Reasoning about Arbitrary Natural Numbers from a Carnapian Perspective

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Abstract

Inspired by Kit Fine's theory of arbitrary objects, we explore some ways in which the generic structure of the natural numbers can be presented. Following a suggestion of Saul Kripke's, we discuss how basic facts and questions about this generic structure can be expressed in the framework of Carnapian quantified modal logic.

Keywords: quantified modal logic, individual concepts, generic structures, arbitrary objects.

Contents

1 Introduction

Kit Fine proposed, defended and explored his approach to reasoning about arbitrary objects in [Fine 1985a] and [Fine 1985b]. The approach taken naturally gives rise to an interesting theory of mathematical structure — which was articulated in [Horsten 2018] (note that Fine discussed a somewhat similar theory in [Fine 1998, pp. 618–619]). Now from the viewpoint of this theory, the natural number structure can be conceived as an 'arbitrary', or 'generic', system of objects, which may be called the *generic ω-sequence*. Then two ways of modelling this generic structure formally suggest themselves:

- one is in the spirit of [Benacerraf 1996], according to which any particular ω-sequence can be thought of as an admissible instance of the natural number structure;
- another is in the spirit of [Halbach & Horsten 2005, Horsten 2012], according to which we are to restrict ourselves to computable ω-sequences;

Both of these will be considered below.

It was proposed in [Kripke 1992, p. 73] that reasoning with arbitrary objects can in effect be represented using Carnapian modal logic with quantifiers over *individual concepts*, i.e. functions from possible worlds to elements of a given domain (cf. [Carnap 1956]) — instead of Fine's own framework. Following Kripke's proposal, we explore how different facts and questions about the generic structure of the natural numbers can be expressed in this setting. In fact, the structure in question is a good test case for the formal investigation of generic structures in general. Note also that a distinctive feature of our approach — which will play a key role in our modelling — is that we supplement the original modal language with a special intensional predicate.

2 Two kinds of generic structure

Roughly speaking, a generic (or arbitrary) structure is an entity that can be in different *states*. In particular we shall be concerned with the generic structure of the natural numbers, with the states representable as various ω-sequence orderings of some fixed underlying countably infinite plurality of objects — so it will also be called the *generic ω-sequence*. In this case we may view any natural number as an arbitrary object too, or rather as an individual concept, i.e. a function from states to elements of the underlying set. It will be seen shortly how both *arbitrary* and *specific* natural numbers arise in this framework.

Next we describe two essentially different versions of this generic ω-sequence. In the present article we shall not adjudicate between these but rather confine ourselves to uncovering some of their logical/metamathematical properties.

2.1 The 'full' version

Without loss of generality, and purely for exposition, we may identify the underlying countably infinite plurality of objects with $\mathbb{N}$ itself. Let

$$\mathbb{G} \ := \ \text{the collection of all permutations of } \mathbb{N}.$$

Now suppose that we follow the slogan 'any ω-sequence will do' from [Benacerraf 1996] — as is also done in [Shapiro 1997]. To be more precise, since every permutation of $\mathbb{N}$ can be viewed as a specific ω-sequence, we define the state space of our generic ω-sequence to be $\mathbb{G}$. Let us think of $\mathbb{G}$ as the *full generic ω-sequence*. Clearly $|\mathbb{G}| = 2^{\aleph_0}$, so there are continuum many states that the full generic ω-sequence can be in.

Take σ to be the signature of Peano arithmetic, i.e. $\{0, \mathsf{s}, +, \times, =\}$, and $\mathfrak{N}$ to be its *standard model*, which serves as the intended interpretation of σ. Naturally, every $\pi \in \mathbb{G}$ induces its own isomorphic copy $\pi[\mathfrak{N}]$ of $\mathfrak{N}$, given by:

$$
\begin{array}{lcl}
\pi[\mathfrak{N}] \models \pi(i) = 0 & \iff & \mathfrak{N} \models i = 0; \\
\pi[\mathfrak{N}] \models \mathsf{s}(\pi(i)) = \pi(j) & \iff & \mathfrak{N} \models \mathsf{s}(i) = j; \\
\pi[\mathfrak{N}] \models \pi(i) + \pi(j) = \pi(k) & \iff & \mathfrak{N} \models i + j = k; \\
\pi[\mathfrak{N}] \models \pi(i) \times \pi(j) = \pi(k) & \iff & \mathfrak{N} \models i \times j = k.^1
\end{array}
$$

— in other words, $\pi[\mathfrak{N}]$ can be obtained by applying π^{-1} to the intended interpretations of the symbols in σ.

Intuitively an arbitrary number in $\mathbb{G}$ is a kind of 'individual concept' which for each state π picks out some position in π, regarded as an ω-sequence. Formally, *arbitrary numbers* in $\mathbb{G}$ can be identified with function from $\mathbb{G}$ to $\mathbb{N}$. Denote by $\mathbb{A}$ the collection of all arbitrary numbers in $\mathbb{G}$. Evidently $|\mathbb{A}| = 2^{2^{\aleph_0}}$.

Roughly, the specific numbers are treated as limiting values of arbitrary numbers. To illustrate how this works, consider the arbitrary number α such that for every $\pi \in \mathbb{G}$,

$$\alpha(\pi) \;=\; \pi(0),$$

i.e. for each state $\pi \in \mathbb{G}$, α picks out the object that plays the role of 0 at π (regarded as an ω-sequence). Naturally, this α may be thought of as the *specific number* 0 in $\mathbb{G}$. Then, in general, we define the *specific number* n in $\mathbb{G}$ to be the arbitrary number α given by

$$\alpha(\pi) \;=\; \pi(n).$$

Denote by $\mathbb{S}$ the collection of all specific numbers. Obviously $|\mathbb{S}| = \aleph_0$.

2.2 The 'computable' version

Of course most of the permutations of $\mathbb{N}$ induce non-computable presentations — e.g. there are continuously many π's in $\mathbb{G}$ such that the interpretation of s in $\pi[\mathfrak{N}]$, and hence also that of $+$, is not Turing computable. Since it is possible to compute with (ordinary) natural numbers, one might want to disqualify these 'incomputable' permutations from being states of our generic ω-sequence; see [Benacerraf 1965, pp. 275–277], [Halbach & Horsten 2005] and [Horsten 2012]. So, one might insist that the structures that instantiate our intended interpretation must represent constructive systems of notations.

This line of reasoning leads to narrowing the state space $\mathbb{G}$ to

$$\mathbb{G}^{\mathsf{c}} \;:=\; \text{the collection of all computable permutations of } \mathbb{N}$$

(whose elements are regarded as constructive systems of notations) — which gives a computable isomorphism type. Let us think of $\mathbb{G}^{\mathsf{c}}$ as the *computable generic ω-sequence.* Evidently there are only countably many Turing machines, hence $|\mathbb{G}^{\mathsf{c}}| = \aleph_0$.

As in the case of the full generic ω-sequence, one can then go on to identify the collection of all arbitrary numbers in $\mathbb{G}^{\mathsf{c}}$ with the set $\mathbb{A}^{\mathsf{c}}$ of all functions from $\mathbb{G}^{\mathsf{c}}$ to $\mathbb{N}$; obviously $|\mathbb{A}^{\mathsf{c}}| = 2^{\aleph_0}$. And the specific numbers in $\mathbb{G}^{\mathsf{c}}$ are treated as limiting values of arbitrary numbers, similarly to how we did for $\mathbb{G}$.

We shall be concerned with both $\mathbb{G}$ and $\mathbb{G}^{\mathsf{c}}$. Occasionally, we shall write G when there is no need to differentiate between these two; let A be the collection of all functions from G to $\mathbb{N}$.

[1] All σ-structures are assumed to be normal, so $=$ has its usual meaning

3 Formal presentation

It was suggested in [Kripke 1992, p. 73] that Fine's reasoning with arbitrary objects can be represented using Carnap's individual concepts. We shall follow this approach in a certain way, by regarding G as essentially a possible worlds model of Carnapian quantified modal logic **S5** with equality (see [Carnap 1956, Chapter V]). Now we present a formal semantics for our framework and also describe some variation on it which will turn out to be useful.

3.1 A framework

Let us elaborate on the ideas from the previous section. In addition to the symbols of the signature σ of Peano arithmetic, our formal language $\mathcal{L}$ includes:

- a countable set $\mathrm{Var} = \{x, y, z, \dots\}$ of *variables*;
- the *connective* symbols $\neg$ and $\vee$;
- the *quantifier* symbol $\exists$;
- the *modal operator* symbol $\Diamond$, read as 'it is possible that __';
- the extra unary predicate symbol Sp, read as '__ is specific'.

Notice that in the context of first-order logic, we shall treat $\wedge$, $\rightarrow$ and $\forall$ as defined, rather than as primitive. The *$\mathcal{L}$-formulas* are then built up in the usual manner:

- if t_1 and t_2 are σ-terms, then $t_1 = t_2$ is an $\mathcal{L}$-formula;
- if Φ is an $\mathcal{L}$-formula, then $\neg\Phi$ is an $\mathcal{L}$-formula;
- if Φ and Ψ are $\mathcal{L}$-formulas, then $\Phi \vee \Psi$ is an $\mathcal{L}$-formula;
- if x is a variable and Φ is an $\mathcal{L}$-formula, then $\exists x\, \Phi$ is an $\mathcal{L}$-formula;
- if Φ is an $\mathcal{L}$-formula, then $\Diamond\Phi$ is an $\mathcal{L}$-formula;
- if t is a σ-term, then $\mathsf{Sp}\,(t)$ is an $\mathcal{L}$-formula.

We abbreviate $\neg\,(\neg\Phi \vee \neg\Psi)$ to $\Phi \wedge \Psi$, $\neg\Phi \vee \Psi$ to $\Phi \rightarrow \Psi$, $\neg\exists x\, \neg\Phi$ to $\forall x\, \Phi$, and lastly $\neg\Diamond\neg\Phi$ to $\Box\Phi$. Of course, x, y, z, … are intended to range over arbitrary natural numbers. However, the $\mathcal{L}$-formula $x = y$ will not express the identity of x and y, but only their 'coincidence' at a given possible world (cf. [Kripke 1992, p. 71]). Still, identity turns out to be expressible, via

$$\simeq (x, y) \;:=\; \Box\,(x = y),$$

as one may expect. It should also be remarked that Sp is the only symbol of $\mathcal{L}$ with intensional meaning, and its presence will play a crucial role for the expressive power of $\mathcal{L}$. Notice that the first-order σ-formulas are a subset of the $\mathcal{L}$-formulas. These formulas will be occasionally called *purely arithmetical.*

Any *possible world* is a permutation of $\mathbb{N}$, which may or may not be computable, depending on what we put for G. Intuitively a possible world represents a specific 'state' in which G might be; on the other hand, there is no 'state' in which G actually is, i.e. no preferred world. As was mentioned earlier, we associate with each $\pi \in \mathrm{G}$ the σ-structure $\pi\,[\mathfrak{N}]$. More precisely,

$$\pi\,[\mathfrak{N}] \;:=\; \langle \mathbb{N}; 0^\pi, \mathsf{s}^\pi, +^\pi, \times^\pi, =^\pi \rangle$$

where $=^\pi$ is the ordinary equality relation on $\mathbb{N}$, and the others are given by:

$$\begin{aligned} 0^\pi &:= \pi(0); \\ \mathsf{s}^\pi(i) &:= \pi\left(\pi^{-1}(i)+1\right); \\ i +^\pi j &:= \pi\left(\pi^{-1}(i)+\pi^{-1}(j)\right); \\ i \times^\pi j &:= \pi\left(\pi^{-1}(i)\times\pi^{-1}(j)\right). \end{aligned}$$

Here 0, s, $+$ and $\times$ on the right sides have their standard meaning, as in $\mathfrak{N}$. Thus, for instance, we define $i +^\pi j$ to be (roughly speaking) the number that plays at π the role of the sum of the role that i plays at π and the role that j plays at π.

By a *valuation* in A we mean simply a function from Var to A. Naturally, every valuation γ in A can be inductively extended to σ-terms:

$$\begin{aligned} \gamma(0) &:= \lambda\pi.[0^\pi]; \\ \gamma(\mathsf{s}(t)) &:= \lambda\pi.[\mathsf{s}^\pi((\gamma(t))(\pi))]; \\ \gamma(t_1+t_2) &:= \lambda\pi.[(\gamma(t_1))(\pi) +_\pi (\gamma(t_2))(\pi)]; \\ \gamma(t_1\times t_2) &:= \lambda\pi.[(\gamma(t_1))(\pi) \times_\pi (\gamma(t_2))(\pi)]. \end{aligned}$$

Here $\lambda\pi.[\ldots\pi\ldots]$ traditionally denotes the function which maps each π in G to $\ldots\pi\ldots$.

Of course the 'standard' interpretation of Sp in G would be

$$\mathrm{S} := \{\lambda\pi.[\pi(n)] \mid n \in \mathbb{N}\}.$$

Nevertheless, it is instructive to allow Sp to be interpreted by other subsets of A as well. Given an $S \subseteq \mathrm{A}$, we define, for any $\mathcal{L}$-formula Φ, valuation γ in A and world $\pi \in \mathrm{G}$, what it means for Φ *to be true in* $\langle \mathrm{G}, S\rangle$ *at* π *under* γ, written $\langle \mathrm{G}, S\rangle \models_\pi \Phi[\gamma]$, as follows:

- $\langle \mathrm{G}, S\rangle \models_\pi t_1 = t_2[\gamma]$ iff $(\gamma(t_1))(\pi) = (\gamma(t_2))(\pi)$;
- $\langle \mathrm{G}, S\rangle \models_\pi \neg\Phi[\gamma]$ iff $\langle \mathrm{G}, S\rangle \not\models_\pi \Phi[\gamma]$;
- $\langle \mathrm{G}, S\rangle \models_\pi \Phi\vee\Psi[\gamma]$ iff $\langle \mathrm{G}, S\rangle \not\models_\pi \Phi[\gamma]$ or $\langle \mathrm{G}, S\rangle \not\models_\pi \Psi[\gamma]$;
- $\langle \mathrm{G}, S\rangle \models_\pi \exists x\,\Phi[\gamma]$ iff there exists $\alpha \in \mathrm{A}$ such that $\langle \mathrm{G}, S\rangle \models_\pi \Phi[\gamma^x_\alpha]$;
- $\langle \mathrm{G}, S\rangle \models_\pi \Diamond\Phi[\gamma]$ iff there exists $\pi' \in \mathrm{G}$ such that $\langle \mathrm{G}, S\rangle \models_{\pi'} \Phi[\gamma]$;
- $\langle \mathrm{G}, S\rangle \models_\pi \mathsf{Sp}(t)[\gamma]$ iff $\gamma(t) \in S$.

Here we use γ^x_α for the valuation which agrees with γ except that $\gamma^x_\alpha(x) = \alpha$, viz.

$$\gamma^x_\alpha(y) := \begin{cases} \gamma(y) & \text{if } y \neq x, \\ \alpha & \text{if } y = x. \end{cases}$$

Clearly if Φ is of the form $\Phi(x_1,\ldots,x_\ell)$, i.e. the free variables of Φ are among $x_1,\ldots,x_\ell$, then it does not matter what values γ assigns to the elements of $\mathrm{Var}\setminus\{x_1,\ldots,x_\ell\}$, so we may write

$$\langle \mathrm{G}, S\rangle \models_\pi \Phi[\gamma(x_1),\ldots,\gamma(x_\ell)],$$

or more explicitly $\langle \mathrm{G}, S\rangle \models_\pi \Phi[x_1/\gamma(x_1),\ldots,x_\ell/\gamma(x_\ell)]$. Further, when Φ is an $\mathcal{L}$-sentence, i.e. no variable occurs free in Φ, this becomes $\langle \mathrm{G}, S\rangle \models_\pi \Phi$.

Proposition 3.1. *Let $S \subseteq \mathrm{A}$ and $\pi \in \mathrm{G}$. For any purely arithmetical $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_\ell)$ and $(\alpha_1, \ldots, \alpha_\ell) \in \mathrm{A}^\ell$,*

$$\langle \mathrm{G}, S \rangle \models_\pi \Phi[\alpha_1, \ldots, \alpha_\ell] \quad \Longleftrightarrow \quad \pi[\mathfrak{N}] \models \Phi[\alpha_1(\pi), \ldots, \alpha_\ell(\pi)].$$

Proof. By an easy induction on the construction of Φ. $\square$

Corollary 3.2. *The collection of all purely arithmetical sentences true in $\langle \mathrm{G}, S \rangle$ at π coincides with the first-order theory of $\mathfrak{N}$.*

Proof. The set of first-order σ-sentences true in $\langle \mathrm{G}, S \rangle$ at π coincides with the first-order theory of $\pi[\mathfrak{N}]$ by the Proposition above. Moreover, since $\pi[\mathfrak{N}]$ and $\mathfrak{N}$ are isomorphic, their first-order theories coincide. So the result follows. $\square$

We say Φ *is generically true in* $\langle \mathrm{G}, S \rangle$ *under* γ, written $\langle \mathrm{G}, S \rangle \models \Phi[\gamma]$, iff $\langle \mathrm{G}, S \rangle \models_\pi \Phi[\gamma]$ for all $\pi \in \mathrm{G}$. As before, in the case where Φ is an $\mathcal{L}$-sentence we can omit γ and write $\langle \mathrm{G}, S \rangle \models \Phi$. We notice that the notion of generic truth does not presuppose there being an 'actual' world in which arbitrary numbers take their 'actual' values.

Corollary 3.3. *The collection of all purely arithmetical sentences true generically in $\langle \mathrm{G}, S \rangle$ coincides with the first-order theory of $\mathfrak{N}$.*

Proof. Immediate. $\square$

Notice that since there is no restriction on π' in the defining condition for $\langle \mathrm{G}, S \rangle \models_\pi \Diamond\Phi[\gamma]$, the intended accessibility relation is, in effect, the Cartesian square of G — and thus $\Diamond$ satisfies the propositional laws of **S5**. One readily verifies that in $\langle \mathrm{G}, S \rangle$, *Leibniz's scheme of identity*

$$\forall x \, \forall y \, (x = y \to (\Phi(x) \to \Phi[x/y]) \wedge (\Phi[x/y] \to \Phi(x)))$$

fails already for the Sp-free $\mathcal{L}$-formulas, but holds for the $\Box$-free $\mathcal{L}$-formulas. Roughly speaking, the basic quantificational principles that hold in $\langle \mathrm{G}, S \rangle$ are, unsurprisingly, those for Carnapian quantified modal logic. It is easy to see that the *Barcan formula/scheme*

$$\forall x \, \Box\Phi(x) \to \Box\forall x \, \Phi(x)$$

and its converse both hold in $\langle \mathrm{G}, S \rangle$.[2] The so-called *Ghilardi formula/scheme*

$$\Box\exists x \, \Phi(x) \to \exists x \, \Box\Phi(x),$$

which can be regarded as a choice principle, fails already for the Sp-free $\mathcal{L}$-formulas (though its converse holds in $\langle \mathrm{G}, S \rangle$, of course).[3] For consider the $\mathcal{L}$-formula

$$\Psi \;:=\; (x = 0) \wedge \neg\Box(x = 0);$$

then $\langle \mathrm{G}, S \rangle \models \Box\exists x \, \Psi$ and yet $\langle \mathrm{G}, S \rangle \not\models \exists x \, \Box\Psi$.

[2] Cf. also [Heylen 2010, p. 357] and [Williamson 2013, Sections 2.1–2.2].

[3] Cf. also [Heylen 2010, Footnote 1 on p. 358] and [Williamson 2013, pp. 54–56].

3.2 A useful variation

Now we are going to give a somewhat simpler yet equivalent semantics for $\mathcal{L}$ — which will turn out to be quite helpful in what follows. We do not suggest that this version is intuitively better than the original one, but rather wish to attract attention to its technical advantages.

The idea is to identify directly each $\pi[\mathfrak{N}]$ with $\mathfrak{N}$. Thus we do not want to think of possible worlds as permutations of $\mathbb{N}$. Let a non-empty set W be given. Then we shall write $\mathcal{W}$ for $\mathbb{N}^W$, viz. the collection of all functions from W to $\mathbb{N}$. In the present context, the elements of W play the role of *possible worlds*, and those of $\mathcal{W}$ the role of *arbitrary numbers*.

Similarly to before, by a *valuation* in $\mathcal{W}$ we understand simply a function γ from Var to $\mathcal{W}$, which can be extended to σ-terms in the natural way:

$$\begin{aligned}
\gamma(0) &:= \lambda w.[0];\\
\gamma(\mathsf{s}(t)) &:= \lambda w.[\mathsf{s}((\gamma(t))(w))];\\
\gamma(t_1 + t_2) &:= \lambda w.[(\gamma(t_1))(w) + (\gamma(t_2))(w)];\\
\gamma(t_1 \times t_2) &:= \lambda w.[(\gamma(t_1))(w) \times (\gamma(t_2))(w)].
\end{aligned}$$

Here w ranges over W, of course.

Clearly the 'standard' interpretation of Sp in the present context would be

$$\mathrm{F} := \{\lambda w.[n] \mid n \in \mathbb{N}\}$$

Nevertheless, we allow Sp to be interpreted by other subsets of $\mathcal{W}$ too. Given an $F \subseteq \mathcal{W}$, let us define $\langle W, F\rangle \Vdash_w \Phi[\gamma]$ exactly as before, but with W and F in place of G and S respectively.

For the rest of this subsection, assume $|\mathrm{G}| = |W|$, viz. the cardinalities of G and W must be equal. Fix a one-to-one function ι from W onto G. For each $\alpha \in \mathrm{A}$, define

$$\alpha^\iota := \lambda w.\Big[(\iota(w))^{-1}(\alpha(\iota(w)))\Big].$$

In particular, if $W = \mathrm{G}$ and $\iota = \lambda w.[w]$, then $\alpha^\iota := \lambda w.\big[w^{-1}(\alpha(w))\big]$. Given an $R \subseteq \mathrm{A}^\ell$, take

$$R^\iota := \{(\alpha_1^\iota, \ldots, \alpha_\ell^\iota) \mid (\alpha_1, \ldots, \alpha_\ell) \in R\}.$$

As might be expected, we have:

Theorem 3.4. *Assuming $|\mathrm{G}| = |W|$, let $S \subseteq \mathrm{A}$. For any $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_\ell)$, $(\alpha_1, \ldots, \alpha_\ell) \in \mathrm{A}^\ell$ and $w \in W$,*

$$\langle \mathrm{G}, S\rangle \models_{\iota(w)} \Phi[\alpha_1, \ldots, \alpha_\ell] \iff \langle W, S^\iota\rangle \Vdash_w \Phi[\alpha_1^\iota, \ldots, \alpha_\ell^\iota].$$

Proof. One can easily verify that for any first-order σ-term $t(x_1, \ldots, x_\ell)$, $(\alpha_1, \ldots, \alpha_\ell) \in \mathrm{A}^\ell$ and $w \in W$,

$$(\gamma(t))(\iota(w)) = (\iota(w))((\gamma(t))^\iota(w)),$$

where γ is a valuation which maps $x_1, \ldots, x_\ell$ to $\alpha_1, \ldots, \alpha_\ell$ respectively — in other words,

$$(\gamma(t))(\iota(w)) \text{ plays the role of } ((\gamma(t))^\iota(w)) \text{ at } \iota(w).$$

From this observation it is a short step to the desired result, which we can now prove by induction of the construction of Φ.

Suppose Φ is atomic. Then the result follows quickly by the observation made above.

The case when $\Phi = \neg\Psi$ is immediate from the inductive hypothesis.

The case when $\Phi = \Psi \vee \Theta$ is also immediate from the inductive hypothesis.

Suppose $\Phi = \exists y\, \Psi(x_1, \ldots, x_\ell, y)$. Note that $\lambda\alpha.[\alpha^\iota]$ is a one-to-one function from A onto $\mathcal{W}$, so in particular, every $\beta \in \mathcal{W}$ has the form α^ι with $\alpha \in \mathrm{A}$. Thus the result follows quickly from the inductive hypothesis.

The case when $\Phi = \Diamond\Psi$ is immediate from the inductive hypothesis — because every $\pi \in \mathrm{G}$ has the form $\iota(w)$ with $w \in W$. □

Of course, things look simpler in $\langle W, S^\iota \rangle$ than in $\langle \mathrm{G}, S \rangle$. For instance, if Sp has its standard meaning in $\langle \mathrm{G}, S \rangle$, then so does Sp in $\langle W, S^\iota \rangle$, and vice versa — more formally,

$$S = \mathrm{S} \iff S^\iota = \mathrm{F},$$

independently of the choice of ι. Moreover, S is not closed under permutations of G, but F (i.e. the set of all constant functions from W to $\mathbb{N}$) is clearly closed under permutations of W.

In fact, we shall be mainly concerned with the standard interpretations of Sp. Then instead of $\langle \mathrm{G}, \mathrm{S} \rangle \models_\pi \Phi[\gamma]$ and $\langle W, \mathrm{F} \rangle \Vdash_w \Phi[\gamma]$ we can write respectively

$$\mathrm{G} \models_\pi \Phi[\gamma] \quad \text{and} \quad W \Vdash_w \Phi[\gamma],$$

or simply $\pi \models \Phi[\gamma]$ and $w \Vdash \Phi[\gamma]$. Let us finish with an easy application of Theorem 3.4 which gives an alternative definition of generic truth for $\mathcal{L}$-sentences in the case $S = \mathrm{S}$.

Corollary 3.5. *Assume* $|\mathrm{G}| = |W|$. *For any* $\mathcal{L}$*-sentence* Φ *and* $\{\pi_1, \pi_2\} \subseteq \mathrm{G}$,

$$\mathrm{G} \models_{\pi_1} \Phi \iff \mathrm{G} \models_{\pi_2} \Phi.$$

Thus $\mathrm{G} \models \Phi$ *iff* $\mathrm{G} \models_\pi \Phi$ *for some* $\pi \in \mathrm{G}$.

Proof. Let Φ be an $\mathcal{L}$-sentence. It is readily seen that the truth of $W \Vdash_w \Phi$ does not depend on the choice of w, i.e. for all $\{w_1, w_2\} \subseteq W$,

$$W \Vdash_{w_1} \Phi \iff W \Vdash_{w_2} \Phi.$$

Now the result follows immediately by this fact and the Theorem above. □

In words, any two possible worlds are indistinguishable by $\mathcal{L}$-sentences provided that Sp has its standard meaning.

4 Model-theoretic aspects

To get a deeper understanding of the behavior of arbitrary numbers in our framework, we need to investigate some basic model-theoretic properties of $\mathcal{L}$.

4.1 Definability

Henceforth by a *specific number* we mean a 'standard' specific number, viz. a function of either of the forms

$$\lambda\pi.[\pi(n)] \quad \text{and} \quad \lambda w.[n]$$

where $n \in \mathbb{N}$, depending on whether G or W is involved.

Given an $S \subseteq \mathrm{A}$, call $R \subseteq \mathrm{A}^\ell$ *definable* in $\langle \mathrm{G}, S\rangle$ iff there is an $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_\ell)$ such that for every $(\alpha_1, \ldots, \alpha_\ell) \in \mathrm{A}^\ell$,

$$(\alpha_1, \ldots, \alpha_\ell) \in R \iff \langle \mathrm{G}, S\rangle \models \Phi[\alpha_1, \ldots, \alpha_\ell].$$

If the same Φ defines R in $\langle \mathrm{G}, S\rangle$ for all $S \subseteq \mathrm{A}$, we say R is *uniformly definable* in G. Similarly for $\langle W, F\rangle$. Remembering the definition of α^ι from the previous secton, we have:

Proposition 4.1. *Assume $|W| = |\mathrm{G}|$, and let $S \subseteq \mathrm{A}$. Then for every $R \subseteq \mathrm{A}^\ell$ and every $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_\ell)$,*

$$\Phi \text{ defines } R \text{ in } \langle \mathrm{G}, S\rangle \iff \Phi \text{ defines } R^\iota \text{ in } \langle W, S^\iota\rangle.$$

Proof. Immediate from Theorem 3.4. $\square$

As usual, one can speak of definability of arbitrary numbers and functions on them:

- by identifying an arbitrary number with its singleton;
- by identifying a function on the arbitrary numbers with its graph.[4]

As has been observed earlier, $\lambda\alpha.[\alpha^\iota]$ maps A one-to-one onto $\mathcal{W}$, so we are free to pass from G to W, and vice versa. To facilitate discussion we mostly deal with W.

For instance, the identity relation on $\mathcal{W}$ is uniformly definable in W by the $\mathcal{L}$-formula

$$x \simeq y \;:=\; \Box(x = y).$$

The 'coordinatewise' addition and multiplication, viz.

$$\begin{aligned} R_+ &:= \{(\alpha_1, \alpha_2, \lambda w.[\alpha_1(w) + \alpha_2(w)]) \mid (\alpha_1, \alpha_2) \in \mathcal{W}^2\} \quad \text{and} \\ R_\times &:= \{(\alpha_1, \alpha_2, \lambda w.[\alpha_1(w) \times \alpha_2(w)]) \mid (\alpha_1, \alpha_2) \in \mathcal{W}^2\}, \end{aligned}$$

can then be uniformly defined in W by

$$\Phi_+(x, y, z) \;:=\; x + y \simeq z \quad \text{and} \quad \Phi_\times(x, y, z) \;:=\; x \times y \simeq z$$

respectively. Let us temporarily pretend that σ includes neither 0 nor s. Still, the specific numbers $\lambda w.[0]$ and $\lambda w.[1]$ will be uniformly definable in W by

$$\Phi_0(x) \;:=\; \forall y\, \Phi_+(y, x, y) \quad \text{and} \quad \Phi_1(x) \;:=\; \forall y\, \Phi_\times(y, x, y)$$

respectively. Hence the 'coordinatewise' successor function, viz.

$$R_{\mathsf{s}} \;:=\; \{(\alpha, \lambda w.[\alpha(w) + 1]) \mid \alpha \in \mathcal{W}\},$$

can be uniformly defined in W by

$$\Phi_{\mathsf{s}}(x, y) \;:=\; \exists z\, (\Phi_1(z) \wedge \Phi_+(x, z, y)).$$

[4]The second convention may look redundant since functions are officially identified with their graphs in ZFC, and we deal with set-theoretic objects. Still, in first-order logic predicate and function symbols are often treated separately, and the notions of definability and representability, for instance, are described in two stages. Also, in computability theory it is common practice to distinguish between functions and their graphs.

Note in passing that as far as definability is concerned there is no significant difference between σ and, say, the smaller signature $\{+, \times, =\}$; in fact any other σ' would do as well provided that the standard models of σ and σ' are first-order interdefinable.

Now for each $n \in \mathbb{N}$, denote $\lambda w.[n]$ by $\mathbf{n}$. Define a sequence $\mathsf{A}_0\,(x)$, $\mathsf{A}_1\,(x)$, $\ldots$ of $\mathcal{L}$-formulas by recursion:

$$\mathsf{A}_n\,(x) \;:=\; \begin{cases} x \simeq 0 & \text{if } \; n = 0, \\ \exists y\,(\mathsf{A}_{n-1}\,(y) \wedge x \simeq \mathsf{s}\,(y)) & \text{if } \; n > 0. \end{cases}$$

Evidently the following holds.

Proposition 4.2. *For each $n \in \mathbb{N}$, the specific number $\mathbf{n}$ is uniformly definable in W by the $\mathcal{L}$-formula $\mathsf{A}_n\,(x)$.*

Proof. By a trivial induction on n. □

Furthermore, no non-specific number is definable in W, as will be seen shortly. However, we need a little more semantic machinery to establish this. Let δ be a permutation of W. For each $\alpha \in \mathcal{W}$ we denote by $\alpha \circ \delta$ the composition of α and δ, i.e.

$$\alpha \circ \delta \;:=\; \lambda w.[\alpha\,(\delta\,(w))];$$

obviously $\alpha \circ \delta \in \mathcal{W}$. Given an $R \subseteq \mathcal{W}^\ell$, take

$$R \circ \delta \;:=\; \{(\alpha_1 \circ \delta, \ldots, \alpha_\ell \circ \delta) \mid (\alpha_1, \ldots, \alpha_\ell) \in R\}.$$

As one would expect, we have:

Proposition 4.3. *Let δ be a permutation of W and $F \subseteq \mathcal{W}$. For any $\mathcal{L}$-formula $\Phi\,(x_1, \ldots, x_\ell)$, $(\alpha_1, \ldots, \alpha_\ell) \in \mathcal{W}^\ell$ and $w \in W$,*

$$\langle W, F\rangle \Vdash_{\delta(w)} \Phi\,[\alpha_1, \ldots, \alpha_n] \quad\Longleftrightarrow\quad \langle W, F \circ \delta\rangle \Vdash_w \Phi\,[\alpha_1 \circ \delta, \ldots, \alpha_n \circ \delta].$$

Proof. By an easy induction on the construction of Φ. □

Remember that we abbreviate $\langle W, \mathrm{F}\rangle \Vdash_w \Phi\,[\gamma]$, where F denotes the standard interpretation of Sp, to $W \Vdash_w \Phi\,[\gamma]$. Likewise 'definable in W' will stand for 'definable in $\langle W, \mathrm{F}\rangle$'.

Corollary 4.4. *For every $R \subseteq \mathcal{W}^\ell$,*

$$R \text{ is definable in } W \quad\Longrightarrow\quad R \circ \delta \,=\, R \;\text{ for all permutations } \delta \text{ of } W.$$

Proof. Assume $R \subseteq \mathcal{W}^\ell$ can be defined in $\langle W, \mathrm{F}\rangle$ by an $\mathcal{L}$-formula Φ. Let δ be a permutation of W. Evidently $\mathrm{F} \circ \delta = \mathrm{F}$. Hence by the Proposition above, for any $(\alpha_1, \ldots, \alpha_\ell) \in \mathcal{W}^\ell$,

$$W \Vdash \Phi\,[\alpha_1, \ldots, \alpha_\ell] \quad\Longleftrightarrow\quad W \Vdash \Phi\,[\alpha_1 \circ \delta, \ldots, \alpha_\ell \circ \delta].$$

Consequently $R \circ \delta = R$, as desired. □

Finally we are ready for the promised result.

Theorem 4.5. *For every $\alpha \in \mathcal{W}$ the following are equivalent:*

1. *α is specific;*

2. *α is uniformly definable in W;*

3. α is definable in W.

Proof. $\boxed{1 \Longrightarrow 2}$ Immediate from Proposition 4.2.

$\boxed{2 \Longrightarrow 3}$ This is obvious.

$\boxed{3 \Longrightarrow 1}$ Assume α is definable in W. So by Corollary 4.4, $\alpha \circ \delta = \alpha$ for all permutations δ of W. Thus α must be a constant function, and hence specific. $\square$

Is it possible to fix the intended interpretation of Sp by means of a single $\mathcal{L}$-sentence? Curiously the answer turns out to be affirmative. Take the *axioms for* Sp to be

$$\begin{aligned}\mathsf{B1} &:= \forall x\,(x \simeq 0 \to \mathsf{Sp}\,(x)),\\ \mathsf{B2} &:= \forall x\,(\mathsf{Sp}\,(x) \to \mathsf{Sp}\,(\mathsf{s}\,(y))) \quad \text{and}\\ \mathsf{B3} &:= \forall x\,(\mathsf{Sp}\,(x) \wedge x \not\simeq 0 \to \exists y\,(\mathsf{Sp}\,(y) \wedge x \simeq \mathsf{s}\,(y))).\end{aligned}$$

We are going to show that the conjunction of these three $\mathcal{L}$-sentences does the job.

Theorem 4.6. *For every $F \subseteq \mathcal{W}$,*

$$\langle W, F\rangle \models \mathsf{B1–B3} \quad \Longleftrightarrow \quad F \;=\; \mathrm{F}.$$

Proof. $\boxed{\Longleftarrow}$ This is obvious.

$\boxed{\Longrightarrow}$ Given an $\alpha \in \mathcal{W}$, take

$$\mathrm{rank}\,(\alpha) \;:=\; \min\,\{\alpha\,(w) \mid w \in W\}.$$

Assume $\langle W, F\rangle \models \mathsf{B1–B3}$. Using $\mathsf{B1–2}$ we easily get $\mathrm{F} \subseteq F$. Suppose $F \neq \mathrm{F}$. Consequently there exists $\alpha \in F \setminus \mathrm{F}$. Clearly $\alpha \neq \mathbf{0}$. So by $\mathsf{B3}$ there must be $\alpha' \in F$ such that $\langle W, F\rangle \models \alpha \simeq \mathsf{s}\,(\alpha')$. Then $\mathrm{rank}\,(\alpha') < \mathrm{rank}\,(\alpha)$, and moreover, $\alpha' \in F \setminus \mathrm{F}$ — because otherwise the successor α of α' would be in F. Continuing in this fashion we obtain an infinite sequence

$$\alpha, \quad \alpha', \quad ,\alpha'', \quad \ldots$$

of elements of $F \setminus \mathrm{F}$ such that

$$\mathrm{rank}\,(\alpha) \;>\; \mathrm{rank}\,(\alpha') \;>\; \mathrm{rank}\,(\alpha') \;>\; \ldots$$

This contradicts the well-foundedness of $\mathbb{N}$ with the usual ordering, of course. $\square$

4.2 Cardinality

Temporarily forget about the condition $|W| = |\mathrm{G}|$ and let W be any non-empty set (of possible worlds). Given an $F \subseteq \mathcal{W}$, by the *$\mathcal{L}$-theory of $\langle W, F\rangle$* we mean

$$\mathrm{Th}\,(W, F) \;:=\; \{\Phi \mid \Phi \text{ is an } \mathcal{L}\text{-sentence and } \langle W, F\rangle \Vdash \Phi\}.$$

Similarly for $\langle \mathrm{G}, S\rangle$. As might be expected, $\mathrm{Th}\,(W, \mathrm{F})$ and $\mathrm{Th}\,(\mathrm{G}, \mathrm{S})$ are abbreviated to $\mathrm{Th}\,(W)$ and $\mathrm{Th}\,(\mathrm{G})$ respectively. More explicitly $\mathrm{Th}\,(W)$ can be written as $\mathrm{Th}\,(\mathfrak{N}; W)$. Obviously every cardinal $\mathfrak{a}$ can itself be viewed as a set of worlds of cardinality $\mathfrak{a}$. Thus, for instance,

$$\mathrm{Th}\,(\mathbb{G}_{\mathsf{c}}) \;=\; \mathrm{Th}\,(\mathfrak{N}; \aleph_0) \quad \text{and} \quad \mathrm{Th}\,(\mathbb{G}) \;=\; \mathrm{Th}\,\big(\mathfrak{N}; 2^{\aleph_0}\big).$$

— remember Theorem 3.4. To facilitate the further exposition, we abbreviate

$$\forall x\,(\mathsf{Sp}\,(x) \to \Phi) \quad \text{and} \quad \exists x\,(\mathsf{Sp}\,(x) \wedge \Phi)$$

to $\forall^{\mathsf{Sp}} x\,\Phi$ and $\exists^{\mathsf{Sp}} x\,\Phi$ respectively.

Proposition 4.7. *There exists an $\mathcal{L}$-sentence Φ such that for every cardinal $\mathfrak{a}$,*

$$\mathfrak{a} \Vdash \Phi \quad \Longleftrightarrow \quad \mathfrak{a} \ = \ \aleph_0.$$

Proof. Consider the $\mathcal{L}$-formula

$$\mathsf{C}\,(x) \ := \ \forall^{\mathsf{Sp}} y \Diamond (x = y) \wedge \forall^{\mathsf{Sp}} y \forall z \, (\Diamond (x = y \wedge z = y) \to \Box (x = y \to z = y)).$$

Clearly for each $\alpha \in \mathcal{W}$,

$$W \Vdash \mathsf{C}\,[\alpha] \quad \Longleftrightarrow \quad \alpha \text{ is a one-to-one function from } W \text{ onto } \mathbb{N}.$$

Take Φ to be $\exists x \, \mathsf{C}\,(x)$. Obviously $W \Vdash \Phi$ iff $|W| = \aleph_0$. □

Corollary 4.8. $\mathrm{Th}\,(\mathfrak{N}; \aleph_0) \neq \mathrm{Th}\,(\mathfrak{N}; 2^{\aleph_0})$*, i.e.* $\mathrm{Th}\,(\mathbb{G}_{\mathsf{c}}) \neq \mathrm{Th}\,(\mathbb{G})$.

Proof. Immediate. □

Then, since $\mathcal{L}$ allows us to distinguish $\aleph_0$ from the other cardinals, the reader may very well ask whether we can do the same for $\aleph_1$, $\aleph_2$, and so on. The answer turns out to be negative, as will be seen below. For the rest of this subsection W, U, etc. stand for sets of worlds. We write $\mathcal{W}$, $\mathcal{U}$, etc. for the corresponding collections of arbitrary numbers, i.e. $\mathbb{N}^W$, $\mathbb{N}^U$, etc.

Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathcal{W}^\ell$. For each $\vec{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$, define

$$[\![\vec{\alpha}; \vec{n}]\!] \ := \ \{w \in W \mid \alpha_i\,(w) = n_i \text{ for all } i \text{ from 1 to } \ell\}.$$

By the *profile* of $\vec{\alpha}$ we mean the function $\mathrm{prof}_{\vec{\alpha}}$ from $\mathbb{N}^\ell$ to $\mathbb{N} \cup \{\aleph_0, \aleph_1\}$ given by

$$\mathrm{prof}_{\vec{\alpha}}\,(\vec{n}) \ := \ \begin{cases} |[\![\vec{\alpha}; \vec{n}]\!]| & \text{if } |[\![\vec{\alpha}; \vec{n}]\!]| \leqslant \aleph_0 \\ \aleph_1 & \text{otherwise.} \end{cases}$$

In particular, if $\ell = 0$, then $\vec{\alpha} = ()$ and $\mathbb{N}^\ell = \{()\}$, hence $\mathrm{prof}_{\vec{\alpha}}$ maps $()$ to $\min\{|W|, \aleph_1\}$. Given $\vec{\alpha} \in \mathcal{W}^\ell$ and $\vec{\beta} \in \mathcal{U}^\ell$, we say that $\vec{\alpha}$ and $\vec{\beta}$ are *congruent*, and write $\vec{\alpha} \cong \vec{\beta}$, iff their profiles coincide — although the underlying sets W and U need not be the same.[5]

Lemma 4.9. *Let $\vec{\alpha} \in \mathcal{W}^\ell$ and $\vec{\beta} \in \mathcal{U}^\ell$. Suppose $\vec{\alpha}$ and $\vec{\beta}$ are congruent. Then for every $\alpha' \in \mathcal{W}$ there exists $\beta' \in \mathcal{U}$ such that $(\vec{\alpha}, \alpha')$ and $(\vec{\beta}, \beta')$ are congruent.*

Proof. Since $\{[\![\vec{\beta}; \vec{n}]\!] \mid \vec{n} \in \mathbb{N}^\ell\}$ is clearly a partition of U, it suffices to define β' on each element of this partition. Consider an arbitrary $\vec{n} \in \mathbb{N}^\ell$. Obviously

$$|[\![\vec{\alpha}; \vec{n}]\!]| \ = \ \sum_{k=0}^{\infty} |[\![(\vec{\alpha}, \alpha'); (\vec{n}, k)]\!]|$$

and therefore

$$\mathrm{prof}_{\vec{\alpha}}\,(\vec{n}) \ = \ \sum_{k=0}^{\infty} \mathrm{prof}_{(\vec{\alpha}, \alpha')}\,(\vec{n}, k).$$

On the other hand, $\mathrm{prof}_{\vec{\alpha}}\,(\vec{n}) = \mathrm{prof}_{\vec{\beta}}\,(\vec{n})$ by assumption. So β' can be defined on $[\![\vec{\beta}; \vec{n}]\!]$ in such a way that for each $k \in \mathbb{N}$,

$$\mathrm{prof}_{(\vec{\beta}, \beta')}\,(\vec{n}, k) \ = \ \mathrm{prof}_{(\vec{\alpha}, \alpha')}\,(\vec{n}, k).$$

In detail, the argument falls into two cases.

[5] However, $\vec{\alpha} \cong \vec{\beta}$ implies $\min\{|W|, \aleph_1\} = \min\{|U|, \aleph_1\}$, as can be readily verified.

i. Suppose $|[\![\vec{\alpha};\vec{n}]\!]| \leqslant \aleph_0$, and hence $|[\![(\vec{\alpha},\alpha');(\vec{n},k)]\!]| \leqslant \aleph_0$ for any $k \in \mathbb{N}$. Then $|[\![\vec{\beta};\vec{n}]\!]| \leqslant \aleph_0$, and we can even define β' on $[\![\vec{\beta};\vec{n}]\!]$ so that for each $k \in \mathbb{N}$,

$$|[\![(\vec{\alpha},\alpha');(\vec{n},k)]\!]| \;=\; |[\![(\vec{\beta},\beta');(\vec{n},k)]\!]|,$$

which is more than enough.

ii. Suppose $|[\![\vec{\alpha};\vec{n}]\!]| \geqslant \aleph_1$, and hence $|[\![(\vec{\alpha},\alpha');(\vec{n},i)]\!]| \geqslant \aleph_1$ for some $i \in \mathbb{N}$. Then $|[\![\vec{\beta};\vec{n}]\!]| \geqslant \aleph_1$ and therefore $[\![\vec{\beta};\vec{n}]\!]$ has a subset B of cardinality $\aleph_1$. Certainly one can define β' on B so that for each $k \in \mathbb{N}$,

$$\min\{|[\![(\vec{\alpha},\alpha');(\vec{n},k)|,\aleph_1\} \;=\; \min\{|B \cap [\![(\vec{\beta},\beta');(\vec{n},k)]\!]|,\aleph_1\},$$

and we get the desired result by setting $\beta'(u) := i$ for all $u \in [\![\vec{\beta};\vec{n}]\!] \setminus B$.

These are the only cases possible, of course. □

We are ready for the key result of this subsection:

Theorem 4.10. *Let $\vec{\alpha} \in \mathcal{W}^\ell$, $\vec{\beta} \in \mathcal{U}^\ell$, $w \in W$ and $u \in U$. Suppose $\vec{\alpha}$ and $\vec{\beta}$ are congruent, with $\vec{\alpha}(w) = \vec{\beta}(u)$. Then for every $\mathcal{L}$-formula $\Phi(x_1,\ldots,x_\ell)$,*

$$W \Vdash_w \Phi[\vec{\alpha}] \quad\Longleftrightarrow\quad U \Vdash_u \Phi[\vec{\beta}].$$

Proof. By induction on the construction of Φ. Note: for every $w' \in W$ there exists $u' \in U$ such that $\vec{\alpha}(w') = \vec{\beta}(u')$ — simply because

$$\mathrm{prof}_{\vec{\beta}}(\vec{\alpha}(w')) \;=\; \mathrm{prof}_{\vec{\alpha}}(\vec{\alpha}(w')) \;=\; \min\{|[\![\vec{\alpha};\vec{\alpha}(w')]\!]|,\aleph_1\} \;\neq\; 0,$$

i.e. $|[\![\vec{\beta};\vec{\alpha}(w')]\!]| \neq 0$, and thus $[\![\vec{\beta};\vec{\alpha}(w')]\!] \neq \varnothing$.

Suppose Φ is atomic and does not contain Sp, so it has the form $t_1 = t_2$ where t_1 and t_2 are σ-terms. Clearly we have

$$w \Vdash \Phi[\vec{\alpha}] \quad\Longleftrightarrow\quad t_1^{\mathfrak{N}}(\vec{\alpha}(w)) = t_2^{\mathfrak{N}}(\vec{\alpha}(w)),$$

and similarly for u and $\vec{\beta}$. Since $\vec{\alpha}(w) = \vec{\beta}(u)$, the result follows.

Suppose Φ is atomic and does contain Sp, so it has the form $\mathsf{Sp}(t)$ with t a σ-term. For the $\Leftarrow$ direction, assume $w \nVdash \Phi[\vec{\alpha}]$. In other words, $t^{\mathfrak{N}}(\vec{\alpha}(w)) \neq t^{\mathfrak{N}}(\vec{\alpha}(w'))$ for some $w' \in W$. Now take u' to be an element of U such that $\vec{\alpha}(w') = \vec{\beta}(u')$. Of course $t^{\mathfrak{N}}(\vec{\beta}(u)) \neq t^{\mathfrak{N}}(\vec{\beta}(u'))$. Thus $u \nVdash \Phi[\vec{\beta}]$. The $\Rightarrow$ direction holds by symmetry.

The case when $\Phi = \neg\Psi$ is immediate from the inductive hypothesis.

The case when $\Phi = \Psi \vee \Theta$ is also immediate from the inductive hypothesis.

Suppose $\Phi = \exists y\,\Psi(x_1,\ldots,x_\ell,y)$. For the $\Rightarrow$ direction, assume $w \Vdash \Phi[\vec{\alpha}]$; thus $w \Vdash \Psi[\vec{\alpha},\alpha']$ for some $\alpha' \in \mathcal{W}$. By Lemma 4.9 there is $\beta' \in \mathcal{U}$ such that $\langle\vec{\alpha},\alpha'\rangle \cong \langle\vec{\beta},\beta'\rangle$, so by the inductive hypothesis we have $u \Vdash \Psi[\langle\vec{\beta},\beta'\rangle]$. Hence $w \Vdash \Phi[\vec{\beta}]$. The $\Leftarrow$ direction holds by symmetry.

Suppose $\Phi = \Diamond\Psi$. For the $\Rightarrow$ direction, assume $w \Vdash \Phi[\vec{\alpha}]$; thus $w' \Vdash \Psi[\vec{\alpha}]$ for some $w' \in W$. Take u' to be an element of U such that $\vec{\alpha}(w') = \vec{\beta}(u')$, so by the inductive hypothesis we have $u' \Vdash \Psi[\vec{\beta}]$. Hence $u \Vdash \Phi[\vec{\beta}]$. The $\Leftarrow$ direction holds by symmetry again. □

Corollary 4.11. *For any cardinals* $\mathfrak{a}$ *and* $\mathfrak{b}$*,*

$$\min\{\mathfrak{a},\aleph_1\} \;=\; \min\{\mathfrak{b},\aleph_1\} \quad\Longrightarrow\quad \mathrm{Th}\,(\mathfrak{N};\mathfrak{a}) \;=\; \mathrm{Th}\,(\mathfrak{N};\mathfrak{b}).$$

In particular, $\mathrm{Th}\,(\mathfrak{N};\aleph_1) = \mathrm{Th}\,(\mathfrak{N};\mathfrak{a})$ *for all cardinals* $\mathfrak{a} \geqslant \aleph_1$*.*

Proof. Immediate. □

For instance, it makes no substantial difference whether we choose $2^{\aleph_0}$ or $\aleph_1$, and clearly no assumption about the Continuum Hypothesis has been used to prove this.[6]

Let us conclude with a remark on finite cardinals. Certainly the condition 'W has exactly n members' is expressible in $\mathcal{L}$. Formally, for each $n \in \mathbb{N}$, consider the $\mathcal{L}$-sentence

$$\begin{aligned}\mathsf{D}_n \;:=\; \exists x_1 \ldots \exists x_n \,\exists y \left(\bigwedge\nolimits_{i=1}^{n-1} \bigwedge\nolimits_{j=i+1}^{n} \Box\,(x_i \neq x_j) \wedge \bigwedge\nolimits_{i=1}^{n} \Diamond\,(y = x_i)\right) \wedge \\ \neg\exists x_1 \ldots \exists x_{n+1} \,\exists y \left(\bigwedge\nolimits_{i=1}^{n} \bigwedge\nolimits_{j=i+1}^{n+1} \Box\,(x_i \neq x_j) \wedge \bigwedge\nolimits_{i=1}^{n+1} \Diamond\,(y = x_i)\right).\end{aligned}$$

It is straightforward to show that for every cardinal $\mathfrak{a}$,

$$\mathfrak{a} \Vdash \mathsf{D}_n \quad\Longleftrightarrow\quad \mathfrak{a} \;=\; n.$$

Thus $\mathcal{L}$ allows us to distinguish, in addition to $\aleph_0$, each finite cardinal (and nothing else).

4.3 Indiscernibility

Although the original semantics for $\mathcal{L}$ provided in Subsection 3.1 may be technically less convenient than its variation presented in Subsection 3.2, it has some advantages for structuralists:

- while $\mathfrak{N}$ is the preferred first-order structure of the natural numbers in Subsection 3.2, we treat it on par with any $\pi\,[\mathfrak{N}]$ in Subsection 3.1;
- while each $n \in \mathbb{N}$ plays the same role at all possible worlds in Subsection 3.2, its role varies with the choice of π in Subsection 3.1.

We are going to discuss indiscernibility in terms of the original semantics. Evidently one could do it in terms of the alternative semantics as well (recalling Theorem 3.4) — but it would seem somewhat unnatural.

For expository purposes, we took the underlying countably infinite plurality of objects to be $\mathbb{N}$ itself in Subsection 3.1. Then every $n \in \mathbb{N}$ can be embedded into G as

$$c_n \;:=\; \lambda\pi.[n].$$

Notice that the constant function c_n is, in effect, a very arbitrary number, and nothing like the specific number n in G. (Moreover, if we pass from c_n to c_n^ι using Theorem 3.4, then c_n^ι will be nothing like a constant function, of course.)

In keeping with the spirit of structuralism, we want the elements of the underlying set to be to the highest possible degree (compatible with the presence of identity) indistinguishable from each other. To this end, for any $\{\alpha_1,\alpha_2\} \subseteq \mathrm{A}$ we let

$$\mathrm{Type}\,(\alpha_1,\alpha_2) \;:=\; \{\Phi\,(x,y) \mid \Phi \text{ is an } \mathcal{L}\text{-formula and } \mathrm{G} \models \Phi\,[\alpha_1,\alpha_2]\}.$$

Now α_1 and α_2 are said to be *relatively distinguishable* iff $\mathrm{Type}\,(\alpha_1,\alpha_2) \neq \mathrm{Type}\,(\alpha_2,\alpha_1)$ — this definition and a discussion of its suitability can be found in [Ladyman et al. 2012].

[6]We assume nothing beyond ZFC (the Axiom of Choice was used in the proof of Lemma 4.9, for example).

Theorem 4.12. *For any* $\{i, j\} \subseteq \mathbb{N}$,

$$\text{Type}\,(c_i, c_j) \;=\; \text{Type}\,(c_j, c_i),$$

i.e. c_i *and* c_j *are relatively indistinguishable.*

Proof. Let $\{i, j\} \subseteq \mathbb{N}$. We need to show that for all $\mathcal{L}$-formulas $\Phi\,(x, y)$,

$$\mathrm{G} \models \Phi\,[c_i, c_j] \quad\Longleftrightarrow\quad \mathrm{G} \models \Phi\,[c_j, c_i]$$

— which is, assuming $|W| = |\mathrm{G}|$ and using the notation of Subsection 3.2, equivalent to

$$W \Vdash \Phi\left[c_i^\iota, c_j^\iota\right] \quad\Longleftrightarrow\quad W \Vdash \Phi\left[c_j^\iota, c_i^\iota\right]$$

(by Theorem 3.4). At the same time it is straightforward to check that:

a. c_i^ι and c_j^ι are congruent;

b. for every $w \in W$ there exists $w' \in W$ such that

$$\left(c_i^\iota\,(w), c_j^\iota\,(w)\right) \;=\; \left(c_j^\iota\,(w'), c_i^\iota\,(w')\right),$$

and vice versa.

Thus by Theorem 4.10 one easily obtains the desired result. □

Intuitively, while Theorem 4.5 guarantees that the specific numbers in G are maximally distinguishable from each other, Theorem 4.12 tells us that the objects in the underlying set — or rather, their representations in G via $\lambda n.[c_n]$ — are at the opposite extreme.

4.4 Compactness

Let W be a non-empty collection of possible worlds. A set Γ of $\mathcal{L}$-formulas whose free variables are among $x_1, \ldots, x_\ell$ is called *satisfiable* in W if there is an $\vec{\alpha} \in \mathcal{W}^\ell$ such that $W \Vdash \Phi\,[\vec{\alpha}]$ for all $\Phi \in \Gamma$, and *finitely satisfiable* if every finite subset of Γ is satisfiable. Similarly for G, which can be viewed as the special case when $|W| = |\mathrm{G}|$ by Theorem 3.4. Not surprising, we have:

Proposition 4.13. *Let* W *be infinite. There exists a set of* $\mathcal{L}$*-formulas which is finitely satisfiable but not satisfiable in* W.

Proof. We prove this for $\ell = 1$. Traditionally, with every $n \in \mathbb{N}$ comes a closed σ-term $\underline{n}$, called its *numeral*, given by:

$$\underline{n} \;:=\; \begin{cases} 0 & \text{if } n = 0, \\ \mathsf{s}\,(\underline{n-1}) & \text{if } n > 0. \end{cases}$$

Now define a sequence $\Phi_0\,(x)$, $\Phi_1\,(x)$, $\ldots$ of $\mathcal{L}$-formulas by recursion:

$$\Phi_n\,(x) \;:=\; \begin{cases} \Diamond\,(x = 0) & \text{if } n = 0, \\ \Phi_{n-1} \wedge \Diamond\,(x = \underline{n}) & \text{if } n > 0. \end{cases}$$

On the other hand, let

$$\Phi_\infty\,(x) \;:=\; \neg\forall^{\mathsf{Sp}} y \,\Diamond\,(x = y).$$

Evidently $\{\Phi_n \mid n \in \mathbb{N}\} \cup \{\Phi_\infty\}$ has the desired properties. □

The failure of compactness for $\mathcal{L}$-formulas may point to complexity beyond first-order arithmetic. In the next section we shall examine certain computational aspects of $\mathcal{L}$.

5 Complexity aspects

We are now going to prove that, in fact, both $\mathrm{Th}(\mathbb{G})$ and $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$ have the same complexity as *complete second-order arithmetic*, i.e. the second-order theory of $\mathfrak{N}$. To make this precise, some basic terminology of computability theory is needed; see e.g. [Soare 2016, Chapter 1].

Let P and Q be subsets of $\mathbb{N}$. We say that P is *m-reducible* to Q, written $P \leqslant_m Q$, iff there exists a computable $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$n \in P \quad \Longleftrightarrow \quad f(n) \in Q.$$

Also, P and Q are called *m-equivalent*, written $P \equiv_m Q$, iff they are m-reducible to each other. The notion of m-reducibility clearly plays a major role in studying complexity of decision problems and expressive power of formal languages in the foundations of mathematics.

Without loss of generality, we can restrict our attention to *monadic* second-order arithmetic (since first-order arithmetic allows us to code elements of $\mathbb{N}^\ell$ as elements of $\mathbb{N}$). Its language $\mathcal{L}_2$ includes two sorts of variables, namely:

- *individual variables* x, y, z, … (intended to range over natural numbers);
- *set variables* X, Y, Z, … (intended to range over sets of natural numbers).[7]

Accordingly one must distinguish between *individual* and *set quantifiers*, viz.

$$\exists x,\ \exists y,\ \exists z,\ \ldots \quad \text{and} \quad \exists X,\ \exists Y,\ \exists Z,\ \ldots$$

The *$\mathcal{L}_2$-formulas* — or *monadic second-order σ-formulas* — are built up from the first-order σ-formulas and the expressions of the form $t \in X$, where t is a σ-term and X is a set variable, by means of the connective symbols and the quantifiers in the usual way. As one would expect, we write $\neg\exists X \neg\Phi$ as shorthand for $\forall X\, \Phi$, and adopt other standard abbreviations.

So we wish to show that both $\mathrm{Th}(\mathbb{G})$ and $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$ are m-equivalent to the set of all $\mathcal{L}_2$-sentences true in $\mathfrak{N}$, denoted by $\mathrm{Th}_2(\mathfrak{N})$.

5.1 Hardness

First we prove that each of $\mathrm{Th}(\mathbb{G})$, $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$ is at least as complex as $\mathrm{Th}_2(\mathfrak{N})$. And in fact, our argument below will demonstrate a bit more.

For convenience we pass from G to W (without imposing any restrictions on the cardinality of W at the moment). We say $P \subseteq \mathbb{N}^\ell$ is *absolutely definable* in W iff there exists an $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_\ell)$ such that for every $(n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$,

$$(n_1, \ldots, n_\ell) \in P \quad \Longleftrightarrow \quad W \Vdash \Phi[\mathbf{n}_1, \ldots, \mathbf{n}_\ell]$$

where $\mathbf{n}_i = \lambda w.[n_i]$ for each $i \in \{1, \ldots, \ell\}$.

Theorem 5.1. *There exists a computable function τ that, given any $\mathcal{L}_2$-formula $\Phi(x_1, \ldots, x_\ell)$, produces an $\mathcal{L}$-formula $\Phi^\tau(y_1, \ldots, y_\ell)$ such that for every infinite W and $(n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$,*

$$\mathfrak{N} \models \Phi[n_1, \ldots, n_\ell] \quad \Longleftrightarrow \quad W \Vdash \Phi^\tau[\mathbf{n}_1, \ldots, \mathbf{n}_\ell],$$

i.e. Φ^τ absolutely defines in W the same set as Φ defines in $\mathfrak{N}$.[8]

[7] Henceforth we shall identify the collection of all individual variables with Var.

[8] We can try to allow free set variables in Φ as well: the translation τ supplied by the proof below shows that just as for each $n \in \mathbb{N}$ its role in W is played by $\mathbf{n}$, for each non-empty $P \subseteq \mathbb{N}$ its role in W can be played by an arbitrary $\alpha \in \mathcal{W}$ that satisfies $P = \{\alpha(w) \mid w \in W\}$. There may be infinitely many such α's, however, and there seems to be no canonical way to choose between them.

Proof. Assume W is infinite. Then given any non-empty $P \subseteq \mathbb{N}$, one can find $\alpha \in \mathcal{W}$ such that

$$P \;=\; \{\alpha(w) \mid w \in W\}$$

— in other words, for all $n \in \mathbb{N}$,

$$n \;\in\; P \quad\Longleftrightarrow\quad \alpha \text{ and } \mathbf{n} \text{ coincide at some } w \in W.$$

Clearly we want the elements of F to play the role of natural numbers, and those of $\mathcal{W}$ the role of non-empty sets of natural numbers. Of course, the empty set is easy to model.

It is straightforward to effectively convert every $\mathcal{L}_2$-formula Φ into a semantically equivalent $\mathcal{L}_2$-formula $\Phi^\star$ satisfying the following conditions:

- each atomic subformula of $\Phi^\star$ is of either of the forms

$$x = y, \quad x = y + z, \quad x = y \times z \quad \text{and} \quad x \in X;$$

- no set or individual quantifier occurs more than once in $\Phi^\star$.

Now let v_0, v_1, ... and V_0, V_1, ... be fixed enumerations of the individual variables and the set variables respectively. Given an $\mathcal{L}_2$-formula Φ, take

$$\Phi^\tau \;:=\; (\Phi^\star)^\tau$$

with $(\Phi^\star)^\tau$ defined inductively as follows:

$$\begin{aligned}
(v_i = v_j)^\tau &:= v_{2i} = v_{2j};\\
(v_i + v_j = v_k)^\tau &:= v_{2i} + v_{2j} = v_{2k};\\
(v_i \times v_j = v_k)^\tau &:= v_{2i} \times v_{2j} = v_{2k};\\
(v_i \in V_j)^\tau &:= \Diamond\,(v_{2i} = v_{2j+1});\\
(\neg\Psi)^\tau &:= \neg\tau\,(\Psi);\\
(\Psi \wedge \Theta)^\tau &:= \Psi^\tau \wedge \Theta^\tau;\\
(\exists v_i\,\Psi)^\tau &:= \exists v_{2i}\,(\mathsf{Sp}\,(v_{2i}) \wedge \Psi^\tau);\\
(\exists V_j\,\Psi)^\tau &:= \exists v_{2j+1}\,\Psi^\tau \vee (\Psi\,[V_j := \varnothing])^\tau
\end{aligned}$$

where $\Psi\,[V_j := \varnothing]$ denotes the result of replacing each $v_i \in V_j$ in Ψ by $v_i \neq v_i$. Observe that for every $\mathcal{L}_2$-formula $\Phi\,(v_{i_1}, \ldots, v_{i_\ell})$ the corresponding $\mathcal{L}$-formula $\Phi^\tau\,(v_{2i_1}, \ldots, v_{2i_\ell})$ absolutely defines in W what is needed, as an easy induction on the construction of Φ shows. $\square$

We remark that in the context of monadic second-order logic the difference between the signature σ of Peano arithmetic and its fragment $\{0, \mathsf{s}, +, =\}$, which is the signature of Presburger arithmetic, turns out to be inessential, because we can eliminate $\times$ from $\mathcal{L}_2$ without any loss of expressiveness; see [Speranski 2013] for details and further references.[9] In view of Theorem 5.1, the same applies to $\mathcal{L}$. However, the reader should bear in mind that although multiplication is absolutely definable using a $\times$-free $\mathcal{L}$-formula, expressing set quantifiers in $\mathcal{L}$ requires $\Diamond$, and in particular, no purely arithmetical $\times$-free formula can do this job.

Corollary 5.2. *$\mathrm{Th}_2\,(\mathfrak{N})$ is m-reducible to both $\mathrm{Th}\,(\mathbb{G})$ and $\mathrm{Th}\,(\mathbb{G}_\mathsf{c})$.*

Proof. By the Theorem above, for each infinite W, $\mathrm{Th}_2\,(\mathfrak{N})$ is m-reducible to $\mathrm{Th}\,(\mathfrak{N}; W)$. Since $\mathrm{Th}\,(\mathbb{G}) = \mathrm{Th}\,(\mathfrak{N}; 2^{\aleph_0})$ and $\mathrm{Th}\,(\mathbb{G}_\mathsf{c}) = \mathrm{Th}\,(\mathfrak{N}; \aleph_0)$, the result follows. $\square$

Thus the expressive power of $\mathcal{L}$ is greater than or equal to that of $\mathcal{L}_2$. In effect, the equality holds, as will be established in the next subsection.

[9]While in the context of first-order logic this is not so, of course.

5.2 Boundedness

Further, the complexity of $\mathrm{Th}(\mathbb{G})$ and $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$ turns out to be bounded by that of $\mathrm{Th}_2(\mathfrak{N})$. In order to prove this, we shall reconsider the technique developed in Subsection 4.2.

Theorem 5.3. *For every W, $\mathrm{Th}(\mathfrak{N}; W)$ is m-reducible to $\mathrm{Th}_2(\mathfrak{N})$.*

Proof. By Corollary 4.11, without loss of generality, we can restrict to W's with $|W| \leqslant \aleph_1$. The most interesting case is where $|W| = \aleph_1$; the other cases are, in fact, much simpler, and may be treated similarly. Assume W has cardinality $\aleph_1$.

Briefly stated, Theorem 4.10 allows us to replace possible worlds by tuples of elements of $\mathbb{N}$, and arbitrary numbers by functions from tuples of elements of $\mathbb{N}$ to elements of $\mathbb{N} \cup \{\aleph_0, \aleph_1\}$ — hence the result follows. Formally, we are going to suitably redefine $\Vdash$.

Throughout the proof by an *ℓ-profile* we shall mean a function from $\mathbb{N}^\ell$ to $\mathbb{N} \cup \{\aleph_0, \aleph_1\}$ that takes $\aleph_1$ as value at least once.[10] Denote by Prof_ℓ the collection of all ℓ-profiles; note that Prof_0 coincides with $\{\mathrm{prof}_{()}\}$, where $\mathrm{prof}_{()}$ is the unique function from $\{()\}$ to $\{\aleph_1\}$. Evidently

$$\mathrm{Prof}_\ell \;=\; \big\{\mathrm{prof}_{\vec{\alpha}} \mid \vec{\alpha} \in \mathcal{W}^\ell\big\}.$$

Remember that by Theorem 4.10, any two elements of $\mathcal{W}^\ell$ with the same profile are indistinguishable by $\mathcal{L}$-formulas.

Moreover, in view of Theorem 4.10, given an $\mathcal{L}$-formula $\Phi(x_1, \dots, x_\ell)$ and $\vec{\alpha} \in \mathcal{W}^\ell$, the only thing we need to know about $w \in W$ is $\vec{\alpha}(w)$. For $\ell \neq 0$ this amounts to switching from worlds to ℓ-tuples that $\vec{\alpha}$ takes as value at least once. Accordingly, for each $f \in \mathrm{Prof}_\ell$ we consider

$$\mathbb{W}_f \;:=\; \big\{\vec{n} \in \mathbb{N}^\ell \mid f(\vec{n}) \neq 0\big\}.$$

Certainly reducing the number of quantifiers in a given $\Phi(x_1, \dots, x_\ell)$ requires passing to tuples of length greater than ℓ. To see how to do it in the present setting, let $f \in \mathrm{Prof}_\ell$, $g \in \mathrm{Prof}_{\ell+1}$, $\vec{n} \in \mathbb{W}_f$ and $\vec{m} \in \mathbb{W}_g$. We say $\langle g, \vec{m}\rangle$ *extends* $\langle f, \vec{n}\rangle$, written $\langle g, \vec{m}\rangle \succcurlyeq \langle f, \vec{n}\rangle$, if:

i. $f(\vec{p}) = \sum_{i=0}^{\infty} g(\vec{p}, i)$ for every $\vec{p} \in \mathbb{N}^\ell$;

ii. $\vec{m} = (\vec{n}, i)$ for some $i \in \mathbb{N}$.

Intuitively, (i) guarantees that g can be obtained by appropriately splitting f (cf. the argument for Lemma 4.9), and by (ii) it can be assumed that we deal with the same world.

Now we are ready to redefine $\Vdash$ in a suitable way. Namely, for any $\mathcal{L}$-formula $\Phi(x_1, \dots, x_\ell)$, $f \in \mathrm{Prof}_\ell$ and $\vec{n} \in \mathbb{W}_f$, define

$$\vec{n} \triangleright \Phi[f]$$

inductively as follows:

- $\vec{n} \triangleright t_1 = t_2[f]$ iff $t_1^{\mathfrak{N}}(\vec{n}) = t_2^{\mathfrak{N}}(\vec{n})$;
- $\vec{n} \triangleright \neg\Psi[f]$ iff it is not the case that $\vec{n} \triangleright \Psi[f]$;
- $\vec{n} \triangleright \Psi \vee \Theta[f]$ iff $\vec{n} \triangleright \Psi[f]$ or $\vec{n} \triangleright \Theta[f]$;
- $\vec{n} \triangleright \exists y\, \Psi[f]$ iff there are $g \in \mathrm{Prof}_{\ell+1}$ and $\vec{m} \in \mathbb{W}_g$ such that $\langle g, \vec{m}\rangle \succcurlyeq \langle f, \vec{n}\rangle$ and $\vec{m} \triangleright \Psi[g]$;
- $\vec{n} \triangleright \Diamond\Psi[f]$ iff there is $\vec{m} \in \mathbb{W}_f$ such that $\vec{m} \triangleright \Psi[f]$;

[10] In other words, the sum of the elements in the range of this function must be $\aleph_1$ — in general, one needs to substitute $|W|$ for $\aleph_1$, of course.

- $\vec{n} \triangleright \mathsf{Sp}(t)[f]$ iff $t^{\mathfrak{N}}(\vec{n}) = t^{\mathfrak{N}}(\vec{m})$ for all $\vec{m} \in \mathbb{W}_f$.

Then using Theorem 4.10, it is straightforward to show that for every $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_\ell)$, $\vec{\alpha} \in \mathcal{W}^\ell$ and $w \in W$,

$$w \Vdash \Phi[\vec{\alpha}] \quad \Longleftrightarrow \quad \vec{\alpha}(w) \triangleright \Phi[\mathrm{prof}_{\vec{\alpha}}].$$

Therefore $\mathrm{Th}(\mathfrak{N}; W)$ coincides with the set of all $\mathcal{L}$-sentences Φ which satisfy $() \triangleright \Phi[\mathrm{prof}_{()}]$. To complete the argument, we make the following observations:

a. each ℓ-profile, being a countable object, can be encoded as a subset of $\mathbb{N}$;

b. $\triangleright$ can be readily expressed within $\mathcal{L}_2$, so there exists a computable function ρ from $\mathcal{L}$-sentences to $\mathcal{L}_2$-sentences such that

$$() \triangleright \Phi[\mathrm{prof}_{()}] \quad \Longleftrightarrow \quad \rho(\Phi) \ \in \ \mathrm{Th}_2(\mathfrak{N}).$$

This gives the desired m-reduction. □

Corollary 5.4. *Both* $\mathrm{Th}(\mathbb{G})$ *and* $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$ *are m-reducible to* $\mathrm{Th}_2(\mathfrak{N})$*.*

Proof. Since $\mathrm{Th}(\mathbb{G}) = \mathrm{Th}\left(\mathfrak{N}; 2^{\aleph_0}\right)$ and $\mathrm{Th}(\mathbb{G}_{\mathsf{c}}) = \mathrm{Th}(\mathfrak{N}; \aleph_0)$, the result follows. □

Finally we get:

Theorem 5.5. $\mathrm{Th}(\mathbb{G})$, $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$ *and* $\mathrm{Th}_2(\mathfrak{N})$ *are m-equivalent to each other.*

Proof. Immediate from Corollaries 5.2 and 5.4. □

6 Directions of further research

It would be interesting to continue the investigation of generic structures and their metamathematical aspects. Let us briefly mention some directions that might profitably be explored.

- In this paper we have been concerned with the model-theoretic and complexity aspects of G, i.e. $\mathbb{G}$ and $\mathbb{G}_{\mathsf{c}}$. On the other hand, one may also wish to develop some proof theory for reasoning about these generic structures. Of course, no computably enumerable system of axioms and rules can capture $\mathrm{Th}(\mathbb{G})$ or $\mathrm{Th}(\mathbb{G}_{\mathsf{c}})$, by Corollary 5.2; however, Corollary 5.4 suggests that one may try to develop proof systems for them in much the same fashion as in second-order arithmetic.

- We have shown that the $\mathcal{L}$-theories of $\mathbb{G}$ and $\mathbb{G}_{\mathsf{c}}$ both have the same complexity as complete second-order arithmetic. One might wonder what happens when we restrict ourselves to reasonable fragments of $\mathcal{L}$. The answer is clear in some cases (e.g., when $\times$ is excluded from the language), but it requires a deeper analysis in others.

- Another issue not touched on here concerns the study of the notions of independence and determinateness within the Carnapian framework we exploited. It would be interesting to investigate the relationship between the present framework and that of the so-called *independence-friendly logic* (which was introduced in [Hintikka & Sandu 1989], and advocated in [Hintikka 1996]) and its variations (see, for instance, [Grädel & Väänänen 2013]).

- In effect we do not have to limit our attention to arithmetic and its reducts — the notion of generic truth can be applied to other structures as well.

All these fall beyond the scope of this paper, and are the subject of future work.

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