Hintikka's independence-friendly logic meets Nelson's realizability

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Abstract

Inspired by Hintikka's ideas on constructivism, we are going to 'effectivize' the game-theoretic semantics (abbreviated GTS) for independence-friendly first-order logic (IF-FOL), but in a somewhat different way than he did in the monograph 'The Principles of Mathematics Revisited'. First we show that Nelson's realizability interpretation — which extends the famous Kleene's realizability interpretation by adding 'strong negation' — restricted to the implication-free first-order formulas can be viewed as an effective version of GTS for FOL. Then we propose a realizability interpretation for IF-FOL, inspired by the so-called 'trump semantics' which was discovered by Hodges, and show that this trump realizability interpretation can be viewed as an effective version of GTS for IF-FOL. Finally we prove that the trump realizability interpretation for IF-FOL appropriately generalises Nelson's restricted realizability interpretation for the implication-free first-order formulas.

1 Introduction

In his famous monograph [6], Hintikka discusses *independence-friendly first-order logic* (IF-FOL[1] for short) and how drastically different the situation in the philosophy of mathematics would be if classical FOL were replaced by IF-FOL. According to Hintikka, the latter emphasises the *descriptive function of logic*, as opposed to its *deductive function*. More formally, IF-FOL extends FOL in the following way:

- syntactically, we need to add expressions of the form $\exists x \backslash X$ with $\{x\} \cup X$ a set of individual variables, called *independence quantifiers*;
- adopting the standard game-theoretic conventions, we pass from games with perfect information to those with *imperfect information*;
- semantically, for each occurrence of $\exists x \backslash X$ we assume that a choice of value for x does not depend on a choice of values for X (including the cases when $\exists x \backslash X$ occurs in the scope of universal quantifiers over variables from X).

Equivalently, IF-FOL can be easily interpreted using skolemisations, so as Skolem terms for occurences of $\exists x \backslash X$ do not contain variables from X. For any given structure — this computably reduces the problem of determining which IF-FOL-sentences are true to the analogous problem for existential second-order sentences. Interestingly, the converse holds too, due to Enderton [5] and Walkoe [18]. Thus

the collection of all IF-FOL-sentences true in $\mathfrak{A}$ is computably equivalent to the existential fragment of the second-order theory of $\mathfrak{A}$,

[1] This logic was introduced in [7].

where $\mathfrak{A}$ is a structure of signature with equality. It leads to some intriguing consequences:

1. the set of valid IF-FOL-formulas is not computably enumerable, and hence IF-FOL is not recursively axiomatisable, or *non-deductive* as one might put it;

2. the set of Gödel numbers[2] of IF-FOL-sentences true in the standard model $\mathfrak{N}$ of arithmetic is definable in $\mathfrak{N}$ by an IF-FOL-formula, in contrast to the case of FOL.

Several other unusual features of IF-FOL are mentioned in [6] as well. However, IF-FOL shares certain nice model-theoretic properties with FOL, see e.g. [12, Chapter 5]. As for modern developments in this area, we refer the reader to [11].

Evidently Hintikka focuses on the game-theoretic semantics (abbreviated GTS) in his book. Furthermore [6, Chapter 10], entitled 'Constructivism Reconstructed', begins as follows:

> The approach represented in this book has a strong spiritual kinship with constructivistic ideas. This kinship can be illustrated in a variety of ways. One of the basic ideas of constructivists like Michael Dummett [3, 4] is that meaning has to be mediated by teachable, learnable, and practicable human activities. This is precisely the job which semantical games do in game-theoretical semantics. ...

A great deal of work has gone into understanding the concepts of learning, teaching, reasoning, belief change, etc. in terms of GTS since the appearance of [6] — see [2] and references therein. In particular, numerous results have been obtained by computer scientists. Now — inspired by Hintikka's ideas on constructivism — we are going to 'effectivize' GTS for IF-FOL, but in a somewhat different way than he did in the book. Here is how, in a nutshell:

i. First we show that Nelson's realizability interpretation — which extends the famous Kleene's realizability interpretation by adding 'strong negation' — restricted to the implication-free first-order formulas can be viewed as an effective version of GTS for FOL.

ii. Then we propose a realizability interpretation for IF-FOL, inspired by the so-called 'trump semantics' which was discovered by Hodges, and show that this trump realizability interpretation can be viewed as an effective version of GTS for IF-FOL.

iii. Finally we prove that the trump realizability interpretation for IF-FOL appropriately generalises Nelson's restricted realizability interpretation for the implication-free first-order formulas.

Surprisingly enough, the relationship between IF-FOL and realizability-like semantics has not been formally analysed before. There is a natural explanation of this, however. In Kleene's realizability interpretation (see [10, § 82]), as well as in intuitionistic logic, $\neg\phi$ is defined as $\phi \to \bot$, i.e. via a reduction to absurdity. On the other hand, here is what Hintikka says about negation on [6, p. 153]:

> All that is involved, so it might seem, is an inversion of truth-values. For instance, in [19] Wittgenstein held that the negation of a pictorially interpreted sentence is not only also a picture, but the same picture, only with its polarity reversed.

In a two-player game the role-reversal of the players may be thought of as the process of reversing 'the polarity'. This agrees perfectly with the concept of 'strong negation' (cf. [14]) but not with that of intuitionistic negation. Actually, the former was developed in order to avoid some non-constructive features of the latter — it leads to Nelson's extension of Kleene's realizability interpretation in which two kinds of constructive procedures are used, corresponding to verification of formulas and falsification of formulas.[3] Under Nelson's approach,

[2] Assume some Gödel numbering for the IF-FOL-formulas and terms in the signature $\{0, \mathsf{s}, +, \times, =\}$.

[3] Natural numbers that encode such procedures are called *positive* and *negative realizations* respectively.

to verify (falsify) the negation of a statement, we need
to falsify (respectively verify) the original statement.

So in particular — thinking of *verifier* and *falsifier* as 'polarities' — one might expect that there is a close relationship between Nelson's realizability and effective versions of GTS for FOL, and hope that it can be expanded to IF-FOL in a suitable way. We shall see that this is indeed the case.

2 Game-theoretic semantics

This section presents GTS for FOL and IF-FOL (see e.g. [12] for details). We begin by defining win-lose extensive games with perfect and imperfect information (cf. [15]).

2.1 Two kinds of game

In a (win-lose extensive) game each player may or may not be allowed to see and remember all previous moves in the play. Roughly speaking, when a player is fully aware of the moves leading up to the current position, he or she has perfect information.

Formally, plays — or 'histories' — are finite sequences in a suitably chosen alphabet Σ (but not necessarily vice versa). Given $\{a_1, \ldots, a_n, a_{n+1}\} \subseteq \Sigma$, define

$$(a_1, \ldots, a_n)^\frown a_{n+1} \ := \ (a_1, \ldots, a_n, a_{n+1}).$$

Let $\prec$ denote the transitive closure of the corresponding successor relation, i.e. for any two finite sequences w_1 and w_2 in Σ we have

$$w_1 \prec w_2 \quad \Longleftrightarrow \quad w_2 = w_1^\frown a_1^\frown a_2^\frown \ldots a_k \ \text{ for some } \{a_1, \ldots, a_k\} \subseteq \Sigma.$$

Actually, there will be no need to mention Σ in what follows, because it can be easily recovered from the context.

By a (*win-lose extensive*) *game with perfect information* we mean a tuple

$$\langle N, H, Z, P, u \rangle$$

where:

- N is a set whose elements are called *players*.
- H is a set of finite sequences, called *histories*, with the propery that for all h_1, h_2, h_3,

$$h_1 \prec h_2 \prec h_3 \ \text{ and } \ \{h_1, h_3\} \subseteq H \quad \Longrightarrow \quad h_2 \in H.$$

- Z is the set of so-called *terminal* histories, defined by

$$Z \ := \ \{h \in H \mid \text{there is no } h' \in H \text{ for which } h \prec h'\}.$$

  Here we require that for every $h \in H \setminus Z$ there exists $h' \in Z$ such that $h \prec h'$.
- P is a function from $H \setminus Z$ to N, called the *player function*.
- u is a function from Z to N, called the *winner function*.

Intuitively, P indicates whose turn it is to move, while u indicates the winner of each terminal history. We think of the transition from a non-terminal h to one of its successors $h^\frown a$ in H as being caused by an 'action' a. So for any $h \in H \setminus Z$ the player $P(h)$ chooses an a' from

$$A(h) \ := \ \{a \mid h^\frown a \in H\}$$

and the game proceeds from $h' := h^\frown a'$. On the other hand, every player $p \in N$ acts on the set of histories where it is p's turn, i.e.

$$H_p \ := \ \{h \in H \setminus Z \mid P(h) = p\}.$$

By a *strategy for* p we simply mean a function δ with domain H_p such that $\delta(h) \in A(h)$ for all $h \in H_p$. Given a history h', we say p *follows* δ *during* h' iff for each h,

$$h \in H_p \ \text{ and } \ h \prec h' \quad \Longrightarrow \quad h^\frown \delta(h) \prec h' \ \text{ or } \ h^\frown \delta(h) = h'.$$

Now consider the three sets of histories:

$$H_\delta \ := \ \{h \in H \mid p \text{ follows } \delta \text{ during } h\},$$
$$Z_\delta \ := \ H_\delta \cap Z \quad \text{and} \quad Z_p \ := \ \{h \in Z \mid u(h) = p\}.$$

A strategy δ for p is called *winning* iff $Z_\delta \subseteq Z_p$ — in words, iff p wins in every terminal history during which he or she follows δ. In fact, we are only interested in how δ acts on H_δ, because using δ the player p cannot reach any history in $H \backslash H_\delta$. Therefore we shall identify two strategies δ and δ' for p whenever $H_\delta = H_{\delta'}$.

By a (*win-lose extensive*) *game with imperfect information* we mean a tuple

$$\langle N, H, Z, P, u, \{\sim_p \mid p \in N\}\rangle$$

where:

- N, H, Z, P and u are as above (and so constitute a game with perfect information).
- $\sim_p$ is an equivalence relation on H_p with the property that for all h_1, h_2,

  $$h_1 \sim_p h_2 \quad \Longrightarrow \quad A(h_1) \ = \ A(h_2).$$

  (The definitions of H_p and $A(h)$ remain unchanged.)

If $h_1 \sim_p h_2$, we say h_1 and h_2 are *indistinguishable for* p — intuitively, the player p cannot tell the difference between the histories h_1 and h_2, and hence p has to act on them in the same way. Consequently, for this kind of game the class of strategies must be suitably restricted. Call δ a *strategy for* p iff it is a strategy for p in $\langle N, H, Z, P, u\rangle$ such that for any h_1 and h_2,

$$h_1 \sim_p h_2 \quad \Longrightarrow \quad \delta(h_1) \ = \ \delta(h_2).$$

The other notions are defined in the same way as before.

2.2 The case of first-order logic

It is convenient to make the following assumptions about FOL:

- the *connective symbols* are $\neg$, $\vee$ and $\rightarrow$;

- the *quantifier symbol* is $\exists$.

Take the set $\mathcal{V}$ of *individual variables* to be $\{v_n \mid n \in \mathbb{N}\}$. Readers who wish to employ $\wedge$ and $\forall$ should use the standard definitions:

$$\psi \wedge \theta \;:=\; \neg(\neg\psi \vee \neg\theta) \quad \text{and} \quad \forall x\, \psi \;:=\; \neg\exists x\, \neg\psi.$$

Let σ be a signature, i.e. a collection of non-logical symbols, each of which has an arity. Given a first-order σ-formula ϕ, define

$$\begin{aligned} FV(\phi) &:= \text{ the set of individual variables that occur free in } \phi, \\ OS(\phi) &:= \text{ the set of occurrences of subformulas of } \phi \text{ in } \phi. \end{aligned}$$

In effect, there are several ways of representing $OS(\phi)$, but the details are not really important here.[4] For Ψ, Θ, etc. in $OS(\phi)$, we use ψ, θ, etc. to denote the corresponding subformulas, and in fact, we shall, without danger of confusion, occasionally identify them.

Consider a σ-structure $\mathfrak{D}$ with domain D. By an *assignment in* $\mathfrak{D}$ we simply mean a mapping s from a finite subset of $\mathcal{V}$ (denoted by $\mathrm{dom}(s)$, as expected) to M. If $x \in \mathcal{V}$ and $d \in D$, we write $s(x/d)$ for the assignment defined by

$$s(x/d)(y) \;:=\; \begin{cases} s(y) & \text{if } y \in \mathrm{dom}(s) \setminus \{x\}, \\ d & \text{if } y = x \end{cases}$$

— so in particular, $\mathrm{dom}(s(x/d)) = \mathrm{dom}(s) \cup \{x\}$. Finally, the expression $\mathfrak{D}, s \models \phi$ is read as ϕ *is true in* $\mathfrak{D}$ *under* s, provided that $FV(\phi) \subseteq \mathrm{dom}(s)$, of course.

Let s be an assignment in $\mathfrak{D}$ and ϕ be an implication-free first-order σ-formula with $FV(\phi) \subseteq \mathrm{dom}(s)$. We define the game $\mathsf{G}(\mathfrak{D}, s, \phi)$ with perfect information as follows.[5]

- There are only two players, Eloise (E) and Abelard (A). Initially one can think of them as playing the roles of 'verifier' and 'falsifier' respectively. During the game they may switch their roles, however — see the next item.

- The set $H := \bigcup \{H_\Psi \mid \Psi \in OS(\phi)\}$ of histories is defined by recursion, along with the functions $\mathsf{ver} : H \to \{\mathsf{E}, \mathsf{A}\}$ and $\mathsf{fals} : H \to \{\mathsf{E}, \mathsf{A}\}$ determining the roles for each history:
  - if $\Psi = \phi$, then $H_\Psi := \{(s, \Psi)\}$, $\mathsf{ver}((s, \Psi)) := \mathsf{E}$ and $\mathsf{fals}((s, \Psi)) := \mathsf{A}$;[6]
  - if $\Psi = \Psi_1 \vee \Psi_2$, then $H_{\Psi_i} := \{h^\frown \Psi_i \mid h \in H_\Psi\}$ where $i \in \{1, 2\}$, and for all $h \in H_\Psi$,

    $$\mathsf{ver}(h^\frown \Psi_i) \;:=\; \mathsf{ver}(h) \quad \text{and} \quad \mathsf{fals}(h^\frown \Psi_i) \;:=\; \mathsf{fals}(h);$$

  - if $\Psi = \exists x\, \Theta$, then $H_\Theta := \{h^\frown(x, d) \mid h \in H_\Psi \text{ and } d \in D\}$, and for every $h \in H_\Psi$ and every $d \in D$,

    $$\mathsf{ver}(h^\frown(x, d)) \;:=\; \mathsf{ver}(h) \quad \text{and} \quad \mathsf{fals}(h^\frown(x, d)) \;:=\; \mathsf{fals}(h);$$

[4] E.g. ϕ can be viewed as an ordered tree, in which case every element of $OS(\phi)$ becomes a subtree of it.

[5] This definition as well as the definition of semantic games for IF-FOL given in the next subsection are natural modifications of respective definitions from [12], where semantics games were defined for formulas in negative normal form. We define games for arbitrary formulas. This slightly complicates definitions, but explicates the sense of negation in GTS.

[6] Naturally we identify the unique occurrence of ϕ in ϕ with ϕ itself.

– if $\Psi = \neg\Theta$, then $H_\Theta := \{h^\frown\Theta \mid h \in H_\Psi\}$, and for all $h \in H_\Psi$,

$$\mathsf{ver}\,(h^\frown\Theta) \;:=\; \mathsf{fals}\,(h) \quad \text{and} \quad \mathsf{fals}\,(h^\frown\Theta) \;:=\; \mathsf{ver}\,(h)$$

(so $\neg$ indicates the role-reversal of the players).

Furthermore, each history h' induces an assignment $s_{h'}$ extending or modifying s:

$$s_{h'} \;:=\; \begin{cases} s & \text{if } h' = (s, \phi), \\ s_h & \text{if } h' = h^\frown\Psi \text{ for some } \Psi \in OS\,(\phi), \\ s_h\,(x/d) & \text{if } h' = h^\frown(x, d) \text{ for some } d \in D. \end{cases}$$

- The terminal histories are those which cannot be continued, i.e.

$$Z \;:=\; \bigcup \{H_\Psi \mid \Psi \text{ is an occurrence of an atomic formula}\}.$$

- By our choice of logical symbols, we can take P to be ver.[7]
- If Ψ is an occurrence of an atomic formula ψ in ϕ, then for all $h \in H_\Psi$,

$$u\,(h) \;:=\; \begin{cases} \mathsf{ver}\,(h) & \text{if } \mathfrak{D}, s_h \models \psi, \\ \mathsf{fals}\,(h) & \text{if } \mathfrak{D}, s_h \not\models \psi. \end{cases}$$

(It works because for any $h \in Z$ there exists a unique $\Psi \in OS\,(\phi)$ such that $h \in H_\Psi$.)

Finally we write $\mathfrak{D}, s \models^+_{\mathtt{GTS}} \phi$ iff Eloise has a winning strategy for $\mathsf{G}\,(\mathfrak{D}, s, \phi)$, and $\mathfrak{D}, s \models^-_{\mathtt{GTS}} \phi$ iff Abelard has a winning strategy for $\mathsf{G}\,(\mathfrak{D}, s, \phi)$. Observe that

$$\begin{aligned} \mathfrak{D}, s \models^+_{\mathtt{GTS}} \neg\phi \quad &\Longleftrightarrow \quad \mathfrak{D}, s \models^-_{\mathtt{GTS}} \phi, \\ \mathfrak{D}, s \models^-_{\mathtt{GTS}} \neg\phi \quad &\Longleftrightarrow \quad \mathfrak{D}, s \models^+_{\mathtt{GTS}} \phi. \end{aligned} \tag{†}$$

Rather surprisingly, this semantics turns out to be equivalent to the compositional one. The following fact is well known.

Theorem 2.1. *For every assignment s in $\mathfrak{D}$ and every implication-free first-order σ-formula ϕ with $FV\,(\phi) \subseteq \mathrm{dom}\,(s)$,*

$$\mathfrak{D}, s \models^+_{\mathtt{GTS}} \phi \quad \Longleftrightarrow \quad \mathfrak{D}, s \models \phi.$$

In particular, the law of excluded middle is valid in the game-theoretic semantics for FOL.

Hence for s and ϕ as above we have, among other things,

$$\begin{aligned} \mathfrak{D}, s \models^+_{\mathtt{GTS}} \phi \quad &\Longleftrightarrow \quad \mathfrak{D}, s \not\models^-_{\mathtt{GTS}} \phi, \\ \mathfrak{D}, s \models^-_{\mathtt{GTS}} \phi \quad &\Longleftrightarrow \quad \mathfrak{D}, s \not\models^+_{\mathtt{GTS}} \phi \end{aligned} \tag{‡}$$

— in words, either Eloise or Abelard has a winning strategy for $\mathsf{G}\,(\mathfrak{D}, s, \phi)$. In fact, this follows immediately from a classical result of game theory known as Gale-Stewart theorem, which may fail if we alter the class of strategies or the notion of a game.

Note, in passing, that GTS does not give a meaning to $\to$, unless we treat $\phi \to \psi$ as $\neg\phi \vee \psi$ (moving away from constructivism).

[7]Evidently the treatment of $\wedge$ and $\forall$ in [12] agrees with the use of $\psi \wedge \theta$ and $\forall x\, \psi$ as abbreviations.

2.3 Adding independence-friendly quantifiers

Let σ be a signature. The *independence-friendly σ-formulas* are built up from the atomic (first-order) σ-formulas by the following rules, for any finite set $X \cup \{x\}$ of individual variables:

- if ψ and θ are independence-friendly σ-formulas, then so are $\neg\psi$, $\psi \vee_{\backslash X} \theta$ and $\exists x \backslash X\, \psi$.[8]

In this context the 'pure' $\vee$ and $\exists$ can be introduced, via

$$\psi \vee \theta \;:=\; \psi \vee_{\backslash \varnothing} \theta \quad \text{and} \quad \exists x\, \psi \;:=\; \exists x \backslash \varnothing\, \psi.$$

Given an independence-friendly σ-formula ϕ, the set $SO(\phi)$ is defined as before, whereas the set $FV(\phi)$ for atomic formulas and negations is defined as above and for other cases as follows:

$$FV(\psi \vee_{\backslash X} \theta) = FV(\psi) \cup FV(\theta) \cup X, \quad FV(\exists x \backslash X\, \psi) = (FV(\psi) \setminus x) \cup X.$$

Consider a σ-structure $\mathfrak{D}$ with domain D. For every finite set X of individual variables, if s_1 and s_2 are assignments in $\mathfrak{D}$ such that $\mathrm{dom}(s_1) = \mathrm{dom}(s_2)$, we write $s_1 \approx_X s_2$ to mean that they coincide on elements of $\mathrm{dom}(s_1) \setminus X$ — i.e. for all $x \in \mathrm{dom}(s_1) \setminus X$, $s_1(x) = s_2(x)$.

Now let s be an assignment in $\mathfrak{D}$ and ϕ be an independence-friendly σ-formula with $FV(\phi) \subseteq \mathrm{dom}(s)$. We define the game $\mathsf{G}^\star(\mathfrak{D}, s, \phi)$ with imperfect information as follows.

- Again the only players are Eloise (E) and Abelard (A).
- H, ver and fals are exactly like those in Subsection 2.2, except that $\Psi_1 \vee \Psi_2$ and $\exists x\, \Theta$ are replaced by $\Psi_1 \vee_{\backslash X} \Psi_2$ and $\exists x \backslash X\, \Theta$ respectively. Further — for each $h \in H$ we get s_h.
- Z, P and u are defined in the same way as in the case of FOL.
- The indistinguishability relations $\sim_{\mathsf{E}}$ and $\sim_{\mathsf{A}}$ are given by the conditions:
  - if $h_1 \sim_{\mathsf{E}} h_2$ or $h_1 \sim_{\mathsf{A}} h_2$, then $\{h_1, h_2\} \subseteq H_\Psi$ for some $\Psi \in OS(\phi)$;
  - for any $\{h_1, h_2\} \subseteq H_\Psi \cap H_{\mathsf{E}}$ with $\Psi = \Psi_1 \vee_{\backslash X} \Psi_2$ or $\Psi = \exists x \backslash X\, \Theta$, we have $h_1 \sim_{\mathsf{E}} h_2$ iff $s_{h_1} \approx_X s_{h_2}$;
  - for any $\{h_1, h_2\} \subseteq H_\Psi \cap H_{\mathsf{E}}$ with $\Psi = \neg\Theta$, we have $h_1 \sim_{\mathsf{E}} h_2$ iff $s_{h_1} = s_{h_2}$;
  - for any $\{h_1, h_2\} \subseteq H_\Psi \cap H_{\mathsf{A}}$ we have $h_1 \sim_{\mathsf{A}} h_2$ iff $s_{h_1} = s_{h_2}$.[9]

  (Recall that H_{E} and H_{A} are the pre-images of E and A under P and that winning strategies for players E and A must be agreed with $\sim_{\mathsf{E}}$ and respectively $\sim_{\mathsf{A}}$.)

Finally we write $\mathfrak{D}, s \models^+_{\mathsf{GTS}^\star} \phi$ iff Eloise has a winning strategy for $\mathsf{G}^\star(\mathfrak{D}, s, \phi)$, and $\mathfrak{D}, s \models^-_{\mathsf{GTS}^\star} \phi$ iff Abelard has a winning strategy for $\mathsf{G}^\star(\mathfrak{D}, s, \phi)$. Observe that the analogue of (†) holds.

On the other hand, the analogue of (‡) fails, because the law of excluded middle is certainly not valid in the game-theoretic semantics for IF-FOL — for instance, if $\mathfrak{D} = \langle \mathbb{N}; = \rangle$, $s = \varnothing$ and $\phi = \forall x\, \exists y \backslash \{x\}\, x = y$, then neither player has a winning strategy for $\mathsf{G}^\star(\mathfrak{D}, s, \phi)$.

For purposes of this article it will be technically convenient to work with formulas ϕ satisfying the following condition:

(✓) for every individual variable x, the string $\exists x$ occurs at most once in ϕ.

Intuitively, one can rename the bound variables of ϕ without changing its intended meaning, so we shall henceforth assume all formulas satisfy (✓).

[8] We remark that, by definition, these formulas do not contain $\to$.

[9] Actually, when the players reach an occurrence of $\neg$, they do not make choices. Instead, they simply reverse things according to the rules of the game. Thus it does not matter how we define P, $\sim_{\mathsf{E}}$ and $\sim_{\mathsf{A}}$ on $H_{\neg\Theta}$.

3 An effective GTS for FOL

Since we aim to study the relationship between GTS and the approach of [14] (cf. also [10, § 82]), we start by recalling the definition of Nelson's realizability interpretation.

Assume some effective enumeration $\{\mu_e\}_{e\in\mathbb{N}}$ of the partial computable functions which satisfies the *s-m-n*-theorem.[10] Denote by [,] your favorite computable pairing function; take π_1 and π_2 to be the projection mappings associated with it. Thus for all $\{n,k\}\subseteq\mathbb{N}$,

$$\pi_1([n,k]) \;=\; n, \quad \pi_2([n,k]) \;=\; k \quad \text{and} \quad [\pi_1(n),\pi_2(n)] \;=\; n.$$

Let $\sigma_{\mathbb{N}}$ and $\mathfrak{N}$ be the signature of Peano arithmetic and its standard model, i.e.

$$\sigma_{\mathbb{N}} \;:=\; \{0,\mathsf{s},+,\times,=\} \quad \text{and} \quad \mathfrak{N} \;:=\; \langle \mathbb{N}; 0^{\mathbb{N}}, \mathsf{s}^{\mathbb{N}}, +^{\mathbb{N}}, \times^{\mathbb{N}}, =^{\mathbb{N}}\rangle.$$

Now for any $e\in\mathbb{N}$, assignment s in $\mathfrak{N}$ and first-order $\sigma_{\mathbb{N}}$-formula ϕ with $FV(\phi)\subseteq \mathrm{dom}(s)$, we inductively define

$$e \,Ⓟ\, s,\phi \quad \text{and} \quad e \,Ⓝ\, s,\phi$$

as follows (where α is a meta-variable standing for an atomic formula).

`Positive realizability Ⓟ:`

$e \,Ⓟ\, s,\alpha$	iff	$e=0$ and $\mathfrak{N},s\models\alpha$;
$e \,Ⓟ\, s,\psi\vee\theta$	iff	either $e=[1,k]$ where $k \,Ⓟ\, s,\psi$, or $e=[2,k]$ where $k \,Ⓟ\, s,\theta$;
$e \,Ⓟ\, s,\psi\to\theta$	iff	for all $n\in\mathbb{N}$, if $n \,Ⓟ\, s,\psi$, then $\mu_e(n) \,Ⓟ\, s,\theta$;
$e \,Ⓟ\, s,\exists x\,\psi$	iff	$e=[n,k]$ where $k \,Ⓟ\, s(x/n),\psi$;
$e \,Ⓟ\, s,\neg\psi$	iff	$e \,Ⓝ\, s,\psi$.

`Negative realizability Ⓝ:`

$e \,Ⓝ\, s,\alpha$	iff	$e=0$ and $\mathfrak{N},s\not\models\alpha$;
$e \,Ⓝ\, s,\psi\vee\theta$	iff	$e=[n,k]$ where $n \,Ⓝ\, s,\psi$ and $k \,Ⓝ\, s,\theta$;
$e \,Ⓝ\, s,\psi\to\theta$	iff	$e=[n,k]$ where $n \,Ⓟ\, s,\psi$ and $k \,Ⓝ\, s,\theta$;
$e \,Ⓝ\, s,\exists x\,\psi$	iff	for all $n\in\mathbb{N}$, $\mu_e(n) \,Ⓝ\, s(x/n),\psi$;
$e \,Ⓝ\, s,\neg\psi$	iff	$e \,Ⓟ\, s,\psi$.

(Keep in mind that if $\mu_e(n) \,Ⓟ\, s,\phi$ or $\mu_e(n) \,Ⓟ\, s,\phi$, then n must be in the domain of μ_e.)

For e, s and ϕ as above, $e \,Ⓟ\, s,\phi$ is read as *e positively realizes ϕ under s* — or *e is a positive realization for ϕ under s*. We call ϕ *positively realizable under s* iff $n \,Ⓟ\, s,\phi$ for some number n. Similarly for Ⓝ, replacing 'positive(ly)' by 'negative(ly)'.

Roughly speaking, each positive (negative) realization of ϕ under s encodes an effective verification (respectively falsification) procedure for ϕ in $\mathfrak{N}$ under s. In fact, purely for exposition, the definitions of Ⓟ and Ⓝ given here differ in minor details from those in [14]:

i. Instead of first-order $\sigma_{\mathbb{N}}$-sentences, we use pairs s,ϕ where s is an assignment in $\mathfrak{N}$ and ϕ is a first-order $\sigma_{\mathbb{N}}$-formula with $FV(\phi)\subseteq\mathrm{dom}(s)$.

[10] In Sections 4 and 5 this theorem, together with Church's Thesis, will often be tacitly used. Readers who wish to know more about enumerations might consult [17, § 1.8].

ii. Nelson took $\wedge$ and $\forall$ as primitive — although his conditions for $\phi \wedge \psi$ and $\forall x\, \phi$ are easily seen to be equivalent to those for $\neg(\neg\phi \vee \neg\psi)$ and $\neg\exists x\, \neg\phi$.[11] We treat them as defined.

Conventionally Ⓟ and Ⓝ are interpreted over $\mathfrak{N}$. However, one can do the same with any computable structure.

Henceforth we shall restrict ourselves to implication-free formulas — because we are concerned with 'effectivizing' game-theoretic semantics, and $\rightarrow$ is not available in GTS.[12]

3.1 Strategies revisited

Of course most functions are not computable, and so cannot be expressed in the form μ_e where $e \in \mathbb{N}$. The reader may well ask:

What happens with Ⓟ and Ⓝ if we drop constructivity?

Let σ be a signature and $\mathfrak{D}$ a σ-structure with domain D. For any assignment s in $\mathfrak{D}$ and implication-free first-order σ-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(s)$, we inductively define the sets

$$S^+(\mathfrak{D}, s, \phi) \quad \text{and} \quad S^-(\mathfrak{D}, s, \phi)$$

as follows (where again α stands for an atomic formula).

<u>`Set-theoretic analogue of Ⓟ without →:`</u>

$$\begin{aligned}
S^+(\mathfrak{D}, s, \alpha) &:= \{0\} \text{ if } \mathfrak{D}, s \models \alpha, \text{ and } \varnothing \text{ otherwise;} \\
S^+(\mathfrak{D}, s, \psi \vee \theta) &:= \big(\{1\} \times S^+(\mathfrak{D}, s, \psi)\big) \cup \big(\{2\} \times S^+(\mathfrak{D}, s, \theta)\big); \\
S^+(\mathfrak{D}, s, \exists x\, \psi) &:= \big\{\langle d, t\rangle \mid d \in D \text{ and } t \in S^+(\mathfrak{D}, s(x/d), \psi)\big\}; \\
S^+(\mathfrak{D}, s, \neg\psi) &:= S^-(\mathfrak{D}, s, \psi).
\end{aligned}$$

<u>`Set-theoretic analogue of Ⓝ without →:`</u>

$$\begin{aligned}
S^-(\mathfrak{D}, s, \alpha) &:= \{0\} \text{ if } \mathfrak{D}, s \not\models \alpha, \text{ and } \varnothing \text{ otherwise;} \\
S^-(\mathfrak{D}, s, \psi \vee \theta) &:= S^-(\mathfrak{D}, s, \psi) \times S^-(\mathfrak{D}, s, \theta); \\
S^-(\mathfrak{D}, s, \exists x\, \psi) &:= \text{the set of all functions } f \text{ with domain } D \text{ such that} \\
& f(d) \in S^-(\mathfrak{D}, s(x/d), \psi) \text{ for each } d \in D; \\
S^-(\mathfrak{D}, s, \neg\psi) &:= S^+(\mathfrak{D}, s, \psi).
\end{aligned}$$

When $\mathfrak{D} = \mathfrak{N}$, we omit $\mathfrak{D}$ and write simply $S^+(s, \phi)$, $S^-(s, \phi)$.

Actually, one can think of elements of $S^+(\mathfrak{D}, s, \phi)$ and $S^-(\mathfrak{D}, s, \phi)$ as winning strategies for Eloise and Abelard in $\mathsf{G}(\mathfrak{D}, s, \phi)$. More precisely, bearing in mind that we identify strategies δ_1 and δ_2 whenever $H_{\delta_1} = H_{\delta_2}$, it is not difficult to show the following.

Theorem 3.1. *For every assignment s in $\mathfrak{D}$ and every implication-free first-order σ-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(s)$, there exist canonical 1-1 functions ι^+ and ι^- such that:*

- *ι^+ maps $S^+(\mathfrak{D}, s, \phi)$ onto the set of winning strategies for E in $\mathsf{G}(\mathfrak{D}, s, \phi)$;*

[11] This does not apply to Kleene's version in which the only negation is intuitionistic, and which does not deal with 'negative realizability'.

[12] Recall that the implication cann't be expressed via other connectives in constructive setting.

- ι^- *maps* $S^-(\mathfrak{D}, s, \phi)$ *onto the set of winning strategies for* A *in* $\mathsf{G}(\mathfrak{D}, s, \phi)$.

Proof. By induction on the complexity of ϕ.

Suppose ϕ is atomic. In this case $H = \{(s, \phi)\} = Z$, so the only possible strategy is $\varnothing$. Now if $\mathfrak{D}, s \models \phi$, then the empty strategy is winning for Eloise but not for Abelard, and $S^+(\mathfrak{D}, s, \phi)$ equals $\{0\}$ while $S^-(\mathfrak{D}, s, \phi)$ has no elements at all — thus ι^+ and ι^- are uniquely determined. Similarly if $\mathfrak{D}, s \not\models \phi$.

Suppose $\phi = \exists x\, \psi$. So by the inductive hypothesis, for every $d \in D$ there exist 1-1 functions ι_d^+ and ι_d^- such that:

- ι_d^+ maps $S^+(\mathfrak{D}, s(x/d), \psi)$ onto the set of winning strategies for E in $\mathsf{G}(\mathfrak{D}, s(x/d), \psi)$;
- ι_d^- maps $S^-(\mathfrak{D}, s(x/d), \psi)$ onto the set of winning strategies for A in $\mathsf{G}(\mathfrak{D}, s(x/d), \psi)$.

Let $t \in S^+(\mathfrak{D}, s, \phi)$. Then $t = \langle d, t' \rangle$ for some $d \in D$ and some $t' \in S^+(\mathfrak{D}, s(x/d), \psi)$. Take

$$\delta'_t \; := \; \iota_d^+(t').$$

Recall that each history of $\mathsf{G}(\mathfrak{D}, s, \phi)$ has the form (s, ϕ) or $(s, \phi, (x, d'), \dots)$ where $d' \in D$, and hence the 1-1 function

$$\kappa_d : \; (s, \phi, (x, d), \dots) \; \mapsto \; (s(x/d), \psi, \dots)$$

maps the set of histories of $\mathsf{G}(\mathfrak{D}, s, \phi)$ in which Eloise chooses d for (the value of) x on her first move onto the set of histories of $\mathsf{G}(\mathfrak{D}, s(x/d), \psi)$.[13] Define the strategy δ_t for Eloise by

$$\delta_t(h) \; := \; \begin{cases} (x, d) & \text{if } h = (s, \phi), \\ \delta'_t(\kappa_d(h)) & \text{otherwise.} \end{cases}$$

Evidently δ_t is winning for $\mathsf{G}(\mathfrak{D}, s, \phi)$. Consider the function ι^+ with domain $S^+(\mathfrak{D}, s, \phi)$ given by the equation $\iota^+(t) = \delta_t$. By construction, ι^+ is 1-1. We claim that ι^+ is onto. For let δ be a winning strategy for Eloise in $\mathsf{G}(\mathfrak{D}, s, \phi)$. Then $\delta((s, \phi)) = (x, d)$ for some $d \in D$. Take δ' to be the strategy for Eloise in $\mathsf{G}(\mathfrak{D}, s(x/d), \psi)$ defined by

$$\delta'(h) \; := \; \delta\left(\kappa_d^{-1}(h)\right).$$

Obviously δ' is winning, and hence there exists $t' \in S^+(\mathfrak{D}, s(x/d), \psi)$ for which $\iota_d^+(t') = \delta'$, i.e. $\delta'_t = \delta'$ where $t = \langle d, t' \rangle$. From this we get $\iota^+(t) = \delta$ — because

- if $h = (s, \phi)$, then $\delta_t(h) = (x, d) = \delta(h)$, and
- if $h \neq (s, \phi)$, then $\delta_t(h) = \delta'_t(\kappa_d(h)) = \delta'(\kappa_d(h)) = \delta\left(\kappa_d^{-1}(\kappa_d(h))\right) = \delta(h)$.

Now we move on to the second part. Let $f \in S^-(\mathfrak{D}, s, \phi)$ — i.e. f is a function with domain D such that for all $d \in D$ we have $f(d) \in S^-(\mathfrak{D}, s(x/d), \psi)$. For any $d \in D$, take

$$\delta_d^f \; := \; \iota_d^-(f(d)).$$

Then we build a winning strategy δ^f for Abelard in $\mathsf{G}(\mathfrak{D}, s, \phi)$ as follows:

$$\delta^f((s, \phi, (x, d), \dots)) \; := \; \delta_d^f((s(x/d), \psi, \dots))$$

[13] Here we identify elements of $OS(\psi)$ with those of $OS(\phi)$ in the obvious way.

(notice that in (s, ϕ) it is Eloise's turn to move, since $\phi = \exists x\, \psi$). Consider the function ι^- with domain $S^-(\mathfrak{D}, s, \phi)$ given by the equation $\iota^-(f) = \delta^f$. By construction, ι^- is 1-1. Furthermore we claim that ι^- is onto. For let δ be a winning strategy for Abelard in $\mathsf{G}(\mathfrak{D}, s, \phi)$. For every d in D, take δ_d to be the strategy for Abelard in $\mathsf{G}(\mathfrak{D}, s(x/d), \psi)$ defined by

$$\delta_d((s(x/d), \psi, \dots)) \ := \ \delta((s, \phi, (x, d), \dots)),$$

which is certainly winning. Obviously the function f that maps each $d \in D$ to the pre-image of δ_d under ι_d^- belongs to $S^-(\mathfrak{D}, s, \phi)$. And we have $\iota^-(f) = \delta$, as can be readily checked.

An analogous argument applies if $\phi = \psi_1 \vee \psi_2$.

Suppose $\phi = \neg\psi$. Notice that since $\neg$ indicates the role-reversal of the players, there are 1-1 functions π^+ and π^- (defined in the obvious way) such that:

- π^+ maps the set of winning strategies for E in $\mathsf{G}(\mathfrak{D}, s, \neg\psi)$ onto the set of winning strategies for A in $\mathsf{G}(\mathfrak{D}, s, \psi)$;
- π^- maps the set of winning strategies for A in $\mathsf{G}(\mathfrak{D}, s, \neg\psi)$ onto the set of winning strategies for E in $\mathsf{G}(\mathfrak{D}, s, \psi)$.

The rest is immediate — remembering the definitions $S^+(\mathfrak{D}, s, \neg\psi)$ and $S^-(\mathfrak{D}, s, \neg\psi)$. $\square$

Recalling Nelson's approach — let $\mathbb{I}^+$ ($\mathbb{I}^-$) denote the set of all triples $\langle s, \phi, e\rangle$ where s is an assignment in $\mathfrak{N}$, ϕ is an implication-free first-order $\sigma_{\mathbb{N}}$-formula with $FV(\phi) \subseteq \mathrm{dom}(s)$ and e is a positive (respectively negative) realization of ϕ under s. We can turn each such triple into an 'effectively realizable' winning strategy, using the functions $\mathtt{D}^+$ and $\mathtt{D}^-$ with domains $\mathbb{I}^+$ and $\mathbb{I}^-$ respectively, defined by the following conditions.

For every $\langle s, \phi, e\rangle$ in $\mathbb{I}^+$:

- if ϕ is atomic, then $\mathtt{D}^+(s, \phi, e) = 0$;
- if $\phi = \psi_1 \vee \psi_2$ and $e = [i, k]$, then $\mathtt{D}^+(s, \phi, e) = \langle i, \mathtt{D}^+(s, \psi_i, k)\rangle$;
- if $\phi = \exists x\, \psi$ and $e = [n, k]$, then $\mathtt{D}^+(s, \phi, e) = \langle n, \mathtt{D}^+(s(x/n), \psi, k)\rangle$;
- if $\phi = \neg\psi$, then $\mathtt{D}^+(s, \phi, e) = \mathtt{D}^-(s, \psi, e)$.

For every $\langle s, \phi, e\rangle$ in $\mathbb{I}^-$:

- if ϕ is atomic, then $\mathtt{D}^-(s, \phi, e) = 0$;
- if $\phi = \psi_1 \vee \psi_2$ and $e = [n, k]$, then $\mathtt{D}^-(s, \phi, e) = \langle \mathtt{D}^-(s, \psi_1, n), \mathtt{D}^-(s, \psi_2, k)\rangle$;
- if $\phi = \exists x\, \psi$, then $\mathtt{D}^-(s, \phi, e)$ is the function that maps each $n \in \mathbb{N}$ to $\mathtt{D}^-(s(x/n), \psi, \mu_e(n))$;
- if $\phi = \neg\psi$, then $\mathtt{D}^-(s, \phi, e) = \mathtt{D}^+(s, \psi, e)$.

As you would expect, this construction produces winning strategies for Eloise and Abelard:

Proposition 3.2. *For $\circ \in \{+, -\}$ and any $\langle s, \phi, e\rangle \in \mathbb{I}^\circ$ we have $\mathtt{D}^\circ(s, \phi, e) \in S^\circ(s, \phi)$.*

Proof. By an easy induction on the complexity of ϕ. $\square$

In the other direction — each intuitively computable winning strategy for $\mathsf{G}(\mathfrak{N}, s, \phi)$ can be brought to the form $\mathtt{D}^\circ(s, \phi, e)$ with e a suitable realization, as we shall shortly see.

3.2 Effective winning strategies

For $\circ \in \{+, -\}$, let $\mathbb{S}^\circ$ denote the collection of all triples $\langle s, \phi, t\rangle$ where s is an assignment in $\mathfrak{N}$, ϕ is an implication-free first-order $\sigma_{\mathbb{N}}$-formula with $FV(\phi) \subseteq \mathrm{dom}(s)$, and $t \in S^\circ(s, \phi)$. We are now ready to 'effectivize' the corresponding version of the game-theoretic semantics for FOL — or rather its implication-free fragment. To this end, we use the functions

$$\mathtt{E}^+ : \mathbb{S}^+ \to \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \mathtt{E}^- : \mathbb{S}^- \to \mathcal{P}(\mathbb{N})$$

defined inductively by the following conditions (recalling those for $S^\pm$).[14]

For every $\langle s, \phi, t\rangle$ in $\mathbb{S}^+$:

- if ϕ is atomic, and so $t = 0$, then $\mathtt{E}^+(s, \phi, t) = \{0\}$;
- if $\phi = \psi_1 \vee \psi_2$, and so $t = \langle i, t'\rangle$ for appropriate i and t', then

$$\mathtt{E}^+(s, \phi, t) \;=\; \big\{[i, k] \mid k \in \mathtt{E}^+(s, \psi_i, t')\big\};$$

- if $\phi = \exists x\, \psi$, and so $t = \langle n, t'\rangle$ for appropriate n and t', then

$$\mathtt{E}^+(s, \phi, t) \;=\; \big\{[n, k] \mid k \in \mathtt{E}^+(s(x/n), \psi, t')\big\};$$

- if $\phi = \neg\psi$, then $\mathtt{E}^+(s, \phi, t) = \mathtt{E}^-(s, \psi, t)$.

For every $\langle s, \phi, t\rangle$ in $\mathbb{S}^-$:

- if ϕ is atomic, and so $t = 0$, then $\mathtt{E}^-(s, \phi, t) = \{0\}$;
- if $\phi = \psi_1 \vee \psi_2$, and so $t = \langle t_1, t_2\rangle$ for appropriate t_1 and t_2, then

$$\mathtt{E}^-(s, \phi, t) \;=\; \big\{[n, k] \mid n \in \mathtt{E}^-(s, \psi_1, t_1) \text{ and } k \in \mathtt{E}^-(s, \psi_2, t_2)\big\};$$

- if $\phi = \exists x\, \psi$, and so $t = f$ for an appropriate f, then

$$\mathtt{E}^-(s, \phi, t) \;=\; \big\{e \in \mathbb{N} \mid \text{for all } n \in \mathbb{N},\ \mu_e(n) \in \mathtt{E}^-(s(x/n), \psi, f(n))\big\};$$

- if $\phi = \neg\psi$, then $\mathtt{E}^-(s, \phi, t) = \mathtt{E}^+(s, \psi, t)$.

In each case being 'appropriate' merely means having the properties required by the definitions of $S^+(s, \phi)$ and $S^-(s, \phi)$.

Let ϕ be a first-order $\sigma_{\mathbb{N}}$-formula not containing $\to$, and s be an assignment in $\mathfrak{N}$, such that $FV(\phi) \subseteq \mathrm{dom}(s)$. For $\circ \in \{+, -\}$, call a strategy $t \in S^\circ(s, \phi)$ *effective* iff $\mathtt{E}^\circ(s, \phi, t) \neq \varnothing$. Now we can 'effectivize' the relations $\models^+_{\mathtt{GTS}}$ and $\models^-_{\mathtt{GTS}}$ from Subsection 2.2: say ϕ is *true* (*false*) under s in the *effective game-theoretic semantics* for FOL, written $s \models^+_{\mathtt{EGTS}} \phi$ (respectively $s \models^-_{\mathtt{EGTS}} \phi$), iff there exists an effective strategy in $S^+(s, \phi)$ (respectively $S^-(s, \phi)$).

Roughly speaking, if $t \in S^\circ(s, \phi)$ where $\circ \in \{+, -\}$, then each number in $\mathtt{E}^\circ(s, \phi, t)$ encodes an algorithm for computing t. Of course this reminds us of positive and negative realizations of ϕ under s. Actually, the two approaches are equivalent (for implication-free formulas):

[14] As usual, we write $\mathcal{P}(\mathbb{N})$ for the powerset of $\mathbb{N}$.

Proposition 3.3. *Let ϕ be a first-order $\sigma_{\mathbb{N}}$-formula not containing $\rightarrow$, and s an assignment in $\mathfrak{N}$, such that $FV(\phi) \subseteq \operatorname{dom}(s)$. For all $e \in \mathbb{N}$ we have*

$$\begin{aligned} e \, ⓟ \, s, \phi \quad &\Longleftrightarrow \quad e \in \mathtt{E}^{+}(s, \phi, t) \ \textit{for some } t \in S^{+}(s, \phi), \\ e \, ⓝ \, s, \phi \quad &\Longleftrightarrow \quad e \in \mathtt{E}^{-}(s, \phi, t) \ \textit{for some } t \in S^{-}(s, \phi). \end{aligned}$$

Moreover, in each case, if the left-hand side holds, then t is uniquely determined by e, s and ϕ.

Proof. It is straightforward to check that for $\circ \in \{+, -\}$ and any $\langle s, \phi, e \rangle \in \mathbb{I}^{\circ}$,

$$e \in \mathtt{E}^{\circ}(s, \phi, \mathtt{D}^{\circ}(s, \phi, e))$$

— keeping in mind that $\mathtt{D}^{\circ}(s, \phi, e) \in S^{\circ}(s, \phi)$ by Proposition 3.2, and hence can be substituted for t in $\mathtt{E}^{\circ}(s, \phi, t)$. This immediately gives the implications from left to right. Further, a simple inductive argument shows that for $\circ \in \{-, +\}$ and any $\{t_1, t_2\} \subseteq S^{\circ}(s, \phi)$,

$$t_1 \neq t_2 \quad \Longrightarrow \quad \mathtt{E}^{\circ}(s, \phi, t_1) \cap \mathtt{E}^{\circ}(s, \phi, t_2) = \varnothing.$$

Thus the uniqueness follows. The right-to-left direction is also straightforward. $\square$

So in particular, for ϕ and s as above we have

$$\begin{aligned} s \models^{+}_{\mathtt{EGTS}} \phi \quad &\Longleftrightarrow \quad \phi \text{ is positively realizable under } s, \\ s \models^{-}_{\mathtt{EGTS}} \phi \quad &\Longleftrightarrow \quad \phi \text{ is negatively realizable under } s. \end{aligned}$$

However, Proposition 3.3 tells us more: the two semantics are not only extensionally equivalent but also intentionally equivalent.[15]

It should be remarked that that for every implication-free first-order $\sigma_{\mathbb{N}}$-sentence ψ,

$$\varnothing \models^{+}_{\mathtt{EGTS}} \psi \quad \Longrightarrow \quad S^{+}(\varnothing, \phi) \neq \varnothing \quad \Longrightarrow \quad \varnothing \models^{+}_{\mathtt{GTS}} \psi.$$

Consequently, if ψ is positively realizable under $\varnothing$, then it is true classically. By contraposition, if ψ is false classically, then it is not positively realizable under $\varnothing$ (just because there are no gluts in classical logic). Let us now consider one curious sentence, suggested by Kleene. Assuming an appropriate coding $\mathtt{M}_0, \mathtt{M}_1, \mathtt{M}_2, \ldots$ of all Turing machines, take $\beta(x)$ to be an existential first-order $\sigma_{\mathbb{N}}$-formula with no connective symbols, such that for any $e \in \mathbb{N}$,

$$\mathfrak{N} \models \beta(e) \quad \Longleftrightarrow \quad \mathtt{M}_e \text{ halts on input } e$$

(cf. [13]). We are going to discuss the constructive content of the first-order $\sigma_{\mathbb{N}}$-sentence

$$\chi := \forall x \, (\beta(x) \vee \neg \beta(x)).$$

Clearly χ is true classically. However, it is not positively realizable under $\varnothing$. For otherwise there would be a computable function f such that for each $e \in \mathbb{N}$ exactly one of the following holds:

- $\pi_1(f(e)) = 1$ and $\mathtt{M}_e$ halts on input e;
- $\pi_1(f(e)) = 2$ and $\mathtt{M}_e$ does not halt on input e.

And this would contradict the undecidability of the self-applicability problem for Turing machines. Here are some futher observations concerning χ:

[15] Again, one could use any computable structure in place of $\mathfrak{N}$.

i. $\neg\chi$ is false classically;

ii. $\neg\chi$ is neither negatively nor positively realizable under $\varnothing$;

iii. $\neg\chi$ becomes positively realizable under $\varnothing$ if we treat $\neg$ intuitionistically.[16]

In particular, the conjunction of (i) and (iii) reveals somewhat counterintuitive features of intuitionistic negation as compared with 'strong negation'.

4 An effective GTS for IF-FOL

At this point it helps to recall the *trump semantics* for IF-FOL discovered by Hodges, which is known to be equivalent to the original game-theoretic semantics for IF-FOL; see [8, 9]. Here we closely follow the treatment in [12, Chapter 4], adapting it to our framework.

Let σ be a signature and $\mathfrak{D}$ a σ-structure with domain D. By a *team in* $\mathfrak{D}$ is meant a set of assignments in $\mathfrak{D}$ that have a common domain. For any team T in $\mathfrak{D}$, mapping f from T to D, and individual variable x, we define

$$\begin{aligned} T[x, D] &:= \{s(x/d) \mid s \in T \text{ and } d \in D\}, \\ T[x, f] &:= \{s(x/f(s)) \mid s \in T\}. \end{aligned}$$

Let T be a team in $\mathfrak{D}$ and X a finite set of individual variables. A subteam T' of T — in other words, an element T' of $\mathcal{P}(T)$ — is said to be X*-closed in* T iff for any $\{s_1, s_2\} \subseteq T$,

$$s_1 \approx_X s_2 \ \text{ and } \ s_1 \in T' \quad \Longrightarrow \quad s_2 \in T'.$$[17]

A cover $\{T_1, T_2\}$ of T — i.e. a family $\{T_1, T_2\} \subseteq \mathcal{P}(T)$ with $T_1 \cup T_2 = T$ — is called X*-uniform* iff both T_1 and T_2 are X-closed in T. A mapping f from T to D is said to be X*-uniform* iff for any $\{s_1, s_2\} \subseteq T$,

$$s_1 \approx_X s_2 \quad \Longrightarrow \quad f(s_1) = f(s_2).$$

We write $\mathcal{C}(T, X)$ for the collection of all X-uniform two-element covers of T and $\mathcal{M}(T, X)$ for the collection of all X-uniform mappings from T to D. Using these terms, for any team T in $\mathfrak{D}$ and independence-friendly first-order σ-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(T)$, we define

$$\mathfrak{D}, T \models_{\mathtt{t}}^{+} \phi \quad \text{and} \quad \mathfrak{D}, T \models_{\mathtt{t}}^{-} \phi$$

inductively as follows (where α ranges over the atomic formulas, as always).

<u>`Teams for Eloise:`</u>

$\mathfrak{D}, T \models_{\mathtt{t}}^{+} \alpha$	iff	$\mathfrak{D}, s \models \alpha$ for each $s \in T$;
$\mathfrak{D}, T \models_{\mathtt{t}}^{+} \psi \vee_{\backslash X} \theta$	iff	$\mathfrak{D}, T_1 \models_{\mathtt{t}}^{+} \psi$ and $\mathfrak{D}, T_2 \models_{\mathtt{t}}^{+} \theta$ for some $\{T_1, T_2\} \in \mathcal{C}(T, X)$;
$\mathfrak{D}, T \models_{\mathtt{t}}^{+} \exists x \backslash X\, \psi$	iff	$\mathfrak{D}, T[x, f] \models_{\mathtt{t}}^{+} \psi$ for some $f \in \mathcal{M}(T, X)$;
$\mathfrak{D}, T \models_{\mathtt{t}}^{+} \neg\psi$	iff	$\mathfrak{D}, T \models_{\mathtt{t}}^{-} \psi$.

[16] Formally, this means that $\neg\theta$ is interpreted as $\theta \to \bot$, or rather as $\theta \to 0 = \mathsf{s}(0)$ (since our language does not contain $\bot$ explicitly).

[17] Remember from Subsection 2.3 that $s_1 \approx_X s_2$ means that s_1 and s_2 agree on $\mathrm{dom}(s_1) \setminus X$.

`Teams for Abelard:`

$$
\begin{array}{lll}
\mathfrak{D}, T \models_{\mathtt{t}}^{-} \alpha & \text{iff} & \mathfrak{D}, s \not\models \alpha \text{ for each } s \in T; \\
\mathfrak{D}, T \models_{\mathtt{t}}^{-} \psi \vee_{\backslash X} \theta & \text{iff} & \mathfrak{D}, T \models_{\mathtt{t}}^{-} \psi \text{ and } \mathfrak{D}, T \models_{\mathtt{t}}^{-} \theta; \\
\mathfrak{D}, T \models_{\mathtt{t}}^{-} \exists x \backslash X\, \psi & \text{iff} & \mathfrak{D}, T\,[x, D] \models_{\mathtt{t}}^{-} \psi; \\
\mathfrak{D}, T \models_{\mathtt{t}}^{-} \neg \psi & \text{iff} & \mathfrak{D}, T \models_{\mathtt{t}}^{+} \psi.
\end{array}
$$

Observe that the analogue of (†) for $\models_{\mathtt{t}}^{\pm}$ holds, obviously.

On the other hand, the definition of $\mathsf{G}^{\star}(\mathfrak{D}, s, \phi)$ given in Subsection 2.3 can be easily extended to deal with teams. Namely, we define $\mathsf{G}^{\star}(\mathfrak{D}, T, \phi)$ just as $\mathsf{G}^{\star}(\mathfrak{D}, s, \phi)$, except that now H_{ϕ} is taken to be $\{(s, \phi) \mid s \in T\}$, rather than $\{(s, \phi)\}$. Thus:

- each pair (s, ϕ) with $s \in T$ is an 'initial' history of $\mathsf{G}^{\star}(\mathfrak{D}, T, \phi)$;
- each history of $\mathsf{G}^{\star}(\mathfrak{D}, T, \phi)$ is a history of $\mathsf{G}^{\star}(\mathfrak{D}, s, \phi)$ for a suitable $s \in T$.

As one may expect, we write $\mathfrak{D}, T \models_{\mathtt{GTS}^{\star}}^{+} \phi$ iff Eloise has a winning strategy for $\mathsf{G}^{\star}(\mathfrak{D}, T, \phi)$ and $\mathfrak{D}, T \models_{\mathtt{GTS}^{\star}}^{-} \phi$ iff Abelard has a winning strategy for $\mathsf{G}^{\star}(\mathfrak{D}, T, \phi)$.

Theorem 4.1 (see [12, Sections 4.3 and 4.4] for details)**.** *For every team T in $\mathfrak{D}$ and every independence-friendly first-order σ-formula ϕ with $FV(\phi) \subseteq \operatorname{dom}(T)$,*

$$
\mathfrak{D}, T \models_{\mathtt{GTS}^{\star}}^{+} \phi \iff \mathfrak{D}, T \models_{\mathtt{t}}^{+} \phi.
$$

So in particular, if $T = \{s\}$, then $\mathfrak{D}, s \models_{\mathtt{GTS}^{\star}}^{+} \phi$ and $\mathfrak{D}, \{s\} \models_{\mathtt{t}}^{+} \phi$ are equivalent.

Note that a subteam T' of T is X-closed in T iff $T \setminus T'$ is so. Further — instead of $\mathcal{C}(T, X)$ one could use the collection $\widehat{\mathcal{C}}(T, X)$ of all disjoint X-uniform two-element covers of T throughout. This modification looks very natural, because our strategies are deterministic, and IF-FOL is not sensitive to whether we choose to work with $\mathcal{C}(T, X)$ or with $\widehat{\mathcal{C}}(T, X)$.

In fact we can obtain some interesting analogues of S^{+} and S^{-} for IF-FOL by 'intentionalizing' the trump semantics.

4.1 Strategies revisited

Let σ and $\mathfrak{D}$ be as usual. For any team T in $\mathfrak{D}$ and independence-friendly first-order σ-formula ϕ with $FV(\phi) \subseteq \operatorname{dom}(T)$, we inductively define the sets

$$
S_{\star}^{+}(\mathfrak{D}, T, \phi) \quad \text{and} \quad S_{\star}^{-}(\mathfrak{D}, T, \phi)
$$

as follows (where again α ranges over the atomic formulas, and $\mathbf{0}_T$ denotes the function from T to $\mathbb{N}$ given by $\mathbf{0}_T(s) = 0$).

`Intentional content underlying` $\models_{\mathtt{t}}^{+}$`:`

$$
\begin{array}{rcl}
S_{\star}^{+}(\mathfrak{D}, T, \alpha) & := & \{\mathbf{0}_T\} \text{ if } \mathfrak{D}, s \models \alpha \text{ for each } s \in T, \text{ and } \varnothing \text{ otherwise}; \\
S_{\star}^{+}\left(\mathfrak{D}, T, \psi \vee_{\backslash X} \theta\right) & := & \bigcup_{\{T_1, T_2\} \in \widehat{\mathcal{C}}(T, X)} \{T_1\} \times S_{\star}^{+}(\mathfrak{D}, T_1, \psi) \times S_{\star}^{+}(\mathfrak{D}, T_2, \theta); \\
S_{\star}^{+}(\mathfrak{D}, T, \exists x \backslash X\, \psi) & := & \left\{\langle f, t\rangle \mid f \in \mathcal{M}(T, X) \text{ and } t \in S_{\star}^{+}(\mathfrak{D}, T\,[x, f], \psi)\right\}; \\
S_{\star}^{+}(\mathfrak{D}, T, \neg \psi) & := & S_{\star}^{-}(\mathfrak{D}, T, \psi).
\end{array}
$$

`Intentional content underlying` $\models_{\mathtt{t}}^{-}$ `:`

$$\begin{aligned}
S_{\star}^{-}(\mathfrak{D},T,\alpha) &:= \{\mathbf{0}_T\} \text{ if } \mathfrak{D},s \not\models \alpha \text{ for each } s\in T, \text{ and } \varnothing \text{ otherwise;}\\
S_{\star}^{-}\left(\mathfrak{D},T,\psi \vee_{\backslash X} \theta\right) &:= S_{\star}^{-}(\mathfrak{D},T,\psi)\times S_{\star}^{-}(\mathfrak{D},T,\theta);\\
S_{\star}^{-}(\mathfrak{D},T,\exists x\backslash X\,\psi) &:= S_{\star}^{-}(\mathfrak{D},T[x,D],\psi);\\
S_{\star}^{-}(\mathfrak{D},T,\neg\psi) &:= S_{\star}^{+}(\mathfrak{D},T,\psi).
\end{aligned}$$

When $\mathfrak{D}=\mathfrak{N}$, we omit $\mathfrak{D}$ and write simply $S_{\star}^{+}(T,\phi)$, $S_{\star}^{-}(T,\phi)$.

It is straightforward to verify the adequacy of this 'intentionalization':

Proposition 4.2. *For T and ϕ as above we have*

$$\mathfrak{D},T\models_{\mathtt{t}}^{+}\phi \quad\Longleftrightarrow\quad S_{\star}^{+}(\mathfrak{D},T,\phi)\neq\varnothing,$$

and similarly with $-$ in place of $+$.

Proof. By an easy induction on the complexity of ϕ. □

Moreover, elements of $S_{\star}^{+}(\mathfrak{D},T,\phi)$ and $S_{\star}^{-}(\mathfrak{D},T,\phi)$ can be thought of as winning strategies for Eloise and Abelard in $\mathsf{G}^{\star}(\mathfrak{D},T,\phi)$, as will be shown shortly. Notice that we again identify strategies δ_1 and δ_2 whenever $H_{\delta_1}=H_{\delta_2}$.

Theorem 4.3. *For every team T in $\mathfrak{D}$ and every independence-friendly first-order σ-formula ϕ with $FV(\phi)\subseteq\operatorname{dom}(T)$, there exist canonical 1-1 functions ν^{+} and ν^{-} such that:*

- *ν^{+} maps $S_{\star}^{+}(\mathfrak{D},T,\phi)$ onto the set of winning strategies for E in $\mathsf{G}^{\star}(\mathfrak{D},T,\phi)$;*
- *ν^{-} maps $S_{\star}^{-}(\mathfrak{D},T,\phi)$ onto the set of winning strategies for A in $\mathsf{G}^{\star}(\mathfrak{D},T,\phi)$.*

Proof. By induction on the complexity of ϕ.

In the case where ϕ is atomic the result follows easily (cf. the proof of Theorem 3.1).

Suppose $\phi=\exists x\backslash X\,\psi$. So by the inductive hypothesis, for every $f\in\mathcal{M}(T,X)$ there exists a 1-1 function ν_f^{+} such that:

- ν_f^{+} maps $S^{+}(\mathfrak{D},T[x,f],\psi)$ onto the set of winning strategies for E in $\mathsf{G}^{\star}(\mathfrak{D},T[x,f],\psi)$.

Let $t\in S_{\star}^{+}(\mathfrak{D},T,\phi)$. Then $t=\langle f,t'\rangle$ for some $f\in\mathcal{M}(T,X)$ and $t'\in S_{\star}^{+}(\mathfrak{D},T[x,f],\psi)$. Take

$$\delta_t' := \nu_f^{+}(t').$$

Recall that every history of $\mathsf{G}^{\star}(\mathfrak{D},T,\phi)$ has the form (s,ϕ) or $(s,\phi,(x,d),\dots)$ where $s\in T$ and $d\in D$. Hence the 1-1 function

$$\lambda_f:\ (s,\phi,(x,f(s)),\dots)\ \mapsto\ (s(x/f(s)),\psi,\dots)$$

maps the set of histories of $\mathsf{G}^{\star}(\mathfrak{D},T,\phi)$ in which Eloise chooses $f(s)$ for (the value of) x on her first move onto the set of histories of $\mathsf{G}^{\star}(\mathfrak{D},T[x,f],\psi)$. Define the strategy δ_t for Eloise by

$$\delta_t(h) := \begin{cases}(x,f(s)) & \text{if } h=(s,\phi) \text{ with } s\in T,\\ \delta_t'(\lambda_f(h)) & \text{otherwise.}\end{cases}$$

Clearly δ_t is winning for $\mathsf{G}^\star(\mathfrak{D}, T, \phi)$. Consider the function ν^+ with domain $S^+_\star(\mathfrak{D}, T, \phi)$ given by the equation $\nu^+(t) = \delta_t$. By construction, ν^+ is 1-1, and we claim that ν^+ is onto. For let δ be a winning strategy for Eloise in $\mathsf{G}^\star(\mathfrak{D}, T, \phi)$. So for each $s \in T$, there exists a unique $d_s \in D$ such that $\delta((s, \phi)) = (x, d_s)$. Furthermore for any $\{s_1, s_2\} \subseteq T$,

$$s_1 \approx_X s_2 \quad \Longrightarrow \quad (s_1, \phi) \sim_{\mathsf{E}} (s_2, \phi) \quad \Longrightarrow \quad \delta((s, \phi)) \;=\; \delta((s, \phi)) \quad \Longrightarrow \quad d_{s_1} \;=\; d_{s_2}$$

(remember the definition of $\sim_{\mathsf{E}}$). Thus the mapping f from T to D given by $f(s) = d_s$ belongs to $\mathcal{M}(T, X)$. Take δ' to be the strategy for Eloise in $\mathsf{G}^\star(\mathfrak{D}, T[x, f], \psi)$ defined by

$$\delta'(h) \;:=\; \delta(\lambda_f^{-1}(h)).$$

This δ' is winning, obviously. Hence there exists a $t' \in S^+_\star(\mathfrak{D}, T[x, f], \psi)$ for which $\nu^+_f(t') = \delta'$, i.e. $\delta'_t = \delta'$ where $t = \langle f, t' \rangle$. So we get $\nu^+(t) = \delta$ — because

- if $h = (s, \phi)$ with $s \in T$, then $\delta_t(h) = (x, f(s)) = (x, d_s) = \delta(h)$, and
- if h is not initial, then $\delta_t(h) = \delta'_t(\lambda_f(h)) = \delta'(\lambda_f(h)) = \delta(\lambda_f^{-1}(\lambda_f(h))) = \delta(h)$.

Now we move on to the second part of the argument for $\phi = \exists x \backslash X\, \psi$. For present purposes the inductive hypothesis ensures the existence of a 1-1 function ν^-_D such that:

- ν^-_D maps $S^-(\mathfrak{D}, T[x, D], \psi)$ onto the set of winning strategies for A in $\mathsf{G}^\star(\mathfrak{D}, T[x, D], \psi)$.

Let $t \in S^-(\mathfrak{D}, T, \phi)$ — i.e. $t \in S^+(\mathfrak{D}, T[x, D], \psi)$. Take

$$\delta^t_D \;:=\; \nu^-_D(t).$$

Then we build a winning strategy δ^t for Abelard in $\mathsf{G}^\star(\mathfrak{D}, T, \phi)$ as follows:

$$\delta^t((s, \phi, (x, d), \dots)) \;:=\; \delta^t_D((s(x/d), \psi, \dots))$$

(bearing in mind that in every initial history it is Eloise's turn to move). Consider the function ν^- with domain $S^-_\star(\mathfrak{D}, T, \phi)$ given by the equation $\nu^-(t) = \delta^t$. By construction, ν^- is 1-1. We claim that ν^- is also onto. For let δ be a winning strategy for Abelard in $\mathsf{G}^\star(\mathfrak{D}, T, \phi)$. Take δ_D to be the strategy for Abelard in $\mathsf{G}(\mathfrak{D}, T[x, D], \psi)$ defined by

$$\delta_D((s(x/d), \psi, \dots)) \;:=\; \delta((s, \phi, (x, d), \dots)),$$

which is evidently winning. Further, the pre-image t of δ_D under ν^-_D belongs to $S^-_\star(\mathfrak{D}, T, \phi)$, which is equal by definition to $S^-_\star(\mathfrak{D}, T[x, D], \psi)$, and we have $\nu^-(t) = \delta$, as can be readily checked.

An analogous argument applies if $\phi = \psi \vee_{\backslash X} \theta$.

The case $\phi = \neg\psi$ is simple (cf. the proof of Theorem 3.1). $\square$

Note, in passing, that Theorem 4.1 (about the equivalence of the trump semantics and GTS for IF-FOL) can now be immediately derived from Proposition 4.2 and Theorem 4.3.

Next, we propose a '*realizability interpretation*' for IF-FOL, inspired by Hodges's semantics. To simplify the exposition we notice that assignments in $\mathfrak{N}$ can be identified with natural numbers (and vice versa).[18] Then for any $e \in \mathbb{N}$, team T in $\mathfrak{N}$ and independence-friendly first-order $\sigma_{\mathbb{N}}$-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(T)$, let

$$e \boxed{\mathtt{P}}\, T, \phi \quad \text{and} \quad e \boxed{\mathtt{N}}\, T, \phi$$

[18]Here we assume some 1-1 numbering of the collection of all assignments in $\mathfrak{N}$, viz. some 1-1 function from $\mathbb{N}$ onto this collection; certainly such numberings exist (remember, assignments have finite domains, by definition). Further, via such a function, teams in $\mathfrak{N}$ can be identified with sets of natural numbers.

be defined inductively as follows (where α ranges over the atomic formulas).

`Positive realizability` $\boxed{\mathtt{P}}$:

$$
\begin{array}{lll}
e \boxed{\mathtt{P}} T, \alpha & \text{iff} & \text{for all } s \in T,\ \mu_e(s) = 0 \text{ and } \mathfrak{N}, s \models \alpha; \\
e \boxed{\mathtt{P}} T, \psi \vee_{\backslash X} \theta & \text{iff} & e = [m, [n, k]] \text{ and there exists } \{T_1, T_2\} \in \widehat{\mathcal{C}}(T, X) \\
 & & \text{such that } \mu_m \restriction_T \text{ is the characteristic function} \\
 & & \text{of } T_1 \text{ in } T,\ n \boxed{\mathtt{P}} T_1, \psi \text{ and } k \boxed{\mathtt{P}} T_2, \theta; \\
e \boxed{\mathtt{P}} T, \exists x \backslash X\, \psi & \text{iff} & e = [n, k],\ T \subseteq \operatorname{dom}(\mu_n),\ \mu_n \restriction_T \in \mathcal{M}(T, X) \\
 & & \text{and } k \boxed{\mathtt{P}} T[x, \mu_n \restriction_T], \psi; \\
e \boxed{\mathtt{P}} T, \neg\psi & \text{iff} & e \boxed{\mathtt{N}} T, \psi.
\end{array}
$$

`Negative realizability` $\boxed{\mathtt{N}}$:

$$
\begin{array}{lll}
e \boxed{\mathtt{N}} T, \alpha & \text{iff} & \text{for all } s \in T,\ \mu_e(s) = 0 \text{ and } \mathfrak{N}, s \not\models \alpha; \\
e \boxed{\mathtt{N}} T, \psi \vee_{\backslash X} \theta & \text{iff} & e = [n, k],\ n \boxed{\mathtt{N}} T, \psi \text{ and } k \boxed{\mathtt{N}} T, \theta; \\
e \boxed{\mathtt{N}} T, \exists x \backslash X\, \psi & \text{iff} & e \boxed{\mathtt{N}} T[x, \mathbb{N}], \psi; \\
e \boxed{\mathtt{N}} T, \neg\psi & \text{iff} & e \boxed{\mathtt{P}} T, \psi.
\end{array}
$$

(Keep in mind that $\operatorname{dom}(\mu_e)$ may be thought of as a set of assignments.)

For e, T and ϕ as above, $e \boxed{\mathtt{P}} T, \phi$ is read as *e positively* `t`*-realizes ϕ under T* — or *e is a positive* `t`*-realization for ϕ under T*. We call ϕ *positively* `t`*-realizable under T* iff $n \boxed{\mathtt{P}} T, \phi$ for some number n. Similarly for $\boxed{\mathtt{N}}$, replacing 'positive(ly)' by 'negative(ly)'.

By analogy with the case of FOL, let $\mathbb{I}^+_\star$ ($\mathbb{I}^-_\star$) denote the set of all triples $\langle T, \phi, e\rangle$ where T is a team in $\mathfrak{N}$, ϕ is an independence-friendly first-order $\sigma_{\mathbb{N}}$-formula with $FV(\phi) \subseteq \operatorname{dom}(T)$ and e is a positive (respectively negative) `t`-realization of ϕ under T. Further, define the functions $\mathtt{D}^+_\star$ and $\mathtt{D}^-_\star$ with domains $\mathbb{I}^+_\star$ and $\mathbb{I}^-_\star$ respectively, by the following conditions.

For every $\langle T, \phi, e\rangle$ in $\mathbb{I}^+_\star$:

- if ϕ is atomic, then $\mathtt{D}^+_\star(T, \phi, e) = \mathbf{0}_T$;
- if $\phi = \psi \vee_{\backslash X} \theta$ and $e = [m, [n, k]]$, then $\mathtt{D}^+_\star(T, \phi, e) = \langle T_1, \mathtt{D}^+_\star(T_1, \psi, n), \mathtt{D}^+_\star(T_2, \theta, k)\rangle$ where T_1 is the subteam of T with characteristic function $\mu_m \restriction_T$ and T_2 is $T \setminus T_1$.
- if $\phi = \exists x \backslash X\, \psi$ and $e = [n, k]$, then $\mathtt{D}^+_\star(T, \phi, e) = \langle \mu_n \restriction_T, \mathtt{D}^+_\star(T[x, \mu_n \restriction_T], \psi, k)\rangle$;
- if $\phi = \neg\psi$, then $\mathtt{D}^+_\star(T, \phi, e) = \mathtt{D}^-_\star(T, \psi, e)$.

For every $\langle T, \phi, e\rangle$ in $\mathbb{I}^-_\star$:

- if ϕ is atomic, then $\mathtt{D}^-_\star(T, \phi, e) = \mathbf{0}_T$;
- if $\phi = \psi \vee_{\backslash X} \theta$ and $e = [n, k]$, then $\mathtt{D}^-_\star(T, \phi, e) = \langle \mathtt{D}^-_\star(T, \psi, n), \mathtt{D}^-_\star(T, \theta, k)\rangle$;
- if $\phi = \exists x \backslash X\, \psi$, then $\mathtt{D}^-_\star(T, \phi, e) = \mathtt{D}^-_\star(T[x, \mathbb{N}], \psi, e)$;
- if $\phi = \neg\psi$, then $\mathtt{D}^-_\star(T, \phi, e) = \mathtt{D}^+_\star(T, \psi, e)$.

Not surprisingly, this construction turns each element of $\mathbb{I}^+_\star$ or $\mathbb{I}^-_\star$ into an ('effectively realizable') winning strategy for Eloise or Abelard:

Proposition 4.4. *For $\circ \in \{+,-\}$ and any $\langle T,\phi,e\rangle \in \mathbb{I}^\circ_\star$ we have $\mathtt{D}^\circ_\star(T,\phi,e) \in S^\circ_\star(T,\phi)$.*

Proof. By an easy induction on the complexity of ϕ. $\square$

In the next section we define effective winning strategies for $\mathsf{G}^\star(\mathfrak{N},T,\phi)$ and prove that every such strategy is expressible as $\mathtt{D}^\circ_\star(T,\phi,e)$ for some suitable $\mathtt{t}$-realization e.

4.2 Effective winning stategies

For $\circ \in \{+,-\}$ we let $\mathbb{S}^\circ_\star$ be the collection of all triples $\langle T,\phi,t\rangle$ where T is a team in $\mathfrak{N}$, ϕ is an independence-friendly first-order $\sigma_\mathbb{N}$-formula with $FV(\phi) \subseteq \mathrm{dom}(T)$, and $t \in S^\circ_\star(T,\phi)$. Now to 'effectivize' the corresponding version of GTS for IF-FOL, we use the functions

$$\mathtt{E}^+_\star : \mathbb{S}^+_\star \to \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \mathtt{E}^-_\star : \mathbb{S}^-_\star \to \mathcal{P}(\mathbb{N})$$

defined inductively by the following conditions (recalling those for $S^\pm_\star$).

<u>For every $\langle T,\phi,t\rangle$ in $\mathbb{S}^+_\star$:</u>

- if ϕ is atomic, and so $t = \mathbf{0}_T$, then $\mathtt{E}^+_\star(T,\phi,t) = \{e \in \mathbb{N} \mid \mu_e(s) = 0 \text{ for all } s \in T\}$;
- if $\phi = \psi \vee_{\backslash X} \theta$, and so $t = \langle T_1,t_1,t_2\rangle$ for appropriate T_1, t_1 and t_2, then

$$\begin{aligned}\mathtt{E}^+_\star(T,\phi,t) \;=\; \{[m,[n,k]] \mid \mu_m\restriction_T &\text{ is the characteristic function of}\\ &T_1 \text{ in } T,\; n \in \mathtt{E}^+_\star(T_1,\psi,t_1) \text{ and } k \in \mathtt{E}^+_\star(T\setminus T_1,\theta,t_2)\big\};\end{aligned}$$

- if $\phi = \exists x \backslash X\, \psi$, and so $t = \langle f,t'\rangle$ for appropriate f and t', then

$$\mathtt{E}^+_\star(T,\phi,t) \;=\; \big\{[n,k] \mid T \subseteq \mathrm{dom}(\mu_n),\; \mu_n\restriction_T = f \text{ and } k \in \mathtt{E}^+_\star(T[x,f],\psi,t')\big\};$$

- if $\phi = \neg\psi$, then $\mathtt{E}^+_\star(T,\phi,t) = \mathtt{E}^-_\star(T,\psi,t)$.

<u>For every $\langle T,\phi,t\rangle$ in $\mathbb{S}^-_\star$:</u>

- if ϕ is atomic, and so $t = \mathbf{0}_T$, then $\mathtt{E}^-_\star(T,\phi,t) = \{e \in \mathbb{N} \mid \mu_e(s) = 0 \text{ for all } s \in T\}$;
- if $\psi \vee_{\backslash X} \theta$, and so $t = \langle t_1,t_2\rangle$ for appropriate t_1 and t_2, then

$$\mathtt{E}^-_\star(T,\phi,t) \;=\; \big\{[n,k] \mid n \in \mathtt{E}^-_\star(T,\psi,t_1) \text{ and } k \in \mathtt{E}^-_\star(T,\theta,t_2)\big\};$$

- if $\phi = \exists x \backslash X\, \psi$, then $\mathtt{E}^-_\star(T,\phi,t) = \mathtt{E}^-_\star(T[x,\mathbb{N}],\psi,t)$;
- if $\phi = \neg\psi$, then $\mathtt{E}^-_\star(T,\phi,t) = \mathtt{E}^+_\star(T,\psi,t)$.

In each case being 'appropriate' merely means having the properties required by the definitions of $S^+_\star(s,\phi)$ and $S^-_\star(s,\phi)$.

Let ϕ be an independence-friendly first-order $\sigma_\mathbb{N}$-formula and T a team in $\mathfrak{N}$ with $FV(\phi) \subseteq \mathrm{dom}(T)$. For $\circ \in \{+,-\}$, call a strategy $t \in S^\circ_\star(T,\phi)$ *effective* iff $\mathtt{E}^\circ_\star(T,\phi,t) \neq \varnothing$. The relations $\models^+_{\mathtt{GTS}^\star}$ and $\models^-_{\mathtt{GTS}^\star}$ from Subsection 2.3 can now be suitably 'effectivized': we say ϕ is *true* (*false*) under T in the *effective game-theoretic semantics* for IF-FOL, written $T \models^+_{\mathtt{EGTS}^\star} \phi$ ($T \models^-_{\mathtt{EGTS}^\star} \phi$), iff there exists an effective strategy in $S^+_\star(T,\phi)$ (respectively $S^-_\star(T,\phi)$).

It is straightforward to show that this semantics agrees perfectly with our 'trump' realizability interpretation:

Proposition 4.5. *Let ϕ be an independence-friendly $\sigma_{\mathbb{N}}$-formula and T a team in $\mathfrak{N}$, such that $FV(\phi) \subseteq \mathrm{dom}(T)$. For all $e \in \mathbb{N}$ we have*

$$\begin{aligned} e \boxed{\mathrm{P}} T, \phi \quad &\iff \quad e \in \mathtt{E}^{+}_{\star}(T, \phi, t) \ \text{ for some } t \in S^{+}_{\star}(T, \phi), \\ e \boxed{\mathrm{N}} T, \phi \quad &\iff \quad e \in \mathtt{E}^{-}_{\star}(T, \phi, t) \ \text{ for some } t \in S^{-}_{\star}(T, \phi). \end{aligned}$$

Moreover, in each case, if the left-hand side holds, then t is uniquely determined by e, T and ϕ.

Proof. Similar to the proof of Proposition 3.3. □

So in particular, for ϕ and T as above we have

$$\begin{aligned} T \models^{+}_{\mathtt{EGTS}^{\star}} \phi \quad &\iff \quad \phi \text{ is positively } \mathtt{t}\text{-realizable under } T, \\ T \models^{-}_{\mathtt{EGTS}^{\star}} \phi \quad &\iff \quad \phi \text{ is negatively } \mathtt{t}\text{-realizable under } T. \end{aligned}$$

Finally — notice that one could use any computable structure instead of $\mathfrak{N}$ throughout.

5 About the connection between these two EGTS's

Remember, implication-free first-order formulas may be viewed as 'pure' independence-friendly first-order formulas, by thinking of $\psi \vee \theta$ and $\exists x \, \psi$ as $\psi \vee_{\backslash \varnothing} \theta$ and $\exists x \backslash \varnothing \, \psi$ respectively. Yet the connection between realizations of such formulas and their $\mathtt{t}$-realizations has to be investigated. In this section we shall see that the trump realizability interpretation restricted to the pure $\sigma_{\mathbb{N}}$-formulas is computably equivalent to Nelson's realizability interpretation restricted to the implication-free first-order $\sigma_{\mathbb{N}}$-formulas.

This is a good place to bring in some additional notation and terminology. If T is a team in $\mathfrak{N}$, m is a natural number and μ_m is the characteristic function of T (so T is computable), then we call m an *index* of T. The following functions will prove to be useful.[19]

- Let $\mathrm{rev}_{\mathsf{A}}$ and $\mathrm{rev}_{\mathsf{E}}$ be computable functions such that for any team T in $\mathfrak{N}$, index m of T, individual variable x not in $\mathrm{dom}(T)$, and natural number e with $\mathrm{dom}(\mu_e) \supseteq T$:
  - $\mathrm{rev}_{\mathsf{A}}(m, x)$ is an index of $T[x, \mathbb{N}]$;
  - $\mathrm{rev}_{\mathsf{E}}(m, x, e)$ is an index of $T[x, \mu_e\restriction_T]$.
- Recall the projection mappings π_1 and π_2 introduced in Section 3. Using these, we can get computable functions $\dot{\pi}_1$ and $\dot{\pi}_2$ such that for any $\{e, m\} \subseteq \mathbb{N}$:

$$\begin{aligned} \mu_{\dot{\pi}_1(e)}(m) &= \pi_1(\mu_e(m)); \\ \mu_{\dot{\pi}_2(e)}(m) &= \pi_2(\mu_e(m)). \end{aligned}$$

- Let sub_1 and sub_2 denote the computable functions such that for all $\{n, k, m\} \subseteq \mathbb{N}$:

$$\mu_{\mathrm{sub}_1(n,k)}(m) = \begin{cases} 1 & \text{if } \mu_n(m) = \mu_k(m) = 1, \\ 0 & \text{if } \mu_n(m) = 0, \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$\mu_{\mathrm{sub}_2(n,k)}(m) = \begin{cases} 1 & \text{if } \mu_n(m) = 1 - \mu_k(m) = 1, \\ 0 & \text{if } \mu_n(m) = 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

[19]The *s-m-n*-theorem ensures that they exist and are computable.

Thus if n is an index of T, and k is such that $T \subseteq \operatorname{dom}(\mu_k)$, then $\mu_{\mathrm{sub}_1(n,k)}$ and $\mu_{\mathrm{sub}_2(n,k)}$ are indexes of $T_1 := \{s \in T \mid \mu_k(s) = 1\}$ and $T_2 := T \setminus T_1$ respectively.

- Take $\dot{\pi}$ and glue to be computable functions such that for any $\{i, j, k, n\} \subseteq \mathbb{N}$:

$$\begin{aligned} \mu_{\dot{\pi}(i,j)}(n) &= [\mu_i(n), \mu_j(n)]; \\ \mu_{\mathrm{glue}(i,j,k)}(n) &= \begin{cases} \mu_i(n) & \text{if } \mu_k(n) = 1, \\ \mu_j(n) & \text{if } \mu_k(n) = 0, \\ \text{undefined} & \text{otherwise.} \end{cases} \end{aligned}$$

Now the proof of the desired equivalence naturally falls into two parts.

5.1 From realizations to t-realizations

Here we describe how t-realizations for 'pure' independence-friendly first-order $\sigma_{\mathbb{N}}$-formulas under computable teams in $\mathfrak{N}$ can be obtained by putting together ordinary — i.e. in the sense of Nelson — realizations in a uniform effective way:

Theorem 5.1. *There exist computable functions $\mathtt{r}_{\mathtt{t}}^{+}$ and $\mathtt{r}_{\mathtt{t}}^{-}$ such that for every $e \in \mathbb{N}$, team T in $\mathfrak{N}$, implication-free first-order $\sigma_{\mathbb{N}}$-formula ϕ with $FV(\phi) \subseteq dom(T)$, and index m of T:*

$$\begin{aligned} \mu_e(s) \, Ⓟ \, s, \phi \text{ for each } s \in T \quad &\Longleftrightarrow \quad \mathtt{r}_{\mathtt{t}}^{+}(e, m, \phi) \, \boxed{\mathrm{P}} \, T, \phi; \\ \mu_e(s) \, Ⓝ \, s, \phi \text{ for each } s \in T \quad &\Longleftrightarrow \quad \mathtt{r}_{\mathtt{t}}^{-}(e, m, \phi) \, \boxed{\mathrm{N}} \, T, \phi. \end{aligned}$$

In particular, if $T = \{s\}$, then the collection of all positive realizations of ϕ under s is computably reducible to the collection of all positive t-realizations of ϕ under $\{s\}$, uniformly in ϕ and s, and similarly with 'negative' in place of 'positive'.

Proof. Take $\gamma_{\mathtt{t}}$ to be a computable function such that for any $\{e, n\} \subseteq \mathbb{N}$,

$$\mu_{\gamma_{\mathtt{t}}(e)}(n) = \begin{cases} 1 & \text{if } \mu_e(n) = 1, \\ 0 & \text{if } \mu_e(n) = 2, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Further — take ξ and η to be computable functions such that for every $e \in \mathbb{N}$, individual variable x and assignment s in $\mathfrak{N}$ with $\operatorname{dom}(s) \subseteq \mathcal{V} \setminus \{x\}$:

– $\mu_{\xi(e,x)}(s(x/\pi_1(\mu_e(s)))) = \pi_2(\mu_e(s))$;

– $\mu_{\eta(e,x)}(s(x/n)) = \mu_{\mu_e(s)}(n)$.[20]

Now for any $\{e, m\} \subseteq \mathbb{N}$ and implication-free first-order $\sigma_{\mathbb{N}}$-formula ϕ, we inductively define

$$\mathtt{r}_{\mathtt{t}}^{+}(e, m, \phi) \quad \text{and} \quad \mathtt{r}_{\mathtt{t}}^{-}(e, m, \phi)$$

by the following conditions.

For Ⓟ and $\boxed{\mathrm{P}}$:

[20]To understand how ξ and η can be constructed, consider for instance η. By the s-m-n-theorem, it suffices to provide an algorithm such that, given an $e \in \mathbb{N}$, an individual variable x and an assignment s' in $\mathfrak{N}$, if s' has the form $s(x/n)$ where $\operatorname{dom}(s) \subseteq \mathcal{V} \setminus \{x\}$, then the algorithm produces $\mu_{\mu_e(s)}(n)$. Thus, we only need to effectively obtain s and n from s' — provided that s' has the form required. In order to do this, first check whether x is in $\operatorname{dom}(s')$ or not, and if 'yes', then let n be $s'(x)$ and s the restriction of s' to $\operatorname{dom}(s') \setminus \{x\}$. Similarly for ξ.

– if ϕ is atomic, then $\mathtt{r}_{\mathtt{t}}^{+}(e,m,\phi)=e$;

– if $\phi=\psi_1\vee\psi_2$, then $\mathtt{r}_{\mathtt{t}}^{+}(e,m,\phi)$ is

$$\left[\gamma_{\mathtt{t}}(\dot{e}_1),\left[\mathtt{r}_{\mathtt{t}}^{+}(\dot{e}_2,\mathrm{sub}_1(m,\gamma_{\mathtt{t}}(\dot{e}_1)),\psi_1),\mathtt{r}_{\mathtt{t}}^{+}(\dot{e}_2,\mathrm{sub}_2(m,\gamma_{\mathtt{t}}(\dot{e}_1)),\psi_2)\right]\right]$$

where $\dot{e}_1$ and $\dot{e}_2$ denote $\dot{\pi}_1(e)$ and $\dot{\pi}_2(e)$ respectively;

– if $\phi=\exists x\,\psi$, then $\mathtt{r}_{\mathtt{t}}^{+}(e,m,\phi)=\left[\dot{e}_1,\mathtt{r}_{\mathtt{t}}^{+}(\xi(e,x),\mathrm{rev}_{\mathsf{E}}(m,x,\dot{e}_1),\psi)\right]$;

– if $\phi=\neg\psi$, then $\mathtt{r}_{\mathtt{t}}^{+}(e,m,\phi)=\mathtt{r}_{\mathtt{t}}^{-}(e,m,\psi)$.

<u>`For` Ⓝ `and` $\boxed{\mathbb{N}}$`:`</u>

– if ϕ is atomic, then $\mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)=e$;

– if $\phi=\psi_1\vee\psi_2$, then $\mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)=\left[\mathtt{r}_{\mathtt{t}}^{-}(\dot{e}_1,m,\psi_1),\mathtt{r}_{\mathtt{t}}^{-}(\dot{e}_2,m,\psi_2)\right]$;

– if $\phi=\exists x\,\psi$, then $\mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)=\mathtt{r}_{\mathtt{t}}^{-}(\eta(e,x),\mathrm{rev}_{\mathsf{A}}(m,x),\psi)$;

– if $\phi=\neg\psi$, then $\mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)=\mathtt{r}_{\mathtt{t}}^{+}(e,m,\psi)$.

It is more or less straightforward to verify that the pair $\mathtt{r}_{\mathtt{t}}^{+}$, $\mathtt{r}_{\mathtt{t}}^{-}$ does the job. We check the item negative realizations of existentialy quantified formulas:

$$\mu_e(s)\,Ⓝ\,s,\exists x\psi\ \text{ for each } s\in T\quad\Longleftrightarrow\quad \mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)\,\boxed{\mathbb{N}}\,T,\exists x\psi,$$

where m is an index of T. By definition $\mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)\,\boxed{\mathbb{N}}\,T,\exists x\psi$ if and only if $\mathtt{r}_{\mathtt{t}}^{-}(e,m,\phi)\,\boxed{\mathbb{N}}\,T[x,\mathbb{N}],\psi$. Thus, we have to check that

$$\mu_e(s)\,Ⓝ\,s,\exists x\psi\ \text{ for each } s\in T\quad\Longleftrightarrow\quad \mathtt{r}_{\mathtt{t}}^{-}(\eta(e,x),\mathrm{rev}_{\mathsf{A}}(m,x),\psi)\,\boxed{\mathbb{N}}\,T[x,\mathbb{N}],\psi.$$

By induction hypothesis we have

$$\mu_e(s)\,Ⓝ\,s,\psi\ \text{ for each } s\in U\quad\Longleftrightarrow\quad \mathtt{r}_{\mathtt{t}}^{-}(e,k,\psi)\,\boxed{\mathbb{N}}\,U,\psi,$$

where k is an index of a team U. If m is an index of T, then $\mathrm{rev}_{\mathsf{A}}(m,x)$ is an index of $T[x,\mathbb{N}]$. Therefore, it remains to check that

$$\mu_e(s)\,Ⓝ\,s,\exists x\psi\ \text{ for each } s\in T\quad\Longleftrightarrow\quad \mu_{\eta(e,x)}(s(x/n))\,Ⓝ\,s(x/n),\psi \text{ for all } s\in T \text{ and } n\in\mathbb{N}.$$

The last equivalence easily follows from the definition of negative realizability and the definition of η. □

5.2 From t-realizations to realizations

On the other hand, t-realizations for 'pure' independence-friendly first-order $\sigma_{\mathbb{N}}$-formulas under computable teams in $\mathfrak{N}$ can be split into ordinary realizations (viz., in the sense of Nelson) in a uniform effective way:

Theorem 5.2. *There exist computable functions* $\mathtt{r}^+$ *and* $\mathtt{r}^-$ *such that for every* $e \in \mathbb{N}$*, team* T *in* $\mathfrak{N}$*, implication-free first-order* $\sigma_{\mathbb{N}}$*-formula* ϕ *with* $FV(\phi) \subseteq dom(T)$*, and index* m *of* T*:*

$$e \boxed{\mathtt{P}} T, \phi \quad \Longleftrightarrow \quad \mu_{\mathtt{r}^+(e,m,\phi)}(s) \text{ Ⓟ } s, \phi \text{ for each } s \in T;$$
$$e \boxed{\mathtt{N}} T, \phi \quad \Longleftrightarrow \quad \mu_{\mathtt{r}^-(e,m,\phi)}(s) \text{ Ⓝ } s, \phi \text{ for each } s \in T.$$

In particular, if $T = \{s\}$*, then the collection of all positive* $\mathtt{t}$*-realizations of* ϕ *under* $\{s\}$ *is computably reducible to the collection of all positive realizations of* ϕ *under* s*, uniformly in* ϕ *and* s*, and similarly with 'negative' in place of 'positive'.*

Proof. Take γ to be a computable function such that for any $\{e, n\} \subseteq \mathbb{N}$,

$$\mu_{\gamma(e)}(n) = \begin{cases} 1 & \text{if } \mu_e(n) = 1, \\ 2 & \text{if } \mu_e(n) = 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Further — take ζ and ρ to be computable functions such that for every $e \in \mathbb{N}$, individual variable x and assignment s in $\mathfrak{N}$ with $\mathrm{dom}(s) \subseteq \mathcal{V} \setminus \{x\}$:

- $\mu_{\zeta(i,x,j)}(s) = [\mu_i(s), \mu_j(s(x/\mu_i(s)))]$;
- $\mu_{\mu_{\rho(e,x)}(s)}(n) = \mu_e(s(x/n))$.[21]

Now for any $\{e, m\} \subseteq \mathbb{N}$ and implication-free first-order $\sigma_{\mathbb{N}}$-formula ϕ, we inductively define

$$\mathtt{r}^+(e, m, \phi) \quad \text{and} \quad \mathtt{r}^-(e, m, \phi)$$

by the following conditions.

<u>`For` $\boxed{\mathtt{P}}$ `and` Ⓟ`:`</u>

- if ϕ is atomic, then $\mathtt{r}^+(e, m, \phi) = e$;
- if $\phi = \psi_1 \vee \psi_2$, then $\mathtt{r}^+(e, m, \phi)$ is

  $$\dot{\pi}\left(\gamma(e_1), \mathrm{glue}\left(\mathtt{r}^+(\pi_1(e_2), \mathrm{sub}_1(m, e_1), \psi_1), \mathtt{r}^+(\pi_2(e_2), \mathrm{sub}_2(m, e_1), \psi_2), e_1\right)\right)$$

  where e_1 and e_2 denote $\pi_1(e)$ and $\pi_2(e)$ respectively;
- if $\phi = \exists x\, \psi$, then $\mathtt{r}^+(e, m, \phi) = \zeta(e_1, x, \mathtt{r}^+(e_2, \mathrm{rev}_{\mathsf{E}}(m, x, e_1), \psi))$;
- if $\phi = \neg\psi$, then $\mathtt{r}^+(e, m, \phi) = \mathtt{r}^-(e, m, \psi)$.

<u>`For` $\boxed{\mathtt{N}}$ `and` Ⓝ`:`</u>

- if ϕ is atomic, then $\mathtt{r}^-(e, m, \phi) = e$;
- if $\phi = \psi_1 \vee \psi_2$, then $\mathtt{r}^-(e, m, \phi) = \dot{\pi}(\mathtt{r}^-(e_1, m, \psi_1), \mathtt{r}^-(e_2, m, \psi_2))$;
- if $\phi = \exists x\, \psi$, then $\mathtt{r}^-(e, m, \phi) = \rho(\mathtt{r}^-(e, \mathrm{rev}_{\mathsf{A}}(m, x), \psi), x)$;
- if $\phi = \neg\psi$, then $\mathtt{r}^-(e, m, \psi) = \mathtt{r}^+(e, m, \psi)$.

[21] To obtain ρ, use the *s-m-n*-theorem twice: first apply it to $\mu_e(s(x/n))$, regarded as a function of e, x, s and n, to get g such that $\mu_{g(e,x,s)}(n) = \mu_e(s(x/n))$; then apply it to g to get ρ such that $\mu_{\rho(e,x)}(s) = g(e, x, s)$.

It is straightforward to verify that the pair $\mathtt{r}^{+}$, $\mathtt{r}^{-}$ does the job. $\square$

We finish the section by noting that the computable reductions appearing in the statements of Theorems 5.1 and 5.2 readily establish the computable equivalences between the corresponding collections of $\mathtt{t}$-realizations/realizations. In this way, EGTS for IF-FOL suitably generalises EGTS for (the implication-free fragment of) FOL. As a bonus, these theorems show that $\mathtt{t}$-realizations for 'pure' independence-friendly $\sigma_{\mathbb{N}}$-formulas under computable teams in $\mathfrak{N}$ can in fact be split into $\mathtt{t}$-realizations under smaller teams, or put together if needed.[22]

6 Conclusion

To sum up, we have established the following:

- Nelson's realizability interpretation restricted to the implication-free fragment of FOL — a traditional constructive semantics for this fragment — can be viewed as an effective version of GTS for FOL.
- A constructive version of Hodges's semantics for IF-FOL — what we call the 'trump realizability interpretation' — can be developed. Its relationship to GTS for IF-FOL turns out to be the same as that of Nelson's restricted realizability to GTS for FOL.
- Actually, we can think of the trump realizability as a generalization of Nelson's restricted realizability.

Naturally, it would be interesting to continue the investigation of the trump realizability and its variations. Let us briefly mention some directions that might profitably be explored.

- We can look at how the set of trump realizations of ϕ under T varies as T varies, and try to computably predict it.
- We can compare the expressive power of the original IF-FOL and that of its effective version, and more generally, study the difference between them at the meta-level.
- We can consider certain modifications of the definition of the trump realizability interpretation, in which the class of all partial computable functions is replaced by a broader class, such as that of all arithmetical functions. In fact, the original formulation allows considerable freedom: we do not require T be computable, and our realizations are not augmented with 'self-verification procedures' (as in some advanced realizability intepretations).
- We can also try to expand our framework by adding intuitionistic implication (along with intuitionistic negation) — and we think that a suitable expansion of the trump realizability interpretation can indeed be defined.[23] However, in this case we will have to deal with higher-order functions, while strategies are intended to be first-order, so the tight connection between GTS and realizability will probably be lost.

All these fall beyond the scope of this paper, and are the subject of future work.

[22] In fact, they can be viewed as providing an effective version of what is knows as the *flatness property*, which states that for every team T in $\mathfrak{D}$ and every implication-free first-order σ-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(T)$,

$$\mathfrak{D}, T \models^{+}_{\mathtt{t}} \phi \quad \iff \quad \mathfrak{D}, s \models \phi \text{ for each } s \in T$$

(cf. [16, Section 3.4]); here 'flatness' derives from [8], where a 'flattening operation' was defined.

[23] Actually, a version of intuitionistic implication has been studied in the context of *dependence logic* (which is another approach to IF-FOL); cf. [1] and [20].

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References

[1] S. Abramsky and J. Väänänen (2009). From IF to BI: A tale of dependence and separation. *Synthese* 167(2), 207–230.

[2] J. van Benthem (2014). *Logic in Games.* MIT Press.

[3] M. Dummett (1978). *Truth and Other Enigmas.* Harvard University Press.

[4] M. Dummett (1993). *The Logical Basis of Metaphysics.* Harvard University Press.

[5] H. B. Enderton (1970). Finite partially-ordered quantifiers. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 16(8), 393–397. DOI: 10.1002/malq.19700160802

[6] J. Hintikka (1996). *The Principles of Mathematics Revisited.* Cambridge University Press. DOI: 10.1017/CBO9780511624919

[7] J. Hintikka and G. Sandu (1989). Informational dependence as a semantical phenomenon. In J. E. Fenstad et al (eds.) *Logic, Methodology, and Philosophy of Science VIII,* 571–589, Studies in Logic and the Foundations of Mathematics, North-Holland.

[8] W. Hodges (1997). Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL* 5(4), 539–563. DOI: 10.1093/jigpal/5.4.539

[9] W. Hodges (1997). Some strange quantifiers. In J. Mycielski et al. (eds.), *Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht*, 51–65. Springer. DOI: 10.1007/3-540-63246-8_4

[10] S.C. Kleene (1952). *Introduction to Metamathematics.* North-Holland Publishing Company.

[11] J. Kontinen, J. Väänänen and D. Westerståhl (eds.) (2013). Special Issue on Dependence and Independence in Logic, *Studia Logica* 101(2).

[12] A. L. Mann, G. Sandu and M. Sevenster (2011). *Independence-Friendly Logic: A Game-Theoretic Approach.* Cambridge University Press.

[13] Yu. V. Matiyasevich (1993). *Hilbert's Tenth Problem.* MIT Press, 1993.

[14] D. Nelson (1949). Constructible falsity. *Journal of Symbolic Logic* 14(1), 16–26. DOI: 10.2307/2268973

[15] M. J. Osborne and A. Rubinstein (1994). *A Course in Game Theory.* MIT Press.

[16] J. Väänänen (2007). *Dependence Logic: A New Approach to Independence Friendly Logic.* Cambridge University Press.

[17] H. Rogers, Jr. (1967). *Theory of Recursive Functions and Effective Computability.* McGraw-Hill Book Company.

[18] W. J. Walkoe (1970). Finite partially-ordered quantification. *Journal of Symbolic Logic* 35(4), 535–555. DOI: 10.2307/2271440

[19] L. Wittgenstein (1922). *Tractatus Logico-Philosophicus.* Routledge & Kegan Paul.

[20] F. Yang (2014). *On Extensions and Variants of Dependence Logic — A Study of Intuitionistic Connectives in the Team Semantics Setting* (PhD Thesis). University of Helsinki. Available online at https://helda.helsinki.fi/handle/10138/43011

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