Notes on the computational aspects of Kripke's theory of truth

Stanislav O. Speranski

St. Petersburg State University, Saint Petersburg, Russia

This is a pre-print version of the article published online in *Studia Logica*. DOI: 10.1007/s11225-016-9694-8

Abstract

The paper contains a survey on the complexity of various truth hierarchies arising in Kripke's theory. I present some new arguments, and use them to obtain a number of interesting generalisations of known results. These arguments are both relatively simple, involving only the basic machinery of constructive ordinals, and very general.

1 Introduction

In formal theories of truth the first-order language $\mathcal{L}$ of Peano arithmetic and its expansion $\mathcal{L}_T$ obtained by adding an extra unary predicate symbol T are usually considered. Intuitively, here T stands for a truth predicate, which — if we assume an untyped glut-free setting — is somehow doomed to be partial and at least three-valued. In particular, this applies to Kripke's approach [8] and subsequent modifications of it (like [6]). In effect, there are also situations where it is convenient to think of T as a free set variable, thus treating $\mathcal{L}_T$ as a fragment of monadic second-order arithmetic.

In [8], Kripke used partial valuation schemes and their jump operators to define various transfinite hierarchies converging to admissible interpretations of T. Since then a number of interesting results on the complexity of such constructions have been obtained. Burgess [1] showed that the least fixed points of the jump operators based on the strong Kleene scheme and certain supervaluation schemes are Π^1_1-complete. Further, by a somewhat different argument, Welch [20] proved the same for Leitgeb's groundedness operator and the associated truth operator (which may be represented in Kripke's framework, cf. [17, Section 5]), along with the Π^1_1-hardness of all non-trivial levels of the corresponding hierarchies.[1] I refer readers to [3] for discussion and applications to axiomatisability. However, the weak Kleene scheme does not necessarily produce a Π^1_1-hard interpretation of T — it depends heavily on the choice of Gödel numbering, as was demonstrated by Cain and Damnjanovic [2].

I shall present some new arguments, and use them to get a number of interesting generalisations of known results. These arguments will turn out to be relatively simple, involving only the basic machinery of constructive ordinals, and surprisingly general.

[1] Earlier Welch mistakenly claimed the Δ^1_1-boundedness of these levels (see [9, Subsection 5.5]), but this situation was corrected by the appearance of [20].

Section 2 consists of preliminary material on Kripke's theory of truth, monadic second-order arithmetic and Kleene's system of notation for constructive ordinals. Section 3 splits into three subsections. We now say a bit more about them.

Subsection 3.1 gives an easy application of effective transfinite recursion. It shows how, for every reasonable valuation scheme V, one can directly obtain complexity upper bounds for all 'constructive' levels of the truth hierarchy for V, in a uniform manner.

Intuitively, we build up an admissible interpretation of T in stages, so that

$$\begin{array}{rl} 0=0 & \text{is true at } \mathsf{stage\ 0} \text{ but not at } \mathsf{stage\ 1}, \\ T(\ulcorner 0=0 \urcorner) & \text{is true at } \mathsf{stage\ 1} \text{ but not at } \mathsf{stage\ 2}, \\ T(\ulcorner T(\ulcorner 0=0 \urcorner) \urcorner) & \text{is true at } \mathsf{stage\ 2} \text{ but not at } \mathsf{stage\ 3}, \\ & \vdots \end{array}$$

— thus the closure ordinal of the corresponding jump operator is at least ω. In Subsection 3.2, I describe how to extend this to all constructive ordinals in a uniform effective way. It follows that each 'well-behaved' truth hierarchy requires (at least) ω_1^{CK} stages to settle the truth of $\mathcal{L}_T$-sentences, and we need a path with limit ω_1^{CK} to reach the least fixed point — but a typical such path is not weaker than Π_1^1; moreover one easily proves the Π_1^1-hardness of the resulting interpretations of T as a corollary.[2] In particular, focusing on some important issues raised in [2], I analyse the case of the weak Kleene scheme.[3]

Subsection 3.3 presents very simple proofs for the results of [20] — including the observation (made by Hjorth and Meadows) about supervaluation schemes. Like Burgess [1] and Welch [20], although in a much more direct manner, I'll exploit suitable definable portions of T to interpret free unary predicates. We shall finish with a discussion of possible generalisations to reasonable fragments of $\mathcal{L}_T$.

In a nutshell, the arguments presented below clarify the structure of various truth hierarchies, offering new insights into the complexity aspects of Kripke's approach. Let us now elaborate on how the ideas involved contribute to a better understanding of the matter.

A. For any valuation scheme V, if the truth hierarchy for V (which we denote by T_V) is 'well-behaved', then the argument of Subsection 3.2 establishes the following:

    I. the least fixed point of the jump operator for V is Π_1^1-hard;
    II. the closure ordinal of the jump operator for V is at least ω_1^{CK}.[4]

   Besides its simplicity and generality, this argument has the advantage that we do not need to examine (I) and (II) separately because the same construction directly yields both. So it makes explicit the connection between the two.

B. Recall, Cain and Damnjanovic showed in [2] that

[2] Among other things, this explains the fact that although all 'constructive' levels of the truth hierarchy for the strong Kleene scheme are uniformly Δ_1^1-bounded, their supremum is known to be Π_1^1-complete.

[3] In [12], Meadows employed infinitary tableau systems and well-founded recursive trees (constructed by means of the diagonal lemma) to get the closure ordinals and the Π_1^1-completeness theorems for the strong Kleene scheme and two supervaluation schemes. His approach is less general and actually more demanding than that of Subsection 3.2, despite some 'structural similarity' between them; for instance, it relies on the existence of appropriate tableaus and sheds no light on the problems discussed in [2].

[4] Note in passing that the least fixed point in (I) coincides with the union of T_V and the closure ordinal in (II) is the stage at which T_V collapses.

(I) and (II) for the weak Kleene scheme depend on the Gödel numbering and the language for $\mathbb{N}$ *we choose.*

The argument of Subsection 3.2 also leads to a deeper understanding of this interesting source of intensionality, which has not been carefully studied after [2], and how it can be avoided. For instance it will turn out that the problem disappears if we add a symbol for the proper subtraction to the signature of Peano arithmetic. Furthermore we shall see that some natural modifications of the weak Kleene semantics (including the scheme employed by Feferman in [5]) do not suffer from this kind of dependence.

C. Recall that the results of [20] show the Π^1_1-hardness of all non-trivial levels of Leitgeb's truth and groundedness hierarchies; and the same applies to the truth hierarchies for supervaluation schemes (as observed by Hjorth and Meadows). In Subsection 3.3, I'll provide a simpler and somewhat more direct argument for this, where neither Kleene normal form nor any coding of sequences is exploited. Then, we shall come across an intensionality phenomenon, which looks a bit like that of (B): given a relatively weak fragment $\mathcal{F}$ of $\mathcal{L}_T$, whether or not an analogous argument works for the relevant hierarchies restricted to $\mathcal{F}$ depends on the choice of Gödel numbering.

Remark: Subsection 3.2 suggests a number of formalisations of the informal notion of well-behaved, used in (A) and implicitly in (B).

2 Preliminaries

2.1 Kripke's theory of truth

Consider the signature of Peano arithmetic and its expansion obtained by adding an extra unary predicate symbol T, viz.

$$\sigma \ := \ \{0, \mathsf{s}, +, \times, =\} \quad \text{and} \quad \sigma_T \ := \ \sigma \cup \{T\}.$$

Throughout this text the following assumptions are in force:

- the *connective symbols* are $\neg$, $\wedge$ and $\vee$;
- the *quantifier symbols* are $\forall$ and $\exists$.

We abbreviate $\neg\varphi \vee \psi$ to $\varphi \to \psi$, $(\varphi \to \psi) \wedge (\psi \to \varphi)$ to $\varphi \leftrightarrow \psi$, etc. Let $\mathcal{L}$ and $\mathcal{L}_T$ be the first-order languages of σ and σ_T respectively. Here is some related notation:

$$\begin{aligned} \mathit{For} \ &:= \ \text{the collection of all } \mathcal{L}\text{-formulas;} \\ \mathit{Sen} \ &:= \ \text{the collection of all } \mathcal{L}\text{-sentences;} \\ \mathit{For}_T \ &:= \ \text{the collection of all } \mathcal{L}_T\text{-formulas;} \\ \mathit{Sen}_T \ &:= \ \text{the collection of all } \mathcal{L}_T\text{-sentences.} \end{aligned}$$

For Kripke's semantic approach the symbols of σ have their usual meaning, as in the standard model $\mathfrak{N}$ of Peano arithmetic. Then if $A \subseteq \mathbb{N}$, we write $\langle \mathfrak{N}, A \rangle$ for the expansion of $\mathfrak{N}$ to σ_T in which T is interpreted as the characteristic function of A.

For each $n \in \mathbb{N}$ we have a closed $\mathcal{L}$-term $\underline{n}$, called the *numeral* for it:

$$\underline{0} \ := \ 0, \quad \underline{1} \ := \ \mathsf{s}(0), \quad \underline{2} \ := \ \mathsf{s}(\mathsf{s}(0)), \quad \ldots$$

Assume some Gödel numbering of $\mathcal{L}_T$ has been chosen. Given $\varphi \in For_T$, define

$$\#\varphi \ := \ \text{the Gödel code of } \varphi \quad \text{and} \quad \ulcorner \varphi \urcorner \ := \ \text{the numeral for } \#\varphi.$$

For instance, by diagonalisation one can obtain a *liar sentence* λ in the language $\mathcal{L}_T$, such that $\lambda \leftrightarrow \neg T(\ulcorner \lambda \urcorner)$ is provable in Peano arithmetic (see e.g. [4] for details). In this context $A \subseteq \mathbb{N}$ is said to be *consistent* iff there exists no $\phi \in Sen_T$ for which $\{\#\phi, \#\neg\phi\} \subseteq A$. We shall sometimes identify $\mathcal{L}_T$-formulas with their codes without danger of confusion.

In [8], Kripke employed *partial interpretations of* T, i.e. pairs of the form $S = \langle S^+, S^- \rangle$ where S^+ and S^- are disjoint subsets of $\mathbb{N}$ — respectively called the *extension of* S and the *anti-extension of* S.[5] A *partial valuation for* σ_T *(or* $\mathcal{L}_T$*)* is a mapping from Sen_T to a superset of $\{0, \frac{1}{2}, 1\}$.

By a *valuation scheme* we mean a function from partial interpretations to partial valuations. To begin with, let $\leqslant_{\mathrm{SK}}$ and $\leqslant_{\mathrm{WK}}$ be the orderings given by

$$0 \ \leqslant_{\mathrm{SK}} \ \tfrac{1}{2} \ \leqslant_{\mathrm{SK}} \ 1 \quad \text{and} \quad \tfrac{1}{2} \ \leqslant_{\mathrm{WK}} \ 0 \ \leqslant_{\mathrm{WK}} \ 1.$$

Define the *strong Kleene valuation scheme* V_{SK} by induction as follows:

- for any closed $\mathcal{L}$-terms t_1 and t_2,

$$V_{\mathrm{SK}}(S)(t_1 = t_2) \ := \ \begin{cases} 1 & \text{if } \mathfrak{N} \models t_1 = t_2, \\ 0 & \text{if } \mathfrak{N} \models t_1 \neq t_2; \end{cases}$$

- for every closed $\mathcal{L}$-term t,

$$V_{\mathrm{SK}}(S)(T(t)) \ := \ \begin{cases} 1 & \text{if } \langle \mathfrak{N}, S^+ \rangle \models T(t), \\ 0 & \text{if } \langle \mathfrak{N}, S^- \cup (\mathbb{N} \setminus \#Sen_T) \rangle \models T(t), \\ \frac{1}{2} & \text{otherwise;} \end{cases}$$

- $V_{\mathrm{SK}}(S)(\neg\varphi) := 1 - V_{\mathrm{SK}}(S)(\varphi)$;
- $V_{\mathrm{SK}}(S)(\varphi \wedge \phi) := \min_{\leqslant_{\mathrm{SK}}} \{V_{\mathrm{SK}}(S)(\varphi), V_{\mathrm{SK}}(S)(\phi)\}$;
- $V_{\mathrm{SK}}(S)(\varphi \vee \phi) := V_{\mathrm{SK}}(S)(\neg(\neg\varphi \wedge \neg\phi))$;
- $V_{\mathrm{SK}}(S)(\forall x\, \varphi(x)) := \min_{\leqslant_{\mathrm{SK}}} \{V_{\mathrm{SK}}(S)(\varphi(t)) \mid t \text{ is a closed } \mathcal{L}\text{-term}\}$;
- $V_{\mathrm{SK}}(S)(\exists x\, \varphi(x)) := V_{\mathrm{SK}}(S)(\neg\forall x\, \neg\varphi(x))$.

To get the *weak Kleene valuation scheme* V_{WK}, replace $\leqslant_{\mathrm{SK}}$ by $\leqslant_{\mathrm{WK}}$. Next we turn to the so-called *supervaluation schemes*, each of which has the form

$$V(S)(\varphi) \ := \ \begin{cases} 1 & \text{if for all } A \subseteq \mathbb{N} \text{ satisfying } [*],\ \langle \mathfrak{N}, A \rangle \models \varphi, \\ 0 & \text{if for all } A \subseteq \mathbb{N} \text{ satisfying } [*],\ \langle \mathfrak{N}, A \rangle \models \neg\varphi, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

[5] Henceforth we shall limit ourselves to partial interpretations of T with consistent extensions. For technical reasons, it may also be convenient to assume the falsity of $T(\underline{n})$ for every $n \in \mathbb{N} \setminus \#Sen_T$ (i.e. add to the anti-extension of a given interpretation all natural numbers which are not codes of $\mathcal{L}_T$-sentences).

The best known such schemes are V_{SV}, V_{VB}, V_{FV} and V_{MC}, given by:

$$\begin{aligned}
V &= V_{\mathrm{SV}} && \iff && \texttt{[*]} = \text{`}S^+ \subseteq A\text{'};\\
V &= V_{\mathrm{VB}} && \iff && \texttt{[*]} = \text{`}S^+ \subseteq A \text{ and } A \cap S^- = \varnothing\text{'};\\
V &= V_{\mathrm{FV}} && \iff && \texttt{[*]} = \text{`}S^+ \subseteq A \text{ and } A \text{ is consistent'};\\
V &= V_{\mathrm{MC}} && \iff && \texttt{[*]} = \text{`}S^+ \subseteq A \text{ and } A \text{ is consistent and complete'}.
\end{aligned}$$

Here '*complete*' means that for each $\phi \in Sen_T$ we have $\#\phi \in A$ or $\#\neg\phi \in A$.

The last scheme emerges from [9], and although Leitgeb did not state explicitly the definition presented below, one can easily extract it from his article (see [17, Section 5]). Say that $\varphi \in Sen_T$ *depends on* $A \subseteq \mathbb{N}$ iff for any $B, C \subseteq \mathbb{N}$,

$$A \cap B \;=\; A \cap C \quad \Longrightarrow \quad (\, \langle \mathfrak{N}, B\rangle \models \varphi \quad \Longleftrightarrow \quad \langle \mathfrak{N}, C\rangle \models \varphi \,)$$

— or equivalently, as was observed in [9], iff for every $B \subseteq \mathbb{N}$,

$$\langle \mathfrak{N}, B\rangle \models \varphi \quad \Longleftrightarrow \quad \langle \mathfrak{N}, B \cap A\rangle \models \varphi.$$

Remark: naturally the dependence relation induces a monotone operator on $\mathcal{P}(\mathbb{N})$, namely the function $\mathcal{D}$ that maps each $A \subseteq \mathbb{N}$ to $\#\{\varphi \in Sen_T \mid \varphi \text{ depends on } A\}$. Define *Leitgeb's valuation scheme* V_{L} by

$$V_{\mathrm{L}}(S)(\varphi) \;:=\; \begin{cases} 1 & \text{if } \varphi \text{ depends on } S^+ \cup S^- \text{ and } \langle \mathfrak{N}, S^+\rangle \models \varphi,\\ 0 & \text{if } \varphi \text{ depends on } S^+ \cup S^- \text{ and } \langle \mathfrak{N}, S^+\rangle \models \neg\varphi,\\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

(Cf. [13] for an interesting connection with V_{FV}.)

Before bringing hierarchies into the picture, let

$$\begin{aligned}
\mathsf{Ord} &:= \text{the class of all ordinals},\\
\mathsf{L\text{-}Ord} &:= \text{the class of all limit ordinals},\\
\mathsf{C\text{-}Ord} &:= \text{the class of all constructive ordinals},\\
\omega_1^{\mathrm{CK}} &:= \text{the least element of } \mathsf{Ord} \setminus \mathsf{C\text{-}Ord}.
\end{aligned}$$

Every valuation scheme V induces a function $\mathcal{J}_V$ from partial interpretations to partial interpretations, called the *Kripke-jump operator for* V, as follows:

$$\begin{aligned}
\mathcal{J}_V(S)^+ &:= \{\#\varphi \mid \varphi \in Sen_T \text{ and } V(S)(\varphi) = 1\},\\
\mathcal{J}_V(S)^- &:= \{\#\varphi \mid \varphi \in Sen_T \text{ and } V(S)(\varphi) = 0\} \cup \{n \in \mathbb{N} \mid n \notin \#Sen_T\}.
\end{aligned}$$

In turn, $\mathcal{J}_V$ generates a transfinite sequence indexed by ordinals:

$$\mathcal{J}_V^{\alpha}(S) \;:=\; \begin{cases} S & \text{if } \alpha = 0,\\ \mathcal{J}_V\left(\mathcal{J}_V^{\beta}(S)\right) & \text{if } \alpha = \beta + 1,\\ \langle \bigcup_{\beta<\alpha} \mathcal{J}_V^{\beta}(S)^+, \bigcup_{\beta<\alpha} \mathcal{J}_V^{\beta}(S)^- \rangle & \text{if } \alpha \in \mathsf{L\text{-}Ord}. \end{cases}$$

We often write T_V^{α} instead of $\mathcal{J}_V^{\alpha}(\varnothing, \varnothing)^+$ — these sets, or rather predicates, constitute the *truth hierarchy for* V.

Furthermore, Kripke's article deals with *monotone* schemes, i.e. those which satisfy the condition that for any partial interpretations S_1 and S_2,

$$S_1^+ \subseteq S_2^+ \ \& \ S_1^- \subseteq S_2^- \quad \Longrightarrow \quad \mathcal{J}_V(S_1)^+ \subseteq \mathcal{J}_V(S_2)^+ \ \& \ \mathcal{J}_V(S)^- \subseteq \mathcal{J}_V(S)^- .$$

For each such V we obtain the least — with respect to the product ordering, as you would expect — fixed point of $\mathcal{J}_V$, by a version of the well-known Knaster–Tarski theorem:

Observation 2.1 (S. Kripke)**.** *For any monotone valuation scheme V there exists an ordinal α with $\mathcal{J}_V^\alpha(\varnothing,\varnothing) = \mathcal{J}_V^{\alpha+1}(\varnothing,\varnothing)$, yielding the least fixed point of $\mathcal{J}_V$.*

It is easy to verify that each $V \in \{V_{\mathrm{SK}}, V_{\mathrm{WK}}, V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}, V_{\mathrm{L}}\}$ is monotone and, moreover, has the following properties:[6]

- if $\mathcal{J}_V(S) = S$, then $V(S)(T(\ulcorner\varphi\urcorner)) = V(S)(\varphi)$;
- $\#\varphi \in \mathcal{J}_V^\alpha(S)^-$ iff $\#\neg\varphi \in \mathcal{J}_V^\alpha(S)^+$, and $\#\varphi \in \mathcal{J}_V^\alpha(S)^+$ iff $\#\neg\varphi \in \mathcal{J}_V^\alpha(S)^-$ — in this way $\mathcal{J}_V^\alpha(S)^-$ can be recovered from $\mathcal{J}_V^\alpha(S)^+$, and vice versa;
- $\mathcal{J}_V$ turns out to be a 'Π_1^1-operator' — so by a well-know theorem of Spector (consult [14] for details), $\mathcal{J}_V^\alpha(\varnothing,\varnothing) = \mathcal{J}_V^{\alpha+1}(\varnothing,\varnothing)$ already for some $\alpha \leqslant \omega_1^{\mathrm{CK}}$, and so we may limit ourselves to constructive ordinals, plus their supremum.

The first two properties are straightforward. The third is a bit more complicated, because it assumes a knowledge of the basic techniques from monadic second-order arithmetic, and in fact the Kripke-jump operators for the Kleene valuation schemes are even Δ_1^1 (compare the proof of Corollary 3.2 below). Cf. [8] for a discussion.

Notice also that certain results on the complexity of the corresponding truth hierarchies quickly imply certain non-axiomatisability results; e.g., if T_V^α is Π_1^1-hard, then there exists no computably enumerable set $\mathcal{A}$ of $\mathcal{L}$-sentences such that for any partial interpretation S,

$$\langle \mathfrak{N}, S^+ \rangle \models \mathcal{A} \quad \Longleftrightarrow \quad S^+ \ = \ \mathrm{T}_V^\alpha$$

— see [3] for a proof of this simple fact and its applications.

Finally, in [9], Leitgeb introduced the *groundedness hieararchy* G_α along with the *associated truth hierarchy* Θ_α. However, as was observed in [17, Section 5],

$$\Theta_\alpha \ = \ \mathcal{J}_{V_{\mathrm{L}}}^\alpha(\varnothing,\varnothing)^+ \quad \text{and} \quad \mathrm{G}_\alpha \ = \ \mathcal{J}_{V_{\mathrm{L}}}^\alpha(\varnothing,\varnothing)^+ \cup \mathcal{J}_{V_{\mathrm{L}}}^\alpha(\varnothing,\varnothing)^- .$$

I shall pay specific attention to both of these in Subsection 3.3.

2.2 Monadic second-order arithmetic

Recall that in monadic second-order arithmetic we have

i. *individual variables* $x, y, z, \ldots$ (intended to range over $\mathbb{N}$) and

ii. *set variables* $X, Y, Z, \ldots$ (intended to range over $\mathcal{P}(\mathbb{N})$).

[6] We use S, α and φ to stand for partial valuations, ordinals and $\mathcal{L}_T$-sentences respectively.

Accordingly we distinguish between *individual* and *set quantifiers*:

$$\forall x, \exists x, \forall y, \exists y, \forall z, \exists z, \dots \quad \text{and} \quad \forall X, \exists X, \forall Y, \exists Y, \forall Z, \exists Z, \dots$$

$\mathcal{L}_2$*-formulas* are built up from $\mathcal{L}$-formulas and expressions of the form $t \in X$, where t is an $\mathcal{L}$-term (in the first-order setting) and X is a set variable, using logical connective symbols and quantifiers in the customary way. An $\mathcal{L}_2$-formula is in Π^1_n (Σ^1_n) iff it has the form

$$\underbrace{\forall X_1 \, \exists X_2 \, \forall X_3 \, \dots \, X_n}_{n-1 \text{ alternations}} \Psi \quad \text{(respectively } \underbrace{\exists X_1 \, \forall X_2 \, \exists X_3 \, \dots \, X_n}_{n-1 \text{ alternations}} \Psi)$$

with $X_1, \dots, X_n$ set variables and Ψ containing no set quantifiers.

Let $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$. We say A is *computably reducible* to B iff there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $A = f^{-1}(B)$, i.e.

$$A \; = \; \{k \in \mathbb{N} \mid f(k) \in B\}.$$

We call A and B *computably equivalent* iff they are computably reducible to each other. A is said to be Π^1_n*-bounded* iff there exists a Π^1_n-formula $\Phi(x)$ in $\mathcal{L}_2$ such that

$$A \; = \; \{k \in \mathbb{N} \mid \mathfrak{N} \models \Phi(k)\}.$$

A is called Π^1_n*-hard* iff any Π^1_n-bounded set is computably reducible to it. Finally A is Π^1_n-*complete* iff it is both Π^1_n-bounded and Π^1_n-hard. Similarly for Σ^1_n. Let

$$\Delta \; := \; \left\{\Pi^1_{n+1}, \Sigma^1_{n+1} \mid n \in \mathbb{N}\right\}.$$

We shall be mainly concerned with the complexity classes corresponding to the elements of Δ. In other words, we focus on second-order, excluding the case of $\Pi^1_0 = \Sigma^1_0$.

Folklore 2.2 (cf. [15, § 16.1])**.** *For every $\delta \in \Delta$ the following hold:*

- *if A is computably reducible to B and B is δ-bounded, then A is δ-bounded;*
- *if A is computably reducible to B and A is δ-hard, then B is δ-hard;*
- *the set of (codes of) δ-sentences true in $\mathfrak{N}$ is δ-complete.*

In addition, Δ^1_n*-bounded* sets (of natural numbers) are characterised as those which are both Π^1_n-bounded and Σ^1_n-bounded. But here no 'Δ^1_n-complete set' exists.

By an $\mathcal{L}_2$*-formula positive in* X we mean an $\mathcal{L}_2$-formula in which no free occurrence of X is in the scope of $\neg$ (remember that we treat $\to$ as defined, not as primitive). Given an $\mathcal{L}_2$-formula $\Phi(x, X)$ positive in X and a set A of natural numbers, let

$$\Gamma_\Phi(A) \; := \; \{n \in \mathbb{N} \mid \mathfrak{N} \models \Phi(n, A)\};$$

in this way Φ induces a monotone operator on $\mathcal{P}(\mathbb{N})$, namely Γ_Φ. Further — starting with some $A \subseteq \mathbb{N}$, we inductively define

$$\mathscr{R}^\alpha(\Phi, A) \; := \; \begin{cases} A & \text{if } \alpha = 0, \\ \Gamma_\Phi\left(\mathscr{R}^\beta(\Phi, A)\right) & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} \mathscr{R}^\beta(\Phi, A) & \text{if } \alpha \in \mathsf{L\text{-}Ord}. \end{cases}$$

Such operators and hierarchies play a central role throughout the paper (the reader should bear in mind that, using coding techniques, each $\mathsf{Q}_i X_i$ with $\mathsf{Q}_i \in \{\forall, \exists\}$ in the definition of Π^1_n/Σ^1_n-$\mathcal{L}_2$-formulas can be replaced by $\mathsf{Q}_i X_i^1 \dots \mathsf{Q}_i X_i^{n_i}$, and vice versa).

2.3 Kleene's $\mathcal{O}$

Kleene's system of notation for C-Ord (see e.g. [15, 16]) consists of:

- a special partial function $\nu_{\mathcal{O}}$ from $\mathbb{N}$ onto C-Ord, with domain $\mathrm{dom}\,(\nu_{\mathcal{O}})$;
- a special ordering relation $<_{\mathcal{O}}$ on $\mathrm{dom}\,(\nu_{\mathcal{O}})$, which mimics $<$ on C-Ord.

Call $n \in \mathbb{N}$ a *notation for* $\alpha \in$ C-Ord iff $\nu_{\mathcal{O}}\,(n) = \alpha$. Fixing one's favourite universal partial computable (two-place) function æ, $\nu_{\mathcal{O}}$ and $<_{\mathcal{O}}$ are defined simultaneously by induction:

- The ordinal 0 receives the only notation, namely 1. Thus $\nu_{\mathcal{O}}^{-1}\,(0) = \{1\}$.
- Suppose all ordinals below α have received their notations. And assume that $<_{\mathcal{O}}$ has been defined on these notations.
  - If $\alpha = \beta + 1$, then α receives the notations $\left\{2^k \mid k \in \nu_{\mathcal{O}}^{-1}\,(\beta)\right\}$. Further, for each $k \in \nu_{\mathcal{O}}^{-1}\,(\beta)$ we set $i <_{\mathcal{O}} 2^k$ if $i = k$ or $i <_{\mathcal{O}} k$.
  - If $\alpha \in$ L-Ord, then α receives the notation 3×5^k for every k such that

$$æ_k\,(0) \;<_{\mathcal{O}}\; æ_k\,(1) \;<_{\mathcal{O}}\; æ_k\,(2) \;<_{\mathcal{O}}\; \ldots \quad \text{and} \quad \bigcup_{i\in\mathbb{N}} \nu_{\mathcal{O}}\,(æ_k\,(i)) \;= \alpha$$

  (so in particular, $æ_k$ must be total and all $æ_k\,(i)$ must belong to $\bigcup_{\beta<\alpha} \nu_{\mathcal{O}}^{-1}\,(\beta)$). Further, for each such k we set $i <_{\mathcal{O}} 3 \times 5^k$ if $i <_{\mathcal{O}} æ_k\,(j)$ for some j.

For convenience we often write $n \in \mathcal{O}$ instead of $n \in \mathrm{dom}\,(\nu_{\mathcal{O}})$.

As a classical application of Kleene's fixed-point theorem we obtain

Folklore 2.3 (cf. [16, Theorems 2.2(ii) and 3.2])**.** *Suppose $f : \mathbb{N} \to \mathbb{N}$ is a computable function with the property that for any $e \in \mathbb{N}$ and $n \in \mathcal{O}$,*

$$æ_e\,(k) \text{ is defined for all } k <_{\mathcal{O}} n \quad \Longrightarrow \quad æ_{f(e)}\,(n) \text{ is defined.}$$

Then there exists a $c \in \mathbb{N}$ such that $æ_c = æ_{f(c)}$ and $æ_c\,(n)$ is defined for every $n \in \mathcal{O}$.

The restriction of $<_{\mathcal{O}}$ to $\{k \mid k <_{\mathcal{O}} n\}$ is computably enumerable uniformly in n:

Folklore 2.4 (cf. [16, Theorem 3.5(i)])**.** *There exists a computable function $f : \mathbb{N} \to \mathbb{N}$ with the property that for each $n \in \mathcal{O}$, $\{k \mid k <_{\mathcal{O}} n\} = \mathrm{dom}\left(æ_{f(n)}\right)$.*

Consequently one can find an $\mathcal{L}$-formula $\eta_<\,(x, y)$ such that for all $n \in \mathcal{O}$ and $k \in \mathbb{N}$,

$$k <_{\mathcal{O}} n \quad \Longleftrightarrow \quad \mathfrak{N} \models \eta_<\,(k, n).$$

In Subsection 3.2 we shall exploit $\eta_<$ in encoding Kleene's $\mathcal{O}$ into least fixed-points.

Another basic fact about $\nu_{\mathcal{O}}$ concerns the complexity of the path leading to ω_1^{CK}.

Folklore 2.5 (cf. [16, Theorems 2.2(i) and 5.4])**.** $\mathrm{dom}\,(\nu_{\mathcal{O}})$ *is Π^1_1-complete.*

Readers who want to know more about constructive ordinals and their notations might consult [15] or [16]. However, for our purposes the above three results will suffice.

3 Computational aspects

3.1 Upper bounds

Throughout this subsection δ, δ', etc. stand for elements of Δ, and we also assume that all formulas are in $\mathcal{L}_2$, unless otherwise indicated.

Evidently positive Π^1_1- and Σ^1_1-operators — recall that for Δ^1_1 we need both — play an important role in Kripke's theory of truth. Given $\delta \in \Delta$, take

$$\delta_\Pi \ := \begin{cases} \Pi^1_n & \text{if } \ \delta = \Pi^1_n, \\ \Pi^1_{n+1} & \text{if } \ \delta = \Sigma^1_n \end{cases} \quad \text{and} \quad \delta_\Sigma \ := \begin{cases} \Sigma^1_n & \text{if } \ \delta = \Sigma^1_n, \\ \Sigma^1_{n+1} & \text{if } \ \delta = \Pi^1_n. \end{cases}$$

It is now easy to see how these arise in the hierarchies starting with δ-sets.

Proposition 3.1. *For any Π^1_1-formula $\Phi(x, X)$ positive in X and any δ-set A, there exists a computable function f such that for every $n \in \mathcal{O}$,*

$$f(n) \text{ is a } \delta_\Pi\text{-formula defining } \mathscr{R}^{\nu_{\mathcal{O}}(n)}(\Phi, A) \text{ in } \mathfrak{N}.$$

Similarly for Σ^1_1 and δ_Σ.

Proof. We shall only consider the case of Π^1_1 and δ_Π. An analogous argument will work for Σ^1_1 and δ_Σ. Notice that each δ-formula can be turned into a logically equivalent δ_Π-formula or δ_Σ-formula by adding 'dummy' quantifiers. Let χ_0 be a δ_Π-formula defining A in $\mathfrak{N}$.

Since X occurs only positively in Φ and the δ_Π-sets are closed under effectively enumerable unions, we obtain computable functions s and u such that:

- for each formula χ, if χ is a δ_Π-formula defining a subset B of $\mathbb{N}$ in $\mathfrak{N}$, then $s(\chi)$ is a δ_Π-formula defining $\Gamma_\Phi(B)$ in $\mathfrak{N}$;
- for each natural number n, if $æ_n(0)$, $æ_n(1)$, $\ldots$ are δ_Π-formulas defining subsets B_0, B_1, $\ldots$ of $\mathbb{N}$ in $\mathfrak{N}$, then $u(n)$ is a δ_Π-formula defining $\bigcup_{i\in\mathbb{N}} B_i$ in $\mathfrak{N}$.

Moreover, from the s-m-n theorem we get an injective computable $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the property that $æ_{h(e,k)}(n) = æ_e(æ_k(n))$ for every $\{e, k, n\} \subseteq \mathbb{N}$.

Take g to be a computable function satisfying for all $\{e, n\} \subseteq \mathbb{N}$,

$$æ_{g(e)}(n) \ = \begin{cases} \chi_0 & \text{if } \ n = 1, \\ s(æ_e(k)) & \text{if } \ n = 2^k \neq 1, \\ u(h(e,k)) & \text{if } \ n = 3 \times 5^k, \\ x \neq x & \text{otherwise.} \end{cases}$$

By Folklore 2.3 there exists c for which $æ_{g(c)} = æ_c$. Thus $f := æ_c$ does the job.[7]

□

Of course we can identify partial interpretations of T with sets of natural numbers, e.g. by redefining $S = \langle S^+, S^- \rangle$ as $S^\star = \{2 \times 3^n \mid n \in S^+\} \cup \{3^n \mid n \in S^-\}$. So in particular:

[7] The function $æ_c$ turns out to be total. For otherwise let n be the least element of $\mathbb{N} \setminus \mathrm{dom}(æ_c)$. Then, by the construction of g, we conclude that n must be of the form 2^k with $k \neq 0$. Hence $æ_c(k)$ is undefined for $k = \log_2 n < n$, contradicting the choice of n. This situation is quite typical.

Corollary 3.2. *Let V be a valuation scheme. Then:*

$$\begin{aligned} V \in \{V_{\mathrm{SK}}, V_{\mathrm{WK}}\} \quad &\Longrightarrow \quad \mathrm{T}_V^{\nu_{\mathcal{O}}(n)} \textit{ is } \Delta^1_1\textit{-bounded uniformly in } n \in \mathcal{O}; \\ V \in \{V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}\} \quad &\Longrightarrow \quad \mathrm{T}_V^{\nu_{\mathcal{O}}(n)} \textit{ is } \Pi^1_1\textit{-bounded uniformly in } n \in \mathcal{O}. \end{aligned}$$

Furthermore, $\mathrm{G}_{\nu_{\mathcal{O}}(n)}$ *and* $\Theta_{\nu_{\mathcal{O}}(n)}$ *are* Π^1_1*-bounded uniformly in* $n \in \mathcal{O}$*.*

Proof. It suffices to show that for every scheme V in our list, the function that maps each $S^\star$ to $\mathcal{J}_V(S)^\star$ is induced by a cleverly chosen $\mathcal{L}_2$-formula $\Phi(x, X)$ positive in X, in which case Proposition 3.1 applies.

$\boxed{1}$ Suppose $V \in \{V_{\mathrm{SK}}, V_{\mathrm{WK}}\}$. It is straightforward to obtain an arithmetical predicate $\mathcal{C}$ and a Δ^1_1-predicate $\mathcal{E}$ such that for any $\{i, n\} \cup X \subseteq \mathbb{N}$:

$$\begin{aligned} \mathcal{C}(X) \quad &\Longleftrightarrow \quad X = S^\star \text{ for some partial interpretation } S \text{ of } T; \\ \mathcal{E}(i, n, X) \quad &\Longleftrightarrow \quad \mathcal{C}(X) \text{ and } 2^i \times 3^n \in \mathcal{J}_V(S)^\star \text{ where } S \text{ is the unique} \\ &\qquad \text{partial interpretation of } T \text{ satisfying } X = S^\star.^8 \end{aligned}$$

Since V is monotone, we have:

$$\begin{aligned} \mathcal{E}(i, n, S^\star) \quad &\Longleftrightarrow \quad \mathfrak{N} \models \forall X\, ((S^\star \subseteq X \land \mathcal{C}(X)) \to \mathcal{E}(i, n, X)) \\ &\Longleftrightarrow \quad \mathfrak{N} \models \forall X\, (\exists x\, (x \in S^\star \land x \notin X) \lor \neg\mathcal{C}(X) \lor \mathcal{E}(i, n, X)); \end{aligned}$$

$$\begin{aligned} \mathcal{E}(i, n, S^\star) \quad &\Longleftrightarrow \quad \mathfrak{N} \models \exists X\, (X \subseteq S^\star \land \mathcal{E}(i, n, X)) \\ &\Longleftrightarrow \quad \mathfrak{N} \models \exists X\, (\forall x\, (x \notin X \lor x \in S^\star) \land \mathcal{E}(i, n, X)). \end{aligned}$$

Hence $\mathcal{E}$ can be expressed by a Π^1_1/Σ^1_1-formula positive in $S^\star$. The rest is easy.

$\boxed{2}$ Assume $V \in \{V_{\mathrm{SV}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}\}$. Notice that the predicates

$$\begin{aligned} \mathcal{R} \; &:= \; \{\langle \#\psi, X\rangle \mid \psi \in \mathit{Sen}_T,\ X \subseteq \mathbb{N} \text{ and } \langle \mathfrak{N}, X\rangle \models \psi\} \quad \text{and} \\ \mathcal{U} \; &:= \; \{\langle 1, \#\psi, X\rangle \mid \langle \#\psi, X\rangle \in \mathcal{R}\} \cup \{\langle 0, \#\psi, X\rangle \mid \langle \#\neg\psi, X\rangle \in \mathcal{R}\} \end{aligned}$$

are Δ^1_1. Moreover, for a suitable arithmetical formula $\Psi(X)$ the following holds:

$$\begin{aligned} 2^i \times 3^n \in \mathcal{J}_V(S)^\star \quad &\Longleftrightarrow \quad \mathfrak{N} \models \forall X\, \big(\big(S^+ \subseteq X \land \Psi(X)\big) \to \mathcal{U}(i, n, X)\big) \\ &\Longleftrightarrow \quad \mathfrak{N} \models \forall X\, \big(\exists x\, \big(x \in S^+ \land x \notin X\big) \lor \neg\Psi(X) \lor \mathcal{U}(i, n, X)\big). \end{aligned}$$

So we get a Π^1_1-formula positive in $S^\star$, as desired.

The argument for $V = V_{\mathrm{VB}}$ is the same as for the other supervaluation schemes, except that we use '$\Psi(X) \land X \cap S^- = \varnothing$' instead of '$\Psi(X)$'. It works because the negation of the new expression is logically equivalent to

$$\neg\Psi(X) \land \exists x\, \big(x \in X \land x \in S^-\big),$$

and this again leads us to a Π^1_1-formula positive in $S^\star$.

$\boxed{3}$ Now let $V = V_{\mathrm{L}}$. First observe that for every $\psi \in \mathit{Sen}_T$,

$$\begin{aligned} \#\psi \in \mathcal{D}(X) \quad &\Longleftrightarrow \quad \mathfrak{N} \models \forall Y\, \forall Z\, (Z \subseteq \mathbb{N} \setminus X \to (\mathcal{R}(\#\psi, Y \setminus Z) \leftrightarrow \mathcal{R}(\#\psi, Y))) \\ &\Longleftrightarrow \quad \mathfrak{N} \models \forall Y\, \forall Z\, (\exists x\, (x \in X \land x \in Z) \lor (\mathcal{R}(\#\psi, Y \setminus Z) \leftrightarrow \mathcal{R}(\#\psi, Y))). \end{aligned}$$

[8] Obviously there can be at most one interpretation S for which $S^\star = X$.

Thus Leitgeb's dependence operator can be expressed by a Π^1_1-formula $\Phi(x, X)$ positive in X. As for the hierarchy of T^α_V's, since V is monotone, we have

$$\begin{aligned}
2^i \times 3^n \in \mathcal{J}_V(S)^\star \quad &\Longleftrightarrow \quad \mathcal{U}\left(i, n, S^+\right) \text{ and } n \in \mathcal{D}\left(S^+ \cup S^-\right) \\
&\Longleftrightarrow \quad \mathfrak{N} \models \forall X \left(S^+ \subseteq X \to \mathcal{U}(i, n, X)\right) \wedge \Phi\left(n, S^+ \cup S^-\right) \\
&\Longleftrightarrow \quad \mathfrak{N} \models \forall X \left(\exists x \left(x \in S^+ \wedge x \notin X\right) \vee \mathcal{U}(i, n, X)\right) \wedge \Phi\left(n, S^+ \cup S^-\right),
\end{aligned}$$

which clearly reduces to a Π^1_1-formula positive in $S^\star$. $\square$

An interesting thing happens when we turn to the first non-constructive ordinal.

Corollary 3.3. *For any* $V \in \{V_{\mathrm{SK}}, V_{\mathrm{WK}}, V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}\}$, $\mathrm{T}_V^{\omega_1^{\mathrm{CK}}}$ *is* Π^1_1*-bounded.*

Proof. Remember that $\mathrm{dom}(\nu_{\mathcal{O}})$ is a Π^1_1-set, by Folklore 2.5. Certainly

$$n \;\in\; \mathrm{T}_V^{\omega_1^{\mathrm{CK}}} \quad \Longleftrightarrow \quad \text{there exists } \; k \in \mathrm{dom}(\nu_{\mathcal{O}}) \; \text{ such that } \; n \in \mathrm{T}_V^{\nu_{\mathcal{O}}(k)}.$$

So using Corollary 3.2, one can write down a Π^1_1-formula defining $\mathrm{T}_V^{\omega_1^{\mathrm{CK}}}$ in $\mathfrak{N}$. $\square$

Finally consider the case of V_{L}. For notational ease let

$$\Theta \;:=\; \Theta_{\omega_1^{\mathrm{CK}}} \quad \text{and} \quad \mathrm{G} \;:=\; \mathrm{G}_{\omega_1^{\mathrm{CK}}}.$$

Now we quickly deduce the analogous result for these sets.

Corollary 3.4. Θ *and* G *are* Π^1_1*-bounded.*

Proof. The argument for Θ is the same as in the previous proof, using V_{L} for V. Take

$$\Theta' \;:=\; \#\{\psi \in \mathit{Sen}_T \mid \#\neg\psi \in \Theta\}.$$

Obviously Θ', being computably reducible to Θ, is Π^1_1-bounded as well. We also know that $\mathrm{G} = \Theta \cup \Theta'$. Thus the Π^1_1-boundedness of G follows. $\square$

The reader may ask whether Π^1_1 is an accurate bound for the Kleene valuation schemes (since all the lower levels are only Δ^1_1). Yes, and roughly speaking, the main reason is that 'copies of $\mathrm{dom}(\nu_{\mathcal{O}})$' cannot be avoided, as we shall see in the next subsection.

3.2 About least fixed-points

Given a valuation scheme V, by the *rank* of an $\mathcal{L}_T$-sentence ψ, denoted by $\mathit{rank}_V(\psi)$, we mean the least ordinal α such that $\psi \in \mathrm{T}_V^{\alpha+1}$. Call V *ordinary* iff for any $\alpha \in \mathsf{Ord}$, $\chi \in \mathit{Sen}$, $\psi \in \mathit{Sen}_T$ and $\varphi(x) \in \mathit{For}_T$ the following conditions hold:

1. $\mathrm{T}^\alpha_V \subseteq \mathrm{T}^{\alpha+1}_V$;

2. $\chi \in \mathrm{T}^\alpha_V$ iff $\alpha \neq 0$ and $\mathfrak{N} \models \chi$;

3. $\psi \in \mathrm{T}^\alpha_V$ iff $T(\ulcorner\psi\urcorner) \in \mathrm{T}^{\alpha+1}_V$;

4. $\forall x\, \varphi(x) \in \mathrm{T}^{\alpha+1}_V$ iff $\{\varphi(\underline{n}) \mid n \in \mathbb{N}\} \subseteq \mathrm{T}^{\alpha+1}_V$;

5. $\chi \wedge \psi \in \mathrm{T}^\alpha_V$ iff $\mathfrak{N} \models \chi$ and $\varphi \in \mathrm{T}^\alpha_V$;

6. if $\chi \vee \psi \in \mathrm{T}_V^\alpha$ and $\mathfrak{N} \models \neg\chi$, then $\psi \in \mathrm{T}_V^\alpha$;

7. if $\mathfrak{N} \models \chi$ and $\alpha \neq 0$, then $\chi \vee \psi \in \mathrm{T}_V^\alpha$.

Notice that (7) fails for the weak Kleene scheme. However, all the other valuation schemes considered above are ordinary, as one readily checks.

Proposition 3.5. *Let V be a valuation scheme satisfying (3–4). Then for each $\mathcal{L}_T$-sentence ψ and each $\mathcal{L}_T$-formula $\varphi(x)$ we have*

$$\begin{aligned} rank_V\left(T\left(\ulcorner\psi\urcorner\right)\right) &= rank_V(\psi)+1 \quad \textit{and} \\ rank_V\left(\forall x\, \varphi(x)\right) &= \sup\{rank_V\left(\varphi(\underline{n})\right) \mid n \in \mathbb{N}\}. \end{aligned}$$

Proof. Certainly $rank_V\left(T\left(\ulcorner\psi\urcorner\right)\right)$ cannot be 0 — because $\mathrm{T}_V^0 = \varnothing$. Moreover, since $\mathrm{T}_V^\alpha = \bigcup_{\beta<\alpha}\mathrm{T}_V^\beta$ for all $\alpha \in \mathsf{L\text{-}Ord}$, it cannot be a limit ordinal, too — for otherwise

$$\begin{aligned} rank_V\left(T\left(\ulcorner\psi\urcorner\right)\right) \;=\; \alpha \quad &\Longrightarrow \quad \psi \;\in\; \mathrm{T}_V^\alpha \quad \Longrightarrow \quad \psi \;\in\; \mathrm{T}_V^\beta \;\text{ for some } \beta<\alpha \\ \Longrightarrow \quad T\left(\ulcorner\psi\urcorner\right) \;\in\; \mathrm{T}_V^{\beta+1} \quad &\Longrightarrow \quad rank_V\left(T\left(\ulcorner\psi\urcorner\right)\right) \;\leqslant\; \beta \;<\; \alpha. \end{aligned}$$

The rest is straightforward. $\square$

Henceforth we shall make use of this simple fact without explicit mention.[9]

Proposition 3.6. *For any ordinary valuation scheme V there exists a computable function ρ_V such that for every $n \in \mathcal{O}$, $rank_V(\rho_V(n)) = \nu_{\mathcal{O}}(n)+1$.*

Proof. Clearly we can find an $\mathcal{L}$-formula $\vartheta(x,y,z)$ defining the relation '$æ_x(y) = z$' — viz. the set $\{(k,i,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} \mid æ_k(i) = j\}$ — in $\mathfrak{N}$. Given $k \in \mathbb{N}$, let

$$\chi_k \;:=\; \forall x\, \forall y\, \exists u\, \exists v\left(\neg x<y \vee \left(\vartheta\left(\underline{k},x,u\right) \wedge \vartheta\left(\underline{k},y,v\right) \wedge \eta_<(u,v)\right)\right)$$

(with $\eta_<$ as in Subsection 2.3). There are two observations to be made concerning χ_k:

i. $\mathfrak{N} \models \chi_k$ implies $\mathfrak{N} \models \forall x\, \exists u\, \vartheta\left(\underline{k},x,u\right)$, i.e. that $æ_k$ is total;

ii. $\mathfrak{N} \models \chi_k$ and $æ_k(\mathbb{N}) \subseteq \mathcal{O}$ jointly imply $3\times 5^k \in \mathcal{O}$, and conversely.[10]

Next we obtain computable functions s and u such that:

a. s maps each $\psi \in Sen_T$ to $T\left(\ulcorner\psi\urcorner\right)$;

b. u maps each $n \in \mathbb{N}$ to $T\left(\ulcorner\forall x\, \forall y\left(\neg\vartheta\left(\underline{n},x,y\right) \vee T(y)\right)\urcorner\right)$.

Finally let h be as in the proof of Proposition 3.1.

Now take g to be a computable function satisfying for all $\{e,n\} \subseteq \mathbb{N}$,

$$æ_{g(e)}(n) \;=\; \begin{cases} T\left(\ulcorner 0=0\urcorner\right) & \text{if } n=1, \\ s\left(æ_e(k)\right) & \text{if } n=2^k\neq 1, \\ \chi_k \wedge u\left(h(e,k)\right) & \text{if } n=3\times 5^k, \\ 0\neq 0 & \text{otherwise.} \end{cases}$$

By Folklore 2.3 there exists c for which $æ_{g(c)} = æ_c$; thus it remains to check that $\rho_V := æ_c$ does the job. By induction on $\alpha \in \mathsf{C\text{-}Ord}$. Consider an arbitrary $n \in \nu_{\mathcal{O}}^{-1}(\alpha)$.

[9] Of course, functions like $rank_V$ commonly occur in the theory of positive inductive definitions (cf. [14, Section 2B]) and its applications (for a recent example see [12], where a notion of rank for infinite tableaus is exploited). They behave similarly, although most of them look more complicated than $rank_V$.

[10] Remember, by definition $3\times 5^k \in \mathcal{O}$ iff $æ_k(0) <_{\mathcal{O}} æ_k(1) <_{\mathcal{O}} æ_k(2) <_{\mathcal{O}} \ldots$

- Suppose $n = 1$, so $\rho_V(n) = T(\ulcorner 0 = 0 \urcorner)$. Then $rank_V(\rho_V(n)) = 1$, as desired.
- Suppose $n = 2^k \neq 1$, so $\rho_V(n) = T(\ulcorner \rho_V(k) \urcorner)$. Then

$$rank_V(\rho_V(n)) \;=\; rank_V(\rho_V(k)) + 1 \;=\; \nu_{\mathcal{O}}(k) + 1 + 1 \;=\; \nu_{\mathcal{O}}(n) + 1.$$

- Suppose $n = 3 \times 5^k$ — in particular $æ_k$ is total. Then

$$\begin{aligned} rank_V(\rho_V(n)) &= \sup\{rank_V(T(\ulcorner \rho_V(æ_k(i)) \urcorner)) \mid i \in \mathbb{N}\} + 1 \\ &= \sup\{rank_V(\rho_V(æ_k(i))) + 1 \mid i \in \mathbb{N}\} + 1 \\ &= \sup\{\nu_{\mathcal{O}}(æ_k(i)) \mid i \in \mathbb{N}\} + 1 \;=\; \nu_{\mathcal{O}}(n) + 1. \end{aligned}$$ $\square$

Corollary 3.7. *For every ordinary valuation scheme V, if $\mathrm{T}_V^{\alpha} = \mathrm{T}_V^{\alpha+1}$ (or equivalently, if $\mathrm{T}_V^{\alpha} = \bigcup_{\beta \in \mathsf{Ord}} \mathrm{T}_V^{\beta}$), then $\alpha \geqslant \omega_1^{\mathrm{CK}}$ and T_V^{α} is Π_1^1-hard.*

Proof. Assume $\mathrm{T}_V^{\alpha} = \mathrm{T}_V^{\alpha+1}$. Hence Proposition 3.6 immediately gives $\alpha \geqslant \omega_1^{\mathrm{CK}}$. As for the Π_1^1-hardness of T_V^{α}, it suffices to show that for all $n \in \mathbb{N}$,

$$n \;\in\; \mathcal{O} \quad \Longleftrightarrow \quad \rho_V(n) \;\in\; \mathrm{T}_V^{\alpha}$$

— then the result will follow by Folklore 2.5. The implication from left to right is obvious. In the other direction, consider

$$S \;:=\; \{n \notin \mathcal{O} \mid \rho_V(n) \in \mathrm{T}_V^{\alpha}\}.$$

Suppose $S \neq \varnothing$, and let β be the least ordinal in $rank_V(\rho_V(S))$. So in particular we have $\beta = rank_V(\rho_V(n))$ for a suitable $n \in S$; thus β must be a successor ordinal and $n \neq 1$.

- If $n = 2^k \neq 1$, then $\rho_V(n) = T(\ulcorner \rho_V(k) \urcorner)$, whence $k \in S$. This contradicts the choice of β, because $rank_V(\rho_V(k)) = \beta - 1 < \beta$.
- If $n = 3 \times 5^k$, then $\mathfrak{N} \models \chi_k$ and $æ_k(i) \in S$ for some $i \in \mathbb{N}$, as one readily checks. But $rank_V(\rho_V(æ_k(i))) \leqslant \beta - 1 < \beta$, a contradiction.

Since $0 \neq 0$ does not belong to T_V^{α}, we conclude $S = \varnothing$, as desired. $\square$

This technique can be applied (with minor modifications) in various other situations as well. Let us see how it works e.g. for the weak Kleene scheme. However, as it was shown in [2], the reader should be warned:

> *Actually certain complexity results for the weak Kleene scheme depend on the Gödel numbering and the language of the 'standard model' of $\mathbb{N}$ we use.*

Of course such facts reveal counter-intuitive features of the construction, so J. Cain and Z. Damnjanovic suggested adding a special function symbol to resolve the conflict. More precisely, assuming an appropriate coding $\mathrm{M}_0, \mathrm{M}_1, \ldots$ of all Turing machines, they introduced a new symbol $\mathfrak{u}$ whose interpretation is given by

$$\mathfrak{u}(n, i, k, j) \;:=\; \begin{cases} l & \text{if } \mathrm{M}_n \text{ halts on input } i \text{ at step } k \text{ with output } l, \\ j & \text{if } \mathrm{M}_n \text{ does not halt on input } i \text{ at step } k. \end{cases}$$

Notice that this function is primitive recursive, and hence representable in Robinson arithmetic. What can we do with $\mathfrak{u}$ in our framework?

Observation 3.8. *If we include* $\mathfrak{u}$ *in* σ*, then Proposition 3.6 and Corollary 3.7 generalise to arbitrary valuation schemes satisfying (1–5).*

Proof. We do not need to exploit disjunctions — simply replace $\forall x\,\forall y\,(\neg\vartheta\,(\underline{n},x,y)\vee T\,(y))$ by $\forall x\,\forall y\,T\,(\mathfrak{u}\,(\underline{n},x,y,\ulcorner\forall x\,x=x\urcorner))$ in the description of u. The rest is routine. $\square$

Indeed $\mathfrak{u}$ looks quite peculiar from a number-theoretic viewpoint, and one may well ask whether a more elegant function has been discovered. Here we propose to add a symbol $\dot{-}$ for the proper subtraction, i.e. $i\,\dot{-}\,j:=\max\{0,i-j\}$.

Observation 3.9. *Similar to Observation 3.8, but with* $\dot{-}$ *instead of* $\mathfrak{u}$*.*

Proof. It is known (cf. [11]) that there is an algorithm which finds, for each natural number n, a pair $\left(p_n^1\,(\vec{x})\,,p_n^1\,(\vec{x})\right)$ of polynomials with coefficients in $\mathbb{N}$, such that

$$\text{the range of } æ_n \;=\; \left\{p_n^1\,(\vec{m})-p_n^2\,(\vec{m})\mid \vec{m}\in\mathbb{N}^* \text{ and } p_n^1\,(\vec{m})\geqslant p_n^2\,(\vec{m})\right\}$$

— let t_n^1 and t_n^2 be the corresponding terms in the language $\{0,\mathsf{s},+,\times\}$. Also, we need the term $t^{\mathfrak{u}}\,(x,y,z):=(x\,\dot{-}\,y)+(z\times(\mathsf{s}\,(0)\,\dot{-}\,(\mathsf{s}\,(x)\,\dot{-}\,y)))$. Clearly

$$\left(t^{\mathfrak{u}}\,(i,k,j)\right)^{\mathbb{N}} \;=\; \begin{cases} i\,\dot{-}\,k & \text{if } i\geqslant k,\\ j & \text{otherwise.}\end{cases}$$

Thus one can replace $\forall x\,\forall y\,(\neg\vartheta\,(\underline{n},x,y)\vee T\,(y))$ by $\forall\vec{x}\,T\left(t^{\mathfrak{u}}\left(t_n^1\,(\vec{x})\,,t_n^2\,(\vec{x})\,,\ulcorner\forall x\,x=x\urcorner\right)\right)$ in the description of u in the proof of Proposition 3.6. $\square$

Another modification, with $\mathcal{L}$ unchanged, concerns existential quantifiers. Consider the following condition (for any $\alpha\in\mathsf{Ord}$, $\theta\,(x)\in \mathit{For}$ and $\varphi\,(x)\in \mathit{For}_T$):

8. $\exists x\,(\theta\,(x)\wedge T\,(x))\in \mathrm{T}_V^\alpha$ iff $\mathfrak{N}\models\theta\,(\underline{n})$ and $T\,(\underline{n})\in\mathrm{T}_V^\alpha$ for some $n\in\mathbb{N}$.

In the presence of (8) we may omit (6–7), i.e. forget about disjunctions.

Observation 3.10. *The analogues of Proposition 3.6 and Corollary 3.7 hold for arbitrary valuation schemes satisfying (1–5) and (8).*

Proof. Use $\forall x\,\exists y\,(\vartheta\,(\underline{n},x,y)\wedge T\,(y))$ instead of $\forall x\,\forall y\,(\neg\vartheta\,(\underline{n},x,y)\vee T\,(y))$ throughout. $\square$

In effect, (8) fails for the weak Kleene scheme, but the customary treatment of $\exists$ in the case of V_{WK} does not seem to be well motivated — see e.g. [10, § 2.3]. Alternatively, we can define V_{WK}^* exactly as V_{WK} except that

$$V_{\mathrm{WK}}^*\,(\exists x\,\varphi\,(x)) \;=\; \begin{cases} 1 & \text{if } V_{\mathrm{WK}}^*\,(\varphi\,(t))=1 \text{ for some closed } \mathcal{L}\text{-term } t,\\ 0 & \text{if } V_{\mathrm{WK}}^*\,(\varphi\,(t))=0 \text{ for all closed } \mathcal{L}\text{-terms } t,\\ \frac{1}{2} & \text{otherwise.}\end{cases}$$

(like in V_{SK}). Now V_{WK}^* satisfies (1–5) and (8), so Observation 3.10 applies.

Remember that we took $\rightarrow$ as an abbreviation in Subsection 2.1. However, interpreting $\varphi\rightarrow\psi$ as $\neg\varphi\vee\psi$ is not always the right choice in every situation. To avoid confusion, and to make things clearer, I add a new connective symbol $\twoheadrightarrow$ to the original three, i.e. to $\neg$, $\wedge$ and $\vee$. Note that *For*, For_T, *Sen* and Sen_T are easily modified to accommodate $\twoheadrightarrow$. Intuitively, even when we treat $\twoheadrightarrow$ as the material conditional on $\{0,1\}$, the meanings of $\varphi\twoheadrightarrow\psi$ and $\neg\varphi\vee\psi$ may differ on $\left\{0,\frac{1}{2},1\right\}$. Consider the following variation on (6–7):

6'. if $\chi \twoheadrightarrow \psi \in \mathrm{T}_V^\alpha$ and $\mathfrak{N} \models \chi$, then $\psi \in \mathrm{T}_V^\alpha$;

7'. if $\mathfrak{N} \models \neg\chi$, then $\chi \twoheadrightarrow \psi \in \mathrm{T}_V^\alpha$

— where χ and ψ range over the modified versions of *Sen* and Sen_T respectively.

Observation 3.11. *If we expand $\mathcal{L}$ and $\mathcal{L}_T$ by adding $\twoheadrightarrow$, then the analogues of Proposition 3.6 and Corollary 3.7 hold for arbitrary valuation schemes satisfying (1–5) and (6'–7').*

Proof. Replace $\forall x \, \forall y \, (\neg\vartheta\,(\underline{n}, x, y) \vee T\,(y))$ by $\forall x \, \forall y \, (\vartheta\,(\underline{n}, x, y) \twoheadrightarrow T\,(y))$ throughout. $\square$

This is closely related to a three-valued scheme employed in [5]. Namely, Feferman (see Section 3 of his article) proposed extending V_{WK} to formulas containing $\twoheadrightarrow$ by setting

$$V'_{\mathrm{WK}}\,(\varphi \twoheadrightarrow \psi) \;:=\; \begin{cases} 1 & \text{if } V'_{\mathrm{WK}}\,(\varphi) = 0 \text{ or } V'_{\mathrm{WK}}\,(\psi) = V'_{\mathrm{WK}}\,(\varphi) = 1, \\ 0 & \text{if } V'_{\mathrm{WK}}\,(\varphi) = 1 \text{ and } V'_{\mathrm{WK}}\,(\psi) = 0, \\ \frac{1}{2} & \text{otherwise;} \end{cases}$$

the other clauses are the same as in the definition of V_{WK}, using V'_{WK} for V_{WK}. One easily checks that V'_{WK} satisfies (1–5) and (6'–7'). Hence Observation 3.11 applies. The complexity results for V'_{WK} thus do not depend on the choice of Gödel numbering.

We finish with a few general remarks:

- it is not necessary to start with the empty set, because for a given scheme V one can take V' such that $V'\,(S)\,(\psi) = 1$ iff $V\,(S)\,(\psi) = 1$ or $\#\psi \in S^+$;
- as a matter of fact, valuation schemes need not be three-valued, e.g. almost the same proofs go through for Fitting's four-valued version suggested in [6];
- analogous arguments work for other naturally arising hierarchies, like those of sets of false $\mathcal{L}_T$-sentences, or of sets of grounded $\mathcal{L}_T$-sentences.

3.3 Some strengthenings

In the case of the supervaluation schemes and Leitgeb's scheme we can, in effect, recognise a Π^1_1-complete problem already at the first level, and moreover one such problem will work for all the successive levels (see [20]); however, according to Proposition 3.6, to get the least fixed-point, we need to continue moving up the ordinals up to ω_1^{CK}.

We shall provide a somewhat more direct proof of this fact (for instance neither Kleene normal form nor any coding of sequences will be involved). Indeed, the basic idea could be extracted from [20] or [1], and is quite simple: use an appropriate collection of ungrounded $\mathcal{L}_T$-sentences to interpret a second-order universal quantifier.

But before doing that, let us make a few remarks. Clearly we can view any $\mathcal{L}_T$-formula as an arithmetical $\mathcal{L}_2$-formula whose only second-order variable is T, and vice versa. Next, given an $\mathcal{L}_T$-sentence ψ and an $\mathcal{L}$-formula $\chi\,(x)$, construct

$$\psi_\chi \;:=\; \text{the result of replacing every } T\,(t) \text{ in } \psi \text{ by } \chi\,(t) \wedge T\,(t).$$

Then $\mathfrak{N} \models \forall T\,(\psi_\chi\,(T) \leftrightarrow \psi\,(T \cap A))$ where A denotes $\{n \in \mathbb{N} \mid \mathfrak{N} \models \chi\,(n)\}$.

Observation 3.12. *Let $\chi\,(x)$ be an $\mathcal{L}$-formula which defines an infinite computable subset of $\mathbb{N}$ in $\mathfrak{N}$. Then $\Pi_\chi := \{\psi_\chi \mid \psi \in Sen_T \text{ and } \mathfrak{N} \models \forall T\,\psi_\chi\,(T)\}$ is Π^1_1-complete.*

Proof. Take $A := \{n \in \mathbb{N} \mid \mathfrak{N} \models \chi(n)\}$. Remember — the set of all true $\mathcal{L}_2$-sentences of the form $\forall X\, \Psi$ with Ψ arithmetical is Π^1_1-complete.

Obviously Π_χ is Π^1_1-bounded. To obtain Π^1_1-hardness, we consider an $\mathcal{L}$-formula $\xi(x, y)$ defining some one-one function f from $\mathbb{N}$ onto A in $\mathfrak{N}$. For $\psi \in Sen_T$, let

$$\psi' \;:=\; \text{the result of replacing each } T(t) \text{ in } \psi \text{ by } \exists y\,(\xi(t, y) \wedge T(y))$$

where y is the first individual variable not ocurring in ψ. Thus

$$\mathfrak{N} \models \forall T\, \psi(T) \iff \mathfrak{N} \models \forall T\, \psi'(f(T)) \iff$$
$$\mathfrak{N} \models \forall T\, (T \subseteq A \rightarrow \psi'(T)) \iff \mathfrak{N} \models \forall T\, (\psi'(T))_\chi$$

(here we identify $T(t)$ with $t \in T$, so $f(T)(t)$ stands for $t \in f(T)$). □

It gives probably the quickest way to get the desired results.

Theorem 3.13 (P. D. Welch, G. Hjorth, T. Meadows)**.** *For every ordinal $\alpha > 0$ and every valuation scheme $V \in \{V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}, V_{\mathrm{L}}\}$, T^α_V is Π^1_1-hard.*

Proof. Assume $V = V_{\mathrm{L}}$. Take $A := \#\{\mu, T(\ulcorner\mu\urcorner), T(\ulcorner T(\ulcorner\mu\urcorner)\urcorner), \dots\}$, where μ denotes the truthteller. Let χ be an $\mathcal{L}$-formula defining A in $\mathfrak{N}$. Then since $A \cap \mathrm{G} = \varnothing$, we obtain

$$\begin{aligned}
\#\psi_\chi \in \mathrm{T}^{\beta+1}_V &\iff \#\psi_\chi \in \mathrm{G}_{\beta+1} \text{ and } \langle \mathfrak{N}, \mathrm{T}^\beta_V \rangle \models \psi_\chi \\
&\iff \mathfrak{N} \models \forall T\, (\psi_\chi(T \cap \mathrm{G}_\beta) \leftrightarrow \psi_\chi(T)) \wedge \psi_\chi(\mathrm{T}^\beta_V) \\
&\iff \mathfrak{N} \models \forall T\, (\psi_\chi(\varnothing) \leftrightarrow \psi_\chi(T)) \wedge \psi_\chi(\varnothing) \\
&\iff \mathfrak{N} \models \forall T\, \psi_\chi(T).
\end{aligned}$$

Obviously $\mathrm{T}^\alpha_V = \bigcup_{\beta<\alpha} \mathrm{T}^{\beta+1}_V$, and so for each $\alpha > 0$ we have: $\#\psi_\chi \in \mathrm{T}^\alpha_V$ iff $\psi_\chi \in \Pi_\chi$. Thus the Π^1_1-hardness of T^α_V follows by Observation 3.12. (Notice that one may use any suitable collection of ungrounded sentences instead of A.)

Perfectly analogous arguments apply to the other schemes. □

Now we quickly deduce the same for Leitgeb's groundedness hierarchy.

Corollary 3.14 (P. D. Welch)**.** *For every ordinal $\alpha > 0$, G_α is Π^1_1-hard.*

Proof. Let A and χ be as in the previous proof. By construction, $\Theta_\alpha \cap \{\psi_\chi \mid \psi \in Sen_T\}$ is Π^1_1-hard (for $\alpha > 0$). Choose $n \notin \mathrm{G} \cup A$, say $n = \#\lambda$. One readily checks that

$$\#\psi_\chi \in \Theta_\alpha \iff \mathfrak{N} \models \forall T\, \psi(T \cap A) \iff$$
$$\mathfrak{N} \models \forall T\, (\psi(T \cap A) \vee T(\underline{n})) \iff \#(\psi_\chi \vee T(\underline{n})) \in \mathrm{G}_\alpha.$$
□

Consequently the corresponding bounds from Subsection 3.1 turn out to be exact.

Further — the Π^1_1-hardness proofs in this subsection do not seem to rely heavily on the strength of $\mathfrak{N}$, so the reader might well ask:

What will happen if we restrict ourselves to a reasonable fragment of $\mathcal{L}_T$?

E.g. consider $V = V_{\mathrm{L}}$. Clearly, for a particular ordinal $\alpha > 0$, in the proof of Theorem 3.13 we can replace A by any infinite computable $B_\alpha \subseteq \mathbb{N}$ with the property that $B_\alpha \cap \mathrm{G}_\beta = \varnothing$ for all $\beta < \alpha$. Let our language be, say, the $\times$-free fragment of $\mathcal{L}_T$, i.e. $\{0, \mathsf{s}, +, =, T\}$. Now by taking $B_1 = \mathbb{N}$ and using the result of [7], it is easy to show that

- $\{\psi \in \mathrm{T}_V^1 \mid \psi$ does not contain $\times\}$ is Π_1^1-complete.

Obviously for $\alpha > 1$, B_α must be a proper subset of $\mathbb{N}$. And if some one-one function from $\mathbb{N}$ onto B_α is first-order definable in $\langle \mathbb{N}; +, = \rangle$, then the same argument goes through; this, however, depends on the Gödel numbering, because each G_β consists of codes of sentences. Similarly for certain other reducts of the standard model of $\mathbb{N}$, using results of [18, 19].

Acknowledgements. I would like to thank all those who provided useful comments and encouraged me with the writing of this article. In particular I am grateful to the members of the Munich Center for Mathematical Philosophy, especially Hannes Leitgeb and Thomas Schindler (who is now at the University of Cambridge), for their interest in this work. The research was partially supported by the Alexander von Humboldt Foundation.

References

[1] J. P. Burgess (1986). The truth is never simple. *Journal of Symbolic Logic* 51(4), 663–681. DOI: 10.2307/2274021

[2] J. Cain and Z. Damnjanovic (1991). On the weak Kleene scheme in Kripke's theory of truth. *Journal of Symbolic Logic* 56(4), 1452–1468. DOI: 10.2307/2275486

[3] M. Fischer, V. Halbach, J. Kriener and J. Stern (2015). Axiomatizing semantic theories of truth? *Review of Symbolic Logic* 8(2), 257–278. DOI: 10.1017/S1755020314000379

[4] P. Hájek and P. Pudlák (1993). *Metamathematics of First-Order Arithmetic*. Springer.

[5] S. Feferman (2008). Axioms for determinateness and truth. *Review of Symbolic Logic* 1(2), 204–217. DOI: 10.1017/S1755020308080209

[6] M. Fitting (1997). A theory of truth that prefers falsehood. *Journal of Philosophical Logic* 26(5), 477–500. DOI: 10.1023/A:1004217812355

[7] J. Y. Halpern (1991). Presburger arithmetic with unary predicates is Π_1^1 complete. *Journal of Symbolic Logic* 56(2), 637–642. DOI: 10.2307/2274706

[8] S. Kripke (1975). Outline of a theory of truth. *The Journal of Philosophy* 72(19), 690–716. DOI: 10.2307/2024634

[9] H. Leitgeb (2005). What truth depends on. *Journal of Philosophical Logic* 34(2), 155–192. DOI: 10.1007/s10992-004-3758-3

[10] O. Magidor (2013). *Category Mistakes*. Oxford University Press.

[11] Yu. V. Matiyasevich (1993). *Hilbert's Tenth Problem*. MIT Press.

[12] T. Meadows (2015). Infinitary tableau for semantic truth. *Review of Symbolic Logic* 8(2), 207–235. DOI: 10.1017/S175502031500012X

[13] T. Meadows (2013). Truth, dependence and supervaluation: living with the ghost. *Journal of Philosophical Logic* 42(2), 221–240. DOI: 10.1007/s10992-011-9219-x

[14] Y. N. Moschovakis (1974). *Elementary Induction on Abstract Structures.* North-Holland Publishing Company.

[15] H. Rogers, Jr. (1967). *Theory of Recursive Functions and Effective Computability.* McGraw-Hill Book Company.

[16] G. E. Sacks (1990). *Higher Recursion Theory.* Springer.

[17] T. Schindler (2015). *Type-Free Truth* (PhD Thesis). Ludwig-Maximilians-Universität München. Available online at https://edoc.ub.uni-muenchen.de/18335/

[18] S. O. Speranski (2013). A note on definability in fragments of arithmetic with free unary predicates. *Archive for Mathematical Logic* 52(5–6), 507–516. DOI: 10.1007/s00153-013-0328-9

[19] S. O. Speranski (2015). Some new results in monadic second-order arithmetic. *Computability* 4(2), 159–174. DOI: 10.3233/COM-150036

[20] P. D. Welch (2014). The complexity of the dependence operator. *Journal of Philosophical Logic* 44(3), 337–340. DOI: 10.1007/s10992-014-9324-8