A note on hereditarily Π^0_1- and Σ^0_1-complete sets of sentences

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Abstract. Many important achievements of formal logic have been concerned with the discovery of incomputability — and thus firmly rooted in the undecidability of the halting problem and its complement. Also, the latter produce influental examples of Σ^0_1- and Π^0_1-complete sets, in modern terminology. Changing the focus from modelling computations to measuring complexity of theories, the paper describes how to obtain Σ^0_1- and Π^0_1-completeness results for a wide range of fragments of theories in a very uniform way, and the reasoning will employ the following concepts. Let σ be a first-order signature and Val_σ the collection of all valid σ-sentences. For $\mathrm{C} \in \left\{\Pi^0_1, \Sigma^0_1\right\}$, call a set Γ of σ-sentences *hereditarily* C*-complete* iff for any C-set Δ, whenever $Val_\sigma \cap \Gamma \subseteq \Delta \subseteq \Gamma$, then Δ is C-complete. Both notions are closely connected with that of being hereditarily undecidable, but unlike their common predecessor, serve the purpose of getting computational complexity results, via employing the two most fundamental levels of the arithmetical hierarchy. This paper presents major tools and main examples in the study of hereditary Π^0_1- and Σ^0_1-completeness, with a discussion of various applications.

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1. Introduction and motivation

Many important achievements of formal logic have been concerned with the discovery of incomputability, and with distinguishing its different kinds, viz.

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what become known as degrees of unsolvability/undecidability (readers may consult [7, 10], for instance). The traces clearly go back to the halting problem and its complement which, in turn, give prominent examples of Σ^0_1- and Π^0_1-complete sets — and so correspond to the two most understood 'undecidable' (associated to incomputable sets) levels of the arithmetical hierarchy. The paper is devoted to revising the classical technique of elementary definability — which has played an important part in establishing undecidability results — for getting computational Π^0_1- and Σ^0_1-compexity results in a very uniform way. Here it is helpful to recall some terminology.

Let A and B be subsets of the collection $\mathbb{N}$ of all natural numbers. We say that A is *effectively reducible* — or *many-one reducible* — to B iff there exists a (total) computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$n \in A \iff f(n) \in B.$$

Further, A and B are *effectively equivalent* — or *many-one equivalent* — iff they are effectively reducible to each other. Now we define:

$$A \text{ is } \Sigma^0_1\textit{-bounded} \text{ (a } \Sigma^0_1\textit{-set}) \quad \text{iff} \quad A \text{ is computably enumerable;}$$
$$A \text{ is } \Pi^0_1\textit{-bounded} \text{ (a } \Pi^0_1\textit{-set}) \quad \text{iff} \quad \overline{A} \text{ is computably enumerable}$$

where $\overline{A}$ denotes $\mathbb{N} \setminus A$, viz. the complement in $\mathbb{N}$ of A. For $\mathrm{C} \in \{\Pi^0_1, \Sigma^0_1\}$, a C-bounded A is called C-*complete* iff every C-bounded B is effectively reducible to A — in a sense, A shall be 'the hardest among the C-bounded sets'. Viewing programs for one-place partial computable functions (say, in terms of Turing machines) as natural numbers in a reasonable manner, consider

$$\mathtt{Halt} := \{n \in \mathbb{N} \mid \text{the program coded by } n \text{ halts on input zero}\},$$

i. e., *the halting problem.* As is well-known, we have:

$$A \text{ is } \Sigma^0_1\textit{-complete} \quad \text{iff} \quad A \text{ is effectively equivalent to } \mathtt{Halt};$$
$$A \text{ is } \Pi^0_1\textit{-complete} \quad \text{iff} \quad \overline{A} \text{ is effectively equivalent to } \overline{\mathtt{Halt}}.$$

As we shall see, such sets arise very naturally in the investigation of undecidable theories — for this purpose, we introduce the notions of being hereditarily Π^0_1- and Σ^0_1-complete, which are central to our exposition, and can be viewed as rooted in the standard concept of effective inseparability. Further, the two are closely related to the familiar notion of being hereditarily undecidable, which can be described in terms of a significantly weaker (but more popular) concept of computable inseparability. And so switching to effective inseparability — whose role in elementary theories seems underestimated — reflects a shift of interest from undecidability to complexity issues.

Some old methods of the area have been modified and adapted to deal with hereditary Π^0_1- and Σ^0_1-completeness, and it should be no surprise that some results may be intuitively 'expected' by particular specialists. The aim is not to derive unexpected new results — but to provide a uniform method for obtaining (hereditary) Π^0_1- and Σ^0_1-completeness results, and so fill a gap between 'intuitively expected ... ' and 'formally proved'. For practically any undecidability proof, the most involved part is to 'invent' an effective translation reducing some known undecidable problem to the one in question. At the same time, the main advantage of the proposed method is that it incorporates the possibility of employing translations from many works on undecidable (fragments of) theories — since the latter often use (variants of) the technique of elementary definability — and it gives us a way to get new Π^0_1- and Σ^0_1-completeness results, and avoid the above-mentioned 'most involved part'. Hopefully, the old proofs can start a new life in this context.

Still, the subject is delicate and should be treated formally — thus the reader may need some time to get used to the machinery we are exploiting, even when the arguments are more or less straightforward. For instance, the idea of interpretability of one class in another is quite simple but its formal analogue — i. e., the notion of elementary definability — is rather technical. The exposition assumes a familiarity with basic computability-theoretic apparatus (e. g., see [10]); of course, a knowledge of the first-order definability technique and related issues (cf. [3]) will also be helpful.

The remainder of this paper is organised as follows. Section 2 provides a review of the formal machinery underlying the approach to hereditary undecidability of first-order theories and their prefix fragments via elementary definability. Section 3 presents major tools and main examples in the study of hereditary Π^0_1- and Σ^0_1-completeness, including also many applications to fragments of first-order theories. We end up with a few words about further applications to languages other than first-order ones.

2. Preliminaries on elementary definability

Let σ be a (first-order) signature. A piece of notation:

$$\begin{aligned} K_\sigma &:= \text{the class of all } \sigma\text{-structures;} \\ K^\circ_\sigma &:= \text{the class of all finite } \sigma\text{-structures;} \\ |\mathfrak{A}| &:= \text{the domain of } \mathfrak{A} \in K_\sigma; \\ Sen_\sigma &:= \text{the set of all } \sigma\text{-sentences;} \\ Val_\sigma &:= \text{the set of all valid } \sigma\text{-sentences.} \end{aligned}$$

Also, for any $n \in \mathbb{N}$ and $\Gamma \subseteq Sen_\sigma$, we write

$$\Sigma_n\text{–}\Gamma \;:=\; \{\Phi \in \Gamma \mid \Phi \text{ is a } \Sigma_n\text{-sentence}\},$$

and similarly with Π_n–Γ. And for ease of reading, we henceforth assume all signatures to be relational and finite (but note that essentially the same will hold with 'decidable' in place of 'relational and finite').

The technique employed deals with prefix fragments, and thus involves a portion of the terminology appearing in [9] — which is a variation on the earlier formalisations (cf. [5, 7]), and its traces go back to [13]. By a *Σ_k-σ_2-scheme $\mathcal{S}$ in σ_1* we mean a list of Σ_k-σ_2-formulas, namely

- $\Phi_U\,(x, \overline{y})$;
- $\Phi_R\,(x_1, \ldots, x_n, \overline{y})$, $\Phi_{\neg R}\,(x_1, \ldots, x_n, \overline{y})$ for each n-ary symbol R in σ_1;

- $\Phi_{=}(x_1, x_2, \overline{y})$ when the equality symbol $=$ does not belong to σ_1.

Now $\mathcal{K}_1 \subseteq K_{\sigma_1}$ is said to be Σ_k-*elementarily definable with parameters* (Σ_k-*e. d. p.* for short) in $\mathcal{K}_2 \subseteq K_{\sigma_2}$ iff some Σ_k-σ_2-scheme $\mathcal{S}$ in σ_1 fulfills the condition: for any $\mathfrak{A} \in \mathcal{K}_1$, there exist $\mathfrak{B} \in \mathcal{K}_2$ and a tuple $\overline{p}$ in $|\mathfrak{B}|$ such that

1. the set $B := \{b \mid b \in |\mathfrak{B}| \text{ and } \mathfrak{B} \vDash \Phi_U(b, \overline{p})\}$ is non-empty;
2. for every $R \in \sigma_1$, the n-ary relation
$$\{(b_1, \dots, b_n) \mid \{b_1, \dots, b_n\} \subseteq B \text{ and } \mathfrak{B} \vDash \Phi_R(b_1, \dots, b_n, \overline{p})\}$$
is the complement in B^n of the n-ary relation
$$\{(b_1, \dots, b_n) \mid \{b_1, \dots, b_n\} \subseteq B \text{ and } \mathfrak{B} \vDash \Phi_{\neg R}(b_1, \dots, b_n, \overline{p})\};$$
3. the set of pairs $E := \{(b_1, b_2) \mid \{b_1, b_2\} \subseteq B \text{ and } \mathfrak{B} \vDash \Phi_{=}(b_1, b_2, \overline{p})\}$ is a congruence relation on the σ_1-structure $\mathfrak{B}'$ with domain B, where each $R \in \sigma_1$ is interpreted via
$$\mathfrak{B}' \vDash R(b_1, \dots, b_n, \overline{p}) \quad \Longleftrightarrow \quad \mathfrak{B} \vDash \Phi_R(b_1, \dots, b_n, \overline{p});$$
4. the quotient structure $\mathfrak{B}'/_E$ is isomorphic to $\mathfrak{A}$.

Whenever $\overline{y}$ in the scheme is empty — and therefore $\overline{p}$ becomes empty, too — we omit the phrase 'with parameters'. In the present context, 'elementarily definable (with parameters)' would stand for 'Σ_k-elementarily definable (with parameters) for some k'. Remark: this description can be easily generalised to allow for the possibility of viewing elements of $\mathfrak{A} \in \mathcal{K}_1$ as l-tuples of elements of $\mathfrak{B} \in \mathcal{K}_2$ (cf. [5, pp. 271–272]) and it will not affect the results below, but the given version demonstrates the idea a bit more explicitly.

Suppose $\mathcal{K}_1 \subseteq K_{\sigma_1}$ is e. d. p. (or e. d.) in $\mathcal{K}_2 \subseteq K_{\sigma_2}$. So — making use of a suitable σ_2-scheme $\mathcal{S}$ in σ_1 — we can construct an effective translation τ which transforms every σ_1-sentence Φ into a σ_2-sentence $\tau(\Phi)$ such that

$$\begin{aligned} \Phi \in Val_{\sigma_1} &\implies \tau(\Phi) \in Val_{\sigma_2}, \\ \Phi \in \mathrm{Th}(\mathcal{K}_1^*) &\iff \tau(\Phi) \in \mathrm{Th}(\mathcal{K}_2) \end{aligned}$$

where $\mathcal{K}_1^* := \{\mathfrak{A} \in K_{\sigma_1} \mid \mathfrak{A} \text{ satisfies Items 1–4 for } \mathcal{S}, \text{ with some } \mathfrak{B} \in \mathcal{K}_2 \text{ and } \overline{p} \text{ in } |\mathfrak{B}|\}$ (see [5, pp. 272–273] for details). Thus

$$\tau(Val_{\sigma_1}) \subseteq Val_{\sigma_2}, \quad \mathcal{K}_1 \subseteq \mathcal{K}_1^*, \quad \text{and}$$
$$\tau(Sen_{\sigma_1} \setminus \mathrm{Th}(\mathcal{K}_1)) \subseteq \tau(Sen_{\sigma_1} \setminus \mathrm{Th}(\mathcal{K}_1^*)) \subseteq Sen_{\sigma_2} \setminus \mathrm{Th}(\mathcal{K}_2).$$

Moreover, in case $\mathcal{K}_1$ is Σ_k-e. d. p. in $\mathcal{K}_2$, a simple analysis shows — cf. the proof of [9, Lemma 3.1] — how to choose τ so that for any $\Phi \in Sen_{\sigma_1}$ and $r \in \{2, 3, \dots\}$, we have

$$\Phi \in \Pi_{r+1}\text{–}Sen_{\sigma_1} \iff \tau(\Phi) \in \Pi_{r+k}\text{–}Sen_{\sigma_2},$$

whence

$$\begin{aligned} \tau(\Pi_{r+1}\text{–}Val_{\sigma_1}) &\subseteq \Pi_{r+k}\text{–}Val_{\sigma_2}, \\ \tau(Sen_{\sigma_1} \setminus \Pi_{r+1}\text{–}\mathrm{Th}(\mathcal{K}_1)) &\subseteq Sen_{\sigma_2} \setminus \Pi_{r+k}\text{–}\mathrm{Th}(\mathcal{K}_2). \end{aligned}$$

Similarly, if $\mathcal{K}_1 \subseteq K_{\sigma_1}$ is Σ_k-e. d. in $\mathcal{K}_2 \subseteq K_{\sigma_2}$ (without parameters), we get τ for which

$$\Phi \in \Sigma_r\text{–}Sen_{\sigma_1} \quad \Longleftrightarrow \quad \tau(\Phi) \in \Sigma_{r+k-1}\text{–}Sen_{\sigma_2},$$

whence

$$\tau(\Sigma_r\text{–}Val_{\sigma_1}) \subseteq \Sigma_{r+k-1}\text{–}Val_{\sigma_2},$$
$$\tau(Sen_{\sigma_1} \setminus \Sigma_r\text{–Th}(\mathcal{K}_1)) \subseteq Sen_{\sigma_2} \setminus \Sigma_{r+k-1}\text{–Th}(\mathcal{K}_2).$$

Further, the technique just described can be readily applied to investigate issues of decidability in first-order logic. Call two disjoint sets A and B of natural numbers *computably inseparable* (or *c. i.* for short) iff there exists no computable set C satisfying $A \subseteq C \subseteq \overline{B}$ — e.g., see [10, p. 93]. A well-known and quite easily verified property is that for each (total) computable function $f : \mathbb{N} \to \mathbb{N}$,

$$\left.\begin{array}{r} A \text{ and } B \text{ are c. i., } C \cap D = \varnothing, \\ f(A) \subseteq C \text{ and } f(B) \subseteq D \end{array}\right\} \quad \Longrightarrow \quad C \text{ and } D \text{ are c. i.}$$

As usual, we identify σ-sentences with natural numbers by means of an appropriate Gödel numbering. Remark that the next notion makes sense both for theories and their fragments: a set Γ of σ-sentences is said to be *hereditarily undecidable* (*h. u.* for short) iff for every Δ,

$$Val_\sigma \cap \Gamma \subseteq \Delta \subseteq \Gamma \quad \Longrightarrow \quad \Delta \text{ is undecidable},$$

i. e., $Val_\sigma \cap \Gamma$ and $Sen_\sigma \setminus \Gamma$ are computably inseparable. Now the foregoing observations about τ suggest a uniform way to transfer hereditary undecidability (cf. [9, Lemma 3.1]): for any $r \in \{2, 3, \dots\}$,

$$\left.\begin{array}{r} \mathcal{K}_1 \text{ is } \Sigma_k\text{-e. d. p. in } \mathcal{K}_2, \\ \Pi_{r+1}\text{–Th}(\mathcal{K}_1) \text{ is h. u.} \end{array}\right\} \quad \Longrightarrow \quad \Pi_{r+k}\text{–Th}(\mathcal{K}_2) \text{ is h. u.;}$$
$$\left.\begin{array}{r} \mathcal{K}_1 \text{ is } \Sigma_k\text{-e. d. in } \mathcal{K}_2, \\ \Sigma_r\text{–Th}(\mathcal{K}_1) \text{ is h. u.} \end{array}\right\} \quad \Longrightarrow \quad \Sigma_{r+k-1}\text{–Th}(\mathcal{K}_2) \text{ is h. u.}$$

Indeed, in the first case, the disjoint sets

$$Val_{\sigma_1} \cap \Pi_{r+1}\text{–Th}(\mathcal{K}_1) = \Pi_{r+1}\text{–}Val_{\sigma_1} \quad \text{and} \quad Sen_{\sigma_1} \setminus \Pi_{r+1}\text{–Th}(\mathcal{K}_1)$$

are c. i. Hence, taking

$$f := \tau, \quad A := \Pi_{r+1}\text{–}Val_{\sigma_1}, \quad B := Sen_{\sigma_1} \setminus \Pi_{r+1}\text{–Th}(\mathcal{K}_1),$$
$$C := \Pi_{r+k}\text{–}Val_{\sigma_2}, \quad D := Sen_{\sigma_2} \setminus \Pi_{r+k}\text{–Th}(\mathcal{K}_2),$$

with τ a suitable translation, we conclude that

$$\Pi_{r+k}\text{–}Val_{\sigma_2} = Val_{\sigma_2} \cap \Pi_{r+k}\text{–Th}(\mathcal{K}_2) \quad \text{and} \quad Sen_{\sigma_2} \setminus \Pi_{r+k}\text{–Th}(\mathcal{K}_2)$$

are c. i., as desired. In the second case, an analogous argument suffices.

Of course, the elementary definability approach has already played an important role in establishing hereditary undecidability (e. g., see [4, 5, 9]) — but here what we are concerned with is to characterise Π_1^0- and Σ_1^0-complete problems arising in the above framework.

3. Hereditarily Π^0_1- and Σ^0_1-complete fragments of theories

The underlying idea for proving hereditary undecidability is to view certain collections of sentences as computably inseparable sets. Still, to provide Π^0_1- and Σ^0_1-complexity results for various theories and their fragments, we shall employ a significantly stronger notion of being 'effectively inseparable'.

Let $\nu : n \mapsto W_n$ (with n ranging over $\mathbb{N}$) be the standard numbering of the family of all Σ^0_1-bounded subsets of $\mathbb{N}$ — viz. W_n denotes the domain of the partial computable function $æ_n : \subseteq \mathbb{N} \to \mathbb{N}$ whose program is coded by n. Call two disjoint sets A and B of natural numbers *effectively inseparable* (*e. i.* for short) iff there exists a (total) computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any $\{n, k\} \subset \mathbb{N}$,

$$A \subseteq W_n, \quad B \subseteq W_k, \quad W_n \cap W_k = \varnothing \quad \Longrightarrow \quad f(n,k) \notin W_n \cup W_k$$

— e. g., see [1, p. 37]. Actually, this precise description won't be crucial, but we do need the five properties:

(i) A and B are e. i. iff B and A are so;
(ii) if A and B are e. i., then A and B are c. i.;
(iii) if $A \subseteq C \subseteq \overline{B}$, A and B are disjoint and e. i., then B and C are e. i.;
(iv) if A and B are disjoint and Σ^0_1-bounded e. i. sets, then each of them is Σ^0_1-complete;
(v) for every computable function $f : \mathbb{N} \to \mathbb{N}$,

$$\left.\begin{array}{c} A \text{ and } B \text{ are e. i., } C \cap D = \varnothing, \\ f(A) \subseteq C \text{ and } f(B) \subseteq D \end{array}\right\} \quad \Longrightarrow \quad C \text{ and } D \text{ are e. i.}$$

Here the items (i–iii) are obvious (by the definition); a proof of (iv) may be found, say, in [10, §§ 7.7, 11.3]; and (v) is straightforward, given some equivalent characterisations of Σ^0_1-bounded sets (see [10, §§ 5.1–5.2] for details). Note: the last property is just like that of computably inseparable sets from Section 2, though these two, of course, hold for different reasons. And adopting the common practice, given a decidable σ, we identify σ-sentences with natural numbers, up to an appropriate Gödel numbering.

As a direct consequence of Kalmar–Suranyi result in conjunction with Gurevich's theorem — cf. [1, Corollary 3.1.24] and [1, Theorem 2.1.39], respectively — we obtain the following

Basic fact (Kalmar, Suranyi, Gurevich). For $\sigma := \{R^2\}$, let

$$\texttt{Fin-sat} := \{\Phi \in Sen_\sigma \mid \neg\Phi \notin \mathrm{Th}(K^\circ_\sigma)\},$$
$$\texttt{Non-sat} := \{\Phi \in Sen_\sigma \mid \neg\Phi \in Val_\sigma\}.$$

Then Π_2–`Fin-sat` and Π_2–`Non-sat` are effectively inseparable.

Now we formally present the pair of central notions. Call a set Γ of σ-sentences *hereditarily* Π^0_1*-complete* (*h.* Π^0_1*-c.* for short) iff for every Δ,

$$Val_\sigma \cap \Gamma \subseteq \Delta \subseteq \Gamma, \quad \Delta \text{ is } \Pi^0_1\text{-bounded} \quad \Longrightarrow \quad \Delta \text{ is } \Pi^0_1\text{-complete;}$$

and similarly for Σ^0_1 in place of Π^0_1. Not very surprisingly, elementary definability (with parameters) preserves hereditary Π^0_1- and Σ^0_1-completeness.

Proposition 3.1. *For any* $r \in \{2, 3, \dots\}$, $\mathrm{C} \in \{\Pi^0_1, \Sigma^0_1\}$, $\mathcal{K}_1 \subseteq K_{\sigma_1}$ *and* $\mathcal{K}_2 \subseteq K_{\sigma_2}$*, we have:*

$$\left.\begin{array}{r}\mathcal{K}_1 \ \textit{is}\ \Sigma_k\text{-}\textit{e. d. p. in}\ \mathcal{K}_2,\\ \Pi_{r+1}\text{–Th}\,(\mathcal{K}_1)\ \textit{is h.}\,\mathrm{C}\text{-}\textit{c.}\end{array}\right\} \implies \Pi_{r+k}\text{–Th}\,(\mathcal{K}_2)\ \textit{is h.}\,\mathrm{C}\text{-}\textit{c.};$$

$$\left.\begin{array}{r}\mathcal{K}_1 \ \textit{is}\ \Sigma_k\text{-}\textit{e. d. in}\ \mathcal{K}_2,\\ \Sigma_r\text{–Th}\,(\mathcal{K}_1)\ \textit{is h.}\,\mathrm{C}\text{-}\textit{c.}\end{array}\right\} \implies \Sigma_{r+k-1}\text{–Th}\,(\mathcal{K}_2)\ \textit{is h.}\,\mathrm{C}\text{-}\textit{c.}$$

Proof. Consider the first implication. Suppose $\mathcal{K}_1$ is e. d. p. in $\mathcal{K}_2$. Then the observations provided in Section 2 ensure the existence of an effective translation $\tau : Sen_{\sigma_1} \to Sen_{\sigma_2}$ such that

$$\begin{array}{rcl}\Phi \in \Pi_{r+1}\text{–}Val_{\sigma_1} & \implies & \tau(\Phi) \in \Pi_{r+k}\text{–}Val_{\sigma_2},\\ \Phi \in \Pi_{r+1}\text{–Th}\,(\mathcal{K}^*_1) & \iff & \tau(\Phi) \in \Pi_{r+k}\text{–Th}\,(\mathcal{K}_2).\end{array}$$

Now let Δ be C-bounded with $\Pi_{r+k}\text{–}Val_{\sigma_2} \subseteq \Delta \subseteq \Pi_{r+k}\text{–Th}\,(\mathcal{K}_2)$. Clearly, we get

$$\tau^{-1}(\Pi_{r+k}\text{–}Val_{\sigma_2}) \subseteq \tau^{-1}(\Delta) \subseteq \tau^{-1}(\Pi_{r+k}\text{–Th}\,(\mathcal{K}_2))$$

and, in addition,

$$\begin{array}{c}\Pi_{r+1}\text{–}Val_{\sigma_1} \subseteq \tau^{-1}(\Pi_{r+k}\text{–}Val_{\sigma_2}),\\ \tau^{-1}(\Pi_{r+k}\text{–Th}\,(\mathcal{K}_2)) = \Pi_{r+1}\text{–Th}\,(\mathcal{K}^*_1) \subseteq \Pi_{r+1}\text{–Th}\,(\mathcal{K}_1).\end{array}$$

Hence

$$\Pi_{r+1}\text{–}Val_{\sigma_1} \subseteq \tau^{-1}(\Delta) \subseteq \Pi_{r+1}\text{–Th}\,(\mathcal{K}_1)$$

where $\tau^{-1}(\Delta)$ turns out to be C-bounded, since the obvious equivalence

$$\Phi \in \tau^{-1}(\Delta) \iff \tau(\Phi) \in \Delta$$

shows how to effectively reduce this set to Δ. So $\tau^{-1}(\Delta)$ is C-complete on the assumption that Π_{r+1}–Th$\,(\mathcal{K}_1)$ is h. C-c. — thus Δ will be C-hard, and therefore also C-complete.

The same sort of argument works for the second implication. □

On the other hand, hereditary Π^0_1- and Σ^0_1-completeness (as well as hereditary undecidability or computable inseparability, discussed in Section 2) arise naturally when analysing effective inseparability.

Proposition 3.2. *For every* $\Gamma \subseteq Sen_\sigma$*, whenever* $Sen_\sigma \setminus \Gamma$ *and* $Val_\sigma \cap \Gamma$ *are effectively inseparable, we have:*

$$\begin{array}{rcl}Val_\sigma \cap \Gamma\ \textit{is}\ \Sigma^0_1\textit{-bounded} & \implies & \Gamma\ \textit{is hereditarily}\ \Pi^0_1\textit{-complete};\\ \Gamma\ \textit{is}\ \Pi^0_1\textit{-bounded} & \implies & \Gamma\ \textit{is hereditarily}\ \Sigma^0_1\textit{-complete.}\end{array}$$

Proof. For the first implication, assume that $Val_\sigma \cap \Gamma$ is Σ^0_1-bounded. Now if $Val_\sigma \cap \Gamma \subseteq \Delta \subseteq \Gamma$ and Δ is Π^0_1-bounded, then

$$Sen_\sigma \setminus \Gamma \subseteq Sen_\sigma \setminus \Delta \subseteq Sen_\sigma \setminus (Val_\sigma \cap \Gamma),$$

and hence the Σ^0_1-sets $Val_\sigma \cap \Gamma$ and $Sen_\sigma \setminus \Delta$ are effectively inseparable by the property (iii) of e. i. sets. And by (iv) this implies the Σ^0_1-completeness of $Sen_\sigma \setminus \Delta$ — i. e., Δ turns out to be Π^0_1-complete.

For the second implication, assume Γ is Π^0_1-bounded. If $Val_\sigma \cap \Gamma \subseteq \Delta \subseteq \Gamma$ and Δ is Σ^0_1-bounded, then Δ separates $Val_\sigma \cap \Gamma$ and $Sen_\sigma \setminus \Gamma$, and hence the Σ^0_1-sets Δ and $Sen_\sigma \setminus \Gamma$ are e. i. by (iii). And by (iv) this implies the Σ^0_1-completeness of Δ. $\square$

Remark that (hereditary) Π^0_1-completeness seems slightly more natural than Σ^0_1-completeness, because for all $\mathcal{K} \subseteq K_\sigma$ and n, the set

$$Val_\sigma \cap \Sigma_n\text{–Th}\,(\mathcal{K}) \;=\; \Sigma_n\text{–}Val_\sigma$$

is contained in Val_σ, and so turns out to be Σ^0_1-bounded. And, in contrast, Σ_n–Th $(\mathcal{K})$ does not necessarily have to be Π^0_1-bounded — thus, intuitively, Σ^0_1-completeness is harder to guarantee. However, in many important cases, $\mathcal{K}$ will consist of suitably chosen finite structures and, indeed, the Π^0_1-boundedness of its theory can be established. Moreover, as we shall see later, one may often avoid the requirement 'Γ is Π^0_1-bounded' in practice.

An example comes from Basic fact, namely

Corollary 3.3. *For $\sigma := \{R^2\}$, the set Σ_2–Th (K°_σ) is hereditarily Π^0_1-complete and hereditarily Σ^0_1-complete.*

Proof. For an obvious translation $\theta : \Phi \mapsto \neg\Phi$ (acting on Sen_σ), we have

$$\theta\,(\Pi_2\text{–}\mathtt{Fin\text{-}sat}) \;\subseteq\; Sen_\sigma \setminus \Sigma_2\text{–Th}\,(K^\circ_\sigma) \quad \text{and} \quad \theta\,(\Pi_2\text{–}\mathtt{Non\text{-}sat}) \;=\; \Sigma_2\text{–}Val_\sigma,$$

and hence $Sen_\sigma \setminus \Sigma_2$–Th (K°_σ) and $Val_\sigma \cap \Sigma_2$–Th $(K^\circ_\sigma) = \Sigma_2$–$Val_\sigma$ are e. i. by Basic fact and (v). Since Σ_2–Th (K°_σ) is easily shown to be Π^0_1-bounded, it only remains to apply Proposition 3.2. $\square$

And further, by looking at effective inseparability from the perspective of elementary definability we get an analogue of Proposition 3.1.

Proposition 3.4. *For any $r \in \{2, 3, \dots\}$, $\mathcal{K}_1 \subseteq K_{\sigma_1}$ and $\mathcal{K}_2 \subseteq K_{\sigma_2}$, we have:*

$$\begin{aligned} &\mathcal{K}_1 \textit{ is } \Sigma_k\textit{-e. d. p. in } \mathcal{K}_2,\ \Pi_{r+1}\text{–}Val_{\sigma_1} \textit{ and } Sen_{\sigma_1} \setminus \Pi_{r+1}\text{–Th}\,(\mathcal{K}_1) \textit{ are e. i.} \\ &\qquad\qquad \Longrightarrow \quad \Pi_{r+k}\text{–}Val_{\sigma_2} \textit{ and } Sen_{\sigma_2} \setminus \Pi_{r+k}\text{–Th}\,(\mathcal{K}_2) \textit{ are e. i.;} \end{aligned}$$

$$\begin{aligned} &\mathcal{K}_1 \textit{ is } \Sigma_k\textit{-e. d. in } \mathcal{K}_2,\ \Sigma_r\text{–}Val_{\sigma_1} \textit{ and } Sen_{\sigma_1} \setminus \Sigma_r\text{–Th}\,(\mathcal{K}_1) \textit{ are e. i.} \\ &\qquad\qquad \Longrightarrow \quad \Sigma_{r+k-1}\text{–}Val_{\sigma_2} \textit{ and } Sen_{\sigma_2} \setminus \Sigma_{r+k-1}\text{–Th}\,(\mathcal{K}_2) \textit{ are e. i.} \end{aligned}$$

Proof. For the first implication, assume $\mathcal{K}_1$ is e. d. p. in $\mathcal{K}_2$. As in Section 2, there exists a suitable translation τ, and taking

$$f \;:=\; \tau, \quad A \;:=\; \Pi_{r+1}\text{–}Val_{\sigma_1}, \quad B \;:=\; Sen_{\sigma_1} \setminus \Pi_{r+1}\text{–Th}\,(\mathcal{K}_1),$$
$$C \;:=\; \Pi_{r+k}\text{–}Val_{\sigma_2}, \quad D \;:=\; Sen_{\sigma_2} \setminus \Pi_{r+k}\text{–Th}\,(\mathcal{K}_2),$$

we conclude that

$$\Pi_{r+k}\text{–}Val_{\sigma_2} \quad \text{and} \quad Sen_{\sigma_2} \setminus \Pi_{r+k}\text{–Th}\,(\mathcal{K}_2)$$

are e. i. by (v). The second implication is similar. $\square$

To help us familiarise ourselves with the machinery, we turn to a fairly simple fact, which shall occasionally be kept in mind. Consider the smallest binary relation $\preccurlyeq$ on

$$\mathbb{L} \ := \ \{\Sigma_r, \Pi_{r+1} \mid r = 2, 3, \dots\}$$

with the property that for any $\{k, l\} \subset \{2, 3, \dots\}$,

$$k \leqslant l \quad \Longrightarrow \quad \Sigma_k \preccurlyeq \Sigma_l,\ \Sigma_k \preccurlyeq \Pi_{l+1},\ \Pi_{k+1} \preccurlyeq \Pi_{l+1},\ \Pi_{k+1} \preccurlyeq \Sigma_{l+2}.$$

Proposition 3.5 (almost folklore). *Let*

$$\{\mathrm{C}_1, \mathrm{C}_2\} \subset \mathbb{L},\ \mathcal{K}_1 \subseteq K_{\sigma_1} \ \textit{and}\ \mathcal{K}_2 \subseteq K_{\sigma_2}$$

be such that

$$\mathrm{C}_1 \preccurlyeq \mathrm{C}_2,\ \sigma_1 \subseteq \sigma_2 \ \textit{and}\ \mathcal{K}_1 \subseteq \{\mathfrak{A} \mid \mathfrak{A} \textit{ is the } \sigma_1\textit{-reduct of some } \mathfrak{B} \in \mathcal{K}_2\}.$$

Suppose C_1, σ_1 *and* $\mathcal{K}_1$ *meet one of the following conditions:*

- *the sets* $\mathrm{C}_1\text{–}Val_{\sigma_1}$ *and* $Sen_{\sigma_1} \setminus \mathrm{C}_1\text{–}\mathrm{Th}\,(\mathcal{K}_1)$ *are e. i.;*
- *the set* $\mathrm{C}_1\text{–}\mathrm{Th}\,(\mathcal{K}_1)$ *is h. u.;*
- *the set* $\mathrm{C}_1\text{–}\mathrm{Th}\,(\mathcal{K}_1)$ *is h.* Π_1^0*-c.;*
- *the set* $\mathrm{C}_1\text{–}\mathrm{Th}\,(\mathcal{K}_1)$ *is h.* Σ_1^0*-c.*

Then C_2, σ_2 *and* $\mathcal{K}_2$ *meet the corresponding condition.*

Proof. The verification would add nothing new, and is left as an exercise for the interested reader. Idea: one can easily produce an effective translation τ which maps Sen_{σ_1} into Sen_{σ_2} and such that for every $\Phi \in Sen_{\sigma_1}$,

$$\Phi \text{ and } \tau(\Phi) \text{ are logically equivalent } \ (\text{viz. } \Phi \leftrightarrow \tau(\Phi) \in Val_{\sigma_2}),$$
$$\Phi \ \in \ \mathrm{C}_1\text{–}Sen_{\sigma_1} \quad \Longleftrightarrow \quad \tau(\Phi) \ \in \ \mathrm{C}_2\text{–}Sen_{\sigma_2},$$

and hence

$$\begin{aligned} \tau(\mathrm{C}_1\text{–}Val_{\sigma_1}) \ &\subseteq \ \mathrm{C}_2\text{–}Val_{\sigma_2}, \\ \tau(Sen_{\sigma_1} \setminus \mathrm{C}_1\text{–}\mathrm{Th}\,(\mathcal{K}_1)) \ &\subseteq \ Sen_{\sigma_2} \setminus \mathrm{C}_2\text{–}\mathrm{Th}\,(\mathcal{K}_2), \\ \tau^{-1}(\mathrm{C}_2\text{–}\mathrm{Th}\,(\mathcal{K}_2)) \ &= \ \mathrm{C}_1\text{–}\mathrm{Th}\,(\mathcal{K}_1^\star) \end{aligned}$$

where $\mathcal{K}_1^\star \ := \ \{\mathfrak{A} \mid \mathfrak{A} \text{ is the } \sigma_1\text{-reduct of some } \mathfrak{B} \in \mathcal{K}_2\}$; the rest is straightforward — cf. Section 2 and the proofs of Propositions 3.1 and 3.4. □

Combining the previous observations with various contributions to hereditarily undecidable theories, we can derive a bunch of principal and useful results about some well-known classes of models. And while the precise descriptions of these classes are not essential for the proofs below, one of them is provided as an example and because it will be mentioned again in Section 4. Remark that a key role in the argument of the next theorem is played by the translations found by different authors — see [2, 4, 5, 7, 9] — as well as by Basic fact and the above observations, of course. Accordingly, the old translations may be employed for obtaining new complexity results.

Let $\sigma_\star$ be $\{R^2\}$ and $\mathcal{G}$ the class of all finite $\sigma_\star$-structures satisfying

$$\forall x, y\, (\neg R(x, x) \wedge (R(x, y) \rightarrow R(y, x))),$$

i. e., the finite undirected irreflexive graphs. Hereafter, where $k \in \{1, 2, \dots\}$ and $\mathcal{K} \subseteq K_\sigma$, we write $\mathcal{K}^{\geqslant k}$ as an abbreviation for

$$\{\mathfrak{A} \in \mathcal{K} \mid \text{the domain } |\mathfrak{A}| \text{ contains at least } k \text{ elements}\}.$$

Theorem 3.6. *For any $i \in \{1, \dots, 10\}$, the sets*

$$\mathrm{C}_i\text{–}Val_{\sigma_i} \quad \textit{and} \quad Sen_{\sigma_i} \setminus \mathrm{C}_i\text{–Th}\,(\mathcal{K}_i),$$

with $\mathcal{K}_i \subseteq K_{\sigma_i}$, are effectively inseparable, where:

- $\mathrm{C}_1 = \Sigma_2$, $\mathcal{K}_1$ *is the class of all finite undirected irreflexive graphs;*
- $\mathrm{C}_2 = \Sigma_3$, $\mathcal{K}_2$ *is the class of all finite models of the theory of two equivalences;*
- $\mathrm{C}_3 = \Sigma_2$, $\mathcal{K}_3$ *is the class of all finite lattices (viewed as partial orders);*
- $\mathrm{C}_4 = \Sigma_2$, $\mathcal{K}_4$ *is the class of all finite partial orders;*
- $\mathrm{C}_5 = \Pi_6$, $\mathcal{K}_5$ *is the class of all free distributive lattices with finitely many generators;*
- $\mathrm{C}_6 = \Sigma_2$, $\mathcal{K}_6$ *is the class of all finite bipartite graphs;*
- $\mathrm{C}_7 \in \{\Sigma_3, \Pi_3\}$, $\mathcal{K}_7$ *is the class of all finite distributive lattices;*
- $\mathrm{C}_8 = \Pi_6$, $\mathcal{K}_8$ *is the class of all finite permutation groups;*
- $\mathrm{C}_9 = \Sigma_3$, $\mathcal{K}_9$ *is the class of all finite commutative associative rings of a (fixed) prime characteristic p in which any product of three elements equals zero — so $\mathcal{K}_9$ is in fact one of the countably many classes;*
- $\mathrm{C}_{10} = \Pi_4$, $\mathcal{K}_{10}$ *is $\{\mathfrak{E}_k \mid k = 1, 2, \dots\}$ where for each $k \in \{1, 2, \dots\}$, $\mathfrak{E}_k$ denotes the lattice of all equivalence relations on $\{1, \dots, k\}$.*

Proof. Warning: at the last step of the proof of every item, Proposition 3.4 must be applied.

$\boxed{i = 1}$ By [9, Theorem 4.2], $\mathcal{K}^{\circ}_{\sigma_\star}$ is Σ_1-e. d. in $\mathcal{K}_1^{\geqslant 3}$ and so in $\mathcal{K}_1 = \mathcal{G}$ (notice: the Σ_4-definability was shown earlier by I. A. Lavrov, cf. [4, Theorem 3.3.3]). It remains to observe that

$$Sen_{\sigma_\star} \setminus \Sigma_2\text{–Th}\left(K^{\circ}_{\sigma_\star}\right) \quad \text{and} \quad Val_{\sigma_\star} \cap \Sigma_2\text{–Th}\left(K^{\circ}_{\sigma_\star}\right) \;=\; \Sigma_2\text{–}Val_{\sigma_\star}$$

are e. i. (from Basic fact — recall the proof of Corollary 3.3).

$\boxed{i = 2}$ As was established in [5, pp. 273–274], $\mathcal{G}$ is Σ_2-e. d. in $\mathcal{K}_2$.

$\boxed{i = 3, 4}$ In view of [7, Appendix A], $\mathcal{G}^{\geqslant 3}$ is Σ_1-e. d. in $\mathcal{K}_3$ (notice: the Σ_2-definability was shown earlier by M. A. Taitslin, cf. [4, Theorem 3.3.4]), and therefore in $\mathcal{K}_4 \supseteq \mathcal{K}_3$.

$\boxed{i = 5}$ As was established in [5, pp. 279–281], $\mathcal{G}$ is Σ_4-e. d. p. in $\mathcal{K}_5$.

$\boxed{i = 6}$ By [9, Corollary 4.5], $\mathcal{K}^{\circ}_{\sigma_\star}$ is Σ_1-e. d. in $\mathcal{K}_6^{*}$, where $\mathcal{K}_6^{*}$ is the collection of all finite bipartite graphs containing at least three elements in each of the two parts, and so in $\mathcal{K}_6$.

$\boxed{i = 7}$ For Σ_3, one can easily verify that $\mathcal{K}_4$ is Σ_2-e. d. in $\mathcal{K}_7$ (see [9, Proposition 4.1]). For Π_3, by [9, Theorem 4.8], $\mathcal{K}_6^{*}$ is Σ_1-e. d. p. in $\mathcal{K}_7$.

$\boxed{i = 8}$ As was established in [5, pp. 283–285], $\mathcal{K}_2$ is Σ_3-e. d. p. in $\mathcal{K}_8$.

$\boxed{i = 9}$ By [4, Theorem 3.3.5] (of M. A. Taitslin and Yu. L. Ershov), $\mathcal{G}$ is Σ_2-e. d. in $\mathcal{K}_9$. Remark that $\mathcal{K}_9$ depends on a chosen prime p.

$\boxed{i = 10}$ By [9, Theorem 4.9], $\mathcal{K}_6^*$ is Σ_2-e. d. p. in $\mathcal{K}_{10}$ (notice: the definability of $\mathcal{K}_2$ in $\mathcal{K}_{10}$ was shown earlier in [2, § 3] but it will not, in effect, give a smaller prefix class). □

Certainly much more results can be obtained in this way — whenever an undecidability proof has been provided by means of the elementary definability technique (cf. bibliography in [4] for examples), tracing backwards to its root, we usually end up with $\mathcal{K}^\circ_{\sigma_\star}$.

Corollary 3.7. *For any $i \in \{1, \ldots, 10\}$ (and any prime p, if $i = 9$), the set C_i–$\mathrm{Th}\,(\mathcal{K}_i)$ is hereditarily Π^0_1-complete and hereditarily Σ^0_1-complete.*

Proof. Fix i (and p if needed). The set

$$Val_{\sigma_i} \cap \mathrm{C}_i\text{–}\mathrm{Th}\,(\mathcal{K}_i) \;=\; \mathrm{C}_i\text{–}Val_{\sigma_i} \;\subset\; Val_{\sigma_i}$$

is clearly Σ^0_1-bounded. In addition, $\mathcal{K}_i$ consists of finite objects with suitable properties, and we can effectively check whether a finite σ_i-structure belongs to $\mathcal{K}_i$; hence C_i–$\mathrm{Th}\,(\mathcal{K}_i)$ is Π^0_1-bounded. Now the desired conclusion follows from Proposition 3.2. □

To be more precise, we pass to concrete fragments of interest.

Corollary 3.8. *For any $i \in \{1, \ldots, 10\}$ (and any prime p, if $i = 9$), the sets C_i–$\mathrm{Th}\,(\mathcal{K}_i)$ and C_i–$\mathrm{Th}\,(\mathcal{K}'_i)$ are, respectively, Π^0_1-complete and Σ^0_1-complete, where $\mathcal{K}'_i$ is the non-finite analogue of $\mathcal{K}_i$ — i. e., obtained by removing the words 'finite' and 'with finitely many generators' in the description of $\mathcal{K}_i$.*

Proof. Fix i (and p if needed). As has been already mentioned, C_i–$\mathrm{Th}\,(\mathcal{K}_i)$ is Π^0_1-bounded. In view of Corollary 3.7, it remains to observe that since $\mathcal{K}'_i$ is computably axiomatisable, C_i–$\mathrm{Th}\,(\mathcal{K}'_i) \subset \mathrm{Th}\,(\mathcal{K}'_i)$ will be Σ^0_1-bounded. □

Also, taking into account what has been said in Section 2, Theorem 3.6 immediately implies the hereditary undecidability of any C_i–$\mathrm{Th}\,(\mathcal{K}_i)$ in the list — and though the cases with $i \in \{1, 6, 7, 10\}$ were previously proved by A. Nies in [9], and those with $i \in \{3, 4\}$ are due to J.H. Schmerl (see [9, Theorem 4.3]), this is still new for $i \in \{2, 5, 8, 9\}$ (and all primes p).

Let us finish with a few remarks which are mainly about Σ^0_1-completeness. If $\mathcal{K} \subseteq K_\sigma$ and the hereditary undecidability of $\mathrm{Th}\,(\mathcal{K})$ was proved via elementary definability, one can very often derive that

$$\mathrm{C}\text{–}Val_\sigma \quad \text{and} \quad Sen_\sigma \setminus \mathrm{C}\text{–}\mathrm{Th}\,(\mathcal{K})$$

are e. i. for an appropriate prefix C. But suppose C–Th $(\mathcal{K})$ turns out to be Σ^0_1-bounded — then it, being undecidable, is not Π^0_1-bounded. And thus we cannot get its (hereditary) Σ^0_1-completeness from Proposition 3.2. The situation may be saved by Proposition 3.1, however. The next example emerges from the proof of [6] (whose concern was the full theory).

Corollary 3.9. *Let $\mathcal{K}$ be the class of all projective planes and σ its intended signature. Then:*

- *the sets Π_4–Val_σ and $Sen_\sigma \setminus \Pi_4$–$\mathrm{Th}\,(\mathcal{K})$ are e. i.;*
- *the set Π_4–$\mathrm{Th}\,(\mathcal{K})$ is h. u., h. Π^0_1-c. and h. Σ^0_1-c.*

In particular, Π_4–$\mathrm{Th}\,(\mathcal{K})$ is Σ^0_1-complete.

Proof. By [6, Theorem 5], $\mathcal{G}$ is Σ_2-e. d. p. in $\mathcal{K}$, and thus the effective inseparability of the two sets follows from Theorem 3.6 and Proposition 3.4.

Clearly, Π_4–$\mathrm{Th}\,(\mathcal{K})$ is h. u. (by the property (ii) of e. i. sets). In view of Proposition 3.2, Π_4–$\mathrm{Th}\,(\mathcal{K})$ is h. Π^0_1-c. On the other hand, Corollary 3.7 and Proposition 3.1 imply that it is h. Σ^0_1-c. as well.

And finally, since $\mathrm{Th}\,(\mathcal{K})$ is computably axiomatisable (cf. [6, § 5]), the set Π_4–$\mathrm{Th}\,(\mathcal{K})$, being Σ^0_1-bounded, is Σ^0_1-complete. □

Yet it seems that for a wide range of such situations, we can replace $\mathcal{K}$ by $\mathcal{K}' \subseteq \mathcal{K}$ consisting of suitably chosen finite structures, with the theory of $\mathcal{K}'$ compelled to be Π^0_1-bounded — and Proposition 3.2 will then suffice.

In exactly the same way, by applying the above method to the proof of [8] we obtain a bunch of additional examples. Recall that a *cancellative groupoid* is a $\{Q^3, =^2\}$-structure $\mathfrak{A}$ satisfying the following conditions:

- $\forall x\,\forall y\,\exists u\,Q\,(x, y, u) \wedge \forall x\,\forall y\,\forall u\,\forall v\,(Q\,(x, y, u) \wedge Q\,(x, y, v) \to u = v)$;
- $\forall x\,\forall y\,\forall u\,\forall v\,(Q\,(x, u, y) \wedge Q\,(x, v, y) \to u = v)$;
- $\forall x\,\forall y\,\forall u\,\forall v\,(Q\,(u, x, y) \wedge Q\,(v, x, y) \to u = v)$.

Note: the first item says '$Q^{\mathfrak{A}}$ represents a function from $|\mathfrak{A}| \times |\mathfrak{A}|$ into $|\mathfrak{A}|$'.

Corollary 3.10. *Let $\sigma := \{Q^3, =^2\}$ and $\sigma' := \sigma \cup \{U^1\}$. For $\mathcal{K} \subseteq K_\sigma$, take*

$$\mathcal{K}' \;:=\; \{\mathfrak{A} \in K_{\sigma'} \mid \textit{the } \sigma\textit{-reduct of } \mathfrak{A} \textit{ belongs to } \mathcal{K}\};$$

$$T' \;:=\; \textit{the set of all } \sigma'\textit{-sentences deducible from } \mathrm{Th}\,(\mathcal{K}).$$

Suppose that some $\mathfrak{A} \in \mathcal{K}$ has a substructure $\mathfrak{B}$ which is an infinite cancellative groupoid. Then:

- *the sets Π_3–$Val_{\sigma'}$ and $Sen_{\sigma'} \setminus \Pi_3$–$\mathrm{Th}\,(\mathcal{K}')$ are e. i.;*
- *the set Π_3–$\mathrm{Th}\,(\mathcal{K}')$ is h. u., h. Π^0_1-c. and h. Σ^0_1-c.*

Consequently, we have:

- *the sets Π_3–$Val_{\sigma'}$ and $Sen_{\sigma'} \setminus \Pi_3$–$T'$ are e. i.;*
- *the set Π_3–T' is h. u., h. Π^0_1-c. and h. Σ^0_1-c.*

And in particular, whenever the theory $\mathrm{Th}\,(\mathcal{K})$ is computably axiomatisable, Π_3–T' is Σ^0_1-complete.

Proof. By [8], $\mathcal{G}$ is Σ_1-e. d. p. in $\mathcal{K}'$, and so the effective inseparability of the two sets follows from Theorem 3.6 and Proposition 3.4.

Clearly, Π_3–$\mathrm{Th}\,(\mathcal{K}')$ is h. u. In view of Proposition 3.2, it is h. Π^0_1-c. On the other hand, Corollary 3.7 and Proposition 3.1 imply that Π_3–$\mathrm{Th}\,(\mathcal{K}')$ is also h. Σ^0_1-c.

A simple (and standard) argument shows that

$$T' \;=\; \mathrm{Th}\,(\mathcal{K}^\star)$$

where

$$\mathcal{K}^\star \ := \ \{\mathfrak{A} \in K_{\sigma'} \mid T' \text{ is true in } \mathfrak{A}\}$$

— because T' is a σ'-theory. Since $\mathcal{K}' \subseteq \mathcal{K}^\star$, we can easily replace $\mathcal{K}'$ by $\mathcal{K}^\star$ in the previous considerations.

Finally, if $\mathrm{Th}\,(\mathcal{K})$ is computably axiomatisable, then so is T', and thus Π_3–T', being Σ^0_1-bounded, must be Σ^0_1-complete. □

Of course, one can immediately get more by expanding prefixes, signatures and classes (remember Proposition 3.5). For instance, if C-$\mathrm{Th}\,(\mathcal{K})$ is a fragment which we have already shown to be Π^0_1- or Σ^0_1-complete, then each element of the family

$$\{\mathrm{C}'\text{-}\mathrm{Th}\,(\mathcal{K}) \mid \mathrm{C}' \in \mathbb{L} \text{ and } \mathrm{C} \preccurlyeq \mathrm{C}'\} \ \cup \ \{\mathrm{Th}\,(\mathcal{K})\}$$

is again Π^0_1- or Σ^0_1-complete, respectively.

Some further discussion

An issue not touched on here concerns applications to languages other than first-order ones. It turns out that the machinery developed so far can, in effect, be used to provide complexity (lower) bounds for decision problems in quantified probability logics, and in particular, for those dealing with prefix fragments of the logic from [11] or its finite-model version — improving the classification in terms of (un)decidability from [12]. But for various reasons, we should leave the formal details for a different paper. And certainly more applications ought to be expected — simply because interpreting classes like $\mathcal{G}$ is a common tool for proving undecidability results in logic.

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