The final publication is available at
http://link.springer.com/article/10.1007/s00153-013-0328-9

A note on definability in fragments of arithmetic with free unary predicates

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This is a version of the article published in
Archive for Mathematical Logic 52: 5–6, 507–516 (2013).

Abstract We carry out a study of definability issues in the standard models of Presburger and Skolem arithmetics (henceforth referred to simply as *Presburger* and *Skolem arithmetics*, for short, because we only deal with these models, not the theories, thus there is no risk of confusion) supplied with free unary predicates—which are strongly related to definability in the monadic SOA (second-order arithmetic) without $\times$ or $+$, respectively. As a consequence, we obtain a very direct proof for Π^1_1-completeness of Presburger, and also Skolem, arithmetic with a free unary predicate, generalize it to all Π^1_n-levels, and give an alternative description of the analytical hierarchy without $\times$ or $+$. Here 'direct' means that one explicitly m-reduces the truth of Π^1_1-formulae in SOA to the truth in the extended structures. Notice that for the case of Presburger arithmetic, the Π^1_1-completeness was already known, but the proof was indirect and exploited some special Π^1_1-completeness results on so-called recurrent nondeterministic Turing machines—for these reasons, it was hardly able to shed any light on definability issues or possible generalizations.

Keywords Definability · Expressiveness · Decidability · Computational complexity · Presburger arithmetic · Skolem arithmetic

Mathematics Subject Classification (2000) 03D35 · 03C40 · 03F30 · 03F35

1 Introduction

The article is devoted to expanding standard arithmetical structures (as well as their fragments) with free unary predicates, a topic which is considerably less developed in comparison with investigation of concrete arithmetical structures (see [2, 11] for a

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survey on the latter). At the same time, the above expansions naturally correspond to Π^1_1-theories and, therefore, it is reasonable to work in a more general setting, passing to reducts of the monadic SOA.

Among other things, we provide a very direct proof for Π^1_1-completeness of Presburger, and also Skolem, arithmetic (by our convention, these are the two standard models) with a free unary predicate, generalize these results to all Π^1_n-levels, and give an alternative description of the analytical hierarchy without $\times$ or $+$. Here 'direct' means that one explicitly m-reduces the truth of Π^1_1-formulae in SOA to the truth in the extended structures (with extra predicates).

Notice that, though Π^1_1-completeness in case of Presburger arithmetic was already known [9] (and appeared to be quite useful in establishing Π^1_1-completeness for various probabilistic languages [1, 16]), the existing proof was indirect and based on m-reducing a specific Π^1_1-complete problem that dealt with recurrent nondeterministic Turing machines [10] (an earlier undecidability result from [4] exploited Turing machines as well—Halting problem was involved). For these reasons, such a proof was too 'special-purpose' and hardly able to shed any light on definability issues or possible generalizations. On the contrary, our new proof is both direct and rather simple, and allows to derive corollaries on first and second order definability. It is not surprising that the 'new proof' essentially stands for a 'new (and deeper) understanding' which is achieved whenever the argument clarifies, becoming more convincing and easier to handle: in effect, the reasons leading to Π^1_1-completeness are quite natural. In addition, we get this (as well as all the other results) by means of classical approach to interpretability and without employing specific facts.

Let us explain briefly why we start with the standard arithmetical structures rather than with their theories (e. g., the classical axioms for Presburger arithmetic, etc). First of all, the standard models are the intended ones, the properties of which are of special interest. Since the theories of such models, as well as of natural classes of structures based on them, often turn out to be enormously complex, one passes to '(computably) axiomatisable approximations' in proof theory (say, Peano system PA approximates the theory of $\langle \mathbb{N}, +, \times \rangle$), even though the latter, in many cases, are incomplete and fail to establish the truth of some principal theorems (it eventually leads to introducing more and more powerful systems—see [15]). Still, investigating definability issues for the above models and classes, and also measuring the actual complexity of the corresponding theories are very important tasks in their own right (clearly, closely connected to each other). Second, the computational complexity for the 'expanded theories' appears to be not so interesting: trivially, if T is an axiomatisable theory in a signature σ, and T' is the collection of consequences of T in an extended decidable signature σ', then T' is axiomatisable, i. e., is at worst Σ^0_1 (typically, whenever T' is undecidable, it will be Σ^0_1-complete; there has been a lot of reseach on decidability for such theories—cf. [5], for instance).

The main emphasis of our work is on definability in the monadic SOA without $\times$ or $+$, respectively, and while some of the statements presented appear to be new and some are probably known (like Halpern's theorem) or even 'folklore' (for particular specialists), the advantage is that all of them are not only brought together and explicitly stated here but also obtained in a very uniform way, which leaves a room for further investigation.

The rest of the paper is organized as follows. Section 2 contains preliminary material on structures with free predicates and the monadic SOA. In Section 3 we study the monadic second-order expansion of Presburger arithmetic, and switch to that of Skolem arithmetic in Section 4. We conclude with a brief discussion of possible applications in Section 5.

2 Preliminaries

Suppose σ is a first-order signature, and $\mathfrak{A}$ is a σ-structure with domain A. Let

$$\overline{\sigma} \ := \ \sigma \cup \{U\}$$

where U stands for a fresh unary predicate symbol. Then $\mathfrak{A}$ *with a free unary predicate* is the class $\mathscr{K}$ of all $\overline{\sigma}$-structures $\mathfrak{B}$ of the form $\langle \mathfrak{A}, U^{\mathfrak{B}} \rangle$ with $U^{\mathfrak{B}} \subseteq A$. Similarly, $\mathfrak{A}$ *with n free predicates of arities* $i_1, \ldots, i_n$, for $\{n, i_1, \ldots, i_n\} \subset \{1, 2, \ldots\}$, are introduced. In what follows, 'definable', by default, stands for 'first-order definable', unless otherwise stated.

Assume $X_1, X_2, \ldots$ are one-place second-order variables (intuitively, they are intended to range over subsets of a given domain). Then, for $n \in \{1, 2, \ldots\}$, a *Π^1_n-σ-formula* is a second-order σ-formula of the sort

$$\forall X_1 \, \exists X_2 \ldots \varphi$$

with $n-1$ quantifier alternations and φ containing no second-order quantifiers. According to this presentation, all second-order variables are unary (hence we are working within the monadic setting), and alternating quantifiers are preferred in the definition of Π^1_n-σ-formulae, instead of blocks of one-type-quantifiers.

Obviously, if $\mathscr{K}$ is $\mathfrak{A}$ with a free unary predicate, for any $\overline{\sigma}$-sentence φ, we have

$$\mathscr{K} \vDash \varphi \quad \Longleftrightarrow \quad \mathfrak{A} \vDash \forall U \, \varphi \, ,$$

so $\mathrm{Th}\,(\mathscr{K})$ is m-equivalent to the collection of all Π^1_1-σ-sentences true in $\mathfrak{A}$ (for the definitions of m-equivalence, m-reducibility, etc., see [14]).

Henceforth by PrA (*Presburger arithmetic*) and SkA (*Skolem arithmetic*) we mean the structures $\langle \mathbb{N}, + \rangle$ and $\langle \mathbb{N}, \times \rangle$, respectively. Notice that the constants 0 and 1 are easily definable in both PrA and SkA, since

$$\begin{array}{lclcl} x = 0 & \Longleftrightarrow & \forall y \, (x + y = y) & \Longleftrightarrow & \forall y \, (x \times y = x) \, , \\ x = 1 & \Longleftrightarrow & \forall y \, (y = 0 \vee \exists z \, (y = x + z)) \wedge x \neq 0 & \Longleftrightarrow & \forall y \, (x \times y = y) \, ; \end{array}$$

and clearly, the relation $x < y$ is defined in PrA by $\exists z \, (x + z = y) \wedge x \neq y$. Here, we make use of the first-order logic with equality—the latter is always included, though omitted in descriptions of signatures (i.e., treated as a logical symbol).

Also, we employ the following notation:

$$\begin{aligned} \mathscr{K}_{+} \ &:= \ \mathsf{PrA} \ \text{with a free unary predicate} \, ; \\ \mathscr{K}_{\times} \ &:= \ \mathsf{SkA} \ \text{with a free unary predicate} \, . \end{aligned}$$

Remark that Π^1_n*-formulae* in SOA (cf. [15]) are precisely Π^1_n-$\{+,\times\}$-formulae, in our terminology. Further, Π^1_n*-sets* are those the characteristic function of which is definable (in $\langle\mathbb{N},+,\times\rangle$) by a Π^1_n-formula (with one free variable). A set is Π^1_n-*hard* iff every Π^1_n-set is m-reducible to it; a Π^1_n-set that turns out to be Π^1_n-hard is called Π^1_n*-complete*. Probably the most prominent example of a Π^1_n-complete set is the collection of all Π^1_n-$\{+,\times\}$-sentences true in the standard model $\langle\mathbb{N},+,\times\rangle$.

In effect, the analytical hierarchy of sets can be described in many different ways (cf. [14]). For instance, without loss of generality, we may consider only Π^1_n-formulae all of whose atomic subformulas are of the sorts

$$x = y, \quad x + y = z \quad \text{and} \quad x \times y = z$$

—henceforth called *special* (Π^1_n*-formulas*): each Π^1_n-formula is effectively transformed into an equivalent special Π^1_n-formula (by introducing new object variables). Say, if we have $x \times y + z = u$, then take a fresh variable v and pass to

$$\exists v\,(v = x \times y \wedge v + z = u)$$

(in fact, the given example illustrates the general method).

3 Revising PrA with unary predicates

For each σ-structure $\mathfrak{A}$, let $\mathrm{Def}\,(\mathfrak{A})$ be the collection of all predicates definable in $\mathfrak{A}$. As was previously noted in [12], we have

$$\mathrm{Def}\,(\mathbb{N},+,\times) \;=\; \mathrm{Def}\,(\mathbb{N},+,\mathrm{Sq}),$$

where $\mathrm{Sq} := \left\{n^2 \mid n \in \mathbb{N}\right\}$: the relation $y = x^2$ is definable in $\langle\mathbb{N},+,\mathrm{Sq}\rangle$ by

$$\mathrm{Sq}\,(y) \wedge \mathrm{Sq}\,(y + x + x + 1) \wedge \neg\exists z\,(\mathrm{Sq}\,(z) \wedge y < z < y + x + x + 1),$$

thus one can introduce $z = x \times y$ via the identity $x^2 + z + z + y^2 = (x + y)^2$. Assume $\Psi_\times$ is a $\{+,U\}$-formula that defines $\times$ in

$$\mathfrak{B} \;:=\; \langle\mathbb{N},+,U^{\mathfrak{B}}\rangle$$

with $U^{\mathfrak{B}} = \mathrm{Sq}$ (i. e., when U is interpreted as the set of all squares).

Proposition 1 *There exists a $\{+,U\}$-sentence Ψ s. t., for any $\mathfrak{B} \in \mathscr{K}_+$,*

$$\mathfrak{B} \vDash \Psi \quad \Longleftrightarrow \quad \Psi_\times \text{ defines multiplication in } \mathfrak{B}.$$

Proof The idea is simple: given $\mathfrak{B}$, if $U^{\mathfrak{B}} = \mathrm{Sq}$, then $\Psi_\times$ defines $\times$ in $\mathfrak{B}$, and we want to go in the opposite direction by writing down the inductive definition of $\times$ in terms of U and $+$. Let Ψ be the conjunction of the following $\{+,U\}$-sentences:

- $\forall x,y\,\exists z\,\Psi_\times\,(x,y,z) \wedge \forall x,y,z,u\,(\Psi_\times\,(x,y,z) \wedge \Psi_\times\,(x,y,u) \rightarrow z = u)$;
- $\forall x,y,z\,(\Psi_\times\,(x,y,z) \rightarrow \Psi_\times\,(y,x,z)) \wedge \forall x\,(\Psi_\times\,(x,0,0) \wedge \Psi_\times\,(x,1,x))$;
- $\forall x,y,z\,(\Psi_\times\,(x,y+1,z) \rightarrow \exists u\,(\Psi_\times\,(x,y,u) \wedge z = u + x))$.

The first condition ensures that $\Psi_\times$ defines a total function f on $|\mathfrak{B}| = \mathbb{N}$. The next condition asserts that $f(x,y) = f(y,x)$ (i. e., f is commutative), $f(x,0) = 0$ and $f(x,1) = x$ (these two provide a basis for induction). Finally, the last condition guarantees the equality $f(x,y+1) = f(x,y) + x$, which is just an inductive step.

The rest is straightforward (by external induction on y). $\square$

Remark that the proof of Proposition 1 also goes through if we take an arbitrary $\{+,U\}$-formula Θ that defines $\times$ in some $\mathfrak{B} \in \mathscr{K}_+$ instead of $\Psi_\times$ itself.

Proposition 2 *There is an effective translation τ transforming each Π^1_n-$\{+,\times\}$-formula*

$$\phi \; = \; \phi(x_1,\dots,x_m,X_1,\dots,X_k)$$

into a Π^1_n-$\{+\}$-formula

$$\tau\phi \; = \; \tau\phi(x_1,\dots,x_m,X_1,\dots,X_k)$$

s. t., for all $\{i_1,\dots,i_m\} \subset \mathbb{N}$ and $\{P_1,\dots,P_k\} \subset 2^{\mathbb{N}}$, we have

$$\langle \mathbb{N},+,\times\rangle \vDash \phi(i_1,\dots,i_m,P_1,\dots,P_k) \quad\Longleftrightarrow\quad \langle \mathbb{N},+\rangle \vDash \tau\phi(i_1,\dots,i_m,P_1,\dots,P_k)\,.$$

Proof Obviously, without loss of generality, we can restrict our attention to special Π^1_n-$\{+,\times\}$-formulae only. Then, the crucial case is that of Π^1_1-$\{+,\times\}$-formulae.

Suppose $\Phi_\times$ is obtained by simultaneously replacing every atom of the sort $U(t)$ in $\Psi_\times$ with $U(2t+1)$. Clearly, if $\mathfrak{B} \in \mathscr{K}_+$ and $\left\{i \mid 2i+1 \in U^{\mathfrak{B}}\right\} = \mathrm{Sp}$, then $\Phi_\times$ defines $\times$ in $\mathfrak{B}$. Hence the variant of Proposition 1 holds, namely there is a $\overline{\sigma}_+$-sentence Φ s. t., for any $\mathfrak{B} \in \mathscr{K}_+$,

$$\mathfrak{B} \vDash \Phi \quad\Longleftrightarrow\quad \Phi_\times \text{ defines multiplication in } \mathfrak{B}$$

(to get Φ, simply take $\Phi_\times$ instead of $\Psi_\times$ in the conjunction from the above proof). Now the advantage is that $U(2t)$ is left undefined in structures satisfying Φ (obviously, it won't be affected by the requirements on $\Phi_\times$) and so can play the role of a new uninterpreted predicate. Consider an arbitrary special Π^1_1-$\{+,\times\}$-formula

$$\phi \; = \; \forall X\, \varphi$$

with free variables $x_1,\dots,x_m,X_1,\dots,X_k$, and let $\hat{\varphi}$ be the result of replacing

- each atom of the sort $x \times y = z$ in φ with $\Phi_\times(x,y,z)$;
- each occurrence of the form $X(t)$ in φ with $U(2t)$.

Next, it's not hard to check the equivalence

$$\langle \mathbb{N},+,\times\rangle \vDash \phi \quad\Longleftrightarrow\quad \langle \mathbb{N},+\rangle \vDash \forall U\,(\Phi \to \hat{\varphi})$$

for all possible choices of $x_1,\dots,x_m,X_1,\dots,X_k$. Thus, one can take

$$\tau\phi \; := \; \forall U\,(\Phi \to \hat{\varphi})\,,$$

for any special Π^1_1-$\{+,\times\}$-sentence ϕ. For $n \in \{2,3,\dots\}$, the proof is analogous: just use the first of the second-order universal quantifiers (in place of $\forall X$) in the foregoing argument. $\square$

Since $\mathrm{Th}(\mathscr{K}_+)$ is m-equivalent to the collection of all true Π^1_1-$\{+\}$-sentences, Proposition 2 trivially implies Π^1_1-completeness of (the theory of) $\mathscr{K}_+$ (previously established in [9]). What is more interesting is that we readily obtain a generalization of this result to all Π^1_n-levels and a new characterization of the analytical hierarchy without $\times$ (these, of course, cannot be extracted from Halpern's proof).

Corollary 1 *1. The set of all Π^1_n-$\{+\}$-sentences true in $\langle \mathbb{N}, +\rangle$ is Π^1_n-complete.*
2. Each Π^1_n-set is definable by a Π^1_n-$\{+\}$-formula.

4 Passing to SkA with unary predicates

We begin with an analogue of Proposition 2.

Proposition 3 *There is an effective translation ρ transforming each Π^1_n-$\{+\}$-formula*

$$\phi \;=\; \phi\,(x_1,\ldots,x_m,X_1,\ldots,X_k)$$

into a Π^1_n-$\{\times\}$-formula

$$\rho\phi \;=\; \rho\phi\,(z,x_1,\ldots,x_m,X_1,\ldots,X_k)$$

s. t., for all prime $p \in \mathbb{N}$, $\{i_1,\ldots,i_m\} \subset \mathbb{N}$ and $\{P_1,\ldots,P_k\} \subset 2^{\mathbb{N}}$, we have

$$\langle \mathbb{N},+\rangle \vDash \phi\,(i_1,\ldots,i_m,P_1,\ldots,P_k) \quad\Longleftrightarrow\quad \langle \mathbb{N},\times\rangle \vDash \rho\phi\left(p,p^{i_1},\ldots,p^{i_m},p^{P_1},\ldots,p^{P_k}\right),$$

where $p^{P_j} := \left\{p^i \mid i \in P_j\right\}$ for any $j \in \{1,\ldots,k\}$.

Proof Obviously, without loss of generality, we can restrict our attention to Π^1_n-$\{+\}$-formulae not containing z. Then, again, the crucial case is that of Π^1_1-$\{+\}$-formulae.

The divisibility relation is given by the $\{\times\}$-formula

$$x \mid y \;:=\; \exists z\,(x \times z = y)\,,$$

i. e., x divides y (without remainder). Trivially, the primality is then definable in SkA by the $\{\times\}$-formula

$$\mathrm{Prim}\,(z) \;:=\; \forall x\,(x \mid z \to (x = 1 \vee x = z)) \wedge x \neq 1\,.$$

Next, we set

$$\mathrm{Deg}\,(y,z) \;:=\; \forall x\,(x \mid y \to (x = 1 \vee z \mid x))$$

which holds for the standard interpretation precisely if $y = z^i$ for some $i \in \mathbb{N}$ (assuming z is a prime, of course). The following is based on a simple observation that

$$\langle \{z^n \mid n \in \mathbb{N}\}\,,\times\rangle \text{ is isomorphic to } \langle \mathbb{N},+\rangle\,.$$

Consider an arbitrary Π^1_1-$\{+\}$-formula

$$\phi \;=\; \forall U\,\varphi$$

with free variables $x_1,\ldots,x_m,X_1,\ldots,X_k$ (and not containing z), and let $\tilde{\varphi}$ be the result of

– replacing each term of the sort $t_1 + t_2$ with $t_1 \times t_2$;
– relativizing all the quantifiers in φ to $\mathrm{Deg}\,(\cdot, z)$.

Now it is easy to see that we can take

$$\rho\phi \;:=\; \forall U\, \tilde{\varphi}\,,$$

for any Π^1_1-$\{+\}$-sentence ϕ (not containing z). For $n \in \{2, 3, \dots\}$, one may use the first of the second-order $\forall$-quantifiers (instead of $\forall U$) in the above argument. □

Notice that $\mathrm{Th}\,(\mathscr{K}_\times)$ is easily seen to be hereditary undecidable by the argument from [7]. But Proposition 3 actually yields much more.

Proposition 4 *The set* $\mathrm{Th}\,(\mathscr{K}_\times)$ *is* Π^1_1*-complete.*

Proof Trivially, $\mathrm{Th}\,(\mathscr{K}_\times)$ is a Π^1_1-set. Let us show its Π^1_1-hardness.

From the proof of Proposition 3, for any prime $p \in \mathbb{N}$ and Π^1_1-$\{+\}$-sentence $\phi = \forall U\, \varphi$, we immediately get

$$\langle \mathbb{N}, + \rangle \vDash \phi \quad\Longleftrightarrow\quad \langle \mathbb{N}, \times \rangle \vDash \forall U\, \tilde{\varphi}\,[z/p]\,,$$

i. e.,

$$\mathscr{K}_+ \vDash \varphi \quad\Longleftrightarrow\quad \mathscr{K}_\times \vDash \tilde{\varphi}\,[z/p]\,.$$

It remains to note that this doesn't depend on the choice of p, whence the above two are both equivalent to

$$\mathscr{K}_\times \vDash \forall z\,(\mathrm{Prim}\,(z) \to \tilde{\varphi})\,.$$

Thus, we have just m-reduced $\mathrm{Th}\,(\mathscr{K}_+)$, which is Π^1_1-complete, to $\mathrm{Th}\,(\mathscr{K}_\times)$. □

Recall the role of the $\{+, U\}$-sentences Ψ and Φ in the results of Section 3: now, their $\{\times, U\}$-translations $\tilde{\Psi}$ and $\tilde{\Phi}$ play a similar role in investigating $\mathscr{K}_\times$. More precisely, from Proposition 4 and the proof of Proposition 3 we (by analogy with PrA) obtain

Corollary 2 *1. The set of all* Π^1_n*-*$\{\times\}$*-sentences true in* $\langle \mathbb{N}, \times \rangle$ *is* Π^1_n*-complete.*
2. *Let* $\mathfrak{B} \in \mathscr{K}_\times$ *and* $\mathfrak{B} \vDash \tilde{\Psi}$*. For any prime* $p \in \mathbb{N}$ *and any arithmetical set* $P \subseteq \mathbb{N}$*, the set* $\{p^i \mid i \in P\}$ *is first-order definable with parameter* p *in* $\mathfrak{B}$*.*
3. *For any prime* $p \in \mathbb{N}$ *and any* Π^1_n*-set* P*, the set* $\{p^i \mid i \in P\}$ *is definable with parameter* p *by a* Π^1_n*-*$\{\times\}$*-formula.*

Naturally, the question appears:

Can we sharpen this representation getting rid of 'modulo $n \mapsto p^n$'
(like it was in the case of Presburger arithmetic)?

Let us briefly explain why such a strengthening looks problematic. We start with the observation that

$$\mathrm{Def}\,(\mathbb{N}, +, \times) \;=\; \mathrm{Def}\,(\mathbb{N}, \times, \mathrm{Fac})\,,$$

where $\mathrm{Fac} := \{n! \mid n \in \mathbb{N}\}$, i. e., the set of all factorials (cf. Item 4(b)9 on p. 125 in [11]). Thus, there exists $\mathfrak{B} := \langle \mathbb{N}, \times, U^{\mathfrak{B}} \rangle$ from $\mathscr{K}_\times$ (take $U^{\mathfrak{B}} := \mathrm{Fac}$) s. t. all arithmetical predicates (and functions) are definable in $\mathfrak{B}$. For the modification of the proof of Proposition 1 to go through, we need to find an arithmetical predicate P (of any arity) satisfying the following two conditions:

– addition $+$ is definable in $\langle \mathbb{N}, \times, P \rangle$;
– P can be rigorously described in terms of $\times$ (in a way similar to that of describing $\times$ in terms of $+$ in the proof of Proposition 1).

For the moment, it is not clear whether the second requirement can be fulfilled for a suitable P. As a consequence, from the definability perspective, PrA (with unary predicates) seems to be more adequate than SkA (with unary predicates).

5 Concluding remarks

Here, we give an interesting example of an application of the technique that was employed in Sections 3–4, and also state an open problem.

We start with a simpler result (and then will pass to its generalization):

$$\text{there is no predicate } P \subseteq \mathbb{N} \text{ s.t. } + \text{ is definable in } \langle \mathbb{N}, s, P \rangle ,$$

where s is successor function. Indeed, if such a predicate P exists, then, taking Ψ_+ to be a $\{s, U\}$-formula that defines $+$ in the corresponding model $\mathfrak{B} = \langle \mathbb{N}, s, U^{\mathfrak{B}} \rangle$ with $U^{\mathfrak{B}} = P$, one can inductively describe $+$ via Ψ_+ in a way similar to that of describing $\times$ in terms of $\Psi_\times$ (cf. the proof of Proposition 1)—exploit the identities

$$x + 0 \;=\; x, \quad x + 1 \;=\; s(x) \quad \text{and} \quad x + s(y) \;=\; s(x + y) .$$

Due to Π^1_1-completeness of $\mathrm{Th}(\mathscr{K}_+)$, this eventually entails Π^1_1-completeness for $\langle \mathbb{N}, s \rangle$ with two free unary predicates—which contradicts to the decidability of the monadic second-order theory of $\langle \mathbb{N}, s \rangle$ (cf. [8]). Slightly modifying the argument leads to a more general and interesting fact:

$$\text{there are no predicates } \{P_1, \ldots, P_n\} \subseteq 2^{\mathbb{N}} \text{ s.t. } + \text{ is definable}$$
$$\text{in } \langle \mathbb{N}, s, P_1, \ldots, P_n \rangle \text{ by a monadic second-order formula.}$$

(The simpler result can be established using Gaifman locality theorem from [6], and the stronger version, in effect, can be derived by means of automata but the proof is clearly non trivial.) Note that s is first-order definable in $\langle \mathbb{N}, \leqslant \rangle$, while $\leqslant$ is definable in $\langle \mathbb{N}, s \rangle$ by a monadic second-order formula, hence we may easily switch from s to $\leqslant$ here. At the same time, these facts are in sharp contrast to the following:

$$\langle \mathbb{N}, s \rangle \text{ with two free predicates of arities 1 and 2 is } \Pi^1_1\text{-complete.}$$

The reason is: $\mathrm{Def}(\mathbb{N}, +, \times)$ coincides with $\mathrm{Def}(\mathbb{N}, s, |)$ [13], so let B be a binary predicate symbol and Ψ_+ be a $\{s, B\}$-formula that defines $+$ in $\mathfrak{B} = \langle \mathbb{N}, s, B^{\mathfrak{B}} \rangle$ with $B^{\mathfrak{B}}$ being the divisibility relation (at this step, you may even go without s, because $\mathrm{Def}(\mathbb{N}, +, \times) = \mathrm{Def}(\mathbb{N}, P)$ for an appropriate $P \subset \mathbb{N}^2$—see Item 1(c)9 on p. 123 in [11]), then employ the above scheme (inductively describe $+$ via Ψ_+, etc). Whence we get yet another alternative representation of the analytical hierarchy. Certainly, other applications of the technique to definability issues (in arithmetical structures) are of interest as well.

An open problem that arises naturally in view of results of Section 4 is:

Can we obtain Π^1_1-completeness for $\langle \mathbb{N}, | \rangle$ with a free unary predicate?

(the divisibility relation | have already played an important part in investigating different fragments of arithmetic—cf. Sections 3–4 in [2], and the paper [3]). For now, it seems that the answer is 'no'—but this is just a personal conjecture. In any case, it will be nice to characterize the precise computational complexity (that is, the *m*-degree) of (the theory of) $\langle \mathbb{N}, | \rangle$ with a free unary predicate.

Acknowledgements I would like to thank Alexis Bès for useful comments and some extra references.

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