sim\Box\sim\neg\Phi$	SN1
8	$\sim\Box\sim\neg\Phi \Leftrightarrow \Diamond\neg\Phi$	M4
9	$\sim\sim\Box\sim\neg\Phi \leftrightarrow \sim\Diamond\neg\Phi$	from 8
10	$\sim\neg\Box\Phi \leftrightarrow \sim\Diamond\neg\Phi$	from 1, 3, 4, 6, 7, 9.

$\boxed{\texttt{M2}^*}$ This case is similar to that of $\texttt{M1}^*$. □

Lemma 3.7. $\texttt{SN6}^*$ *and* $\texttt{SN7}^*$ *are derivable in* QBK.

Proof. $\boxed{\texttt{SN6}^*}$ Let us show that $\sim\sim\forall x\,\Phi \leftrightarrow \sim\exists x\sim\Phi$ belongs to QBK:

1	$\sim\sim\forall x\,\Phi \leftrightarrow \forall x\,\Phi$	SN1
2	$\Phi \leftrightarrow \sim\sim\Phi$	SN1
3	$\forall x\,\Phi \leftrightarrow \forall x\sim\sim\Phi$	from 2 by PR
4	$\forall x\sim\sim\Phi \leftrightarrow \sim\exists x\sim\Phi$	SN7
5	$\sim\sim\forall x\,\Phi \leftrightarrow \sim\exists x\sim\Phi$	from 1, 3, 4.

$\boxed{\texttt{SN7}^*}$ This case is similar to that of $\texttt{SN6}^*$. □

At the same time, we cannot replace $\leftrightarrow$ in SN4 by $\Leftrightarrow$ (this can be formally justified by means of a suitable possible world semantics). However, instead of SN4 one can employ the formula scheme

$$\sim(\Phi \to \Psi) \Leftrightarrow (\neg\neg\Phi \wedge \sim\Psi)\,, \qquad (\texttt{SN4}')$$

which is not hard to obtain by using the above results:

1	$(\Phi \to \Psi) \Leftrightarrow (\neg\Phi \vee \Psi)$	A2
2	$\sim(\Phi \to \Psi) \Leftrightarrow \sim(\neg\Phi \vee \Psi)$	from 1 by WR
3	$\sim(\neg\Phi \vee \Psi) \Leftrightarrow (\sim\neg\Phi \wedge \sim\Psi)$	$\texttt{SN3}^*$
4	$\sim\neg\Phi \Leftrightarrow \neg\neg\Phi$	A1
5	$(\sim\neg\Phi \wedge \sim\Psi) \Leftrightarrow (\neg\neg\Phi \wedge \sim\Psi)$	from 4 by WR
6	$\sim(\Phi \to \Psi) \Leftrightarrow (\neg\neg\Phi \wedge \sim\Psi)$	from 2, 3, 5.

Now we are ready to establish the following.

Theorem 3.8 ((on negation normal form, strong version))**.** *For every σ-formula Φ there exists an n.n.f. $\overline{\Phi}$ such that $\Phi \Leftrightarrow \overline{\Phi} \in \mathsf{QBK}_\sigma$. Moreover, there is an algorithm that constructs, given any σ-formula Φ, a suitable n.n.f. $\overline{\Phi}$.*

Proof. By an easy induction on the complexity of Φ, where we use the rule WR and the schemes $\texttt{SN1}^*$, $\texttt{SN2}^*$, $\texttt{SN3}^*$, $\texttt{SN4}'$, $\texttt{SN6}^*$, $\texttt{SN7}^*$, M3 and M4. Here it is convenient to rewrite M3 and M4 as:

$\texttt{M3}'$. $\sim\Box\Phi \Leftrightarrow \Diamond\sim\Phi$;

$\mathtt{M4}'$. $\sim\Diamond\Phi \Leftrightarrow \Box{\sim}\Phi$. $\square$

Remark 3.9. For Nelson's constructive logics (propositional or quantified), the analogue of Theorem 3.8 does not hold: there the implication connective has a much more complex, intuitionistic character, and we can obtain only a weak version of the n.n.f. theorem.

Finally, by QBK-*extensions* we mean supersets of QBK closed under formula substitutions and all the inference rules (including the modal ones).[10] Given a QBK-extension L, define

$$\Gamma \vdash_L \Delta \quad :\Longleftrightarrow \quad L \cup \Gamma \vdash \Delta.$$

If L is a QBK-extension, and $\mathtt{S}_1, \ldots, \mathtt{S}_n$ are formula schemes, then we shall denote by $L{+}\{\mathtt{S}_1, \ldots, \mathtt{S}_n\}$ the least QBK-extension containing L and all the substitution instances of $\mathtt{S}_1$, …, $\mathtt{S}_n$. In fact, all these notions formally depend on the choice of σ, but it will always be clear from the context which signature we are talking about, or we shall explicitly write L_σ instead of L.

It is not hard to see that the results of the present section will remain true if we replace QBK by an arbitrary QBK-extension in their formulations.

4 Possible world semantics

As usual, by a *frame* is meant an ordered pair of the form $\mathcal{W} = \langle W, R\rangle$, where W is a non-empty set, whose elements are called *possible worlds*, and R is a binary relation on W. Given a frame $\mathcal{W}$ and two families of σ-structures

$$\mathscr{A}^+ \;=\; \langle \mathfrak{A}^+_w : w \in W\rangle \quad \text{and} \quad \mathscr{A}^- \;=\; \langle \mathfrak{A}^-_w : w \in W\rangle,$$

the corresponding triple

$$\mathcal{M} \;=\; \langle \mathcal{W}, \mathscr{A}^+, \mathscr{A}^-\rangle$$

is called a QBK$_\sigma$-*model* if for any $u, v \in W$:

- $A^+_u = A^-_u$;
- $c^{\mathfrak{A}^+_u} = c^{\mathfrak{A}^-_u}$ for all $c \in \mathrm{Const}_\sigma$;
- uRv implies $A^+_u \subseteq A^+_v$;
- uRv implies $c^{\mathfrak{A}^+_u} = c^{\mathfrak{A}^+_v}$ for all $c \in \mathrm{Const}_\sigma$.

Here A^+_u and A^-_u are the domains of σ-structures $\mathfrak{A}^+_u$ and $\mathfrak{A}^-_u$ respectively. We shall usually write A_u instead of A^+_u (which coincides with A^-_u) and $c^{\mathfrak{A}_u}$ instead of $c^{\mathfrak{A}^+_u}$ (which coincides with $c^{\mathfrak{A}^-_u}$). Nevertheless, it is worth remembering that the interpretations of the symbols of Pred_σ in $\mathfrak{A}^+_u$ and $\mathfrak{A}^-_u$ may differ significantly.

Let $\mathcal{M}$ be a QBK$_\sigma$-model. In what follows $R(u)$ will denote $\{v \in W \mid uRv\}$, i.e. the image of $\{u\}$ under R. For any $w \in W$ and A_w-sentense Φ we define

$$\mathcal{M}, w \Vdash^+ \Phi \quad \text{and} \quad \mathcal{M}, w \Vdash^- \Phi$$

[10] A rigorous definition of what it means for a set of formulas to be closed under formula substitutions can be found in [6, Chapter 2].

by induction on the complexity of Φ:

$$
\begin{array}{rcl}
\mathcal{M}, w \Vdash^{+} P(t_1, \ldots, t_n) & \iff & \mathfrak{A}_w^{+} \Vdash P(t_1, \ldots, t_n); \\
\mathcal{M}, w \Vdash^{-} P(t_1, \ldots, t_n) & \iff & \mathfrak{A}_w^{-} \Vdash P(t_1, \ldots, t_n); \\
\mathcal{M}, w \Vdash^{+} \Psi \wedge \Theta & \iff & \mathcal{M}, w \Vdash^{+} \Psi \quad \text{and} \quad \mathcal{M}, w \Vdash^{+} \Theta; \\
\mathcal{M}, w \Vdash^{-} \Psi \wedge \Theta & \iff & \mathcal{M}, w \Vdash^{-} \Psi \quad \text{or} \quad \mathcal{M}, w \Vdash^{-} \Theta; \\
\mathcal{M}, w \Vdash^{+} \Psi \vee \Theta & \iff & \mathcal{M}, w \Vdash^{+} \Psi \quad \text{or} \quad \mathcal{M}, w \Vdash^{+} \Theta; \\
\mathcal{M}, w \Vdash^{-} \Psi \vee \Theta & \iff & \mathcal{M}, w \Vdash^{-} \Psi \quad \text{and} \quad \mathcal{M}, w \Vdash^{-} \Theta; \\
\mathcal{M}, w \Vdash^{+} \Psi \rightarrow \Theta & \iff & \mathcal{M}, w \nVdash^{+} \Psi \quad \text{or} \quad \mathcal{M}, w \Vdash^{+} \Theta; \\
\mathcal{M}, w \Vdash^{-} \Psi \rightarrow \Theta & \iff & \mathcal{M}, w \Vdash^{+} \Psi \quad \text{and} \quad \mathcal{M}, w \Vdash^{-} \Theta; \\
\mathcal{M}, w \Vdash^{+} \perp & \iff & 0 \neq 0; \\
\mathcal{M}, w \Vdash^{-} \perp & \iff & 0 = 0; \\
\mathcal{M}, w \Vdash^{+} \sim \Psi & \iff & \mathcal{M}, w \Vdash^{-} \Psi; \\
\mathcal{M}, w \Vdash^{-} \sim \Psi & \iff & \mathcal{M}, w \Vdash^{+} \Psi; \\
\mathcal{M}, w \Vdash^{+} \Box \Psi & \iff & \mathcal{M}, w \Vdash^{+} \Psi \quad \text{for all } u \in R(w); \\
\mathcal{M}, w \Vdash^{-} \Box \Psi & \iff & \mathcal{M}, w \Vdash^{-} \Psi \quad \text{for some } u \in R(w); \\
\mathcal{M}, w \Vdash^{+} \Diamond \Psi & \iff & \mathcal{M}, w \Vdash^{+} \Psi \quad \text{for some } u \in R(w); \\
\mathcal{M}, w \Vdash^{-} \Diamond \Psi & \iff & \mathcal{M}, w \Vdash^{-} \Psi \quad \text{for all } u \in R(w); \\
\mathcal{M}, w \Vdash^{+} \forall x\, \Psi & \iff & \mathcal{M}, w \Vdash^{+} \Psi(x/\underline{a}) \quad \text{for all } a \in A_w; \\
\mathcal{M}, w \Vdash^{-} \forall x\, \Psi & \iff & \mathcal{M}, w \Vdash^{-} \Psi(x/\underline{a}) \quad \text{for some } a \in A_w; \\
\mathcal{M}, w \Vdash^{+} \exists x\, \Psi & \iff & \mathcal{M}, w \Vdash^{+} \Psi(x/\underline{a}) \quad \text{for some } a \in A_w; \\
\mathcal{M}, w \Vdash^{-} \exists x\, \Phi & \iff & \mathcal{M}, w \Vdash^{-} \Psi(x/\underline{a}) \quad \text{for all } a \in A_w.
\end{array}
$$

In particular, we always have $\mathcal{M}, w \nVdash^{+} \perp$ and $\mathcal{M}, w \Vdash^{-} \perp$. Informally, $\Vdash^{+}$ is responsible for *verifiability*, and $\Vdash^{-}$ is for *falsifiability*. When it is clear from the context which model $\mathcal{M}$ we are talking about, we write $w \Vdash^{\circ} \Phi$ instead of $\mathcal{M}, w \Vdash^{\circ} \Phi$, where $\circ \in \{+, -\}$. Finally, the notation $\mathcal{W} \Vdash \Phi$ means that $\mathcal{M}, w \Vdash^{+} \lambda\Phi$ for any QBK_σ-model $\mathcal{M}$ based on $\mathcal{W}$, $w \in W$ and A_w-substitution λ.

The semantics for QBK, as well as its propositional version from [17], is locally four-valued. More precisely, four situations are potentially possible:

1. $\mathcal{M}, w \Vdash^{+} \Phi$ and $\mathcal{M}, w \nVdash^{-} \Phi$;
2. $\mathcal{M}, w \nVdash^{+} \Phi$ and $\mathcal{M}, w \Vdash^{-} \Phi$;
3. $\mathcal{M}, w \nVdash^{+} \Phi$ and $\mathcal{M}, w \nVdash^{-} \Phi$;
4. $\mathcal{M}, w \Vdash^{+} \Phi$ and $\mathcal{M}, w \Vdash^{-} \Phi$.

The first situation corresponds to the value 'true', the second is for 'false', the third is for 'undefined', and the fourth is for 'overdefined'.

Remark 4.1. Models for QBK may be viewed as 'modalized' versions of models for the quantified Belpan–Dunn logic; cf. [21]. Although the language of the latter does not contain $\perp$ and $\rightarrow$, they can be easily added; see [17, Section 4].

Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$. Say that Δ *follows semantically from* Γ, and write $\Gamma \vDash \Delta$, if for any QBK_σ-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^+, \mathscr{A}^- \rangle$, $w \in W$ and ground A_w-substitution λ,

$$\mathcal{M}, w \Vdash^+ \Phi \quad \text{for all} \quad \Phi \in \Gamma \quad \Longrightarrow \quad \mathcal{M}, w \Vdash^+ \lambda\Psi \quad \text{for some} \quad \Psi \in \Delta.$$

By analogy with $\vdash$, when $\Delta = \{\Phi\}$, we usually write $\Gamma \vDash \Phi$ instead of $\Gamma \vDash \{\Phi\}$. We shall call a σ-formula Φ *valid* if $\vDash \Phi$. As is easily verified, the following semantic analogue of the deduction theorem holds.

Theorem 4.2. *For any* $\Gamma \cup \{\Phi\} \subseteq \mathrm{Sent}_\sigma$ *and* $\Psi \in \mathrm{Form}_\sigma$*,*

$$\Gamma \cup \{\Phi\} \vDash \Psi \quad \Longleftrightarrow \quad \Gamma \vDash \Phi \to \Psi.$$

Our next goal is to show that QBK is sound with respect to the above semantics.

Lemma 4.3. *For every* $\Phi \in \mathrm{Form}_\sigma$*,*

$$\vdash \Phi \quad \Longrightarrow \quad \vDash \Phi.$$

In other words, the derivability of a σ-formula implies its validity.

Proof. Suppose that $\vdash \Phi$, i.e. there exists a finite sequence

$$\Phi_0, \quad \Phi_1, \quad \ldots, \quad \Phi_n \;=\; \Phi$$

of σ-formulas such that for each $i \in \{0, \ldots, n\}$ one of the following conditions is satisfied:

a. Φ_i is an axiom;

b. Φ_i is obtained from previous Φ_j and Φ_k by `MP`;

c. Φ_i is obtained from a previous Φ_j by `MB` or `MD`;

d. Φ_i is obtained from a previous Φ_j by the rule `BR1` or `BR2`.

Let $\mathcal{M}$ be a QBK_σ-model. By induction on i, we shall establish that $\mathcal{M}, w \Vdash^+ \lambda\,(\Phi_i)$ for all $w \in W$ and ground A_w-substitutions λ.

If Φ_i is an axiom of classical logic, then we can argue as in classical first-order logic.

If Φ_i is a 'propositional' axiom for strong negation, then we can argue as in BK; see [18, Section 4]. For example, let Φ_i be an axiom of type `SN4`, i.e. of the form

$$\sim(\Psi \to \Theta) \leftrightarrow (\Psi \wedge \sim\Theta).$$

We need to show that for any $w \in W$ and ground A_w-substitution λ,

$$w \Vdash^+ \lambda{\sim}(\Psi \to \Theta) \quad \Longleftrightarrow \quad w \Vdash^+ \lambda\,(\Psi \wedge \sim\Theta).$$

This is done as follows:

$$\begin{aligned}
w \Vdash^+ \lambda{\sim}(\Psi \to \Theta) \quad &\Longleftrightarrow \quad w \Vdash^+ \sim(\lambda\Psi \to \lambda\Theta)\\
&\Longleftrightarrow \quad w \Vdash^- \lambda\Psi \to \lambda\Theta\\
&\Longleftrightarrow \quad w \Vdash^+ \lambda\Psi \quad \text{and} \quad w \Vdash^- \lambda\Theta\\
&\Longleftrightarrow \quad w \Vdash^+ \lambda\Psi \quad \text{and} \quad w \Vdash^+ \sim\lambda\Theta
\end{aligned}$$

$$\begin{aligned} &\iff \quad w \Vdash^{+} \lambda\Psi \wedge {\sim}\,\lambda\Theta \\ &\iff \quad w \Vdash^{+} \lambda\,(\Psi \wedge {\sim}\,\Theta). \end{aligned}$$

Next, consider the 'quantifier' axioms for strong negation. Let Φ_i be an axiom of type SN6, i.e. of the form

$${\sim}\,\forall x\,\Psi \leftrightarrow \exists x\,{\sim}\,\Psi.$$

We need to show that for any $w \in W$ and ground A_w-substitution λ,

$$w \Vdash \lambda{\sim}\,\forall x\,\Psi \quad\iff\quad w \Vdash \lambda\exists x\,{\sim}\,\Psi.$$

This is done as follows:

$$\begin{aligned} w \vDash \lambda{\sim}\,\forall x\,\Psi \quad &\iff \quad w \Vdash^{+} {\sim}\,\forall x\,\lambda^x_x\Phi \\ &\iff \quad w \Vdash^{-} \forall x\,\lambda^x_x\Phi \\ &\iff \quad w \Vdash^{-} (\lambda^x_x\Phi)\,(x/\underline{a}) \quad \text{for some } \ a \in A_w \\ &\iff \quad w \Vdash^{+} {\sim}\,(\lambda^x_x\Phi)\,(x/\underline{a}) \quad \text{for some } \ a \in A_w \\ &\iff \quad w \Vdash^{+} \exists x\,{\sim}\,\lambda^x_x\Phi \\ &\iff \quad w \Vdash^{+} \lambda\exists x\,{\sim}\,\Psi, \end{aligned}$$

where λ^x_x denotes $(\lambda \setminus \{(x, \lambda\,(x))\}) \cup \{(x, x)\}$. Similarly for SN7.

If Φ_i is either an axiom for $\Box$ or a modal interaction axiom, then we can argue as in BK.

If Φ_i is obtained from previous Φ_j and Φ_k by MP, BR1 or BR2, then we can argue as in classical predicate logic, and if Φ_i is obtained from a previous Φ_j by MB or MD, then we can argue as in BK. $\Box$

Theorem 4.4 ((on the soundness of QBK)). *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash \Delta \quad\Longrightarrow\quad \Gamma \vDash \Delta.$$

Proof. Suppose that $\Gamma \vdash \Delta$. So there will be a finite $\Lambda \subseteq \Gamma$ and $\Phi \in \mathrm{Disj}\,(\Delta)$ such that $\Lambda \vdash \Phi$. We consider two cases separately:

- Assume $\Lambda = \varnothing$. Then $\vDash \Phi$ by Lemma 4.3, whence $\Gamma \vDash \Delta$.
- Assume $\Lambda = \{\Psi_0, \ldots, \Psi_n\}$. So $\Psi_0 \wedge \ldots \wedge \Psi_n \vdash \Phi$, which is equivalent to

  $$\vdash \Psi_0 \wedge \ldots \wedge \Psi_n \to \Phi$$

  by Theorem 3.2. Then $\vDash \Psi_0 \wedge \ldots \wedge \Psi_n \to \Phi$ by Lemma 4.3, which is equivalent to

  $$\Psi_0 \wedge \ldots \wedge \Psi_n \vDash \Phi$$

  by Theorem 4.2. Hence $\Gamma \vDash \Delta$. $\Box$

Let L be a QBK-extension. Denote by $\vDash_L$ the relativization of $\vDash$ to the class of frames

$$\mathcal{K}_L \ := \ \{\mathcal{W} \mid \mathcal{W} \Vdash \Phi \text{ for all } \Phi \in L\}.$$

It is easy to see that the results of the present section will remain true if we replace $\vdash$ by $\vdash_L$ and $\vDash$ by $\vDash_L$ in their formulations, where L is an arbitrary QBK-extension.

5 Strong completeness theorem

We shall call a σ-theory Γ *saturated* if:

- $\Gamma \neq \mathrm{Sent}_\sigma$;
- $\{\Phi \in \mathrm{Sent}_\sigma \mid \Gamma \vdash \Phi\} \subseteq \Gamma$;
- $\Phi \vee \Psi \in \Gamma$ implies $\Phi \in \Gamma$ or $\Psi \in \Gamma$;
- $\exists x\, \Phi \in \Gamma$ implies $\Phi\,(x/c) \in \Gamma$ for some $c \in \mathrm{Const}_\sigma$.

Here the first two conditions are responsible for being non-trivial and deductively closed, and the third and fourth are for having the 'disjunctive' and 'existential' properties. Moreover, saturated theories enjoy two other useful properties:

Proposition 5.1. *Let Γ be a saturated σ-theory. Then:*

i. for every $\Phi \in \mathrm{Form}_\sigma$ we have either $\Phi \in \Gamma$ or $\neg\Phi \in \Gamma$;

ii. if $\Phi\,(x/c) \in \Gamma$ for all $c \in \mathrm{Const}_\sigma$, then $\forall x\, \Phi \in \Gamma$.

The proof is similar to the case of classical first-order logic.

For any set S, take

$$\sigma_S \;:=\; \sigma \cup \{\underline{s} \mid s \in S\},$$

where all $\underline{s}$'s are new constant symbols.

Lemma 5.2. *Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$ be such that $\Gamma \nvdash \Delta$. Then for each set S of cardinality $|\mathrm{Sent}_\sigma|$ there exists a saturated σ_S-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

The proof is similar to the case of classical first-order logic.

Now fix some set $S^\star$ of cardinality $|\mathrm{Sent}_\sigma|$. It will play the role of a 'potential' universe in our canonical model for QBK_σ. We shall call $S \subseteq S^\star$ *admissible* if $|S^\star \setminus S| = |S^\star|$.

Lemma 5.3. *Let $S \subseteq S^\star$ be admissible, and let $\Gamma \subseteq \mathrm{Sent}_{\sigma_S}$ and $\Delta \subseteq \mathrm{Form}_{\sigma_S}$ be such that $\Gamma \nvdash \Delta$. Then there exist an admissible $S' \supseteq S$ and a saturated $\sigma_{S'}$-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

Proof. Since $|S^\star \setminus S| = |\mathrm{Sent}_\sigma|$, we can find an admissible $S' \supseteq S$ such that

$$|S' \setminus S| \;=\; |\mathrm{Sent}_\sigma|.$$

Moreover, we have $|\mathrm{Sent}_\sigma| = |\mathrm{Sent}_{\sigma_S}|$, because $|S| \leqslant |\mathrm{Sent}_\sigma|$. It remains to apply Lemma 5.2 with $\sigma := \sigma_S$ and $S := S' \setminus S$. □

Lemma 5.4. *Let $S \subseteq S^\star$ be admissible, and let $\Gamma \subseteq \mathrm{Sent}_{\sigma_S}$ and $\Delta \subseteq \mathrm{Form}_{\sigma_S}$ be such that $\Gamma \nvdash \Delta$. Then there exists a saturated $\sigma_{S^\star}$-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

Proof. We just need to apply Lemma 5.2 with $\sigma := \sigma_S$ and $S = S^\star \setminus S$. □

Finally, we are ready to adapt the method of canonical models for QBK and its extensions. Informally speaking, we need to combine the canonical model for BK from [18, Section 4] and the canonical model for classical quantified modal logic (see [6, Section 6]) in a natural way. We shall sometimes drop the index σ, but it will always be clear from the context which signature we are talking about. For any set S, let

$$\mathrm{Sat}_S \ := \ \text{the collection of all saturated } \sigma_S\text{-theories.}$$

In addition, for an arbitrary set Γ of formulas, denote by $\mathrm{Const}\,(\Gamma)$ the collection of all constant symbols that occur in elements of Γ. Note that for each Γ in Sat_S we shall have $\mathrm{Const}\,(\Gamma) = \mathrm{Const}_{\sigma_S}$: obviously, $\mathrm{Const}_{\sigma_S}$ includes $\mathrm{Const}\,(\Gamma)$; on the other hand, for every $c \in \mathrm{Const}_{\sigma_S}$ we can easily construct $\Psi_c \in \mathsf{QBK}_{\sigma_S} \cap \mathrm{Sent}_{\sigma_S}$ such that c occurs in Ψ_c, and hence we have $\mathrm{Const}_{\sigma_S} \subseteq \mathrm{Const}\,(\Gamma)$ (in view of $\mathsf{QBK}_{\sigma_S} \subseteq \Gamma$). Next, with any saturated σ_S-theory Γ we associate two σ_S-structures $\left(\mathfrak{A}_\Gamma^S\right)^+$ and $\left(\mathfrak{A}_\Gamma^S\right)^-$ with the same domain

$$A_\Gamma^S \quad := \quad \mathrm{Const}\,(\Gamma)$$

such that all constant symbols of σ_S are interpreted as themselves in these structures, and for every atomic σ_S-sentence Φ:

$$\begin{aligned} \left(\mathfrak{A}_\Gamma^S\right)^+ \Vdash \Phi \quad &:\Longleftrightarrow \quad \Phi \ \in \ \Gamma, \\ \left(\mathfrak{A}_\Gamma^S\right)^- \Vdash \Phi \quad &:\Longleftrightarrow \quad {\sim}\Phi \ \in \ \Gamma. \end{aligned}$$

Denote by $\mathfrak{A}_\Gamma^+$ and $\mathfrak{A}_\Gamma^-$ the σ-reducts of $\left(\mathfrak{A}_\Gamma^S\right)^+$ and $\left(\mathfrak{A}_\Gamma^S\right)^-$ respectively. Clearly, every A_Γ-sentence has the form

$$\Phi\left(x_1/\underline{c_1}, \ldots, x_n/\underline{c_n}\right),$$

where $\{c_1, \ldots, c_n\} \subseteq \mathrm{Const}\,(\Gamma)$; therefore, for convenience, we shall often identify it with the σ_S-sentence $\Phi\,(x_1/c_1, \ldots, x_n/c_n)$. Now take

$$W^{\mathsf{QBK}} \ := \ \bigcup \{\mathrm{Sat}_S \mid S \text{ is an admissible subset of } S^\star\}.$$

By the *canonical frame for* QBK we mean a frame $\mathcal{W}^{\mathsf{QBK}} = \langle W^{\mathsf{QBK}}, R^{\mathsf{QBK}} \rangle$, where the relation R^{QBK} is defined in a standard way:

$$R^{\mathsf{QBK}} \ := \ \left\{(\Gamma, \Delta) \in W^{\mathsf{QBK}} \times W^{\mathsf{QBK}} \mid \Gamma_\Box \subseteq \Delta\right\}.^{11}$$

The *canonical model for* QBK is

$$\mathcal{M}^{\mathsf{QBK}} \ = \ \langle \mathcal{W}^{\mathsf{QBK}}, \left(\mathscr{A}^{\mathsf{QBK}}\right)^+, \left(\mathscr{A}^{\mathsf{QBK}}\right)^- \rangle,$$

where

$$\begin{aligned} \left(\mathscr{A}^{\mathsf{QBK}}\right)^+ &:= \ \langle \mathfrak{A}_\Gamma^+ : \Gamma \in W^{\mathsf{QBK}} \rangle, \\ \left(\mathscr{A}^{\mathsf{QBK}}\right)^- &:= \ \langle \mathfrak{A}_\Gamma^- : \Gamma \in W^{\mathsf{QBK}} \rangle. \end{aligned}$$

As we shall verify shortly, this construction is correct.

Lemma 5.5. *$\mathcal{M}^{\mathsf{QBK}}$ is a* QBK*-model.*

[11] As usual, given $\Gamma \subseteq \mathrm{Sent}_\sigma$, we denote $\{\Phi \mid \Box\Phi \in \Gamma\}$ by $\Gamma_\Box$.

Proof. Suppose that $\Gamma R^{\mathsf{QBK}}\Delta$. Let σ_S be the signature of Γ. Clearly, for every $c \in \mathrm{Const}_{\sigma_S}$ we can construct $\Psi_c \in \mathsf{QBK}_{\sigma_S} \cap \mathrm{Sent}_{\sigma_S}$ such that c occurs in Ψ_c; then $\Box\Psi_c \in \mathsf{QBK}_{\sigma_S} \cap \mathrm{Sent}_{\sigma_S}$ (by `RN`), and therefore $\Box\Psi_c \in \Gamma$, whence we obtain $\Psi_c \in \Delta$. As a consequence, $\mathrm{Const}\,(\Delta)$ includes $\mathrm{Const}\,(\Gamma)$, which coincides with $\mathrm{Const}_{\sigma_S}$. Furthermore, the interpretation of the symbols of Const_σ will, obviously, be preserved when passing from $\mathfrak{A}^+_\Gamma$ to $\mathfrak{A}^+_\Delta$. $\Box$

A key role in the proof of the strong completeness theorem is played by:

Lemma 5.6 ((on the canonical model for QBK))**.** *For any* $\Gamma \in W^{\mathsf{QBK}}$ *and* A_Γ*-sentence* Φ*:*

$$\begin{array}{lcl} \mathcal{M}^{\mathsf{QBK}}, \Gamma \Vdash^+ \Phi & \iff & \Phi \in \Gamma; \\ \mathcal{M}^{\mathsf{QBK}}, \Gamma \Vdash^- \Phi & \iff & {\sim}\Phi \in \Gamma. \end{array}$$

Proof. By induction on the complexity of Φ.

The case when Φ is atomic is trivial.

Suppose $\Phi = \exists x\, \Psi$. Let us look at verification first:

$$\begin{array}{lcl} \mathcal{M}^{\mathsf{QBK}}, \Gamma \Vdash^+ \exists x\, \Psi & \iff & \mathcal{M}^{\mathsf{QBK}}, \Gamma \Vdash^+ \Psi\,(x/\underline{a}) \quad \text{for some } \ a \in \mathrm{Const}\,(\Gamma) \\ & \iff & \Psi\,(x/\underline{a}) \in \Gamma \quad \text{for some } \ a \in \mathrm{Const}\,(\Gamma) \\ & \iff & \exists x\, \Psi \in \Gamma. \end{array}$$

Here the last equivalence requires explanation: the direct implication is obtained by using `Q2`, and the converse employs the existential property. Now let us look at falsification:

$$\begin{array}{lcl} \mathcal{M}^{\mathsf{QBK}}, \Gamma \Vdash^- \exists x\, \Psi & \iff & \mathcal{M}^{\mathsf{QBK}}, \Gamma \Vdash^- \Psi\,(x/\underline{a}) \quad \text{for all } \ a \in \mathrm{Const}\,(\Gamma) \\ & \iff & {\sim}\,\Psi\,(x/\underline{a}) \in \Gamma \quad \text{for all } \ a \in \mathrm{Const}\,(\Gamma) \\ & \iff & \forall x \,{\sim}\, \Psi \in \Gamma \\ & \iff & {\sim}\,\exists x\, \Psi \in \Gamma. \end{array}$$

In the third equivalence, the direct implication is obtained by using `Q1`, and the converse uses Proposition 5.1 (ii); the last equivalence is guaranteed by `SN7`.

Similarly for $\Phi = \forall x\, \Psi$.

The remaining cases are handled as in BK (see [18, Section 4] or [17, Section 2]), although we need to use Lemma 5.3 instead of the propositional version of the extension lemma. $\Box$

Theorem 5.7 ((on the strong completeness of QBK))**.** *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash \Delta \quad \iff \quad \Gamma \vDash \Delta.$$

Proof. $\boxed{\Longrightarrow}$ See Theorem 4.4.

$\boxed{\Longleftarrow}$ Suppose that $\Gamma \nvdash \Delta$. Fix some admissible $S \subseteq S^\star$ of cardinality $\aleph_0$ (thus $|S| = |\mathrm{Var}|$). Let λ be a one-one function from Var onto $\{\underline{s} \mid s \in S\}$. Take

$$\Delta' \;:=\; \{\lambda\Psi \mid \Psi \in \Delta\}.$$

It is easy to verify that $\Gamma \nvdash \Delta'$. By Lemma 5.3 (with $\sigma := \sigma_S$ and $S := S^\star \setminus S$), there will be $\Gamma' \in W^{\mathsf{QBK}}$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash \Delta'$. Clearly, λ may be viewed as a ground $A_{\Gamma'}$-substitution. So by Lemma 5.6, we have $\mathcal{M}^{\mathsf{QBK}}, \Gamma' \Vdash \Phi$ for all $\Phi \in \Gamma$ and also $\mathcal{M}^{\mathsf{QBK}}, \Gamma' \nVdash \lambda\Psi$ for all $\Psi \in \Delta$. Hence $\Gamma \nvDash \Delta$. $\Box$

Let L be a QBK-extension. Then the *canonical frame for* L and the *canonical model for* L, denoted by $\mathcal{W}^L$ and $\mathcal{M}^L$ respectively, can be defined in the same way as for QBK, but with Sat_S replaced by

$$\mathrm{Sat}_S^L \;:=\; \{\Gamma \in \mathrm{Sat}_S \mid L \cap \mathrm{Sent}_{\sigma_S} \subseteq \Gamma\}.$$

Further, as is easy to understand, the analogue of Lemma 5.6 for L will hold. However, the analogue of Theorem 5.7 may fail if there are non-canonical models based on $\mathcal{W}^L$ in which formulas from L are refuted, i.e. when $\mathcal{W}^L$ does not belong to $\mathcal{K}_L$.

6 Some natural extensions

Using a variant of the canonical model method from the previous section, it is not difficult to establish the strong completeness of some natural QBK-extensions (cf. [18, Section 4]). Here we shall consider two basic types of such extensions:

i. those obtained by excluding 'undefined' or 'overdefined';

ii. those obtained by imposing additional restrictions — expressed by means of modal formulas — on accessibility relations in frames.

We start with extensions of type (i). At the axiomatic level, the following axiom schemes correspond to excluding 'overdefined' or 'undefined':

Exp. $\sim\Phi \to (\Phi \to \Psi)$;

ExM. $\Phi \vee \sim\Phi$.

Here Exp is an abbreviation for 'explosion', and ExM is for 'excluded middle'. In fact, it suffices to have Exp and ExM for all atomic Φ and Ψ; furthermore, Exp will be equivalent to the scheme '$\sim\Phi \to \neg\Phi$'. Let

$$\mathsf{QBK}^\circ \;:=\; \mathsf{QBK} + \{\mathtt{ExM}\} \quad \text{and} \quad \mathsf{QB3K} \;:=\; \mathsf{QBK} + \{\mathtt{Exp}\}.$$

A QBK_σ-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^+, \mathscr{A}^- \rangle$ will be called:

- a $\mathsf{QB3K}_\sigma$-*model* if for any $w \in W$ and atomic A_w-sentence Φ,

$$\mathfrak{A}_w^+ \nVdash \Phi \quad \text{or} \quad \mathfrak{A}_w^- \nVdash \Phi;$$

- a $\mathsf{QBK}_\sigma^\circ$-*model* if for any $w \in W$ and atomic A_w-sentence Φ,

$$\mathfrak{A}_w^+ \Vdash \Phi \quad \text{or} \quad \mathfrak{A}_w^- \Vdash \Phi.$$

Denote by $\vDash^3_{\mathsf{QBK}}$ and $\vDash^\circ_{\mathsf{QBK}}$ the relativizations of $\vDash$ to the corresponding classes of models. It is worth noting that we avoid the notations $\vDash_{\mathsf{QB3K}}$ and $\vDash_{\mathsf{QBK}^\circ}$, since otherwise there will arise a conflict with how $\vDash_L$ was defined in Section 4.

Theorem 6.1. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$,

$$\Gamma \vdash_{\mathsf{QB3K}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash^3_{\mathsf{QBK}} \Delta.$$

Proof. $\boxed{\Longrightarrow}$ Here the argument is similar to the proof of Theorem 4.4. In the present case the verifiability of Exp in all $\mathsf{QB3K}_\sigma$-models is established by an easy induction on the complexity of Φ.

$\boxed{\Longleftarrow}$ First we check that $\mathcal{M}^{\mathsf{QB3K}}$ is a $\mathsf{QB3K}_\sigma$-model. Let $\Gamma \in W^{\mathsf{QB3K}}$ and σ_S be the signature of Γ. Consider an arbitrary atomic A_Γ-sentence Φ; it can also be viewed as a σ_S-sentence. Assume, by way of contradiction, that $\mathfrak{A}^+_\Gamma \Vdash \Phi$ and $\mathfrak{A}^-_\Gamma \Vdash \Phi$, i.e.

$$\mathcal{M}^{\mathsf{QB3K}}, \Gamma \Vdash^+ \Phi \quad \text{and} \quad \mathcal{M}^{\mathsf{QB3K}}, \Gamma \Vdash^- \Phi.$$

By the analogue of Lemma 5.6 for QB3K, this is equivalent to $\Phi \in \Gamma$ and ${\sim}\Phi \in \Gamma$. At the same time we have ${\sim}\Phi \to (\Phi \to \Psi) \in \Gamma$ for all $\Psi \in \mathrm{Sent}_{\sigma_S}$. Hence we easily get $\Gamma = \mathrm{Sent}_{\sigma_S}$ – a contradiction. Then one can argue as in the proof of Theorem 5.7. $\square$

Theorem 6.2. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash_{\mathsf{QBK}^\circ} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash^\circ_{\mathsf{QBK}} \Delta.$$

The proof is similar to the proof of Theorem 6.1.

Now we proceed by considering extensions of type (ii). For each such extension L we shall assume that:

a. $\mathcal{W}^L$ belongs to $\mathcal{K}_L$, i.e. $\mathcal{W}^L \Vdash \Phi$ for all $\Phi \in L$;

b. for every frame $\mathcal{W} = \langle W, R\rangle$,

$$\mathcal{W} \in \mathcal{K}_L \quad \Longleftrightarrow \quad R \text{ has certain properties.}$$

Then the strong completeness theorem for L can be obtained in the same way as for QBK. As an illustration, consider three axiom schemes:

D. $\Box\Phi \to \Diamond\Phi$;

T. $\Box\Phi \to \Phi$;

4. $\Box\Phi \to \Box\Box\Phi$.

As in classical modal logic, it is easy to show the following.

- D, T and 4 express seriality, reflexivity and transitivity respectively, i.e. for every frame $\mathcal{W} = \langle W, R\rangle$:

$$\begin{aligned} \mathcal{W} \Vdash^+ \mathtt{D} &\iff R \text{ is serial;}\\ \mathcal{W} \Vdash^+ \mathtt{T} &\iff R \text{ is reflexive;}\\ \mathcal{W} \Vdash^+ \mathtt{4} &\iff R \text{ is transitive.}\end{aligned}$$

- For any QBK-extension L:

$$\begin{aligned} L \text{ includes } \mathtt{D} &\implies R^L \text{ is serial;}\\ L \text{ includes } \mathtt{T} &\implies R^L \text{ is reflexive;}\\ L \text{ includes } \mathtt{4} &\implies R^L \text{ is transitive.}\end{aligned}$$

Let us introduce some related notation:

$$\mathsf{QBD} := \mathsf{QBK} + \{\mathtt{D}\}, \quad \mathsf{QBT} := \mathsf{QBK} + \{\mathtt{T}\}, \quad \mathsf{QBK4} := \mathsf{QBK} + \{\mathtt{4}\},$$
$$\mathsf{QBD4} := \mathsf{QBK} + \{\mathtt{D}, \mathtt{4}\} \quad \text{and} \quad \mathsf{QBS4} := \mathsf{QBK} + \{\mathtt{T}, \mathtt{4}\}.$$

In addition, for convenience, we denote by $\mathscr{L}$ the collection of all these logics.

Theorem 6.3. *Let $L \in \mathscr{L}$. Then for any $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$,*

$$\Gamma \vdash_L \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_L \Delta.$$

The proof is similar to the proof of Theorem 5.7 (taking into account the above remarks).

Of course, the two types of extension can be combined. For example, take

$$\mathsf{QB3S4} := \mathsf{QBS4} + \{\mathtt{Exp}\}.$$

Hence 'undefined' must be excluded in QB3S4-*models* (as in QB3K), and the accessibility relations must be preorderings (as in QBS4). Denote by $\vDash^3_{\mathsf{QBS4}}$ the relativization of $\vDash$ to the models of this kind. Clearly, $\mathcal{M}^{\mathsf{QB3S4}}$ will turn out to be a QB3S4-model. So $\vdash_{\mathsf{QB3S4}}$ will coincide with $\vDash^3_{\mathsf{QBS4}}$, i.e. the completeness theorem for QB3S4 will hold.

The logics QBS4 and QB3S4 will play an important role in Section 8, which is devoted to faithful embeddings of Nelson's quantified logics.

7 Constant domain variant

Consider the following two variants of the *Barcan scheme*:

$\mathtt{Ba}^\Box$. $\forall x\, \Box\Phi \to \Box\forall x\, \Phi$;

$\mathtt{Ba}^\Diamond$. $\Diamond\exists x\, \Phi \to \exists x\, \Diamond\Phi$.

Note that the scheme corresponding to the converse to $\mathtt{Ba}^\Box$ is derivable in QBK:

1	$\forall x\, \Phi \to \Phi$	$\mathtt{Q1}$
2	$\Box\forall x\, \Phi \to \Box\Phi$	from 1 by $\mathtt{MD}$
3	$\Box\forall x\, \Phi \to \forall x\, \Box\Phi$	from 2 by $\mathtt{BR1}$.

The converse scheme for $\mathtt{Ba}^\Diamond$ is obtained similarly. Denote

$$\mathsf{QBK}^\sharp_\Box := \mathsf{QBK} + \{\mathtt{Ba}^\Box\} \quad \text{and} \quad \mathsf{QBK}^\sharp_\Diamond := \mathsf{QBK} + \{\mathtt{Ba}^\Diamond\}.$$

Although the proof of the proposition below is exactly similar to the case of classical quantified modal logic, we provide it for expository purposes.

Proposition 7.1. $\mathsf{QBK}^\sharp_\Box$ *and* $\mathsf{QBK}^\sharp_\Diamond$ *coincide.*

Proof. Let us show that $\mathtt{Ba}^\Diamond$ is derivable in $\mathsf{QBK}^\sharp_\Box$:

1	$\forall x\, \Box\neg\Phi \to \Box\forall x\, \neg\Phi$	$\mathtt{Ba}^{\Box}$
2	$\neg\Box\forall x\, \neg\Phi \to \neg\forall x\, \Box\neg\Phi$	from 1
3	$\Diamond\neg\forall x\, \neg\Phi \leftrightarrow \neg\Box\forall x\, \neg\Phi$	M1
4	$\exists x\, \Phi \leftrightarrow \neg\forall x\, \neg\Phi$	classical logic
5	$\Diamond\exists x\, \Phi \leftrightarrow \Diamond\neg\forall x\, \neg\Phi$	from 4 by PR
6	$\neg\forall x\, \Box\neg\Phi \leftrightarrow \exists x\, \neg\Box\neg\Phi$	classical logic
7	$\neg\Box\neg\Phi \leftrightarrow \Diamond\neg\neg\Phi$	M1
8	$\neg\neg\Phi \leftrightarrow \Phi$	classical logic
9	$\Diamond\neg\neg\Phi \leftrightarrow \Diamond\Phi$	from 8 by PR
10	$\neg\Box\neg\Phi \leftrightarrow \Diamond\Phi$	from 7, 9
11	$\exists x\, \neg\Box\neg\Phi \leftrightarrow \exists x\, \Diamond\Phi$	from 10 by PR
12	$\Diamond\exists x\, \Phi \to \exists x\, \Diamond\Phi$	from 5, 3, 2, 6, 11.

In a similar way, one can show that $\mathtt{Ba}^{\Box}$ is derivable in $\mathsf{QBK}^{\sharp}_{\Diamond}$. □

In what follows we shall write $\mathsf{QBK}^{\sharp}$ instead of $\mathsf{QBK}^{\sharp}_{\Box}$ and $\mathsf{QBK}^{\sharp}_{\Diamond}$, and also $\vdash^{\sharp}$ instead of $\vdash_{\mathsf{QBK}^{\sharp}}$. Furthermore, we shall restrict ourselves to at most countable signatures, since dealing with uncountable signatures gives rise to some problems even for the variant of QK with constant domains; cf. [6, Lemma 7.1.2], where the proof makes significant use of the countability of the signature.

We shall call a QBK_{σ}-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^{+}, \mathscr{A}^{-} \rangle$ a $\mathsf{QBK}^{\sharp}_{\sigma}$*-model* if $A_u = A_v$ for all $u, v \in W$. In other words, $\mathsf{QBK}^{\sharp}_{\sigma}$-models are QBK_{σ}-models with constant domains. In fact, the Barcan scheme only guarantees that for any $u, v \in W$,

$$u\, R\, v \quad \Longrightarrow \quad A_v \;=\; A_u.$$

However, this is not crucial, since for a given $u \in W$ one can always pass to the generated submodel whose set of worlds is $R\,(u)$. Denote by $\vDash^{\sharp}$ the relativization of $\vDash$ to the class of all $\mathsf{QBK}^{\sharp}_{\sigma}$-models.

Theorem 7.2 ((on the soundness of $\mathsf{QBK}^{\sharp}_{\sigma}$))**.** *For any* $\Gamma \subseteq \mathrm{Sent}_{\sigma}$ *and* $\Delta \subseteq \mathrm{Form}_{\sigma}$*,*

$$\Gamma \vdash^{\sharp} \Delta \quad \Longrightarrow \quad \Gamma \vDash^{\sharp} \Delta.$$

The proof is similar to the proof of Theorem 4.4. In the present case the verifiability of $\mathtt{Ba}^{\Box}$ (or $\mathtt{Ba}^{\Diamond}$) in all $\mathsf{QBK}^{\sharp}_{\sigma}$-models is established as in classical quantified modal logic.

As in Section 5, fix some set $S^{\star}$ of cardinality $|\mathrm{Sent}_{\sigma}|$. For brevity, we shall write $\sigma_{\star}$ instead of $\sigma_{S^{\star}}$. Take

$$W^{\sharp} \;:=\; \text{the collection of all saturated } \sigma_{\star}\text{-theories.}$$

Obviously, $\mathrm{Const}\,(\Gamma) = \mathrm{Const}_{\sigma_{\star}}$ for all $\Gamma \in W^{\sharp}$. By the *canonical frame for* $\mathsf{QBK}^{\sharp}$ and the *canonical model for* $\mathsf{QBK}^{\sharp}$ we mean

$$\mathcal{W}^{\sharp} \;=\; \langle W^{\sharp}, R^{\sharp} \rangle \quad \text{and} \quad \mathcal{M}^{\sharp} \;=\; \langle \mathcal{W}^{\sharp}, \left(\mathscr{A}^{\sharp}\right)^{+}, \left(\mathscr{A}^{\sharp}\right)^{-} \rangle,$$

where the components $R^{\sharp}$, $\left(\mathscr{A}^{\sharp}\right)^{+}$ and $\left(\mathscr{A}^{\sharp}\right)^{-}$ are defined in the same way as R^{QBK}, $\left(\mathscr{A}^{\mathsf{QBK}}\right)^{+}$ and $\left(\mathscr{A}^{\mathsf{QBK}}\right)^{-}$, but with W^{QBK} replaced by $W^{\sharp}$.

Lemma 7.3. *For any* $\Gamma \in W^{\sharp}$ *and* $\Phi \in \mathrm{Sent}_{\sigma^{\star}}$*:*

$$\Box\Phi \;\in\; \Gamma \quad \Longleftrightarrow \quad \textit{for every } \Delta \in W^{\sharp}, \;\; \textit{if } \Gamma_{\Box} \subseteq \Delta, \;\; \textit{then } \Phi \in \Delta;$$
$$\Diamond\Phi \;\in\; \Gamma \quad \Longleftrightarrow \quad \textit{there exists } \Delta \in W^{\sharp} \;\; \textit{such that } \Gamma_{\Box} \subseteq \Delta \;\; \textit{and } \Phi \in \Delta.$$

The proof is exactly similar to the case of classical quantified modal logic; see, for example, [6, Lemma 7.1.2].

From this we easily obtain:

Lemma 7.4 ((on the canonical model for $\mathsf{QBK}^\sharp$)). *For any $\Gamma \in W^\sharp$ and A_Γ-sentence Φ:*

$$\begin{array}{lcl} \mathcal{M}^\sharp, \Gamma \Vdash^+ \Phi & \Longleftrightarrow & \Phi \ \in \ \Gamma; \\ \mathcal{M}^\sharp, \Gamma \Vdash^- \Phi & \Longleftrightarrow & \sim\Phi \ \in \ \Gamma. \end{array}$$

Proof. By induction on the complexity of Φ.

Of course, since all worlds deal with the same domain $\mathrm{Const}_{\sigma_\star}$, we cannot use Lemma 5.3. However, the cases when Φ does not begin with $\Box$ or $\Diamond$ are handled as in the proof of Lemma 5.6 (because they do not require Lemma 5.3).

Suppose $\Phi = \Box\Psi$. Obviously, the equivalence for $\Vdash^+$ follows from Lemma 7.3 (and the inductive hypothesis). Let us now look at falsification:

$$\begin{array}{lcl} \mathcal{M}^\sharp, \Gamma \Vdash^- \Box\Psi & \Longleftrightarrow & \text{there exists } \ \Delta \in W^\sharp \ \text{ such that} \\ & & \Gamma_\Box \subseteq \Delta \ \text{ and } \ \mathcal{M}^\sharp, \Delta \Vdash^- \Psi \\ & \Longleftrightarrow & \text{there exists } \ \Delta \in W^\sharp \ \text{ such that} \\ & & \Gamma_\Box \subseteq \Delta \ \text{ and } \ \sim\Psi \in \Delta \\ & \Longleftrightarrow & \Diamond\sim\Psi \ \in \ \Gamma \\ & \Longleftrightarrow & \sim\Box\Psi \ \in \ \Gamma. \end{array}$$

Here the third equivalence is guaranteed by Lemma 7.3, and the latter follows from the fact that $\Diamond\sim\Psi \Leftrightarrow \sim\Box\Psi$ is derivable in QBK:

1	$\Diamond\sim\Psi \Leftrightarrow \sim\Box\sim\sim\Psi$	`M4`
2	$\sim\sim\Psi \Leftrightarrow \Psi$	`SN1`*
3	$\sim\Box\sim\sim\Psi \Leftrightarrow \sim\Box\Psi$	from 2 by `WR`
4	$\Diamond\sim\Psi \Leftrightarrow \sim\Box\Psi$	from 1, 3.

Similarly for $\Phi = \Diamond\Psi$. ☐

Theorem 7.5 ((on the strong completeness of $\mathsf{QBK}^\sharp$)). *For any $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$,*

$$\Gamma \vdash^\sharp \Delta \quad \Longleftrightarrow \quad \Gamma \vDash^\sharp \Delta.$$

Proof. $\boxed{\Longrightarrow}$ See Theorem 7.2.

$\boxed{\Longleftarrow}$ Suppose that $\Gamma \nvdash^\sharp \Delta$. Fix some admissible $S \subseteq S^\star$ of cardinality $\aleph_0$. Let λ be a one-one function from Var onto $\{\underline{s} \mid s \in S\}$. Take

$$\Delta' \ := \ \{\lambda\Psi \mid \Psi \in \Delta\}.$$

Clearly, $\Gamma \nvdash^\sharp \Delta'$. By Lemma 5.4, there will be $\Gamma' \in W^\sharp$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash^\sharp \Delta'$. Obviously, λ can be viewed as a ground $A_{\Gamma'}$-substitution. So by Lemma 7.4, we have $\mathcal{M}^\sharp, \Gamma' \Vdash \Phi$ for all $\Phi \in \Gamma$ and also $\mathcal{M}^\sharp, \Gamma' \nVdash \lambda\Psi$ for all $\Psi \in \Delta$. Hence $\Gamma \nvDash^\sharp \Delta$. ☐

Of course, by analogy with QBK, we may consider various natural extensions of $\mathsf{QBK}^\sharp$. In particular, as is easily verified, for extensions obtained from logics in $\mathscr{L}$, $\mathsf{QB3K}$ or QBK° by adding $\mathtt{Ba}^\Box$ the corresponding strong completeness theorems will hold.

8 Faithful embedding of quantified Nelson's logics

Let QN4 be the quantified version of Nelson's constructive logic, QN3 be its extension obtained by excluding 'overdefined', i.e. adding the relativization of the scheme Exp to the language of QN4; see [11, 2], and also [7, 12].

Recall that syntactically, the language of QN4 is simply the non-modal fragment of the language of QBK. However, there are fundamental differences between QN4 and QBK at the semantic level: $\to$ and $\forall$ are verified in QN4 by analogy with intuitionistic logic, and not with classical logic, as in QBK. Intuitively, QN4 enriches quantified intuitionistic logic, QInt, by adding strong negation.

Below, by *constructive σ-formulas* we shall mean σ-formulas in the language of QN4, i.e. those containing no occurrences of $\Box$ and $\Diamond$. Negation normal forms in the language of QN4 are defined in the same way as in QBK. It is well-known that a result similar to Theorem 3.8 holds for QN4, but with weak equivalence instead of strong equivalence:

Theorem 8.1 ((see, for example, [12])). *For every constructive σ-formula Φ there exists an n.n.f. $\overline{\Phi}$ such that $\Phi \leftrightarrow \overline{\Phi} \in \mathsf{QN4}_\sigma$. Moreover, there is an algorithm that constructs, given any constructive σ-formula Φ, a suitable n.n.f. $\overline{\Phi}$.*[12]

Next, we define a translation τ that associates with each n.n.f. in the language of $\mathsf{QN4}_\sigma$ a formula in the language of QBK_σ:

$$
\begin{array}{rcl}
\tau\,(P\,(t_1,\ldots,t_n)) & := & \Box P\,(t_1,\ldots,t_n); \\
\tau\,({\sim}\,P\,(t_1,\ldots,t_n)) & := & {\sim}\,\Diamond P\,(t_1,\ldots,t_n); \\
\tau\,(\Phi\wedge\Psi) & := & \tau\,(\Phi)\wedge\tau\,(\Psi); \\
\tau\,(\Phi\vee\Psi) & := & \tau\,(\Phi)\vee\tau\,(\Psi); \\
\tau\,(\Phi\to\Psi) & := & \Box\,(\tau\,(\Phi)\to\tau\,(\Psi)); \\
\tau\,(\bot) & := & \bot; \\
\tau\,({\sim}\,\bot) & := & \bot\to\bot; \\
\tau\,(\forall x\,\Phi) & := & \Box\forall x\,\tau\,(\Phi); \\
\tau\,(\exists x\,\Phi) & := & \exists x\,\tau\,(\Phi).
\end{array}
$$

The mapping τ can be naturally extended to the set of all constructive σ-formulas: if Φ is not an n.n.f., then take

$$\tau\,(\Phi) \;:=\; \tau\,\big(\overline{\Phi}\big).$$

Formally, we should write τ_σ instead of τ, but it will always be clear from the context which signature we are talking about.

Clearly, the restriction of τ to formulas not containing $\sim$ coincides with the Gödel–McKinsey–Tarski translation, which faithfully embeds QInt into the modal logic QS4. Furthermore, τ may be viewed as a (quantified) extension of the propositional translation proposed earlier in [18, Section 7.1].

Remark 8.2. If we fix the usual algorithm for reducing constructive σ-formulas to n.n.f.'s, then the mapping τ can be described as follows:

$$\tau\,(P\,(t_1,\ldots,t_n)) \;:=\; \Box P\,(t_1,\ldots,t_n); \qquad \tau\,({\sim}\,P\,(t_1,\ldots,t_n)) \;:=\; {\sim}\,\Diamond P\,(t_1,\ldots,t_n);$$

[12]Unlike in Theorem 3.8, we cannot replace $\leftrightarrow$ by $\Leftrightarrow$ here. In particular, it can be shown that there exists no propositional n.n.f. φ such that ${\sim}\,(p\to q) \Leftrightarrow \varphi \in \mathsf{N4}$.

$$
\begin{aligned}
\tau(\Phi \wedge \Psi) &:= \tau(\Phi) \wedge \tau(\Psi); & \tau(\sim(\Phi \wedge \Psi)) &:= \tau(\sim \Phi) \vee \tau(\sim \Psi);\\
\tau(\Phi \vee \Psi) &:= \tau(\Phi) \vee \tau(\Psi); & \tau(\sim(\Phi \vee \Psi)) &:= \tau(\sim \Phi) \wedge \tau(\sim \Psi);\\
\tau(\Phi \to \Psi) &:= \Box(\tau(\Phi) \to \tau(\Psi)); & \tau(\sim(\Phi \to \Psi)) &:= \tau(\Phi) \wedge \tau(\sim \Psi);\\
\tau(\bot) &:= \bot; & \tau(\sim \bot) &:= \bot \to \bot;\\
\tau(\forall x\, \Phi) &:= \Box\forall x\, \tau(\Phi); & \tau(\sim \forall x\, \Phi) &:= \exists x\, \tau(\sim \Phi);\\
\tau(\exists x\, \Phi) &:= \exists x\, \tau(\Phi); & \tau(\sim \exists x\, \Phi) &:= \Box\forall x\, \tau(\sim \Phi);\\
& & \tau(\sim\sim \Phi) &:= \tau(\Phi).
\end{aligned}
$$

Say that a QBK_σ-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^+, \mathscr{A}^- \rangle$ is a $\mathsf{QN4}_\sigma$*-model* if R is a preordering on W, and for any $u, v \in W$,

$$u\, R\, v \quad \Longrightarrow \quad P^{\mathfrak{A}_u^\pm} \subseteq P^{\mathfrak{A}_v^\pm} \quad \text{for all } \ P \in \mathrm{Pred}_\sigma.$$

Further, a $\mathsf{QN4}_\sigma$-model $\mathcal{M}$ is called a $\mathsf{QN3}_\sigma$*-model* if for every $w \in W$,

$$P^{\mathfrak{A}_w^+} \cap P^{\mathfrak{A}_w^-} \ = \ \varnothing \quad \text{for all } \ P \in \mathrm{Pred}_\sigma.$$

The verifiability and falsifiability relations for $\mathsf{QN4}_\sigma$ are defined in the natural way:

$$
\begin{aligned}
\mathcal{M}, w \Vdash^+ P(t_1, \ldots, t_n) &\iff \mathfrak{A}_w^+ \Vdash P(t_1, \ldots, t_n);\\
\mathcal{M}, w \Vdash^- P(t_1, \ldots, t_n) &\iff \mathfrak{A}_w^- \Vdash P(t_1, \ldots, t_n);\\
\mathcal{M}, w \Vdash^+ \Phi \wedge \Psi &\iff \mathcal{M}, w \Vdash^+ \Phi \ \text{ and } \ \mathcal{M}, w \Vdash^+ \Psi;\\
\mathcal{M}, w \Vdash^- \Phi \wedge \Psi &\iff \mathcal{M}, w \Vdash^- \Phi \ \text{ or } \ \mathcal{M}, w \Vdash^- \Psi;\\
\mathcal{M}, w \Vdash^+ \Phi \vee \Psi &\iff \mathcal{M}, w \Vdash^+ \Phi \ \text{ or } \ \mathcal{M}, w \Vdash^+ \Psi;\\
\mathcal{M}, w \Vdash^- \Phi \vee \Psi &\iff \mathcal{M}, w \Vdash^- \Phi \ \text{ and } \ \mathcal{M}, w \Vdash^- \Psi;\\
\mathcal{M}, w \Vdash^+ \Phi \to \Psi &\iff \text{for every } \ u \in R(w),\\
&\qquad \text{if } \ \mathcal{M}, u \Vdash^+ \Phi, \ \text{ then } \ \mathcal{M}, u \Vdash^+ \Psi;\\
\mathcal{M}, w \Vdash^- \Phi \to \Psi &\iff \mathcal{M}, w \Vdash^+ \Phi \ \text{ and } \ \mathcal{M}, w \Vdash^- \Psi;\\
\mathcal{M}, w \Vdash^+ \bot &\iff 0 \neq 0;\\
\mathcal{M}, w \Vdash^- \bot &\iff 0 = 0;\\
\mathcal{M}, w \Vdash^+ \sim \Phi &\iff \mathcal{M}, w \Vdash^- \Phi;\\
\mathcal{M}, w \Vdash^- \sim \Phi &\iff \mathcal{M}, w \Vdash^+ \Phi;\\
\mathcal{M}, w \Vdash^+ \forall x\, \Phi &\iff \mathcal{M}, u \Vdash^+ \Phi(x/\underline{a}) \quad \text{for all } \ u \in R(w) \ \text{ and } \ a \in A_u;\\
\mathcal{M}, w \Vdash^- \forall x\, \Phi &\iff \mathcal{M}, w \Vdash^- \Phi(x/\underline{a}) \quad \text{for some } \ a \in A_w;\\
\mathcal{M}, w \Vdash^+ \exists x\, \Phi &\iff \mathcal{M}, w \Vdash^+ \Phi(x/\underline{a}) \quad \text{for some } \ a \in A_w;\\
\mathcal{M}, w \Vdash^- \exists x\, \Phi &\iff \mathcal{M}, u \Vdash^- \Phi(x/\underline{a}) \quad \text{for all } \ u \in R(w) \ \text{ and } \ a \in A_u.
\end{aligned}
$$

Here, instead of $\Vdash^+$ and $\Vdash^-$, it would be more accurate to write $\Vdash^+_{\mathsf{QN4}}$ and $\Vdash^-_{\mathsf{QN4}}$, but in what follows it will always be clear from the context what kind of logic we are talking about. Note that $\mathcal{M}$ is a $\mathsf{QN3}_\sigma$-model if and only if there are no $w \in W$ and atomic A_w-sentence Φ such that $\mathfrak{A}_w^+ \Vdash \Phi$ and $\mathfrak{A}_w^- \Vdash \Phi$; cf. the definition of $\mathsf{QB3K}_\sigma$-model in Section 6.

The *semanical consequence* relations for QN4 and QN3 are defined by analogy with $\models_{\mathsf{QBK}}$ and $\models^3_{\mathsf{QBK}}$; denote them by $\models_{\mathsf{QN4}}$ and $\models^3_{\mathsf{QN4}}$ respectively.

Theorem 8.3 ((see [12] and [7])). *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$*:*

$$\begin{aligned} \Gamma \vdash_{\mathsf{QN4}} \Delta \quad &\Longleftrightarrow \quad \Gamma \vDash_{\mathsf{QN4}} \Delta; \\ \Gamma \vdash_{\mathsf{QN3}} \Delta \quad &\Longleftrightarrow \quad \Gamma \vDash^3_{\mathsf{QN4}} \Delta, \end{aligned}$$

where $\vdash_{\mathsf{QN4}}$ *and* $\vdash_{\mathsf{QN3}}$ *denote the derivability relations for* QN4 *and* QN3 *respectively.*

Let $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^+, \mathscr{A}^- \rangle$ be an arbitrary $\mathsf{QBS4}_\sigma$-model, i.e. a QBK_σ-model in which the accessibility relation is reflexive and transitive. Associate with it the triple

$$\mathcal{M}' \;:=\; \langle \mathcal{W}, (\mathscr{A}^+)', (\mathscr{A}^-)' \rangle$$

such that for any $w \in W$ and atomic A_w-sentence Φ:

$$\begin{aligned} \left(\mathfrak{A}^+_w\right)' \Vdash \Phi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \Box\Phi; \\ \left(\mathfrak{A}^-_w\right)' \Vdash \Phi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^- \Diamond\Phi. \end{aligned}$$

Here we set $(A_w)' = A_w$ for all $w \in W$. It is easy to understand that $\mathcal{M}'$ will turn out to be a $\mathsf{QN4}_\sigma$-model. Moreover, if $\mathcal{M}$ is a $\mathsf{QB3S4}_\sigma$-model, then $\mathcal{M}'$ will be a $\mathsf{QN3}_\sigma$-model. The following statement is similar to Lemma 14 in [18].

Lemma 8.4. *Let* $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^+, \mathscr{A}^- \rangle$ *be a* $\mathsf{QBS4}_\sigma$*-model. Then for any* $w \in W$ *and constructive* A_w*-sentence* Φ*,*

$$\mathcal{M}', w \Vdash^+ \Phi \quad \Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \tau(\Phi).$$

Proof. By Theorem 8.1, we can assume that Φ is an n.n.f.

By induction on the complexity of Φ.

The case when Φ is atomic is trivial.

The case when $\Phi = \sim P(t_1, \dots, t_n)$ is slightly more complicated:

$$\begin{aligned} \mathcal{M}', w \Vdash^+ \sim P(t_1, \dots, t_n) \quad &\Longleftrightarrow \quad \mathcal{M}', w \Vdash^- P(t_1, \dots, t_n) \\ &\Longleftrightarrow \quad \left(\mathfrak{A}^-_w\right)' \Vdash P(t_1, \dots, t_n) \\ &\Longleftrightarrow \quad \mathcal{M}, w \Vdash^- \Diamond P(t_1, \dots, t_n) \\ &\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \sim \Diamond P(t_1, \dots, t_n) \\ &\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \tau(\sim P(t_1, \dots, t_n)) \end{aligned}$$

Suppose $\Phi = \forall x\, \Psi$. Then

$$\begin{aligned} \mathcal{M}', w \Vdash^+ \forall x\, \Psi \quad &\Longleftrightarrow \quad \mathcal{M}', u \Vdash^+ \Psi(x/\underline{a}) \quad \text{for all } u \in R(w) \text{ and } a \in A_u \\ &\Longleftrightarrow \quad \mathcal{M}, u \Vdash^+ \tau(\Psi(x/\underline{a})) \quad \text{for all } u \in R(w) \text{ and } a \in A_u \\ &\Longleftrightarrow \quad \mathcal{M}, u \Vdash^+ \tau(\Psi)(x/\underline{a}) \quad \text{for all } u \in R(w) \text{ and } a \in A_u \\ &\Longleftrightarrow \quad \mathcal{M}, u \Vdash^+ \forall x\, \tau(\Psi) \quad \text{for all } u \in R(w) \\ &\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \Box\forall x\, \tau(\Psi) \\ &\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \tau(\Phi). \end{aligned}$$

Suppose $\Phi = \exists x\, \Psi$. Then

$$\mathcal{M}', w \Vdash^+ \exists x\, \Psi \quad \Longleftrightarrow \quad \mathcal{M}', w \Vdash^+ \Psi(x/\underline{a}) \quad \text{for some } a \in A_w$$

$$
\begin{aligned}
&\iff \quad \mathcal{M}', w \Vdash^{+} \tau\left(\Psi\left(x/\underline{a}\right)\right) \quad \text{for some } \ a \in A_w \\
&\iff \quad \mathcal{M}', w \Vdash^{+} \tau\left(\Psi\right)\left(x/\underline{a}\right) \quad \text{for some } \ a \in A_w \\
&\iff \quad \mathcal{M}, w \Vdash^{+} \exists x\, \tau\left(\Psi\right) \\
&\iff \quad \mathcal{M}, w \Vdash^{+} \tau\left(\Phi\right).
\end{aligned}
$$

The remaining cases are handled as in BK; see [18, Section 7]. □

Remark 8.5. In the formulation of Lemma 8.4, we cannot replace $\Vdash^{+}$ by $\Vdash^{-}$. Indeed, suppose $\Phi = \Psi \to \Theta$. Then, as is easily verified,

$$
\mathcal{M}', w \Vdash^{-} \Psi \to \Theta \quad \Longrightarrow \quad \mathcal{M}, w \Vdash^{-} \tau\left(\Psi \to \Theta\right),
$$

but the converse implication may in general fail. A similar problem arises for $\Phi = \forall x\, \Psi$.

From this one easily obtains:

Theorem 8.6. *For every constructive σ-sentence Φ,*

$$
\Phi \ \in \ \mathsf{QN4}_{\sigma} \quad \iff \quad \tau\left(\Phi\right) \ \in \ \mathsf{QBS4}_{\sigma}.
$$

In other words, τ faithfully embeds $\mathsf{QN4}$ *into* $\mathsf{QBS4}$.

Proof. In view of the completeness theorems for $\mathsf{QN4}$ and $\mathsf{QBS4}$, we need to show that

$$
\vDash_{\mathsf{QN4}} \Phi \quad \iff \quad \vDash_{\mathsf{QBS4}} \tau\left(\Phi\right)
$$

(see Theorems 8.3 and 6.3 respectively).

$\boxed{\Longrightarrow}$ Suppose that $\vDash_{\mathsf{QN4}} \Phi$. Consider arbitrary $\mathsf{QBS4}_{\sigma}$-model $\mathcal{M}$ and $w \in W$. Since $\mathcal{M}', w \Vdash^{+} \Phi$, we have $\mathcal{M}, w \Vdash^{+} \tau\left(\Phi\right)$ by Lemma 8.4. Hence $\vDash_{\mathsf{QBS4}} \tau\left(\Phi\right)$.

$\boxed{\Longleftarrow}$ Suppose that $\vDash_{\mathsf{QBS4}} \tau\left(\Phi\right)$. Consider arbitrary $\mathsf{QN4}_{\sigma}$-model $\mathcal{M}$ and $w \in W$. Obviously, $\mathcal{M}$ can be thought of as a $\mathsf{QBS4}_{\sigma}$-model; in this case $\mathcal{M}'$ will coincide with $\mathcal{M}$. So, since $\mathcal{M}, w \Vdash^{+} \tau\left(\Phi\right)$, we have $\mathcal{M}, w \Vdash^{+} \Phi$ by Lemma 8.4. Hence $\vDash_{\mathsf{QN4}} \Phi$. □

Furthermore, we have:

Theorem 8.7. *For every constructive σ-sentence Φ,*

$$
\Phi \ \in \ \mathsf{QN3}_{\sigma} \quad \iff \quad \tau\left(\Phi\right) \ \in \ \mathsf{QB3S4}_{\sigma}.
$$

In other words, τ faithfully embeds $\mathsf{QN3}$ *into* $\mathsf{QB3S4}$.

The proof follows from Theorem 8.6, taking into account the completeness theorems for $\mathsf{QN4}$, $\mathsf{QBS4}$, $\mathsf{QN3}$ and $\mathsf{QB3S4}$.

Remark 8.8. Theorems 8.6 and 8.7 could be reformulated in terms of derivability or semantical consequence relations (with non-empty sets of hypotheses). In fact, such formulations will turn out to be equivalent to the original ones in view of the strong completeness theorems for $\mathsf{QN4}$, $\mathsf{QBS4}$, $\mathsf{QN3}$ and $\mathsf{QB3S4}$.

A result similar to Theorem 8.7 holds for the logic $\mathsf{QN4}^\circ$, which is obtained from $\mathsf{QN4}$ by excluding 'undefined', i.e. adding the relativization of the scheme $\mathtt{ExM}$ to the language of $\mathsf{QN4}$. More precisely, take

$$\mathsf{QBS4}^\circ \;:=\; \mathsf{QBS4} + \{\mathtt{ExM}\}.$$

Then τ will embed $\mathsf{QN4}^\circ$ into $\mathsf{QBS4}^\circ$. In this case, formally, we shall need the strong completeness theorem for $\mathsf{QN4}^\circ$, which can be proved by analogy with Theorem 6.2, using the canonical model for $\mathsf{QN4}$.

Finally, all the results of the present section are easily relativized to the case of constant domains. Here the role of the Barcan scheme for $\mathsf{QN4}$-extensions will be played by

$\mathtt{CD}$. $\forall x\,(\Phi \vee \Psi) \to \Phi \vee \forall x\,\Psi$, where x does not occur free in Φ.

However, this kind of relativization may be criticized from a constructive point of view, since $\mathtt{CD}$ is refuted in Kleene's realizability semantics (let alone Nelson's semantics).

9 Interpolation properties

It is known that some extensions of classical quantified modal logic, QK, have Craig's interpolation property; see, for example, [5]. In this section, using the technique of [15, Section 2], we shall show how interpolation results for QK-extensions can be transferred to QBK-extensions.

For each σ-formula Φ, take

$$\mathrm{O}\,(\Phi) \;:=\; \{\varepsilon \in \sigma \mid \varepsilon \text{ occurs in } \Phi\} \cup \mathrm{FV}\,(\Phi).$$

Let L be a QK-extension. Say that L has *Craig's interpolation property*, or *CIP* for short, if for any signature σ and $\Phi \to \Psi \in L_\sigma$ there exists $\Theta \in \mathrm{Form}_\sigma$ such that

$$\{\Phi \to \Theta, \Theta \to \Psi\} \;\subseteq\; L \quad \text{and} \quad \mathrm{O}\,(\Theta) \;\subseteq\; \mathrm{O}\,(\Phi) \cap \mathrm{O}\,(\Psi).$$

Here Θ is called a *Craig interpolant for* $\Phi \to \Psi$ *in* L. This terminology will be used for QBK-extensions as well.

Remark 9.1. In fact, the requirement that $\mathrm{O}\,(\Phi)$ includes $\mathrm{FV}\,(\Phi)$ is not crucial, since, if necessary, all free variables of Φ can be replaced by new constant symbols.

In addition, a more subtle version of CIP for QBK-extensions is worth discussing; cf. [19, § 4]. By Theorem 3.8, we can restrict ourselves to n.n.f.'s. Given a σ-n.n.f. Φ and $P \in \mathrm{Pred}_\sigma$, call an occurrence of P in Φ *positive* if it is not inside the scope of $\sim$, and *negative* otherwise. For each σ-n.n.f. Φ we set:

$$\begin{aligned}
\mathrm{D}\,(\Phi) \;&:=\; \{c \in \mathrm{Const}_\sigma \mid c \text{ occurs in } \Phi\} \cup \mathrm{FV}\,(\Phi);\\
\mathrm{P}^+\,(\Phi) \;&:=\; \{P \in \mathrm{Pred}_\sigma \mid P \text{ occurs positively in } \Phi\};\\
\mathrm{P}^-\,(\Phi) \;&:=\; \{P \in \mathrm{Pred}_\sigma \mid P \text{ occurs negatively in } \Phi\}.
\end{aligned}$$

Let L be a QBK-extension. We shall say that L has the *strong interpolation property*, or *SIP* for short, if for any signature σ and σ-n.n.f. $\Phi \to \Psi$ in L there exists a σ-n.n.f. Θ such that

$$\{\Phi \to \Theta, \Theta \to \Psi\} \;\subseteq\; L, \quad \mathrm{D}\,(\Theta) \;\subseteq\; \mathrm{D}\,(\Phi) \cap \mathrm{D}\,(\Psi),$$

$$\mathrm{P}^+(\Theta) \subseteq \mathrm{P}^+(\Phi) \cap \mathrm{P}^+(\Psi) \quad \text{and} \quad \mathrm{P}^-(\Theta) \subseteq \mathrm{P}^-(\Phi) \cap \mathrm{P}^-(\Psi).$$

Here Θ is called a *strong interpolant for* $\Phi \to \Psi$ *in* L. It is easy to see that SIP is stronger than CIP: if L has SIP, then it has CIP.

Obviously, to each QBK-extension L in $\mathscr{L}$ (from the formulation of Theorem 6.3) there will correspond the QK-extension $\underline{L}$ (without 'B' in the name). It is known that all such $\underline{L}$'s have CIP; see [5]. Our next goal is to obtain SIP for all logics in $\mathscr{L}$.

For any signature σ, take

$$\underline{\sigma} := \sigma \cup \{\underline{P} \mid P \in \mathrm{Pred}_\sigma\},$$

where $\underline{P}$ is a new predicate symbol of the same arity as P. Consider arbitrary L in $\mathscr{L}$ and signature σ. With each L_σ-model $\mathcal{M}$ we associate the $\underline{L}_{\underline{\sigma}}$-model

$$\underline{\mathcal{M}} := \langle \mathcal{W}, \underline{\mathscr{A}} \rangle,$$

where $\underline{\mathscr{A}}$ denotes the family $\langle \underline{\mathfrak{A}}_w : w \in W \rangle$ such that for every $w \in W$:

- $\underline{A}_w = A_w$;
- $c^{\underline{\mathfrak{A}}_w} = c^{\mathfrak{A}_w^+}$ for all $c \in \mathrm{Const}_\sigma$;
- $P^{\underline{\mathfrak{A}}_w} = P^{\mathfrak{A}_w^+}$ and $\underline{P}^{\underline{\mathfrak{A}}_w} = P^{\mathfrak{A}_w^-}$ for all $P \in \mathrm{Pred}_\sigma$.

Obviously, $\underline{\mathcal{M}}$ is a $\underline{L}_{\underline{\sigma}}$-model. Moreover, for every $\underline{L}_{\underline{\sigma}}$-model $\mathcal{N}$ there exists a (unique) L_σ-model $\mathcal{M}$ such that $\underline{\mathcal{M}} = \mathcal{N}$.

Further, define a translation ρ from the set of all σ-n.n.f.'s to the set of all $\underline{\sigma}$-formulas without strong negation as follows (cf. [15, Section 2] and [19, Section 4]):

$$\begin{aligned}
\rho(P(t_1, \ldots, t_n)) &:= P(t_1, \ldots, t_n); \\
\rho({\sim} P(t_1, \ldots, t_n)) &:= \underline{P}(t_1, \ldots, t_n); \\
\rho(\Phi_1 \odot \Phi_2) &:= \rho(\Phi_1) \odot \rho(\Phi_2); \\
\rho(\bot) &:= \bot; \\
\rho({\sim}\bot) &:= \bot \to \bot; \\
\rho(\heartsuit\Phi) &:= \heartsuit\rho(\Phi); \\
\rho(\mathrm{Q}x\,\Phi) &:= \mathrm{Q}x\,\rho(\Phi),
\end{aligned}$$

where $\odot \in \{\wedge, \vee, \to\}$, $\heartsuit \in \{\Box, \Diamond\}$, $\mathrm{Q} \in \{\forall, \exists\}$ and $x \in \mathrm{Var}$. Of course, if necessary, the mapping ρ can be extended to all σ-formulas by setting

$$\rho(\Phi) := \rho\left(\overline{\Phi}\right).$$

Formally, we should write ρ_σ instead of ρ, but it will always be clear from the context which signature we are talking about.

Lemma 9.2. *For any L_σ-model $\mathcal{M}$, $w \in W$ and σ_{A_w}-sentence Φ,*

$$\mathcal{M}, w \Vdash^+ \Phi \quad \Longleftrightarrow \quad \underline{\mathcal{M}}, w \Vdash \rho(\Phi),$$

where $\Vdash$ denotes truth in classical quantified modal logic.[13]

[13]By definition, $\rho(\Phi)$ is a sentence in the signature $\underline{\sigma}_{A_w}$.

Proof. By Theorem 3.8, we can assume that Φ is an n.n.f. Then the lemma is proved by an easy induction on the complexity of Φ. □

Theorem 9.3. *For every σ-formula Φ,*

$$\Phi \in L \quad \Longleftrightarrow \quad \rho(\Phi) \in \underline{L}.$$

In other words, ρ faithfully embeds L into $\underline{L}$.

Proof. In view of the completeness theorems for L and $\underline{L}$, we need to show that

$$\vDash_L \Phi \quad \Longleftrightarrow \quad \vDash_{\underline{L}} \tau(\Phi)$$

(see Theorem 6.3 and, for example, [6, Section 6]). Then we can argue by analogy with the proof of Theorem 8.6, using Lemma 9.2 instead of Lemma 8.4. □

From this, using some interpolation theorems for QK-extensions, we can obtain the desired result:

Theorem 9.4. *All logics in $\mathscr{L}$ have SIP.*

Proof. Consider arbitrary L in $\mathscr{L}$ and signature σ.

Suppose that a σ-n.n.f. $\Phi \to \Psi$ belongs to L. By Theorem 9.3, we have

$$\rho(\Phi) \to \rho(\Psi) \in \underline{L}.$$

Since $\underline{L}$ has CIP, there exists a Craig interpolant Θ for $\rho(\Phi) \to \rho(\Psi)$ in $\underline{L}$. Take

$$\Theta' \;:=\; \text{the result of replacing each } \underline{P} \text{ in } \Theta \text{ by } {\sim} P.$$

Then Θ' is a σ-n.n.f., and $\rho(\Theta') = \Theta$. Applying Theorem 9.3 again, we get

$$\Phi \to \Theta' \in L \quad \text{and} \quad \Theta' \to \Psi \in L.$$

Here by construction,

$$\begin{aligned} \mathrm{D}(\Theta') &= (\mathrm{O}(\Theta') \cap \mathrm{Const}_\sigma) \cup \mathrm{FV}(\Theta') \\ &= (\mathrm{O}(\Theta) \cap \mathrm{Const}_\sigma) \cup \mathrm{FV}(\Theta) \\ &\subseteq \mathrm{D}(\rho(\Phi)) \cap \mathrm{D}(\rho(\Psi)) \\ &= \mathrm{D}(\Phi) \cap \mathrm{D}(\Psi). \end{aligned}$$

Moreover, it is easy to check that

$$\mathrm{P}^+(\Theta) \subseteq \mathrm{P}^+(\Phi) \cap \mathrm{P}^+(\Psi) \quad \text{and} \quad \mathrm{P}^-(\Theta) \subseteq \mathrm{P}^-(\Phi) \cap \mathrm{P}^-(\Psi).$$

So Θ' is a strong interpolant for $\Phi \to \Psi$ in L. □

Let us now discuss QBK-extensions of type (i), i.e. QB3K and QBK°. Define the translations ρ^3 and ρ° exactly as ρ, but with the following modifications (cf. [15, Section 2]):

$$\begin{aligned} \rho^3({\sim} P(t_1, \ldots, t_n)) &:= \underline{P}(t_1, \ldots, t_n) \wedge \neg P(t_1, \ldots, t_n); \\ \rho^\circ({\sim} P(t_1, \ldots, t_n)) &:= \underline{P}(t_1, \ldots, t_n) \vee \neg P(t_1, \ldots, t_n). \end{aligned}$$

Then we have:

Lemma 9.5. *For any* $\mathsf{QB3K}_\sigma$*-model* $\mathcal{M}$*,* $w \in W$ *and* A_w*-sentence* Φ*,*

$$\mathcal{M}, w \Vdash^{+} \Phi \quad \Longleftrightarrow \quad \underline{\mathcal{M}}, w \Vdash \rho^3(\Phi).$$

Similarly for QBK° *and* ρ°*.*

Proof. Note that for every $\mathsf{QB3K}_\sigma$-model $\mathcal{M}$ and atomic A_w-sentence $P(t_1, \ldots, t_n)$,

$$\begin{aligned} \mathcal{M}, w \Vdash^{+} \sim P(t_1, \ldots, t_n) \quad &\Longleftrightarrow \quad \underline{\mathcal{M}}, w \Vdash \underline{P}(t_1, \ldots, t_n) \\ &\Longleftrightarrow \quad \underline{\mathcal{M}}, w \Vdash \underbrace{\underline{P}(t_1, \ldots, t_n) \wedge \neg P(t_1, \ldots, t_n)}_{\rho^3(\sim P(t_1, \ldots, t_n))}. \end{aligned}$$

Therefore we can argue in the same way as in the proof of Lemma 9.2.

Similarly for QBK° and ρ°. □

On the other hand, for $L \in \{\mathsf{QBK}^\circ, \mathsf{QB3K}\}$ it is impossible, for each $\mathsf{QK}_{\underline{\sigma}}$-model $\mathcal{N}$, to find a L_σ-model $\mathcal{M}$ such that $\underline{\mathcal{M}} = \mathcal{N}$. Therefore we need to work more accurately than with $L \in \mathscr{L}$. With each $\mathsf{QK}_{\underline{\sigma}}$-model $\mathcal{N} = \langle \mathcal{W}, \mathscr{A} \rangle$ we associate the $\mathsf{QB3K}_\sigma$-model

$$\mathcal{N}^3 \ := \ \langle \mathcal{W}, \mathscr{A}^{3,+}, \mathscr{A}^{3,-} \rangle,$$

where $\mathscr{A}^{3,+}$ and $\mathscr{A}^{3,-}$ denote the families $\langle \mathfrak{A}_w^{3,+} : w \in W \rangle$ and $\langle \mathfrak{A}_w^{3,+} : w \in W \rangle$ such that for every $w \in W$:

- $A_w^{3,+} = A_w^{3,-} = A_w$;
- $c^{\mathfrak{A}_w^{3,+}} = c^{\mathfrak{A}_w^{3,-}} = c^{\mathfrak{A}_w}$ for all $c \in \mathrm{Const}_\sigma$;
- $P^{\mathfrak{A}_w^{3,+}} = P^{\mathfrak{A}_w}$ for all $P \in \mathrm{Pred}_\sigma$;
- $P^{\mathfrak{A}_w^{3,-}} = \underline{P}^{\mathfrak{A}_w} \cap \left(A_w^n \setminus P^{\mathfrak{A}_w}\right)$ for any n-ary $P \in \mathrm{Pred}_\sigma$.

Obviously, $\mathcal{N}^3$ is a $\mathsf{QB3K}_\sigma$-model. Similarly, the $\mathsf{QBK}^\circ_\sigma$-model

$$\mathcal{N}^\circ \ := \ \langle \mathcal{W}, \mathscr{A}^{\circ,+}, \mathscr{A}^{\circ,-} \rangle$$

is defined, but we need to replace $\cap$ by $\cup$ in the last item:

- $P^{\mathfrak{A}_w^{\circ,-}} = \underline{P}^{\mathfrak{A}_w} \cup \left(A_w^n \setminus P^{\mathfrak{A}_w}\right)$ for any n-ary $P \in \mathrm{Pred}_\sigma$.

The following statement will play the role of a converse to Lemma 9.5.

Lemma 9.6. *For any* $\mathsf{QK}_{\underline{\sigma}}$*-model* $\mathcal{N}$*,* $w \in W$ *and* σ_{A_w}*-sentence* Φ*,*

$$\mathcal{N}, w \Vdash \rho^3(\Phi) \quad \Longleftrightarrow \quad \mathcal{N}^3, w \Vdash^{+} \Phi.$$

Similarly for ρ° *and* $\mathcal{N}^\circ$*.*

The proof is obtained by an easy induction on the complexity of Φ.

Theorem 9.7. *For every σ-formula Φ,*

$$\Phi \in \mathsf{QB3K}_\sigma \iff \rho^3(\Phi) \in \mathsf{QK}_{\underline{\sigma}}.$$

Similarly for QBK° *and* ρ°*. In other words,* ρ^3 *and* ρ° *faithfully embed* $\mathsf{QB3K}$ *and* QBK° *respectively into* QK*.*

This follows from Lemmas 9.6 (for the direct implication) and 9.5 (for the converse implication), taking into account the completeness theorems for $\mathsf{QB3K}$, QBK° and QK.

From this we can already get CIP for extensions of type (i):

Theorem 9.8. $\mathsf{QB3K}$ *and* QBK° *have CIP.*

Proof. Consider an arbitrary signature σ.

Suppose that $\Phi \to \Psi \in \mathsf{QB3K}_\sigma$. By Theorem 9.7, we have

$$\rho^3(\Phi) \to \rho^3(\Psi) \in \underline{L}.$$

Since QK has CIP, there exists a Craig interpolant Θ for $\rho^3(\Phi) \to \rho^3(\Psi)$ in $\mathsf{QB3K}$. Clearly, Θ does not necessarily have the form $\rho^3(\Theta')$ (because there could be 'free-standing' $\underline{P}(t_1, \dots, t_n)$'s in Θ, which are not connected with $\neg P(t_1, \dots, t_n)$ by means of conjunction). Take

$$\widetilde{\Theta} := \begin{array}{c}\text{the result of replacing each free-standing}\\ \underline{P}(t_1,\dots,t_n) \text{ in } \Theta \text{ by } \underline{P}(t_1,\dots,t_n) \wedge \neg P(t_1,\dots,t_n).\end{array}$$

It is easy to show that for any $\mathsf{QB3K}_\sigma$-model $\mathcal{M}$, $w \in W$ and A_w-substitution λ,

$$\underline{\mathcal{M}}, w \Vdash \lambda\Theta \iff \underline{\mathcal{M}}, w \Vdash \lambda\widetilde{\Theta}.$$

Obviously, $\widetilde{\Theta} = \rho^3(\Theta')$ for a suitable σ-formula Θ'. Taking into account the soundness theorem for QK, this implies that for any $\mathsf{QB3K}_\sigma$-model $\mathcal{M}$, $w \in W$ and A_w-substitution λ,

$$\underline{\mathcal{M}}, w \Vdash \lambda\left(\rho^3(\Phi) \to \rho^3(\Theta')\right) \quad \text{and} \quad \underline{\mathcal{M}}, w \Vdash \lambda\left(\rho^3(\Theta') \to \rho^3(\Psi)\right),$$

which, in view of Lemma 9.5, is equivalent to

$$\mathcal{M}, w \Vdash \lambda(\Phi \to \Theta') \quad \text{and} \quad \mathcal{M}, w \Vdash \lambda(\Theta' \to \Psi).$$

By the completeness theorem for $\mathsf{QB3K}$, the latter means that

$$\Phi \to \Theta' \in \mathsf{QB3K}_\sigma \quad \text{and} \quad \Theta' \to \Psi \in \mathsf{QB3K}_\sigma.$$

In addition, it is not hard to check that $\mathrm{O}(\Theta) \subseteq \mathrm{O}(\Phi) \cap \mathrm{O}(\Psi)$. So Θ' is a Craig interpolant for $\Phi \to \Psi$ in $\mathsf{QB3K}$.

In the case of QBK°, we argue similarly. $\square$

Remark 9.9. The logics $\mathsf{QB3K}$ and QBK° do not have SIP. Indeed, in the case of $\mathsf{QB3K}$, take

$$\Phi := P(x_1, \dots, x_n) \quad \text{and} \quad \Psi := \neg\sim P(x_1, \dots, x_n),$$

where P is some n-ary predicate symbol. Then $\Phi \to \Psi \in \mathsf{QB3K}$, but there is no strong interpolant for $\Phi \to \Psi$ in $\mathsf{QB3K}$. In the case of QBK°, one can take

$$\Phi := \neg P(x_1, \dots, x_n) \quad \text{and} \quad \Psi := \sim P(x_1, \dots, x_n).$$

Then $\Phi \to \Psi \in \mathsf{QBK}^\circ$, but there is no strong interpolant for $\Phi \to \Psi$ in QBK°.

Of course, the proof of Theorem 9.8 is easily modified to obtain CIP for $\mathsf{QB3S4}$, for example.

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