Belnap–Dunn Modal Logics: Truth Constants vs. Truth Values

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Abstract

We shall be concerned with the modal logic BK — which is based on the Belnap–Dunn four-valued matrix, and can be viewed as being obtained from the least normal modal logic K by adding 'strong negation'. Though all four values 'truth', 'falsity', 'neither' and 'both' are employed in its Kripke semantics, only the first two are expressible as terms. We show that expanding the original language of BK to include constants for 'neither' or/and 'both' leads to quite unexpected results. To be more precise, adding one of these constants has the effect of eliminating the respective value at the level of BK-extensions. In particular, if one adds both of these, then the corresponding lattice of extensions turns out to be isomorphic to that of ordinary normal modal logics.

Keywords: many-valued modal logic, first-degree entailment, strong negation, algebraic logic.

1 Introduction

This article will be concerned with the modal logic BK, which was originally introduced in [14]. The non-modal base of BK is the Belnap–Dunn 'useful' four-valued matrix augmented with the constant *falsity* ($\bot$) and the so-called *weak implication* ($\rightarrow$). In effect, BK can be viewed as the expansion of the least normal modal logic K obtained by adding *strong negation* ($\sim$). Certainly this naturally leads to a Kripke semantics for BK analogous to that for K, but now at each possible world we have four truth values:

1. T, pronounced *truth*;
2. F, pronounced *falsity*;
3. N, pronounced *neither* (which intuitively stands for 'neither true nor false');
4. B, pronounced *both* (which intuitively stands for 'both true and false').

Of course $\bot$ is always assigned F. Furthermore, $\sim\bot$ defines T. Therefore F and T are explicitly expressible as terms in the language of BK. It is easy to show that N and B do not have this property, however (even though they are implicitly available in the semantics). We are going to investigate how expanding the original language of BK to include constants for N or B modifies the structure of the BK-extensions.

It should be remarked that neither the principle of explosion (*ex falso quodlibet*) nor that of excluded middle (*tertium non datur*) is provable in BK; in other words, BK is both 'gappy' and 'glutty' with respect to $\sim$. Let us consider the BK-extensions

$$\mathsf{B3K} := \mathsf{BK} + \{\sim p \to (p \to q)\}, \quad \mathsf{BK}^\circ := \mathsf{BK} + \{p \vee \sim p\}$$
$$\text{and} \quad \mathsf{B3K}^\circ := \mathsf{BK} + \{\sim p \to (p \to q),\, p \vee \sim p\}.$$

As we shall see, in a sense B3K and BK° are three-valued, while $\mathsf{B3K}^\circ$ is two-valued:

- B3K has a natural Kripke semantics using only the truth values T, F and N;
- BK° has a natural Kripke semantics using only the truth values T, F and B;
- $\mathsf{B3K}^\circ$ has a natural Kripke semantics using only the truth values T and F.

Now expanding the original language of BK to include constants for N or B leads to rather surprising results:

- if one adds a constant for N, then the corresponding lattice of extensions in the expanded language turns out to be isomorphic to that of BK°-extensions;
- if one adds a constant for B, then the corresponding lattice of extensions in the expanded language turns out to be isomorphic to that of B3K-extensions;
- if we add constants for N and B, then the corresponding lattice of extensions turns out to be isomorphic to that of $\mathsf{B3K}^\circ$-extensions.

Notice — as was proved in [13], the lattice of $\mathsf{B3K}^\circ$-extensions, in turn, is isomorphic to that of K-extensions, i.e. consisting of ordinary normal modal logics.

At this point it is worth giving some historical background for our work. Since BK plays the same role for K that $\mathsf{N4}^\perp$ — the version of Nelson's constructive logic N4 augmented with $\perp$ — plays for intuitionistic logic Int, let us briefly discuss $\mathsf{N4}^\perp$ here. Take

$$\mathsf{N3} := \mathsf{N4} + \{\sim p \to (p \to q)\} \quad \text{and} \quad \mathsf{N4}^\circ := \mathsf{N4}^\perp + \{((p \vee \sim p) \to \perp) \to \perp\}.^{1}$$

It has been known for a long time that in N3 the *strong implication* $\Rightarrow$, defined by

$$\phi \Rightarrow \psi := (\phi \to \psi) \wedge (\sim \psi \to \sim \phi),$$

has substructural properties; see e.g. [21]. In particular,

$$p \Rightarrow (p \Rightarrow q) \quad \text{and} \quad p \Rightarrow q$$

are not equivalent in N3, so contraction fails already for N3. Using the prover OTTER M. Spinks and R. Veroff [19] showed syntactically that the variety of N3-lattices — providing an algebraic semantics for N3 — is definitionally equivalent to a suitable variety of residuated lattices. Thus N3 can in fact be treated as an axiomatic extension of the full Lambek calculus with exchange and weakening (see [6]). A rather short semantical proof for the result of Spinks and Veroff was given in [5]. Attempting to generalize this to N4 M. Busaniche and R. Cignoli had to pass from N4-lattices — providing an algebraic semantics for N4 — to their expansions with a constant b specified by

$$\mathtt{b} = \sim \mathtt{b} \quad \text{and} \quad \mathtt{b} \to \mathtt{b} = \mathtt{b}.$$

[1] We do not need to add $\perp$ to the language of N3 because $\perp$ can be introduced into N3 via $\sim (p \to p)$; cf. [9].

It was shown in [4] that the variety of these expansions is definitionally equivalent to a suitable variety of residuated lattices with involution.[2] Notice that one may wish to consider a constant $\mathtt{n}$ specified by

$$\mathtt{n} \;=\; \sim \mathtt{n} \quad \text{and} \quad \neg \mathtt{n} \to \neg \mathtt{n} \;=\; \neg \mathtt{n}$$

in the same vein. In [10, 11] the corresponding logics

$$\mathsf{bN4}^{\perp} \;:=\; \text{the version of } \mathsf{N4}^{\perp} \text{ augmented with } \mathtt{b} \quad \text{and}$$
$$\mathsf{nN4}^{\perp} \;:=\; \text{the version of } \mathsf{N4}^{\perp} \text{ augmented with } \mathtt{n}$$

were introduced, and it was proved that the lattices of $\mathsf{bN4}^{\perp}$- and $\mathsf{nN4}^{\perp}$-extensions turn out to be isomorphic to those of $\mathsf{N3}$- and $\mathsf{N4}^{\circ}$-extensions respectively.

The situation with the FDE-based modal logic BK appears to be somewhat more symmetric than that with Nelson's logics. We are going to understand how adding constants for N or B to the language of BK has the effect of eliminating 'gaps' or 'gluts' at the metalevel of BK-extensions. Moreover, since $\mathsf{N4}^{\perp}$ is faithfully embedded into

$$\mathsf{BS4} \;:=\; \mathsf{BK} + \{\Box p \to p,\, \Box p \to \Box\Box p\}$$

by means of a translation similar to the well-known Gödel–McKinsey–Tarski translation of Int into the normal modal logic $\mathsf{S4}$ (see [14, Section 7.1] for details), our work could be viewed as a generalisation of [10, 11].[3]

2 Preliminaries

The logic BK was originally defined and developed in the language

$$\mathcal{L} \;:=\; \{\vee, \wedge, \to, \perp, \sim, \Box, \Diamond\}.$$

Although some alternatives are possible, we shall continue to use $\mathcal{L}$ because it allows us to pass from formulas to their 'negation normal forms' (cf. [9]) in a direct way.

2.1 The lattice of BK-extensions

Let a countable set $\mathrm{Prop} := \{p_0, p_1, p_2, \dots\}$ of *propositional variables* be given. Then by the set $\mathrm{For}_{\mathcal{L}}$ of *$\mathcal{L}$-formulas* is meant the set of all expressions that can be built up from Prop using the symbols of $\mathcal{L}$ in the customary way; similarly for fragments of $\mathcal{L}$ and its expansions. To simplify the presentation we shall employ some standard abbreviations:

$$\neg\phi \;:=\; \phi \to \perp, \quad \varphi \leftrightarrow \psi \;:=\; (\varphi \to \psi) \wedge (\psi \to \varphi)$$
$$\text{and} \quad \varphi \Leftrightarrow \psi \;:=\; (\varphi \leftrightarrow \psi) \wedge (\sim \varphi \leftrightarrow \sim \psi).$$

Define an *$\mathcal{L}$-logic* to be a collection of $\mathcal{L}$-formulas closed under the substitution rule, *modus ponens* and the monotonicity rules for $\Box$ and $\Diamond$, i.e. under

$$\frac{\varphi\,(p_1, \dots, p_n)}{\varphi\,(\psi_1, \dots, \psi_n)}\ (\mathtt{S}), \quad \frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathtt{MP}), \quad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\ (\Box\mathtt{M}) \quad \text{and} \quad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\,\psi}\ (\Diamond\mathtt{M}).$$

[2]In effect, their syntactical translations are substantionally different from those presented in [19].

[3]Obviously this nice connection may be lost if we interpret $\to$ and $\perp$ differently or drop one of them. On the algebraic side, we want to have an implication which is easier to handle than the intuitionistic one, thus shifting our attention to $\Box$, and the addition of a falsity constant tends to make lattices of logics somewhat more regular (cf. [9]). Note that $\mathsf{N4}^{\perp}$ — not $\mathsf{N4}$ — is a conservative extension of Int, and similarly for BK and K, $\mathsf{BS4}$ and $\mathsf{S4}$ (see [9, 14]). So, in fact, the translation of $\mathsf{N4}^{\perp}$ into $\mathsf{BS4}$ directly extends that of Int into $\mathsf{S4}$.

For an $\mathcal{L}$-logic L and $\Gamma \cup \{\phi\} \subseteq \mathrm{For}_{\mathcal{L}}$, we write $\Gamma \vdash_L \phi$ iff ϕ can be obtained from $\Gamma \cup L$ by MP. Evidently the intersection of any set of $\mathcal{L}$-logics is again an $\mathcal{L}$-logic. Given $X, Y \subseteq \mathrm{For}_{\mathcal{L}}$, take

$$X + Y \;:=\; \text{the intersection of all } \mathcal{L}\text{-logics containing } X \cup Y.$$

For each $\mathcal{L}$-logic L, denote the class of all $\mathcal{L}$-logics extending L by $\mathcal{E}L$. One readily verifies that $\mathcal{E}L$ with operations $\cap$ and $+$ is a lattice, in which the ordering coincides with set inclusion. We let BK be the least $\mathcal{L}$-logic containing the following axioms:

A1. the axioms of the classical propositional logic CL stated in the language $\{\vee, \wedge, \rightarrow, \bot\}$;

A2. the five strong negation axioms

$$\begin{array}{ll} \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q), & \sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q), \\ \sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q), & \sim\sim p \leftrightarrow p \quad \text{and} \quad \sim\bot; \end{array}$$

A3. the two pure modal axioms

$$(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q) \qquad \text{and} \qquad \Box(p \rightarrow p);$$

A4. the four modal interaction axioms

$$\begin{array}{ll} \neg\Box p \leftrightarrow \Diamond\neg p, & \Box p \Leftrightarrow \sim\Diamond\!\sim p, \\ \neg\Diamond p \leftrightarrow \Box\neg p, & \Diamond p \Leftrightarrow \sim\Box\!\sim p. \end{array}$$

Thus we have a Hilbert-style calculus for BK. As was proved in [14], it turns out to be strongly complete with respect to a possible world semantics that is much like the standard semantics of K but with four-valued valuations instead of two-values ones. Furthermore, like Nelson's logics, BK is not closed under the (ordinary) replacement rule, but only under its 'positive' and 'weak' versions, i.e.

$$\frac{\varphi_1 \leftrightarrow \psi_1, \;\ldots,\; \varphi_n \leftrightarrow \psi_n}{\chi(\varphi_1, \ldots, \varphi_n) \leftrightarrow \chi(\psi_1, \ldots, \psi_n)}\ (\mathtt{PR}) \quad \text{and} \quad \frac{\varphi_1 \Leftrightarrow \psi_1, \;\ldots,\; \varphi_n \Leftrightarrow \psi_n}{\theta(\varphi_1, \ldots, \varphi_n) \leftrightarrow \theta(\psi_1, \ldots, \psi_n)}\ (\mathtt{WR})$$

where χ does not contain $\sim$; cf. [14] and also [9].

An $\mathcal{L}$-formula ϕ is said to be a *negation normal form* (*nnf* for short) iff all occurrences of $\sim$ in ϕ immediately precede propositional variables and constants. One quickly deduces

Proposition 2.1. *For every $\phi \in \mathrm{For}_{\mathcal{L}}$ there exists a nnf $\overline{\phi}$ such that $\phi \leftrightarrow \overline{\phi} \in \mathsf{BK}$.*

Proof. Notice — from the axioms $p \leftrightarrow \sim\sim p$, $\Box p \Leftrightarrow \sim\Diamond\!\sim p$ and $\Diamond p \Leftrightarrow \sim\Box\!\sim p$ of BK, and the transitivity of $\leftrightarrow$, it is easy to deduce that

$$\sim\Box p \Leftrightarrow \Diamond\!\sim p \quad \text{and} \quad \sim\Diamond p \Leftrightarrow \Box\!\sim p \tag{$\star$}$$

are in BK. For each $\mathcal{L}$-formula ϕ we define a nnf $\overline{\phi}$ as follows:

- if $\phi \in \mathrm{Prop} \cup \{\bot\}$, then $\overline{\phi} := \phi$;
- if $\phi = \varphi * \psi$ where $* \in \{\vee, \wedge, \rightarrow\}$, then $\overline{\phi} := \overline{\varphi} * \overline{\psi}$;
- if $\phi = \sim\varphi$ where $\varphi \in \mathrm{Prop} \cup \{\bot\}$, then $\overline{\phi} := \phi$;

- if $\phi = \sim(\varphi \wedge \psi)$, then $\overline{\phi} := \overline{\sim\varphi} \vee \overline{\sim\psi}$;
- if $\phi = \sim(\varphi \vee \psi)$, then $\overline{\phi} := \overline{\sim\varphi} \wedge \overline{\sim\psi}$;
- if $\phi = \sim(\varphi \rightarrow \psi)$, then $\overline{\phi} := \overline{\varphi} \wedge \overline{\sim\psi}$;
- if $\phi = \sim\Box\varphi$, then $\overline{\phi} := \Diamond\overline{\sim\varphi}$;
- if $\phi = \sim\Diamond\varphi$, then $\overline{\phi} := \Box\overline{\sim\varphi}$;
- if $\phi = \sim\sim\varphi$, then $\overline{\phi} := \overline{\varphi}$.

Using A2 and ($\star$) together with the admissibility of PR in BK, it is straightforward to derive the desired logical equivalence. □

Given $\phi \in \mathrm{For}_{\mathcal{L}}$, we let the *negation normal form of* ϕ be the nnf $\overline{\phi}$ constructed in the proof of Proposition 2.1. Note in passing that $\overline{\phi}$ can be effectively computed from ϕ.

2.2 About three- and two-valued BK-extensions

At this point we want to discuss the possible world semantics for BK in more detail, and adapt it to certain BK-extensions. Recall that the Belnap–Dunn four-valued matrix BD4 has domain

$$\mathbf{4} \;:=\; \{\mathrm{T}, \mathrm{F}, \mathrm{N}, \mathrm{B}\}$$

whose elements may be viewed as subsets of $\{0, 1\}$, by taking

$$\mathrm{T} \;:=\; \{1\}, \quad \mathrm{F} \;:=\; \{0\}, \quad \mathrm{N} \;:=\; \varnothing \quad \text{and} \quad \mathrm{B} \;:=\; \{0, 1\}.$$

In the present context it is convenient to identify every $S \subseteq \{0, 1\}$ with its *characteristic vector* (S_1, S_0) where for each $\varepsilon \in \{0, 1\}$,

$$S_\varepsilon \;:=\; \begin{cases} 1 & \text{if } \varepsilon \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Thus T, F, N and B become $(1, 0)$, $(0, 1)$, $(0, 0)$ and $(1, 1)$ respectively. The usual operations on $\mathbf{4}$ for BD4 can then be defined as follows:

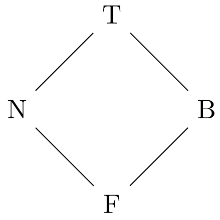
$$(a, b) \vee (c, d) \;:=\; (a \vee c, b \wedge d), \quad (a, b) \wedge (c, d) \;:=\; (a \wedge c, b \vee d) \quad \text{and} \quad \sim(a, b) \;:=\; (b, a).$$

Further — we expand BD4 to $\mathtt{BD4}_{\bot}^{\rightarrow}$ by adding

$$(a, b) \rightarrow (c, d) \;:=\; (a \rightarrow c, a \wedge d) \quad \text{and} \quad \bot \;:=\; (0, 1).$$ [4]

The reader should keep in mind that in the above defining equations, $\vee$, $\wedge$ and $\rightarrow$ on the right-hand sides denote the corresponding operations on $\{0, 1\}$ for classical logic. The truth values T and B are said to be *designated* in both BD4 and $\mathtt{BD4}_{\bot}^{\rightarrow}$. Finally, the so-called *truth ordering* $\leqslant_t$ on $\mathbf{4}$ is given by

[4] Roughly speaking, verifying $(a, b) \rightarrow (c, d)$ means that verifying (a, b) implies verifying (c, d), while falsifying it means verifying (a, b) and falsifying (c, d); so falsifying (a, b) plays no role here, and hence b does not occur on the right-hand side of the defining equation for $\rightarrow$. This is characteristic of Nelson-style bilateral semantics.



Observe that $\leqslant_t$ can be alternatively introduced via

$$(a,b) \leqslant_t (c,d) \quad \Longleftrightarrow \quad a \leqslant c \ \text{ and } \ d \leqslant b$$

where $\leqslant$ denotes the natural ordering of $\{0,1\}$.

It is time to bring Kripke-style structures for BK into the picture. By a BK-*model* we mean a triple $\mathcal{M} = \langle W, R, V \rangle$ where:

- W is a non-empty set, whose elements are called *possible worlds*;
- R is a subset of $W \times W$, called the *accessibility relation*;
- V is a function from $\mathrm{Prop} \times W$ to $\mathbf{4}$, called the *valuation function*.

As one may expect, we extend V to $\mathrm{For}_{\mathcal{L}} \times W$ as follows:

$$\begin{array}{lcl}
V(\varphi \vee \psi, w) & := & V(\varphi, w) \vee V(\psi); \\
V(\varphi \wedge \psi, w) & := & V(\varphi, w) \wedge V(\psi); \\
V(\varphi \to \psi, w) & := & V(\varphi, w) \to V(\psi); \\
V(\bot, w) & := & \mathrm{F}; \\
V({\sim}\, \varphi, w) & := & {\sim}\, V(\varphi, w); \\
V(\Box\varphi, w) & := & \inf_{\leqslant_t} \{V(\varphi, u) \mid wRu\}; \\
V(\Diamond\varphi, w) & := & \sup_{\leqslant_t} \{V(\varphi, u) \mid wRu\}.
\end{array}$$

Here, $\vee$, $\wedge$, $\to$ and $\sim$ on the right sides denote the corresponding operations for $\mathtt{BD4}_{\bot}^{\to}$. Given a BK-model $\mathcal{M} = \langle W, R, V \rangle$, $\phi \in \mathrm{For}_{\mathcal{L}}$ and $w \in W$, we say that ϕ *is true in* $\mathcal{M}$ *at* w — and write $\mathcal{M} \Vdash_w \phi$ — iff $V(\phi, w) \in \{\mathrm{T}, \mathrm{B}\}$.

Now consider the following subsets of $\mathbf{4}$:

$$\underline{\mathbf{3}} \ := \ \{\mathrm{T}, \mathrm{F}, \mathrm{N}\}, \quad \overline{\mathbf{3}} \ := \ \{\mathrm{T}, \mathrm{F}, \mathrm{B}\} \quad \text{and} \quad \mathbf{2} \ := \ \{\mathrm{T}, \mathrm{F}\}.$$

Note that each of these is closed under all of $\mathtt{BD4}_{\bot}^{\to}$'s operations, and moreover, no other proper subset of $\mathbf{4}$ has this property.

Proposition 2.2. *Let $\mathcal{M} = \langle W, R, V \rangle$ be a BK-model, and $S \in \{\underline{\mathbf{3}}, \overline{\mathbf{3}}, \mathbf{2}\}$. Suppose $V(p, w) \in S$ for all $p \in \mathrm{Prop}$ and $w \in W$. Then $V(\phi, w) \in S$ for all $\phi \in \mathrm{For}_{\mathcal{L}}$ and $w \in W$.*

Proof. By an easy induction on the complexity of ϕ. □

This justifies the following: call a BK-model $\mathcal{M} = \langle W, R, V \rangle$ a $\mathsf{B3K}$-*model* if $V[\mathrm{Prop} \times W] \subseteq \underline{\mathbf{3}}$, a BK°-*model* if $V[\mathrm{Prop} \times W] \subseteq \overline{\mathbf{3}}$, and a $\mathsf{B3K}^{\circ}$-*model* if $V[\mathrm{Prop} \times W] \subseteq \mathbf{2}$.[5] Next, for any $\mathcal{M}$ and $\Gamma \cup \{\phi\} \subseteq \mathrm{For}_{\mathcal{L}}$ we define $\Gamma \vDash_{\mathcal{M}} \phi$ to mean that for all $w \in W$,

$$\mathcal{M} \Vdash_w \psi \ \text{ for every } \ \psi \in \Gamma \quad \Longrightarrow \quad \mathcal{M} \Vdash_w \phi.$$

[5]For a function f from A to B, $f[A]$ denotes the range of f, i.e. $\{f(a) \mid a \in A\}$.

Also, we write $\Gamma \vDash_{\mathsf{BK}} \phi$ iff $\Gamma \vDash_{\mathcal{M}} \phi$ for all BK-models $\mathcal{M}$; similarly for $\vDash_{\mathsf{B3K}}$, $\vDash_{\mathsf{BK}^\circ}$ and $\vDash_{\mathsf{B3K}^\circ}$. In fact, the four semantical relations correspond to BK and its three special extensions:

$$\mathsf{B3K} := \mathsf{BK} + \{\sim p \to (p \to q)\}, \quad \mathsf{BK}^\circ := \mathsf{BK} + \{p \vee \sim p\}$$
$$\text{and} \quad \mathsf{B3K}^\circ := \mathsf{BK} + \{\sim p \to (p \to q),\, p \vee \sim p\}.$$

The strong completeness results for these logics can be easily obtained by what is known as the 'canonical model method', as will be seen shortly.

Say that $\Gamma \subseteq \mathrm{For}_{\mathcal{L}}$ *has the disjunction property* iff for each $\{\varphi, \psi\} \subseteq \mathrm{For}_{\mathcal{L}}$,

$$\varphi \vee \psi \in \Gamma \implies \varphi \in \Gamma \text{ or } \psi \in \Gamma.$$

Given $L \in \mathcal{E}\mathsf{BK}$, by a *prime L-theory* we traditionally mean a proper subset Γ of $\mathrm{For}_{\mathcal{L}}$ that has the disjunction property, contains L and is closed under MP. In a standard way one can prove

Lemma 2.3 (cf. [14]). *Let $L \in \mathcal{E}\mathsf{BK}$ and $\Gamma \cup \{\phi\} \subseteq \mathrm{For}_{\mathcal{L}}$. Suppose $\Gamma \nvdash_L \phi$. Then there exists a prime L-theory Δ such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash_L \phi$.*

Define the *canonical model for L* to be the BK-model $\mathcal{M}^L = \langle W^L, R^L, V^L \rangle$ where:

i. W^L is the collection of all prime L-theories;

ii. R^L is the set of all $(\Gamma, \Delta) \in W^L \times W^L$ for which $\{\phi \mid \Box\phi \in \Gamma\} \subseteq \Delta$;

iii. V^L is the unique function from $\mathrm{Prop} \times W^L$ to $\mathbf{4}$ such that for any $p \in \mathrm{Prop}$ and $\Gamma \in W^L$,

$$1 \in V^L(p, \Gamma) \iff p \in \Gamma,$$
$$0 \in V^L(p, \Gamma) \iff \sim p \in \Gamma.$$

In effect, the condition in (iii) continues to hold when we extend V^L to $\mathrm{For}_{\mathcal{L}} \times W$:

Lemma 2.4 (cf. [14]). *Let $L \in \mathcal{E}\mathsf{BK}$. Then for any $\phi \in \mathrm{For}_{\mathcal{L}}$ and $\Gamma \in W^L$:*

$$1 \in V^L(\phi, \Gamma) \iff \phi \in \Gamma;$$
$$0 \in V^L(\phi, \Gamma) \iff \sim \phi \in \Gamma.$$

Furthermore, the canonical models for B3K, BK° and $\mathsf{B3K}^\circ$ behave as desired:

Lemma 2.5. *Let $L \in \{\mathsf{B3K}, \mathsf{BK}^\circ, \mathsf{B3K}^\circ\}$. Then $\mathcal{M}^L$ is an L-model.*

Proof. Let $\Gamma \in W^{\mathsf{B3K}}$. Suppose $V^{\mathsf{B3K}}(p, \Gamma) \notin \underline{\mathbf{3}}$ for some $p \in \mathrm{Prop}$. Then certainly $V^{\mathsf{B3K}}(p, \Gamma) = V^{\mathsf{B3K}}(\sim p, \Gamma) = \mathrm{B}$, hence $\{p, \sim p\} \subseteq \Gamma$. On the other hand $\sim p \to (p \to \psi) \in \mathsf{B3K} \subseteq \Gamma$ for every $\psi \in \mathrm{For}_{\mathcal{L}}$. Since Γ is closed under MP, we conclude that $\psi \in \Gamma$ for all $\psi \in \mathrm{For}_{\mathcal{L}}$, i.e. $\Gamma = \mathrm{For}_{\mathcal{L}}$ — a contradiction.

Let $\Gamma \in W^{\mathsf{BK}^\circ}$. For every $p \in \mathrm{Prop}$ we have $p \vee \sim p \in \mathsf{BK}^\circ \subseteq \Gamma$, and hence, $p \in \Gamma$ or $\sim p \in \Gamma$ (by the disjunction property for Γ), so $V^{\mathsf{BK}^\circ}(p, \Gamma) \neq \mathrm{N}$, i.e. $V^{\mathsf{BK}^\circ}(p, \Gamma) \in \overline{\mathbf{3}}$.

Let $\Gamma \in W^{\mathsf{B3K}^\circ}$. By the above reasoning, $V^{\mathsf{B3K}^\circ}(p, \Gamma) \in \underline{\mathbf{3}} \cap \overline{\mathbf{3}} = \mathbf{2}$ for any $p \in \mathrm{Prop}$. □

Now we quickly deduce a generalisation of the completeness result from [14]:

Theorem 2.6. *Let $L \in \{\mathsf{BK}, \mathsf{B3K}, \mathsf{BK}^\circ, \mathsf{B3K}^\circ\}$. Then for every $\Gamma \cup \{\phi\} \subseteq \mathrm{For}_{\mathcal{L}}$,*

$$\Gamma \vdash_L \phi \iff \Gamma \vDash_L \phi.$$

Proof. This was shown for the case $L = \mathsf{BK}$ in [14]. Assume $L \in \{\mathsf{B3K}, \mathsf{BK}^\circ, \mathsf{B3K}^\circ\}$.

$\boxed{\Longrightarrow}$ Suppose $\Gamma \vdash_L \phi$, which is equivalent to $\Gamma \cup L \vdash_{\mathsf{BK}} \phi$. Then $\Gamma \cup L \vDash_{\mathsf{BK}} \phi$ by the soundness result for BK. One readily verifies that for any $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$:

$$\begin{array}{rcl} V(p,w) \in \{\mathrm{T},\mathrm{F},\mathrm{N}\} & \Longrightarrow & V(\sim p \to (p \to q), w) \in \{\mathrm{T}\}; \\ V(p,w) \in \{\mathrm{T},\mathrm{F},\mathrm{B}\} & \Longrightarrow & V(p \vee \sim p, w) \in \{\mathrm{T},\mathrm{B}\}. \end{array}$$

So, in particular, $\Gamma \cup L \vDash_{\mathsf{BK}} \phi$ implies $\Gamma \vDash_L \phi$. Therefore $\Gamma \vDash_L \phi$.

$\boxed{\Longleftarrow}$ Suppose $\Gamma \nvdash_L \phi$. Take Δ to be a prime L-theory such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash_L \phi$, which exists by Lemma 2.3. Then by Lemma 2.4, we have $\mathcal{M}^L \Vdash_\Delta \psi$ for all $\psi \in \Gamma$, but $\mathcal{M}^L \nVdash_\Delta \phi$. So $\Gamma \nvDash_L \phi$, because $\mathcal{M}^L$ is an L-model by Lemma 2.5. □

2.3 Algebraic semantics

Recall that an algebra $\mathfrak{D} = \langle D; \vee, \wedge, \neg, \Box\rangle$ is said to be a *modal algebra* if it satisfies the following conditions:

i. its reduct $\langle D; \vee, \wedge, \neg\rangle$ is a Boolean algebra with least element 0 and greatest element 1;

ii. $\Box 1 = 1$, and $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $\{a, b\} \subseteq D$.

For expository purposes we employ some standard abbreviations:

$$\Diamond a := \neg\Box\neg a, \quad a \to b := \neg a \vee b \quad \text{and} \quad a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

Note in passing that (ii) is equivalent to

ii'. $\Diamond 0 = 0$, and $\Diamond(a \vee b) = \Diamond a \wedge \Diamond b$ for any $\{a, b\} \subseteq D$.

Also, we traditionally write $a \leqslant b$ iff $a \wedge b = a$ — or equivalently, $a \vee b = b$.

Next, since the $\{\vee, \wedge\}$-reducts of modal algebras are lattices of a special kind, we can adapt the notions of lattice filter and lattice ideal. Given a modal algebra $\mathfrak{D} = \langle D; \vee, \wedge, \neg, \Box\rangle$, we call $S \subseteq D$ a $\Box$-*filter* (a $\Diamond$-*ideal* respectively) on $\mathfrak{D}$ iff it satisfies the following conditions:

i. S is a filter (ideal) on $\langle D; \vee, \wedge\rangle$;

ii. $\Box a \in S$ ($\Diamond a \in S$) for every $a \in S$.

Denote by $\mathscr{F}^\Box(\mathfrak{D})$ ($\mathscr{I}^\Diamond(\mathfrak{D})$ respectively) the class of all $\Box$-filters ($\Diamond$-ideals) on $\mathfrak{D}$. In fact, it is well known that $\mathscr{F}^\Box(\mathfrak{D})$ and $\mathscr{I}^\Diamond(\mathfrak{D})$ can be naturally viewed as lattices — both of which turn out to be isomorphic to the lattice of all congruences on $\mathfrak{D}$ (see e.g. [8, Theorem 4.1.10]).

For the rest of the paper, unless otherwise indicated, we use $\mathfrak{D}$ to stand for a modal algebra with greatest element 1 and least element 0, ∇ for a $\Box$-filter on $\mathfrak{D}$ and Δ for a $\Diamond$-ideal on $\mathfrak{D}$.

Define the *full twist-structure over* $\mathfrak{D}$ to be the $\mathcal{L}$-algebra

$$\mathfrak{D}^{\bowtie} = \langle D \times D; \vee, \wedge, \to, \bot, \sim, \Box, \Diamond\rangle$$

where the operations are given by:

$$\begin{array}{c} (a,b) \vee (c,d) := (a \vee c, b \wedge d); \quad (a,b) \wedge (c,d) := (a \wedge c, b \vee d); \\ (a,b) \to (c,d) := (a \to c, a \wedge d); \quad \bot := (0,1); \quad \sim(a,b) := (b,a); \\ \Box(a,b) := (\Box a, \Diamond b); \quad \Diamond(a,b) := (\Diamond a, \Box b). \end{array}$$

By a *twist-structure over* $\mathfrak{D}$ we shall understand a subalgebra $\mathfrak{A}$ of $\mathfrak{D}^{\bowtie}$ such that $\pi_1[A] = D$ — or equivalently, $\pi_2[A] = D$.[6] Denote the collection of all twist-structures over $\mathfrak{D}$ by $S^{\bowtie}(\mathfrak{D})$. To see how $\Box$-filters and $\Diamond$-ideals work, for any $\nabla \in \mathscr{F}^{\Box}(\mathfrak{D})$ and $\Delta \in \mathscr{I}^{\Diamond}(\mathcal{D})$, consider

$$[\nabla, \Delta] \ := \ \{(a,b) \in D \times D \mid a \vee b \in \nabla \ \text{ and } \ a \wedge b \in \Delta\}.$$

As was remarked in [12], this set is closed under every operation of $\mathfrak{D}^{\bowtie}$, and moreover, its image under π_1 coincides with D. Let $\mathrm{Tw}(\mathfrak{D}, \nabla, \Delta)$ be the twist-structure over $\mathfrak{D}$ with domain $[\nabla, \Delta]$ — i.e. the $\mathcal{L}$-algebra obtained from $\mathfrak{D}^{\bowtie}$ by restricting its operations to $[\nabla, \Delta]$.

Proposition 2.7 (see [12]). $S^{\bowtie}(\mathfrak{D}) = \{\mathrm{Tw}(\mathfrak{D}, \nabla, \Delta) \mid \nabla \in \mathscr{F}^{\Box}(\mathfrak{D})$ *and* $\Delta \in \mathscr{I}^{\Diamond}(\mathfrak{D})\}$.

Further, an $\mathcal{L}$-algebra is called a BK-*lattice* if it is isomorphic to a twist-structure over some modal algebra. Take $\mathcal{V}$ to be the class of all BK-lattices.

Theorem 2.8 (see [12]). $\mathcal{V}$ *is a variety.*

Given $\mathfrak{A} \in \mathcal{V}$ and $\phi \in \mathrm{For}_{\mathcal{L}}$, we write $\mathfrak{A} \vDash \phi$ iff $\neg\phi = \bot$ holds in $\mathfrak{A}$, i.e. belongs to the equational theory of $\mathfrak{A}$. One readily checks that for any $\mathfrak{A} \in S^{\bowtie}(\mathfrak{D})$ and $\phi \in \mathrm{For}_{\mathcal{L}}$,

$$\mathfrak{A} \vDash \varphi \quad \Longleftrightarrow \quad \pi_1(v(\phi)) = 1 \ \text{ for every valuation } v \text{ in } \mathfrak{A}.$$

Now for each class $\mathcal{K}$ of BK-lattices and each set Γ of $\mathcal{L}$-formulas containing BK, take

$$\begin{aligned} \mathbf{L}(\mathcal{K}) \ &:= \ \{\phi \in \mathrm{For}_{\mathcal{L}} \mid \mathfrak{A} \vDash \phi \text{ for all } \mathfrak{A} \in \mathcal{K}\}, \\ \mathbf{V}(\Gamma) \ &:= \ \{\mathfrak{A} \in \mathcal{V} \mid \mathfrak{A} \vDash \phi \text{ for all } \phi \in \Gamma\}. \end{aligned}$$

This leads to an algebraic semantics adequate for studying BK-extensions:

Theorem 2.9 (see [12]). **L** *and* **V** *induce mutually inverse dual isomorphisms between the lattice of all subvarieties of* $\mathcal{V}$ *and* $\mathcal{E}$BK.

Moreover the $\mathcal{L}$-algebras in $\mathbf{V}(\mathsf{B3K})$, $\mathbf{V}(\mathsf{BK}^{\circ})$ and $\mathbf{V}(\mathsf{B3K}^{\circ})$ can be characterised up to isomorphism as follows:

Proposition 2.10 (see [13]).

1. $\mathrm{Tw}(\mathfrak{D}, \nabla, \Delta) \vDash \mathsf{BK}^{\circ}$ *iff* $\nabla = \{1\}$.
2. $\mathrm{Tw}(\mathfrak{D}, \nabla, \Delta) \vDash \mathsf{B3K}$ *iff* $\Delta = \{0\}$.
3. $\mathrm{Tw}(\mathfrak{D}, \nabla, \Delta) \vDash \mathsf{B3K}^{\circ}$ *iff* $\nabla = \{1\}$ *and* $\Delta = \{0\}$.

Some words about quotient structures are in order here. Take $\mathfrak{A}$ to be $\mathrm{Tw}(\mathfrak{D}, \nabla, \Delta)$. Consider an arbitrary congruence relation θ on $\mathfrak{D}$. Let

$$\theta^{\bowtie} \ := \big\{(x,y) \in A^2 \mid (\pi_1(x \Leftrightarrow y), 1) \in \theta\big\}.$$

It is straightforward to check that for any $\{(a_1,b_1),(a_2,b_2)\} \subseteq A$,

$$((a_1,b_1),(a_2,b_2)) \in \theta^{\bowtie} \quad \Longleftrightarrow \quad (a_1,a_2) \in \theta \ \text{ and } \ (b_1,b_2) \in \theta,$$

hence $\theta^{\bowtie}$ turns out to be a congruence relation on $\mathfrak{A}$. On the other hand, define

$$\nabla_{/\theta} \ := \ \{[a]_\theta \mid a \in \nabla\} \quad \text{and} \quad \Delta_{/\theta} \ := \ \{[a]_\theta \mid a \in \Delta\}$$

where $[a]_\theta$ denotes the equivalence class of a modulo θ. One can easily verify that $\nabla_{/\theta}$ and $\Delta_{/\theta}$ are respectively a $\Box$-filter and a $\Diamond$-ideal on the quotient algebra $\mathfrak{D}_{/\theta}$ of $\mathfrak{D}$ modulo θ.

Proposition 2.11 (see [18]). *Let* $\mathfrak{A} = \mathrm{Tw}(\mathfrak{D}, \nabla, \Delta)$. *For each congruence relation* θ *on* $\mathfrak{D}$, *the quotient algebra* $\mathfrak{A}_{/\theta^{\bowtie}}$ *of* $\mathfrak{A}$ *modulo* $\theta^{\bowtie}$ *and* $\mathrm{Tw}\left(\mathfrak{D}_{/\theta}, \nabla_{/\theta}, \Delta_{/\theta}\right)$ *are isomorphic.*

[6] For $i \in \{1,2\}$, by π_i we mean the i-th projection function from $D \times D$ onto D; thus $\pi_i(a_1,a_2) = a_i$ for each $(a_1,a_2) \in D \times D$.

2.4 Adding constants

We shall be concerned with three extensions of $\mathcal{L}$:

$$\mathcal{L}_{\mathtt{n}} \;:=\; \mathcal{L} \cup \{\mathtt{n}\}, \quad \mathcal{L}^{\mathtt{b}} \;:=\; \mathcal{L} \cup \{\mathtt{b}\} \quad \text{and} \quad \mathcal{L}^{\mathtt{b}}_{\mathtt{n}} \;:=\; \mathcal{L} \cup \{\mathtt{n}, \mathtt{b}\}.$$

So the $\mathcal{L}$-logic BK turns respectively into

$$\mathsf{BK}_{\mathtt{n}} \;:=\; \mathsf{BK} + \{\mathtt{n} \to p, \sim \mathtt{n} \to p\}, \quad \mathsf{BK}^{\mathtt{b}} \;:=\; \mathsf{BK} + \{\mathtt{b}, \sim \mathtt{b}\}$$
$$\text{and} \quad \mathsf{BK}^{\mathtt{b}}_{\mathtt{n}} \;:=\; \mathsf{BK} + \{\mathtt{n} \to p, \sim \mathtt{n} \to p, \mathtt{b}, \sim \mathtt{b}\}.$$

Here and below the machinery developed previously for $\mathcal{L}$ is suitably adapted to $\mathcal{L}_{\mathtt{n}}$, $\mathcal{L}^{\mathtt{b}}$ and $\mathcal{L}^{\mathtt{b}}_{\mathtt{n}}$ (unless otherwise stated). In particular, we can define various lattices of $\mathcal{L}_{\mathtt{n}}$-, $\mathcal{L}^{\mathtt{b}}$- and $\mathcal{L}^{\mathtt{b}}_{\mathtt{n}}$-logics. Also, the analogues of Proposition 2.1 for $\mathsf{BK}_{\mathtt{n}}$, $\mathsf{BK}^{\mathtt{b}}$ and $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$ are certainly true.

Let $\mathfrak{A}$ be a BK-lattice. We call an expansion $\mathfrak{B}$ of $\mathfrak{A}$ to $\mathcal{L}_{\mathtt{n}}$ — i.e. an $\mathcal{L}_{\mathtt{n}}$-algebra $\mathfrak{B}$ whose $\mathcal{L}$-reduct is $\mathfrak{A}$ — a $\mathsf{BK}_{\mathtt{n}}$-*lattice* iff

$$\neg \mathtt{n} \;=\; \top \quad \text{and} \quad \sim \mathtt{n} \;=\; \mathtt{n} \tag{$\flat$}$$

hold in $\mathfrak{B}$. Dually, an expansion $\mathfrak{B}$ of $\mathfrak{A}$ to $\mathcal{L}^{\mathtt{b}}$ is called a $\mathsf{BK}^{\mathtt{b}}$-*lattice* iff

$$\neg \mathtt{b} \;=\; \bot \quad \text{and} \quad \sim \mathtt{b} \;=\; \mathtt{b} \tag{$\sharp$}$$

hold in $\mathfrak{B}$. Naturally, say that an expansion $\mathfrak{B}$ of $\mathfrak{A}$ to $\mathcal{L}^{\mathtt{b}}_{\mathtt{n}}$ is a $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$-*lattice* iff ($\flat$) and ($\sharp$) hold in $\mathfrak{B}$. We use the following notation:

$$\begin{aligned} \mathcal{V}_{\mathtt{n}} \;&:=\; \text{the class of all } \mathsf{BK}^{\mathtt{n}}\text{-lattices}, \\ \mathcal{V}^{\mathtt{b}} \;&:=\; \text{the class of all } \mathsf{BK}^{\mathtt{b}}\text{-lattices}, \\ \mathcal{V}^{\mathtt{b}}_{\mathtt{n}} \;&:=\; \text{the class of all } \mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}\text{-lattices}. \end{aligned}$$

Clearly $\mathcal{V}_{\mathtt{n}}$, $\mathcal{V}^{\mathtt{b}}$ and $\mathcal{V}^{\mathtt{b}}_{\mathtt{n}}$ are varieties.

Proposition 2.12. *Let $\mathfrak{A} \in S^{\bowtie}(\mathfrak{D})$. Then for every $(x, y) \in A$:*

$$\begin{aligned} \neg (x, y) = (1, 0) \;\textit{and}\; \sim (x, y) = (x, y) \quad &\Longleftrightarrow \quad x \;=\; y \;=\; 0; \\ \neg (x, y) = (0, 1) \;\textit{and}\; \sim (x, y) = (x, y) \quad &\Longleftrightarrow \quad x \;=\; y \;=\; 1. \end{aligned}$$

Proof. Immediate from the definitions. □

We write $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$ for the expansion of $\mathrm{Tw}(\mathfrak{D}, D, \Delta)$ to $\mathcal{L}_{\mathtt{n}}$ in which $\mathtt{n}$ is interpreted as $(0, 0)$.[7] Similarly with $^{\mathtt{b}}$ and $^{\mathtt{b}}_{\mathtt{n}}$. Now we quickly deduce

Proposition 2.13. *Let $\mathfrak{A}$ be a BK-lattice. Suppose $f : \mathfrak{A} \xrightarrow{\sim} \mathrm{Tw}(\mathfrak{D}, \nabla, \Delta)$.*[8]

1. *If $\mathfrak{B}$ is a $\mathsf{BK}_{\mathtt{n}}$-lattice whose $\mathcal{L}$-reduct is $\mathfrak{A}$, then $\nabla = D$ and $f : \mathfrak{B} \xrightarrow{\sim} \mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$.*

2. *If $\mathfrak{B}$ is a $\mathsf{BK}^{\mathtt{b}}$-lattice whose $\mathcal{L}$-reduct is $\mathfrak{A}$, then $\Delta = D$ and $f : \mathfrak{B} \xrightarrow{\sim} \mathrm{Tw}^{\mathtt{b}}(\mathfrak{D}, \nabla, D)$.*

3. *If $\mathfrak{B}$ is a $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$-lattice whose $\mathcal{L}$-reduct is $\mathfrak{A}$, then $\nabla = \Delta = D$ and $f : \mathfrak{B} \xrightarrow{\sim} \mathrm{Tw}^{\mathtt{b}}_{\mathtt{n}}(\mathfrak{D}, D, D)$.*

[7] Clearly $(0, 0)$ is in the domain of $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$, because $0 \vee 0 = 0 \wedge 0 = 0$ and $0 \in \Delta$.

[8] Here $\xrightarrow{\sim}$ stands for 'maps isomorphically', thus indicating that f must be an isomorphism.

Proof. $\boxed{1.}$ Take $\mathfrak{C}$ to be $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, \nabla, \Delta)$. By Proposition 2.12, f maps the interpretation of $\mathtt{n}$ in $\mathfrak{B}$ to $(0,0)$, so $(0,0)$ is in C. Consequently $0 \in \nabla$, and therefore $\nabla = D$. Evidently $f : \mathfrak{B} \xrightarrow{\sim} \mathfrak{C}$.

$\boxed{2.}$ Similar to (1).

$\boxed{3.}$ Immediate from (1) and (2). ☐

Note that $\mathbf{L}$ and $\mathbf{V}$ are easily modified to accommodate $\mathtt{n}$ and $\mathtt{b}$. For instance, for any class $\mathcal{K}$ of $\mathsf{BK}_{\mathtt{n}}$-lattices and set Γ of $\mathcal{L}_{\mathtt{n}}$-formulas containing $\mathsf{BK}_{\mathtt{n}}$ we define

$$\begin{aligned}\mathbf{L}_{\mathtt{n}}(\mathcal{K}) &:= \{\phi \in \mathrm{For}_{\mathcal{L}_{\mathtt{n}}} \mid \mathfrak{A} \vDash \phi \text{ for all } \mathfrak{A} \in \mathcal{K}\},\\ \mathbf{V}_{\mathtt{n}}(\Gamma) &:= \{\mathfrak{A} \in \mathcal{V}_{\mathtt{n}} \mid \mathfrak{A} \vDash \phi \text{ for all } \phi \in \Gamma\}.\end{aligned}$$

Similarly with $^{\mathtt{b}}$ and $^{\mathtt{b}}_{\mathtt{n}}$.

Theorem 2.14. $\mathbf{L}_{\mathtt{n}}$ *and* $\mathbf{V}_{\mathtt{n}}$ *induce mutually inverse dual isomorphisms between the lattice of all subvarieties of* $\mathcal{V}_{\mathtt{n}}$ *and* $\mathcal{E}\mathsf{BK}_{\mathtt{n}}$. *Similarly with* $^{\mathtt{b}}$ *and* $^{\mathtt{b}}_{\mathtt{n}}$.

Proof. This is a minor modification of the proof of Theorem 2.9. ☐

3 $\mathcal{E}\mathsf{BK}_{\mathtt{n}}$ vs. $\mathcal{E}\mathsf{BK}^{\circ}$

In this section we explore the connection between $\mathsf{BK}_{\mathtt{n}}$-extensions and BK°-extensions. It should be mentioned that the situation for BK^{b} and B3K is perfectly analogous, and the corresponding results can be obtained in exactly the same way. So it suffices to provide detailed proofs for the present case.

Define the translation $\lambda_{\mathtt{n}} : \mathrm{For}_{\mathcal{L}} \to \mathrm{For}_{\mathcal{L}_{\mathtt{n}}}$ by

$$\lambda_{\mathtt{n}}(\phi) := \bigwedge_{p \in \mathrm{Var}(\phi)} (p \vee \sim p) \to \phi$$

where $\mathrm{Var}(\phi)$ denotes the collection of all propositional variables that occur in ϕ. It extends to $\Lambda_{\mathtt{n}} : \mathcal{E}\mathsf{BK}^{\circ} \to \mathcal{E}\mathsf{BK}_{\mathtt{n}}$ by

$$\Lambda_{\mathtt{n}}(L) := \mathsf{BK}_{\mathtt{n}} + \lambda_{\mathtt{n}}[L] = \mathsf{BK}_{\mathtt{n}} + \{\lambda_{\mathtt{n}}(\phi) \mid \phi \in L\}.$$

We are going to show among other things that BK° is faithfully embedded into $\mathsf{BK}_{\mathtt{n}}$ via $\lambda_{\mathtt{n}}$.

Lemma 3.1. *For every* $\phi \in \mathrm{For}_{\mathcal{L}}$,

$$\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta) \vDash \phi \iff \mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta) \vDash \lambda_{\mathtt{n}}(\phi).$$

Proof. Take $\mathfrak{A}$ and $\mathfrak{B}$ to be $\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta)$ and $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$ respectively.

$\boxed{\Longleftarrow}$ Assume $\mathfrak{B} \vDash \lambda_{\mathtt{n}}(\phi)$. Let v be a valuation in $\mathfrak{A}$. For any $p \in \mathrm{Prop}$ we then have

$$\pi_1(v(p \vee \sim p)) = \pi_1(v(p)) \vee \pi_2(v(p)) = 1.$$

Clearly $A \subseteq B$, so v is also a valuation in $\mathfrak{B}$, and moreover, $\mathfrak{A}$ is a subalgebra of the $\mathcal{L}$-reduct of $\mathfrak{B}$. Hence

$$1 = \pi_1(v(\lambda_{\mathtt{n}}(\phi))) = \neg \bigwedge\nolimits_{p \in \mathrm{Var}(\phi)} \pi_1(v(p \vee \sim p)) \vee \pi_1(v(\phi)) = \pi_1(v(\phi)).$$

Thus $\mathfrak{A} \vDash \phi$.

$\boxed{\Longrightarrow}$ Assume $\mathfrak{B} \nvDash \lambda_{\mathtt{n}}(\phi)$. Let v be a valuation in $\mathfrak{B}$ such that $\pi_1(v(\lambda_{\mathtt{n}}(\phi))) \neq 1$ — without loss of generality suppose that $v(p) = (0,1)$ for all $p \in \mathrm{Prop} \setminus \mathrm{Var}(\phi)$. Take

$$a \ := \ \bigwedge\nolimits_{p \in \mathrm{Var}(\phi)} \pi_1(v(p \vee \sim p)) \quad \text{and} \quad b \ := \ \pi_1(v(\phi)).$$

Then $\neg a \vee b \neq 1$, i.e. $a \nleqslant b$. Now consider

$$\approx \ := \ \{(x,y) \in D^2 \mid x \wedge a \leqslant y \text{ and } y \wedge a \leqslant x\}.$$

One easily verifies that $1 \approx a \not\approx b$, and moreover, $\approx$ is a congruence relation on $\mathfrak{D}$. If $x \in D$, we write $[x]$ for the equivalence class of x modulo $\approx$. Define

$$\mathfrak{A}' \ := \ \mathrm{Tw}\left(\mathfrak{D}_{/\approx}, \{1\}_{/\approx}, \Delta_{/\approx}\right) \quad \text{and} \quad \mathfrak{B}' \ := \ \mathrm{Tw}_{\mathtt{n}}\left(\mathfrak{D}_{/\approx}, D_{/\approx}, \Delta_{/\approx}\right).$$

Clearly $A' \subseteq B'$, so $\mathfrak{A}'$ is a subalgebra of the $\mathcal{L}$-reduct of $\mathfrak{B}'$. Let v' be the valuation in $\mathfrak{B}'$ given by

$$v'(p) \ := \ ([\pi_1(v(p))], [\pi_2(v(p))]).$$

It follows by an easy induction that for every $\varphi \in \mathrm{For}_{\mathcal{L}_{\mathtt{n}}}$,

$$\pi_1(v'(\varphi)) \ = \ [\pi_1(v(\varphi))] \quad \text{and} \quad \pi_2(v'(\varphi)) \ = \ [\pi_2(v(\varphi))].$$

This gives two consequences:

1. $\bigwedge_{p \in \mathrm{Var}(\phi)} \pi_1(v'(p \vee \sim p)) = [a] = [1]$;
2. $\pi_1(v'(\phi)) = [b] \neq [1]$.

By (1), for any $p \in \mathrm{Prop}$ we have $\pi_1(v'(p)) \vee \pi_2(v'(p)) = [1]$. Hence v' is also a valuation in $\mathfrak{A}'$. By (2), $\mathfrak{A}' \nvDash \phi$, and since $\mathfrak{A}'$ is, as we know, isomorphic to the quotient-algebra of $\mathfrak{A}$ modulo an appropriate congruence relation, we get $\mathfrak{A} \nvDash \phi$. □

We note, in passing, that $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$ in the statement of Lemma 3.1 may be replaced by $\mathrm{Tw}(\mathfrak{D}, D, \Delta)$, simply because $\lambda_{\mathtt{n}}[\mathrm{For}_{\mathcal{L}}] \subseteq \mathrm{For}_{\mathcal{L}}$.

Theorem 3.2. *For every $\phi \in \mathrm{For}_{\mathcal{L}}$ and $L \in \mathcal{E}\mathsf{BK}^{\circ}$,*

$$\phi \ \in \ L \quad \Longleftrightarrow \quad \lambda_{\mathtt{n}}(\phi) \ \in \ \Lambda_{\mathtt{n}}(L).$$

Proof. $\boxed{\Longrightarrow}$ This is trivial.

$\boxed{\Longleftarrow}$ Assume $\phi \notin L$. Then by the completeness result, there exists a BK-lattice $\mathfrak{A}$ such that $\mathfrak{A} \vDash L$ but $\mathfrak{A} \nvDash \phi$. Without loss of generality we suppose that $\mathfrak{A} = \mathrm{Tw}(\mathfrak{D}, \nabla, \Delta)$ for some modal algebra $\mathfrak{D}$, $\nabla \in \mathscr{F}^{\Box}(\mathfrak{D})$ and $\Delta \in \mathscr{I}^{\Diamond}(\mathfrak{D})$. Since $\mathsf{BK}^{\circ} \subseteq L$, we have $\nabla = \{1\}$ by Proposition 2.10 (see (1)). Take $\mathfrak{B}$ to be $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$. By Lemma 3.1, $\mathfrak{B} \vDash \lambda_{\mathtt{n}}[L]$ but $\mathfrak{B} \nvDash \lambda_{\mathtt{n}}(\phi)$. Therefore by the completeness result, $\lambda_{\mathtt{n}}(\phi) \notin \Lambda_{\mathtt{n}}(L)$. □

Corollary 3.3. *For every $\phi \in \mathrm{For}_{\mathcal{L}}$,*

$$\phi \ \in \ \mathsf{BK}^{\circ} \quad \Longleftrightarrow \quad \lambda_{\mathtt{n}}(\phi) \ \in \ \mathsf{BK}_{\mathtt{n}},$$

i.e. $\lambda_{\mathtt{n}}$ faithfully embeds BK° into $\mathsf{BK}_{\mathtt{n}}$.

Proof. It suffices to check that $\mathsf{BK}_{\mathtt{n}} = \Lambda_{\mathtt{n}}(\mathsf{BK}^{\circ})$.

$\boxed{\subseteq}$ This is obvious.

$\boxed{\supseteq}$ Let $\phi \in \mathsf{BK}^{\circ}$. So for any modal algebra $\mathfrak{D}$ and $\Delta \in \mathscr{I}^{\diamond}(\mathfrak{D})$ we have $\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta) \vDash \phi$ by Proposition 2.10 (see (1)), and hence $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta) \vDash \lambda_{\mathtt{n}}(\phi)$ by Lemma 3.1. Remember, each $\mathsf{BK}_{\mathtt{n}}$-lattice is isomorphic to a twist-structure of the form $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$ by Proposition 2.13 (see (1)). Thus it follows by the completeness result that $\lambda_{\mathtt{n}}(\phi) \in \mathsf{BK}_{\mathtt{n}}$. □

Corollary 3.4. *$\Lambda_{\mathtt{n}}$ is an embedding of $\mathcal{E}\mathsf{BK}^{\circ}$ into $\mathcal{E}\mathsf{BK}_{\mathtt{n}}$.*

Proof. Let $\{L_1, L_2\} \subseteq \mathcal{E}\mathsf{BK}^{\circ}$. Clearly if $L_1 \subseteq L_2$, then $\Lambda_{\mathtt{n}}(L_1) \subseteq \Lambda_{\mathtt{n}}(L_2)$. Furthermore, in view of Theorem 3.2, for every $\phi \in \mathrm{For}_{\mathcal{L}}$ and $i \in \{1, 2\}$,

$$\phi \in L_i \setminus L_{3-i} \quad \Longrightarrow \quad \lambda_{\mathtt{n}}(\phi) \in \Lambda_{\mathtt{n}}(L_i) \setminus \Lambda_{\mathtt{n}}(L_{3-i}).$$

Consequently $L_1 \neq L_2$ implies $\Lambda_{\mathtt{n}}(L_1) \neq \Lambda_{\mathtt{n}}(L_2)$. □

In fact, $\Lambda_{\mathtt{n}}$ can be shown to be onto — giving an isomorphism between $\mathcal{E}\mathsf{BK}^{\circ}$ and $\mathcal{E}\mathsf{BK}_{\mathtt{n}}$. To this end we introduce a new translation $\lambda^{\circ} : \mathrm{For}_{\mathcal{L}_{\mathtt{n}}} \to \mathrm{For}_{\mathcal{L}}$.

- If $\phi = \overline{\phi}$, then $\lambda^{\circ}(\phi)$ is defined inductively as follows:
  - $\lambda^{\circ}(p_i) := p_{2i}$ and $\lambda^{\circ}(\sim p_i) := p_{2i+1} \wedge \sim p_{2i}$;
  - $\lambda^{\circ}(\bot) := \bot$ and $\lambda^{\circ}(\sim \bot) := \bot \to \bot$;
  - $\lambda^{\circ}(\varphi * \psi) := \lambda^{\circ}(\varphi) * \lambda^{\circ}(\psi)$ where $* \in \{\vee, \wedge, \to\}$;
  - $\lambda^{\circ}(*\varphi) := *\lambda^{\circ}(\varphi)$ where $* \in \{\Box, \Diamond\}$;
  - $\lambda^{\circ}(\mathtt{n}) := \bot$ and $\lambda^{\circ}(\sim \mathtt{n}) := \bot$.
- If $\phi \neq \overline{\phi}$, then $\lambda^{\circ}(\phi)$ is defined to be $\lambda^{\circ}\left(\overline{\phi}\right)$.

(Recall, $\overline{\phi}$ denotes our preferred negation normal form for ϕ.) Similarly to before, λ° extends to $\Lambda^{\circ} : \mathcal{E}\mathsf{BK}_{\mathtt{n}} \to \mathcal{E}\mathsf{BK}^{\circ}$ by

$$\Lambda^{\circ}(L) := \mathsf{BK}^{\circ} + \lambda^{\circ}[L] = \mathsf{BK}^{\circ} + \{\lambda^{\circ}(\phi) \mid \phi \in L\}.$$

It will turn out that $\mathsf{BK}_{\mathtt{n}}$ is faithfully embedded into BK° via λ°.

Lemma 3.5. *For every $\phi \in \mathrm{For}_{\mathcal{L}_{\mathtt{n}}}$,*

$$\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta) \vDash \lambda^{\circ}(\phi) \quad \Longleftrightarrow \quad \mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta) \vDash \phi.$$

Proof. Take $\mathfrak{A}$ and $\mathfrak{B}$ to be $\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta)$ and $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta)$ respectively. Notice that since ϕ and $\overline{\phi}$ are equivalent over BK, we may suppose that $\phi = \overline{\phi}$.

$\boxed{\Longleftarrow}$ Assume $\mathfrak{A} \vDash \lambda^{\circ}(\phi)$. Let v be a valuation in $\mathfrak{B}$. Consider the valuation v' given by

$$\begin{aligned} v'(p_{2i}) &:= (\pi_1(v(p_i)), \neg\pi_1(v(p_i)) \vee \pi_2(v(p_i))), \\ v'(p_{2i+1}) &:= (\pi_2(v(p_i)), \neg\pi_2(v(p_i))). \end{aligned}$$

We then have:

- $\pi_1(v'(p_{2i+1})) \vee \pi_2(v'(p_{2i+1})) = 1$;

– $\pi_1(v'(p_{2i+1})) \wedge \pi_2(v'(p_{2i+1})) = 0$;

– $\pi_1(v'(p_{2i})) \vee \pi_2(v'(p_{2i})) = 1 \vee \pi_2(v(p_i)) = 1$;

– $\pi_1(v'(p_{2i})) \wedge \pi_2(v'(p_{2i})) = 0 \vee (\pi_1(v(p_i)) \wedge \pi_2(v(p_i))) = \pi_1(v(p_i)) \wedge \pi_2(v(p_i))$.

Clearly 0 is always in Δ, and $\pi_1(v(p_i)) \wedge \pi_2(v(p_i)) \in \Delta$ by the choice of v. So v' is a valuation in $\mathfrak{A}$. Furthermore, we obtain:

$$\begin{aligned}\pi_1(v'(\lambda^\circ(p_i))) &= \pi_1(v'(p_{2i})) = \pi_1(v(p_i));\\ \pi_1(v'(\lambda^\circ(\sim p_i))) &= \pi_1(v'(p_{2i+1} \wedge \sim p_{2i})) = \pi_1(v'(p_{2i+1})) \wedge \pi_2(v'(p_{2i}))\\ &= 0 \vee \pi_2(v(p_i)) = \pi_2(v(p_i)) = \pi_1(v(\sim p_i)).\end{aligned}$$

It follows by an easy induction that for every $\varphi \in \mathrm{For}_{\mathcal{L}_\mathsf{n}}$ in negation normal form,

$$\pi_1(v'(\lambda^\circ(\varphi))) = \pi_1(v(\varphi)).$$

Consequently $\pi_1(v(\phi)) = \pi_1(v'(\lambda^\circ(\phi))) = 1$. Thus $\mathfrak{B} \vDash \phi$.

$\boxed{\Longrightarrow}$ Assume $\mathfrak{A} \nvDash \lambda^\circ(\phi)$. Let v be a valuation in $\mathfrak{A}$ such that $\pi_1(v(\lambda^\circ(\phi))) \neq 1$. Now consider the valuation v' given by

$$v'(p_i) := (\pi_1(v(p_{2i})), \pi_2(v(p_{2i})) \wedge \pi_1(v(p_{2i+1}))).$$

We then have

$$\pi_1(v'(p_i)) \wedge \pi_2(v'(p_i)) \leqslant \pi_1(v(p_{2i})) \wedge \pi_2(v(p_{2i})).$$

Therefore $\pi_1(v'(p_i)) \wedge \pi_2(v'(p_i))$ is in Δ, because $\pi_1(v(p_{2i})) \wedge \pi_2(v(p_{2i})) \in \Delta$ by the choice of v. So v' is a valuation in $\mathfrak{B}$. Furthermore, we obtain:

$$\begin{aligned}\pi_1(v(\lambda^\circ(p_i))) &= \pi_1(v(p_{2i})) = \pi_1(v'(p_i));\\ \pi_1(v(\lambda^\circ(\sim p_i))) &= \pi_1(v(p_{2i+1})) \wedge \pi_2(v(p_{2i})) = \pi_2(v'(p_i)) = \pi_1(v'(\sim p_i)).\end{aligned}$$

It follows by an easy induction that for every $\varphi \in \mathrm{For}_{\mathcal{L}_\mathsf{n}}$ in negation normal form,

$$\pi_1(v(\lambda^\circ(\varphi))) = \pi_1(v'(\varphi)).$$

Consequently $\pi_1(v'(\phi)) = \pi_1(v(\lambda^\circ(\phi))) \neq 1$. Thus $\mathfrak{B} \nvDash \phi$. □

Theorem 3.6. *For every $\phi \in \mathrm{For}_{\mathcal{L}_\mathsf{n}}$ and $L \in \mathcal{E}\mathsf{BK}_\mathsf{n}$,*

$$\phi \in L \iff \lambda^\circ(\phi) \in \Lambda^\circ(L).$$

Proof. The argument is analogous to that for Theorem 3.2.

$\boxed{\Longrightarrow}$ This is trivial.

$\boxed{\Longleftarrow}$ Assume $\phi \notin L$. So, there exists a BK_n-lattice $\mathfrak{B}$ such that $\mathfrak{B} \vDash L$ but $\mathfrak{B} \nvDash \phi$. Without loss of generality we suppose that $\mathfrak{B} = \mathrm{Tw}_\mathsf{n}(\mathfrak{D}, D, \Delta)$ for appropriate $\mathfrak{D}$ and Δ. Take $\mathfrak{A}$ to be $\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta)$. By Lemma 3.5, $\mathfrak{A} \vDash \lambda^\circ[L]$ but $\mathfrak{A} \nvDash \lambda^\circ(\phi)$. Therefore, $\lambda^\circ(\phi) \notin \Lambda^\circ(L)$. □

Corollary 3.7. *For every $\phi \in \mathrm{For}_{\mathcal{L}_\mathsf{n}}$,*

$$\phi \in \mathsf{BK}_\mathsf{n} \iff \lambda^\circ(\phi) \in \mathsf{BK}^\circ,$$

i.e. λ° faithfully embeds BK_n into BK°.

Proof. By analogy with Corollary 3.3, it suffices to check that $\mathsf{BK}^{\circ} = \Lambda^{\circ}(\mathsf{BK}_{\mathtt{n}})$.

$\boxed{\subseteq}$ This is obvious.

$\boxed{\supseteq}$ Let $\phi \in \mathsf{BK}_{\mathtt{n}}$. So for any modal algebra $\mathfrak{D}$ and $\Delta \in \mathscr{I}^{\diamond}(\mathfrak{D})$ we have $\mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta) \vDash \phi$, and hence $\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta) \vDash \lambda^{\circ}(\phi)$ by Lemma 3.5. Thus it follows that $\lambda^{\circ}(\phi) \in \mathsf{BK}^{\circ}$. ☐

Corollary 3.8. Λ° *is an embedding of* $\mathcal{E}\mathsf{BK}_{\mathtt{n}}$ *into* $\mathcal{E}\mathsf{BK}^{\circ}$.

Proof. Similar to Corollary 3.4. ☐

Finally, we are ready to prove that the two lattices are in fact isomorphic.

Theorem 3.9. $\Lambda_{\mathtt{n}}$ *and* Λ° *are mutually inverse isomorphisms between* $\mathcal{E}\mathsf{BK}^{\circ}$ *and* $\mathcal{E}\mathsf{BK}_{\mathtt{n}}$.

Proof. Let $L \in \mathcal{E}\mathsf{BK}^{\circ}$. For any modal algebra $\mathfrak{D}$ and $\Delta \in \mathscr{I}^{\diamond}(\mathfrak{D})$,

$$\mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta) \vDash L \iff \mathrm{Tw}_{\mathtt{n}}(\mathfrak{D}, D, \Delta) \vDash \Lambda_{\mathtt{n}}(L) \iff \mathrm{Tw}(\mathfrak{D}, \{1\}, \Delta) \vDash \Lambda^{\circ}(\Lambda_{\mathtt{n}}(L))$$

(by Lemmas 3.1 and 3.5). So we have $\mathbf{V}(L) = \mathbf{V}(\Lambda^{\circ}(\Lambda_{\mathtt{n}}(L)))$, and therefore $L = \Lambda^{\circ}(\Lambda_{\mathtt{n}}(L))$. Similarly, for all $L' \in \mathcal{E}\mathsf{BK}_{\mathtt{n}}$ we get $\mathbf{V}_{\mathtt{n}}(L') = \mathbf{V}_{\mathtt{n}}(\Lambda_{\mathtt{n}}(\Lambda^{\circ}(L')))$, whence $L' = \Lambda_{\mathtt{n}}(\Lambda^{\circ}(L'))$. Thus $\Lambda_{\mathtt{n}}$ and Λ° are mutually inverse, and the result follows by Corollaries 3.4 and 3.8. ☐

4 $\mathcal{E}\mathsf{BK}^{\mathsf{b}}$ vs. $\mathcal{E}\mathsf{B3K}$

We now state the analogous results for BK^{b} and $\mathsf{B3K}$; the proofs are omitted because — as was mentioned earlier — they are almost the same as those given in the previous section, where the use of the first items of Propositions 2.10 and 2.13 is replaced by that of the second ones.

Define the translation $\lambda^{\mathsf{b}} : \mathrm{For}_{\mathcal{L}} \to \mathrm{For}_{\mathcal{L}^{\mathsf{b}}}$ by

$$\lambda^{\mathsf{b}}(\phi) := \bigwedge_{p \in \mathrm{Var}(\phi)} (\neg(p \wedge \sim p)) \to \phi.$$

It extends to $\Lambda^{\mathsf{b}} : \mathcal{E}\mathsf{B3K} \to \mathcal{E}\mathsf{BK}^{\mathsf{b}}$ by

$$\Lambda^{\mathsf{b}}(L) := \mathsf{BK}^{\mathsf{b}} + \lambda^{\mathsf{b}}[L] = \mathsf{BK}^{\mathsf{b}} + \left\{\lambda^{\mathsf{b}}(\phi) \mid \phi \in L\right\}.$$

We then have:

Lemma 4.1. *For every* $\phi \in \mathrm{For}_{\mathcal{L}}$,

$$\mathrm{Tw}(\mathfrak{D}, \nabla, \{0\}) \vDash \phi \iff \mathrm{Tw}^{\mathsf{b}}(\mathfrak{D}, \nabla, D) \vDash \lambda^{\mathsf{b}}(\phi).$$

We note, in passing, that $\mathrm{Tw}^{\mathsf{b}}(\mathfrak{D}, \nabla, D)$ in the statement of Lemma 4.1 may be replaced by $\mathrm{Tw}(\mathfrak{D}, \nabla, D)$, simply because $\lambda^{\mathsf{b}}[\mathrm{For}_{\mathcal{L}}] \subseteq \mathrm{For}_{\mathcal{L}}$.

Theorem 4.2. *For every* $\phi \in \mathrm{For}_{\mathcal{L}}$ *and* $L \in \mathcal{E}\mathsf{B3K}$,

$$\phi \in L \iff \lambda^{\mathsf{b}}(\phi) \in \Lambda^{\mathsf{b}}(L).$$

Corollary 4.3. λ^{b} *faithfully embeds* $\mathsf{B3K}$ *into* BK^{b}.

Corollary 4.4. Λ^{b} *is an embedding of* $\mathcal{E}\mathsf{B3K}$ *into* $\mathcal{E}\mathsf{BK}^{\mathsf{b}}$.

For the other direction we introduce a new translation $\lambda_3 : \mathrm{For}_{\mathcal{L}^{\mathsf{b}}} \to \mathrm{For}_{\mathcal{L}}$.

- If $\phi = \overline{\phi}$, then $\lambda_3(\phi)$ is defined inductively as follows:
  - $\lambda_3(p_i) := p_{2i}$ and $\lambda_3(\sim p_i) := p_{2i+1} \vee \sim p_{2i}$;
  - $\lambda_3(\bot) := \bot$ and $\lambda_3(\sim \bot) := \bot \to \bot$;
  - $\lambda_3(\varphi * \psi) := \lambda_3(\varphi) * \lambda_3(\psi)$ where $* \in \{\vee, \wedge, \to\}$;
  - $\lambda_3(*\varphi) := *\lambda_3(\varphi)$ where $* \in \{\Box, \Diamond\}$;
  - $\lambda_3(\mathtt{b}) := \bot \to \bot$ and $\lambda_3(\sim \mathtt{b}) := \bot \to \bot$.[9]
- If $\phi \neq \overline{\phi}$, then $\lambda_3(\phi)$ is defined to be $\lambda_3\left(\overline{\phi}\right)$.

Similarly to before, λ_3 extends to $\Lambda_3 : \mathcal{E}\mathsf{BK}^{\mathsf{b}} \to \mathcal{E}\mathsf{B3K}$ by

$$\Lambda_3(L) \;:=\; \mathsf{B3K} + \lambda_3[L] \;=\; \mathsf{B3K} + \{\lambda_3(\phi) \mid \phi \in L\}.$$

We then have:

Lemma 4.5. *For every $\phi \in \mathrm{For}_{\mathcal{L}^{\mathsf{b}}}$,*

$$\mathrm{Tw}(\mathfrak{D}, \nabla, \{1\}) \vDash \lambda_3(\phi) \quad \Longleftrightarrow \quad \mathrm{Tw}^{\mathsf{b}}(\mathfrak{D}, \nabla, D) \vDash \phi.$$

Theorem 4.6. *For every $\phi \in \mathrm{For}_{\mathcal{L}^{\mathsf{b}}}$ and $L \in \mathcal{E}\mathsf{BK}^{\mathsf{b}}$,*

$$\phi \;\in\; L \quad \Longleftrightarrow \quad \lambda_3(\phi) \;\in\; \Lambda_3(L).$$

Corollary 4.7. *λ_3 faithfully embeds BK^{b} into $\mathsf{B3K}$.*

Corollary 4.8. *Λ_3 is an embedding of $\mathcal{E}\mathsf{BK}^{\mathsf{b}}$ into $\mathcal{E}\mathsf{B3K}$.*

Finally, we get:

Theorem 4.9. *Λ^{b} and Λ_3 are mutually inverse isomorphisms between $\mathcal{E}\mathsf{B3K}$ and $\mathcal{E}\mathsf{BK}^{\mathsf{b}}$.*

5 $\mathcal{E}\mathsf{BK}^{\mathsf{b}}_{\mathtt{n}}$ vs. $\mathcal{E}\mathsf{B3K}^{\circ}$

Again this is very similar to what we did earlier, except that the arguments for the corresponding lemmas can now be simplified.

Define the translation $\lambda^{\mathsf{b}}_{\mathtt{n}} : \mathrm{For}_{\mathcal{L}} \to \mathrm{For}_{\mathcal{L}^{\mathsf{b}}_{\mathtt{n}}}$ by

$$\lambda^{\mathsf{b}}_{\mathtt{n}}(\phi) \;:=\; \lambda^{\mathsf{b}}(\lambda_{\mathtt{n}}(\phi)).$$ [10]

It extends to $\Lambda^{\mathsf{b}}_{\mathtt{n}} : \mathcal{E}\mathsf{B3K}^{\circ} \to \mathcal{E}\mathsf{BK}^{\mathsf{b}}_{\mathtt{n}}$ by

$$\Lambda^{\mathsf{b}}_{\mathtt{n}}(L) \;:=\; \mathsf{BK}^{\mathsf{b}}_{\mathtt{n}} + \lambda^{\mathsf{b}}_{\mathtt{n}}[L] \;=\; \mathsf{BK}^{\mathsf{b}}_{\mathtt{n}} + \left\{\lambda^{\mathsf{b}}_{\mathtt{n}}(\phi) \mid \phi \in L\right\}.$$

As you would expect, we quickly deduce:

[9] In fact, it is exactly like the definition of λ° except that we use $\wedge$ instead of $\vee$ in the description of $\lambda_3(\sim p_i)$. Further — the proof of Lemma 4.5 below can be easily obtained from that of Lemma 3.5 by replacing $\vee$ by $\wedge$ in the descriptions of v' for both $\boxed{\Longleftarrow}$ and $\boxed{\Longrightarrow}$.

[10] This definition makes sense, since $\lambda_{\mathtt{n}}(\phi) \in \mathrm{For}_{\mathcal{L}}$. Similarly, we could have defined $\lambda^{\mathsf{b}}_{\mathtt{n}}(\phi)$ to be $\lambda_{\mathtt{n}}\left(\lambda^{\mathsf{b}}(\phi)\right)$ — there is no essential difference between the two approaches.

Lemma 5.1. *For every* $\phi \in \mathrm{For}_{\mathcal{L}}$,

$$\mathrm{Tw}\left(\mathfrak{D},\{1\},\{0\}\right) \vDash \phi \quad \Longleftrightarrow \quad \mathrm{Tw}^{\mathtt{b}}_{\mathtt{n}}\left(\mathfrak{D},D,D\right) \vDash \lambda^{\mathtt{b}}_{\mathtt{n}}\left(\phi\right).$$

Proof. By Lemmas 3.1 and 3.5, we have

$$\mathrm{Tw}\left(\mathfrak{D},\{1\},\{0\}\right) \vDash \phi \quad \Longleftrightarrow \quad \mathrm{Tw}\left(\mathfrak{D},D,\{0\}\right) \vDash \lambda_{\mathtt{n}}\left(\phi\right) \quad \Longleftrightarrow \quad \mathrm{Tw}^{\mathtt{b}}_{\mathtt{n}}\left(\mathfrak{D},D,D\right) \vDash \lambda^{\mathtt{b}}\left(\lambda_{\mathtt{n}}\left(\phi\right)\right)$$

(notice that since $\lambda^{\mathtt{b}}\left(\lambda_{\mathtt{n}}\left(\phi\right)\right) \in \mathrm{For}_{\mathcal{L}}$, it makes no difference whether we evaluate this formula in $\mathrm{Tw}^{\mathtt{b}}\left(\mathfrak{D},D,D\right)$ or in $\mathrm{Tw}^{\mathtt{b}}_{\mathtt{n}}\left(\mathfrak{D},D,D\right)$). □

Now using Lemma 3.5 one can obtain the following results — which we state without proof, because the corresponding arguments are perfectly analogous to those given above.

Theorem 5.2. *For every* $\phi \in \mathrm{For}_{\mathcal{L}}$ *and* $L \in \mathcal{E}\mathsf{B3K}^{\circ}$,

$$\phi \ \in \ L \quad \Longleftrightarrow \quad \lambda^{\mathtt{b}}_{\mathtt{n}}\left(\phi\right) \ \in \ \Lambda^{\mathtt{b}}_{\mathtt{n}}\left(L\right).$$

Corollary 5.3. $\lambda^{\mathtt{b}}_{\mathtt{n}}$ *faithfully embeds* $\mathsf{B3K}^{\circ}$ *into* $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$.

Corollary 5.4. $\Lambda^{\mathtt{b}}_{\mathtt{n}}$ *is an embedding of* $\mathcal{E}\mathsf{B3K}^{\circ}$ *into* $\mathcal{E}\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$.

For the other direction we introduce a new translation $\lambda^{\circ}_{3} : \mathrm{For}_{\mathcal{L}^{\mathtt{b}}_{\mathtt{n}}} \to \mathrm{For}_{\mathcal{L}}$.

- If $\phi = \overline{\phi}$, then $\lambda^{\circ}_{3}\left(\phi\right)$ is defined inductively as follows:
  - $\lambda^{\circ}_{3}\left(p_i\right) := p_{2i}$ and $\lambda^{\circ}_{3}\left(\sim p_i\right) := p_{2i+1}$;
  - $\lambda^{\circ}_{3}\left(\bot\right) := \bot$ and $\lambda^{\circ}_{3}\left(\sim \bot\right) := \bot \to \bot$;
  - $\lambda^{\circ}_{3}\left(\varphi * \psi\right) := \lambda^{\circ}_{3}\left(\varphi\right) * \lambda^{\circ}_{3}\left(\psi\right)$ where $* \in \{\vee, \wedge, \to\}$;
  - $\lambda^{\circ}_{3}\left(*\varphi\right) := *\lambda^{\circ}_{3}\left(\varphi\right)$ where $* \in \{\Box, \Diamond\}$;
  - $\lambda^{\circ}_{3}\left(\mathtt{n}\right) := \bot$ and $\lambda^{\circ}_{3}\left(\sim \mathtt{n}\right) := \bot$;
  - $\lambda^{\circ}_{3}\left(\mathtt{b}\right) := \bot \to \bot$ and $\lambda^{\circ}_{3}\left(\sim \mathtt{b}\right) := \bot \to \bot$.
- If $\phi \neq \overline{\phi}$, then $\lambda_{3}\left(\phi\right)$ is defined to be $\lambda_{3}\left(\overline{\phi}\right)$.

Similarly to before, λ°_{3} extends to $\Lambda^{\circ}_{3} : \mathcal{E}\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}} \to \mathcal{E}\mathsf{B3K}^{\circ}$ by

$$\Lambda^{\circ}_{3}\left(L\right) \ := \ \mathsf{B3K}^{\circ} + \lambda^{\circ}_{3}\left[L\right] \ = \ \mathsf{B3K}^{\circ} + \{\lambda^{\circ}_{3}\left(\phi\right) \mid \phi \in L\}.$$

As might be expected, we come to:

Lemma 5.5. *For every* $\phi \in \mathrm{For}_{\mathcal{L}^{\mathtt{b}}_{\mathtt{n}}}$,

$$\mathrm{Tw}\left(\mathfrak{D},\{0\},\{1\}\right) \vDash \lambda^{\circ}_{3}\left(\phi\right) \quad \Longleftrightarrow \quad \mathrm{Tw}^{\mathtt{b}}_{\mathtt{n}}\left(\mathfrak{D},D,D\right) \vDash \phi.$$

Proof. Take $\mathfrak{A}$ and $\mathfrak{B}$ to be $\mathrm{Tw}\left(\mathfrak{D},\{1\},\{0\}\right)$ and $\mathrm{Tw}^{\mathtt{b}}_{\mathtt{n}}\left(\mathfrak{D},D,D\right)$ respectively. Notice that since ϕ and $\overline{\phi}$ are equivalent over BK, we may suppose that $\phi = \overline{\phi}$.

$\boxed{\Longleftarrow}$ Assume that $\mathfrak{A} \vDash \lambda^{\circ}_{3}\left(\phi\right)$. Let v be a valuation in $\mathfrak{B}$. Consider then the valuation v' in $\mathfrak{A}$ given by

$$\begin{aligned} v'\left(p_{2i}\right) &:= \left(\pi_1\left(v\left(p_i\right)\right), \neg\pi_1\left(v\left(p_i\right)\right)\right), \\ v'\left(p_{2i+1}\right) &:= \left(\pi_2\left(v\left(p_i\right)\right), \neg\pi_2\left(v\left(p_i\right)\right)\right). \end{aligned}$$

It follows by an easy induction that for every $\varphi \in \mathrm{For}_{\mathcal{L}^{\mathsf{b}}_{\mathsf{n}}}$ in negation normal form,

$$\pi_1 \left(v' \left(\lambda^{\circ}_3 \left(\varphi \right) \right) \right) \;=\; \pi_1 \left(v \left(\varphi \right) \right).$$

Consequently $\pi_1 \left(v \left(\phi \right) \right) = \pi_1 \left(v' \left(\lambda^{\circ} \left(\phi \right) \right) \right) = 1$. Thus $\mathfrak{B} \models \phi$.

$\boxed{\Longrightarrow}$ Assume that $\mathfrak{A} \not\models \lambda^{\circ}_3 \left(\phi \right)$. Let v be a valuation in $\mathfrak{A}$ such that $\pi_1 \left(v \left(\lambda^{\circ} \left(\phi \right) \right) \right) \neq 1$. Consider the valuation v' in $\mathfrak{B}$ given by

$$v' \left(p_i \right) \;:=\; \left(\pi_1 \left(v \left(p_{2i} \right) \right), \pi_1 \left(v \left(p_{2i+1} \right) \right) \right).$$

It follows by an easy induction that for every $\varphi \in \mathrm{For}_{\mathcal{L}^{\mathsf{b}}_{\mathsf{n}}}$ in negation normal form,

$$\pi_1 \left(v \left(\lambda^{\circ} \left(\varphi \right) \right) \right) \;=\; \pi_1 \left(v' \left(\varphi \right) \right).$$

Consequently $\pi_1 \left(v' \left(\phi \right) \right) = \pi_1 \left(v \left(\lambda^{\circ} \left(\phi \right) \right) \right) \neq 1$. Thus $\mathfrak{B} \not\models \phi$. □

Using Lemma 5.5 one can obtain the following results — which we also state without proof, because they are derived in exactly the same way as before.

Corollary 5.6. λ°_3 *faithfully embeds* $\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$ *into* $\mathsf{B3K}^{\circ}$.

Corollary 5.7. Λ°_3 *is an embedding of* $\mathcal{E}\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$ *into* $\mathcal{E}\mathsf{B3K}^{\circ}$.

Furthermore, arguing like before yields:

Theorem 5.8. $\Lambda^{\mathsf{b}}_{\mathsf{n}}$ *and* Λ°_3 *are mutually inverse isomorphisms between* $\mathcal{E}\mathsf{B3K}^{\circ}$ *and* $\mathcal{E}\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$.

This, in effect, leads to an interesting result concerning the relationship between $\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$-extensions and ordinary normal modal logics:

Corollary 5.9. $\mathcal{E}\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$ *and* $\mathcal{E}\mathsf{K}$ *are isomorphic.*

Proof. Remember, $\mathcal{E}\mathsf{B3K}^{\circ}$ is isomorphic to simply $\mathcal{E}\mathsf{K}$, as was proved already in [13]. □

6 Conclusion

As we know, each extension of the FDE-based modal logic BK corresponds to a suitable class of twist-structures over modal algebras — or rather to the universal closure of it. Further, given a modal algebra $\mathfrak{D}$, every twist-structure $\mathfrak{A}$ over $\mathfrak{D}$ is uniquely determined by

$$\nabla \left(\mathfrak{A} \right) \;:=\; \{ a \vee b \mid (a, b) \in A \} \quad \text{and} \quad \Delta \left(\mathfrak{A} \right) \;:=\; \{ a \wedge b \mid (a, b) \in A \}$$

called its *invariants* (see [12, Proposition 6.2]). Roughly speaking, these two are responsible for 'gaps' and 'gluts' respectively. Now in a sense expanding the original language of BK by adding constants for N or B has the effect of collapsing the first or second invariant — and hence leads to eliminating the respective value at the metalevel of BK-extensions. Thus, in particular, if we pass from BK to $\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$, then we arrive at the class of all full twist-structures over modal algebras (in the expanded language $\mathcal{L}^{\mathsf{b}}_{\mathsf{n}}$), and therefore the lattice of $\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$-extensions eventually turns out to be isomorphic to that of normal modal logics.[11]

These results should be useful for studying various FDE-based modal logics (cf. [15]). As an example, consider the modal bilattice logic MBL suggested in [17, 7], which has the logic $\mathsf{GBL}_{\supset}$

[11] In fact, although our twist-structures were defined over arbitrary modal algebras — which provide an algebraic semantics for K — it is possible to start with a smaller variety of underlying modal algebras.

of logical bilattices [1] as its non-modal base. While the modal operators in MBL are defined in a substantially different way from what we have for BK, it is well known that bilattices too can be represented as full twist-structures over lattices of a special kind. Moreover, it was shown in [2] that in the context of logical bilattices expanding the language of De Morgan algebras — as a fragment of the language of $\mathsf{GBL}_{\supset}$ — to include constants for N and B allows us to introduce the lattice operations with respect to the so-called *knowledge ordering*, given by

$$(a, b) \leqslant_k (c, d) \quad \Longleftrightarrow \quad a \leqslant c \ \text{ and } \ b \leqslant d$$

where $\leqslant$ denotes the ordering in the underlying lattice. Clearly these observations motivate the task of comparing the non-modal base of $\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$ and various bilattice logics.[12] Also they motivate the problem of describing the lattices of extensions for MBL as well as for its versions that have weaker non-modal bases.

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[12] Although expanding logics with truth constants may seem a bit ad hoc, there are different ways to arrive at logics of this kind. For instance, as one of the referees has remarked, the non-modal base of BK^{b} is equivalent to the version of the connexive logic MC (see [22]) augmented with $\bot$, and further, the non-modal base of $\mathsf{BK}^{\mathsf{b}}_{\mathsf{n}}$ has the same expressive power as dBD (see [16]) — viz. the expansion of the Belnap–Dunn logic obtained by adding Boolean complementation and connexive conditional. Cf. also [3].

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