Negation as a modality in a quantified setting

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Abstract

The idea of treating negation as a modality manifests itself in various logical systems, especially in Došen's propositional logic N, whose negation is weaker than that of Johansson's minimal logic. Among the interesting extensions of N are the propositional logics N^* and Hype; the former was proposed in [1], while the latter has recently been advocated in [12], but was first introduced in [13]. I shall develop predicate versions of N and N^*, and provide a simple Routley-style semantics for the predicate version of Hype. The corresponding strong completeness results will be proved by means of a useful general technique. It should be remarked that this work can also be seen as a starting point for the investigation of intuitionistic predicate modal logics.

Keywords modal negation · intuitionistic modal logics · quantification · Heyting–Ockham logic · Hype · Routley star

Contents

1 Introduction

The idea of treating negation as a modality manifests itself in various propositional logics. Of special interest are those of them whose positive fragments coincide with that of (propositional) intuitionistic logic, and it is reasonable to develop their predicate versions — taking into account the importance of predicate intuitionistic logic.

Recall that in intuitionistic logic, proving Φ provides a verification of Φ, while proving $\neg\Phi$ gives us a demonstration that

each verification of Φ yields a verification of $\bot$,

and since $\bot$ must not be verified, it implies that Φ cannot be verified. Došen proposed to take seriously the idea that

negation can be thought of as a negative modality

— or more precisely, as a modal operator of impossibility; see [2, 3], and also Došen's entry in [7]. While in the standard possible world semantics for intuitionistic logic we have

$$w \Vdash \neg\Phi \quad \Longleftrightarrow \quad u \nVdash \Phi \text{ for all } u \geqslant w,$$

this appears to be too strong for Došen's proposal, so $\leqslant$ is replaced by an accessibility relation R which agrees with $\leqslant$ in a suitable way. This leads to a propositional logic N; it plays a key role in developing Došen's perspective on intuitionistic modal logics. Axiomatically, N employs the *contraposition rule*, which is rendered as

$$\frac{\Phi \to \Psi}{\neg\Psi \to \neg\Phi} \quad (\mathtt{CR})$$

Clearly, it can be viewed as an antimonotonicity rule. As in modal logic, CR — although trivially admissible — should not be derivable, viz.

$$(\Phi \to \Psi) \to (\neg\Psi \to \neg\Phi) \qquad (\mathtt{CR}')$$

should not belong to our logic. Thus even Johansson's minimal logic is too strong, because it treats $\neg\Phi$ as $\Phi \to \bot$, and hence contains $\mathtt{CR}'$.

Next, N can be extended in various ways. Among the interesting extensions of N are:

- the so-called *Heyting–Ockham logic* N^*, which was was proposed in [1] as a framework for studying foundations of well-founded semantics for logic programs with negation (cf. [8]);
- the logic Hype, which has been recently advocated in [12] as a system suitable for dealing with hyperintensional contexts.[1]

It is known from [1] that N^* has an attractive Routley-style semantics, in which we have

$$w \Vdash \neg\Phi \quad \iff \quad w^* \nVdash \Phi$$

where $*$ is an anti-monotone function on the set of all worlds (cf. also [16]). More precisely, N^* turns out to be strongly complete with respect to the class of all 'starred frames'. On the other hand, Odintsov has recently observed that Hype can be obtained from N^* by adding the laws of double negation introduction and elimination.[2]

I shall develop predicate versions of N and N^*, and provide an appropriate Routley-style semantics for QHype. In particular, there will be two predicate versions of N^*, one of which does not seem to have a Routley-style semantics, only a Došen-style semantics, while the other has both. The corresponding strong completeness results will be proved by means of a general technique, which can be used for related systems.

It should be remarked that the Routley-style semantics for QHype looks much simpler than the semantics given in [12]. Furthermore, only a sketch of proof for the weak completeness result for QHype has been provided in [12], which relies largely on [9].

Finally, in [12], Leitgeb mistakenly claimed that QHype has the disjunction property, but his argument is known to be flawed; cf. [15, 5]. I shall provide a simple argument which shows that both the disjunction property and the existential property fail for predicate logics of a certain kind, including QHype and the two quantified versions of N^*.

2 Preliminaries

Let σ be a signature, i.e. a collection of non-logical symbols, each of which has an arity. As far as expanding domain semantics is concerned, there are well-known problems with equality and function symbols.[3] In this context we shall assume, for simplicity, that $=$ is not in σ, and every function symbol in σ is of arity 0, and hence represents a constant. Define Pred_σ to be the set of predicate symbols in σ and Const_σ to be the set of constant symbols in σ. The latter will be interpreted rigidly: each constant will have the same value at all accessible worlds.

Fix once and for all a countable collection Var of *variables.* Then let Term_σ be the set of σ-terms. Thus $\mathrm{Term}_\sigma = \mathrm{Const}_\sigma \cup \mathrm{Var}$. Our logical vocabulary includes:

- the *connective symbols* $\to$, $\wedge$, $\vee$ and $\neg$;
- the *quantifier symbols* $\forall$ and $\exists$.

[1] H. Wansing has informed me that Hype was first introduced in [13]. On the other hand, as far as I know, the predicate version QHype of Hype has been presented only in [12] (without reference to [13]), and the application to semantic paradoxes in [12] requires QHype.

[2] The reader may consult [15] for further discussion.

[3] One of the key problems is that if we treat $=$ as a binary predicate symbol, and $a_1 \neq a_2$ at a world w, then it can happen that $a_1 = a_2$ at some world accessible from w; the usual trick of turning equivalence relations into identity relations (as in classical predicate logic) may also break down. Clearly, this implies some problems with function symbols as well. See [6] for more information.

Denote by Form_σ the set of (first-order) σ-formulas, and by Sent_σ the set of σ-sentences, which are σ-formulas with no free variable occurrences, as usual. For convenience, we shall abbreviate $(\Phi \to \Psi) \wedge (\Phi \to \Psi)$ to $\Phi \leftrightarrow \Psi$.

By a *σ-substitution of terms*, or just *σ-substitution*, we mean a function from Var to Term_σ. If Φ is a σ-formula in which only $x_1, \ldots, x_n$ may occur free, and λ is a σ-substitution, we write $\lambda\Phi$ for the result of simultaneously substituting $\lambda(x_1), \ldots, \lambda(x_n)$ for all free occurrences of x_1, $\ldots, x_n$ respectively in Φ; in the case when

$$\lambda \;=\; \{(x,t)\} \cup \{(y,y) \mid y \in \mathrm{Var} \text{ and } y \neq x\}$$

(where $x \in \mathrm{Var}$ and $t \in \mathrm{Term}_\sigma$), the notation $\Phi\,(x/t)$ will be used instead of $\lambda\Phi$.[4] Call a σ-substitution λ *ground* iff $\lambda(x) \in \mathrm{Const}_\sigma$ for every $x \in \mathrm{Var}$.

A σ-structure provides a non-empty set and appropriate interpretations of the symbols of σ over it. We generally use $\mathfrak{A}$, $\mathfrak{B}$, etc. to represent σ-structures with domains A, B, etc. Now let $\mathfrak{A}$ be an arbitrary σ-structure. For each $\varepsilon \in \sigma$, take

$$\varepsilon^{\mathfrak{A}} \;:=\; \text{the interpretation of } \varepsilon \text{ in } \mathfrak{A}.$$

Sometimes it will be convenient to expand σ to

$$\sigma_A \;:=\; \sigma \cup \{\underline{a} \mid a \in A\}$$

where $\underline{a}$'s are new constant symbols, and pass from $\mathfrak{A}$ to its σ_A-expansion $\mathfrak{A}^*$ such that

$$\underline{a}^{\mathfrak{A}^*} \;:=\; a \quad \text{for any } a \in A.$$

Then σ_A-formulas will also be called *A-formulas*. Further, if Φ is an A-sentence, we shall often write $\mathfrak{A} \models \Phi$ instead of $\mathfrak{A}^* \models \Phi$. By an *$A$-substitution* will be meant a σ_A-substitution.

Finally, when concerned exclusively with non-empty subsets, we shall abbreviate the phrase 'S is a non-empty subset of T' as $S \sqsubseteq T$.

3 A quantified version QN of Došen's N

We begin by expanding Došen's propositional logic N, which yields the weakest quantified logic we shall be concerned with. It should be remarked that the machinery presented in this section will be used heavily later on.

3.1 A Hilbert-type calculus

Our predicate calculus naturally expands the propositional Hilbert-type system for N described in [3]. It employs the following axiom schemata:

I1. $\Phi \to (\Psi \to \Phi)$;

I2. $(\Phi \to (\Psi \to \Theta)) \to ((\Phi \to \Psi) \to (\Phi \to \Theta))$;

C1. $\Phi \wedge \Psi \to \Phi$;

[4] Naturally, we want t to be free for x in Φ, i.e. either $t \in \mathrm{Const}_\sigma$, or $t \in \mathrm{Var}$ and no free occurrence of x in Φ is within the scope of a t-quantifier. Similarly in the general case.

C2. $\Phi \wedge \Psi \to \Psi$;

C3. $\Phi \to (\Psi \to \Phi \wedge \Psi)$;

D1. $\Phi \to \Phi \vee \Psi$;

D2. $\Psi \to \Phi \vee \Psi$;

D3. $(\Phi \to \Theta) \to ((\Psi \to \Theta) \to (\Phi \vee \Psi \to \Theta))$;

N. $\neg\Phi \wedge \neg\Psi \to \neg(\Phi \vee \Psi)$;

Q1. $\forall x\, \Phi \to \Phi\,(x/t)$ where t is free for x in Φ;

Q2. $\Phi\,(x/t) \to \exists x\, \Phi$ where t is free for x in Φ.

Thus we have the 'positive' axioms of intuitionistic predicate logic plus all instances of N. Also, we employ four inference rules:

MP. *modus ponens*, i.e.

$$\frac{\Phi \qquad \Phi \to \Psi}{\Psi}\ ;$$

CR. the *contraposition rule*, which is expressed as

$$\frac{\Phi \to \Psi}{\neg\Psi \to \neg\Phi}\ ;$$

BR1. the *Bernays rule for* $\forall$, which is rendered as

$$\frac{\Phi \to \Psi}{\Phi \to \forall x\, \Psi} \qquad \text{provided that } x \text{ is not free in } \Phi;$$

BR2. the *Bernays rule for* $\exists$, which is rendered as

$$\frac{\Phi \to \Psi}{\exists x\, \Phi \to \Psi} \qquad \text{provided that } x \text{ is not free in } \Psi.$$

These are the usual rules of intuitionistic predicate logic plus CR. Observe that if we treat $\neg$ as an impossibility operator, then CR can be thought of as a modal rule. Clearly, $\neg$ is weaker than intuitionistic negation, and even that of Johansson's minimal logic.

Let QN_σ denote the least set of σ-formulas containing the axioms of our calculus and closed under its rules of inference. Certainly, QN_σ depends on σ, but the QN_σ's can be viewed as representing the same logic, QN. For every $\Gamma \subseteq \mathrm{Form}_\sigma$, define

$$\mathrm{Disj}\,(\Gamma) \ := \ \{\Phi_0 \vee \ldots \vee \Phi_n \mid n \in \mathbb{N} \text{ and } \{\Phi_0, \ldots, \Phi_n\} \subseteq \Gamma\}.^5$$

Given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$, we write $\Gamma \vdash \Delta$ iff some element of $\mathrm{Disj}\,(\Delta)$ can be obtained from elements of $\Gamma \cup \mathsf{QN}_\sigma$ by means of MP, BR1 and BR2. As might be expected, $\Phi \vdash \Delta$ and $\Gamma \vdash \Phi$ abbreviate $\{\Phi\} \vdash \Delta$ and $\Gamma \vdash \{\Psi\}$ respectively.

[5] When $n = 0$, we have $\Phi_0 \vee \ldots \vee \Phi_n = \Phi_0$. Thus $\mathrm{Disj}\,(\Gamma)$ contains non-empty disjunctions only.

Theorem 3.1 (Deduction Theorem)**.** *For any* $\Gamma \cup \{\Phi\} \subseteq \mathrm{Sent}_\sigma$ *and* $\Psi \in \mathrm{Form}_\sigma$*,*

$$\Gamma \cup \{\Phi\} \vdash \Psi \quad \Longleftrightarrow \quad \Gamma \vdash \Phi \to \Psi.$$

Proof. Exactly as in intuitionistic predicate logic. □

Here is another useful observation.

Theorem 3.2 (Replacement)**.** *Let* $\{\Phi, \Psi, \Psi'\} \subseteq \mathrm{Form}_\sigma$*, and suppose* Φ' *is obtained from* Φ *by replacing some occurrence of* Ψ *by* Ψ'*. Then* $\vdash \Psi \leftrightarrow \Psi'$ *implies* $\vdash \Phi \leftrightarrow \Phi'$*.*

Proof. By induction on the complexity of Φ.

In the case where Φ is atomic the result is immediate.

Suppose $\Phi = \neg\Psi$. The result then follows by the inductive hypothesis, `CR`, and `C1–3`.

In the other cases one can argue as in intuitionistic predicate logic. □

As in the propositional case, we can go on to define a general notion of logic. Roughly, by a (*normal*) *logic* is meant a collection of formulas closed under the four rules above and substitutions. To make this precise, one can adopt the machinery of [6, Chapter 2]. Each logic that includes QN is called a QN-*extension.* Then, given a QN-extension L, we define

$$\Gamma \vdash_L \Delta \quad :\Longleftrightarrow \quad L \cup \Gamma \vdash \Delta.$$

So Theorems 3.1 and 3.2 and many results below will generalise readily to QN-extensions; most importantly, the canonical model construction from Section 3.5 can be suitably modified.

Evidently, for every $\{\Phi, \Psi\} \subseteq \mathrm{Form}_\sigma$ we have $\vdash (\Phi \to \Phi) \leftrightarrow (\Psi \to \Psi)$; thus by Theorem 3.2, $\Phi \to \Phi$ and $\Psi \to \Psi$ are practically interchangeable. Now take

$$\top \;:=\; \widetilde{\Phi} \to \widetilde{\Phi} \quad \text{and} \quad \bot \;:=\; \neg\top$$

where $\widetilde{\Phi}$ is a fixed σ-sentence. It should be remarked that this paper focuses on logics in which there is a significant difference between $\neg\Phi$ and $\Phi \to \bot$, which represent two sorts of negation. We shall occasionally abbreviate $\Phi \to \bot$ to $-\Phi$.

3.2 A possible world semantics

As in [3], by a *frame* we mean a triple $\mathcal{W} = \langle W, \leqslant, R\rangle$ where:

- W is a non-empty set;
- $\leqslant$ is a preordering on W;
- R is a binary relation on W such that $\leqslant \circ R \subseteq R \circ \leqslant^{-1}$.[6]

Next, by a *system of* σ*-structures over* $\mathcal{W}$ we mean an indexed family $\mathscr{A} = \langle \mathfrak{A}_w \mid w \in W\rangle$ of σ-structures, i.e. a function from W to σ-structures. Finally, given a frame $\mathcal{W}$ and a system $\mathscr{A}$ of σ-structures over it, call the pair $\mathcal{M} = \langle \mathcal{W}, \mathscr{A}\rangle$ an *expanding domain model for* QN_σ—or simply a QN_σ*-model*—iff for all $u, v \in W$ the following hold:

- if $u \leqslant v$ or uRv, then $A_u \subseteq A_v$;

[6]Here $M \circ N$ denotes the composition of M and N, while M^{-1} denotes the inverse of M.

- if $u \leqslant v$ or uRv, then $c^{\mathfrak{A}_u} = c^{\mathfrak{A}_v}$ for any $c \in \mathrm{Const}_\sigma$;
- if $u \leqslant v$, then $P^{\mathfrak{A}_u} \subseteq P^{\mathfrak{A}_v}$ for any $P \in \mathrm{Pred}_\sigma$.

Now let $\mathcal{M}$ be a QN_σ-model. For every $w \in W$ and every A_w-sentence Φ,

$$\mathcal{M}, w \Vdash \Phi$$

is defined inductively as follows:

- $\mathcal{M}, w \Vdash \Psi$ iff $\mathfrak{A}_w \models \Psi$, provided that Ψ is atomic;
- $\mathcal{M}, w \Vdash \Psi \wedge \Theta$ iff $\mathcal{M}, w \Vdash \Psi$ and $\mathcal{M}, w \Vdash \Theta$;
- $\mathcal{M}, w \Vdash \Psi \vee \Theta$ iff $\mathcal{M}, w \Vdash \Psi$ or $\mathcal{M}, w \Vdash \Theta$;
- $\mathcal{M}, w \Vdash \Psi \to \Theta$ iff for each $u \in W$, if $w \leqslant u$, then $\mathcal{M}, u \Vdash \Psi$ implies $\mathcal{M}, u \Vdash \Theta$;
- $\mathcal{M}, w \Vdash \neg\Psi$ iff for each $u \in W$, if wRu, then $\mathcal{M}, u \nVdash \Psi$;
- $\mathcal{M}, w \Vdash \exists x\, \Psi$ iff $\mathcal{M}, w \Vdash \Psi\,(x/\underline{a})$ for some $a \in A_w$;
- $\mathcal{M}, w \Vdash \forall x\, \Psi$ iff for each $u \in W$, if $w \leqslant u$, then $\mathcal{M}, u \Vdash \Psi\,(x/\underline{a})$ for all $a \in A_u$.

We shall often write $w \Vdash \Phi$ instead of $\mathcal{M}, w \Vdash \Phi$ when $\mathcal{M}$ is fixed by the context.

As in the propositional version of QN, it is easy to obtain:

Lemma 3.3. *Let $\mathcal{M}$ be a QN_σ-model. For any $w \in W$ and A_w-sentence Φ,*

$$\mathcal{M}, w \Vdash \Phi \quad \Longrightarrow \quad \mathcal{M}, u \Vdash \Phi \quad \textit{for all } u \geqslant w.$$

Proof. By induction on the complexity of Φ.

In the case where Φ is atomic the result is immediate.

Suppose $\Phi = \neg\Psi$. Assume $w \Vdash \Phi$ and $w \leqslant u$. Recall, $\leqslant \circ R \subseteq R \circ \leqslant^{-1}$. So for every $v \in W$, if uRv, then there exists $t \in W$ such that wRt and $v \leqslant t$, hence $t \nVdash \Psi$ and $v \nVdash \Psi$ (for otherwise $t \Vdash \Psi$ by the inductive hypothesis). Thus $u \Vdash \Phi$.

In the other cases one can argue as in intuitionistic predicate logic. □

Given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$, we write $\Gamma \vDash \Delta$ iff for any QN_σ-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$, $w \in W$ and ground A_w-substitution λ,

$$\mathcal{M}, w \Vdash \Phi \quad \text{for all } \Phi \in \Gamma \quad \Longrightarrow \quad \mathcal{M}, w \Vdash \lambda\Psi \quad \text{for some } \Psi \in \Delta.$$

Using Lemma 3.3, a semantic analogue of the Deduction Theorem can be readily established:

Theorem 3.4. *For any $\Gamma \cup \{\Phi\} \subseteq \mathrm{Sent}_\sigma$ and $\Psi \in \mathrm{Form}_\sigma$,*

$$\Gamma \cup \{\Phi\} \vDash \Psi \quad \Longleftrightarrow \quad \Gamma \vDash \Phi \to \Psi.$$

Proof. Exactly as in intuitionistic logic. □

One might wish to consider frame conditions which are stronger than $\leqslant \circ R \subseteq R \circ \leqslant^{-1}$. For instance, the condition

$$\leqslant \circ R \;\subseteq\; R$$

arises naturally in applications—it corresponds to so-called ‘condensed frames’ in [3] and other works of Došen. Furthermore, frames that satisfy the stronger condition

$$\leqslant \circ R \;\subseteq\; R \circ \leqslant^{-1} \;\subseteq\; R,$$

are called ‘strictly condensed’ in [3]. (Cf. also discussion in [11, Section 3].)

3.3 Soundness

The soundness argument for QN combines the argument for N with that for intuitionistic predicate logic in a straightforward way. It is instructive to supply some details, however.[7]

Lemma 3.5. *For every* $\Phi \in \mathrm{Form}_\sigma$*,*

$$\vdash \Phi \quad \Longrightarrow \quad \vDash \Phi.$$

Proof. Suppose $\vdash \Phi$. Then $\Phi \in \mathsf{QN}_\sigma$, i.e. there exists a finite sequence

$$\Phi_0, \quad \Phi_1, \quad \ldots, \quad \Phi_n \;=\; \Phi$$

of σ-formulas such that for each $i \in \{0, \ldots, n\}$ one of the following conditions holds:

a. Φ_i is an axiom;

b. Φ_i is obtained from earlier Φ_j and Φ_k by `MP`;

c. Φ_i is obtained from earlier Φ_j by `CR`;

d. Φ_i is obtained from earlier Φ_j by `BR1` or `BR2`.

Let $\mathcal{M}$ be a QN_σ-model. We are going to prove inductively that $\mathcal{M}, w \Vdash \lambda\Phi_i$ for all $w \in W$ and ground A_w-substitutions λ.

Suppose Φ_i is an axiom of the type `N`, i.e. has the form $\neg\Psi \wedge \neg\Theta \to \neg\,(\Psi \vee \Theta)$. Evidently, it suffices to show that for any $w \in W$ and ground A_w-substitution λ,

$$w \Vdash \neg\lambda\Psi \wedge \neg\lambda\Theta \quad \Longrightarrow \quad w \Vdash \neg\,(\lambda\Psi \vee \lambda\Theta).$$

Let w and λ be as above, and assume $w \Vdash \neg\lambda\Psi \wedge \neg\lambda\Theta$. For each $u \in W$, if wRu, then $u \nVdash \lambda\Psi$ and $u \nVdash \lambda\Theta$, hence $u \nVdash \lambda\Psi \vee \lambda\Theta$. Thus $w \Vdash \neg\,(\lambda\Psi \vee \lambda\Theta)$ as desired. For the other axioms one can argue as in intuitionistic predicate logic.

Suppose Φ_i is obtained from earlier Φ_j by `CR`; so for some Ψ and Θ we have

$$\Phi_j \;=\; \Psi \to \Theta \quad \text{and} \quad \Phi_i \;=\; \neg\Theta \to \neg\Psi.$$

By the inductive hypothesis, $v \Vdash \lambda\Psi \to \lambda\Theta$ for all $v \in W$ and ground A_v-substitutions λ. Evidently, it is enough to show that for any $w \in W$ and ground A_w-substitution λ,

$$w \Vdash \neg\lambda\Theta \quad \Longrightarrow \quad w \Vdash \neg\lambda\Psi.$$

Let w and λ be as above, and assume $w \Vdash \neg\lambda\Theta$. For each $u \in W$, if wRu, then $u \nVdash \lambda\Theta$, which implies $u \nVdash \lambda\Psi$, since otherwise $u \nVdash \lambda\Psi \to \lambda\Theta$. Thus $w \Vdash \neg\lambda\Psi$ as desired.

For the other rules one can argue as in intuitionistic predicate logic. □

In view of the compactness of $\vdash$, this leads to the following.

Theorem 3.6. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash \Delta \quad \Longrightarrow \quad \Gamma \vDash \Delta.$$

Proof. Assume $\Gamma \vdash \Delta$, i.e. $\Gamma \vdash \Phi$ for some $\Phi \in \mathrm{Disj}\,(\Delta)$. So there must be a finite subset Λ of Γ such that $\Lambda \vdash \Phi$. The argument now divides into two cases.

Suppose $\Lambda = \varnothing$. Then $\vDash \Phi$ by Lemma 3.5. Consequently $\Gamma \vDash \Delta$.

Suppose $\Lambda = \{\Psi_0, \ldots, \Psi_n\}$. Clearly, we then have $\Psi_0 \wedge \cdots \wedge \Psi_n \vdash \Phi$, which is equivalent to $\vdash \Psi_0 \wedge \cdots \wedge \Psi_n \to \Phi$ by Theorem 3.1. Therefore $\vDash \Psi_0 \wedge \cdots \wedge \Psi_n \to \Phi$ by Lemma 3.5—which is equivalent to $\Psi_0 \wedge \cdots \wedge \Psi_n \vDash \Phi$ by Theorem 3.4. Consequently $\Gamma \vDash \Delta$. □

[7] In [3], the soundness proof for N was omitted altogether.

3.4 Remarks on new constants

To implement Henkin's approach to completeness, we often need to enrich the original signature to include enough 'witnesses'. The results in this section will help us avoid unwanted effects. The proofs are exactly as in classical or intuitionistic predicate logic, so we omit them. For any set S, take

$$\sigma_S \;:=\; \sigma \cup \{\underline{s} \mid s \in S\}$$

where $\underline{s}$'s are new constant symbols. Given a set Γ of formulas, we shall write $\mathrm{Const}\,(\Gamma)$ for the collection of all constant symbols occurring in elements of Γ.

In fact, $\vdash$ depends on the choice of σ. Occasionally, when we wish to emphasise that QN_σ is employed, $\vdash_\sigma$ is used instead of $\vdash$. Notice that new constant symbols do no harm:

Proposition 3.7. *Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$. Then for any set S,*

$$\Gamma \vdash_\sigma \Delta \quad\Longleftrightarrow\quad \Gamma \vdash_{\sigma_S} \Delta.$$

Next, we have:

Proposition 3.8. *Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$. Then for each variable x and each constant symbol c not in* $\mathrm{Const}\,(\Gamma \cup \Delta)$*,*

$$\Gamma \vdash \Delta \quad\Longleftrightarrow\quad \Gamma \vdash \Delta\,(x/c)$$

where $\Delta\,(x/c)$ denotes $\{\Phi\,(x/c) \mid \Phi \in \Delta\}$.

Among other things, this allows us to safely add witnesses:

Proposition 3.9. *Let $\Gamma \cup \{\exists x\, \Psi\} \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$. For each constant symbol c not in* $\mathrm{Const}\,(\Gamma \cup \{\exists x\, \Psi\} \cup \Delta)$*,*

$$\Gamma \cup \{\exists x\, \Psi\} \vdash \Delta \quad\Longleftrightarrow\quad \Gamma \cup \{\exists x\, \Psi, \Psi\,(x/c)\} \vdash \Delta.$$

Now we turn to a canonical model construction appropriate for QN-extensions.

3.5 Completeness

Call a σ-theory Γ *prime* iff it has the following properties:

- $\{\Phi \in \mathrm{Sent}_\sigma \mid \Gamma \vdash \Phi\} \subseteq \Gamma$;
- if $\Phi \vee \Psi \in \Gamma$, then $\Phi \in \Gamma$ or $\Psi \in \Gamma$;
- if $\exists x\, \Phi \in \Gamma$, then $\Phi\,(x/c) \in \Gamma$ for some constant symbol c.

As in intuitionistic predicate logic, we can obtain:

Lemma 3.10. *Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$ be such that $\Gamma \nvdash \Delta$. Then for each set S of cardinality $|\mathrm{Sent}_\sigma|$ there exists a prime σ_S-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

Proof. Take $\kappa := |\mathrm{Sent}_\sigma|$. Fix an arbitrary set S of cardinality κ. Throughout this proof $\vdash$ will stand for $\vdash_{\sigma_S}$. Clearly,

$$|\mathrm{Sent}_{\sigma_S}| \;=\; \max\{|\sigma_S|, \aleph_0\} \;=\; |\sigma_S| \;=\; |S|.$$

Thus the σ_S-sentences can be arranged into a transfinite sequence of length κ:

$$\mathrm{Sent}_{\sigma_S} \;=\; \langle \Phi_\alpha : \alpha \in \kappa \rangle.$$

Now define $\langle \Gamma_\alpha : \alpha \in \kappa \rangle$ by transfinite recursion as follows.

- Suppose $\alpha = 0$. Then $\Gamma_\alpha := \Gamma$.
- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \nvdash \Delta$ where Φ_β does not start with $\exists$. Then

$$\Gamma_\alpha \ := \ \Gamma_\beta \cup \{\Phi_\beta\}.$$

- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \nvdash \Delta$ where Φ_β has the form $\exists x\, \Psi$. Then choose some c in $\{\underline{s} \mid s \in S\} \setminus \mathrm{Const}\,(\Gamma_\beta \cup \{\Phi_\beta\} \cup \Delta)$, and take

$$\Gamma_\alpha \ := \ \Gamma_\beta \cup \{\Phi_\beta, \Psi\,(x/c)\}.$$[8]

- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \vdash \Delta$. Then $\Gamma_\alpha := \Gamma_\beta$.
- Suppose α is a limit ordinal. Then $\Gamma_\alpha := \bigcup_{\beta\in\alpha} \Gamma_\beta$.

Using Propositions 3.7 and 3.9, it is straightforward to check that $\Gamma' := \bigcup_{\alpha\in\kappa} \Gamma_\alpha$ has the desired properties. □

Sometimes a stronger notion is needed. More precisely, we call a prime σ-theory Γ *strongly prime* iff it has the additional property:

- if $\Phi\,(x/c) \in \Gamma$ for all constant symbols c, then $\forall x\, \Phi \in \Gamma$.

In particular, this kind of theory has been used in building the canonical model for intuitionistic predicate logic with 'constant domains', which includes the following axiom scheme:

`CD`. $\forall x\,(\Phi \vee \Psi) \to \Phi \vee \forall x\, \Psi$ for x not free in Φ.

Throughout this article we shall assume that every strongly prime theory contains all instances of `CD`. As might be expected, the analogue of Lemma 3.10 holds (cf. also [6, Lemma 6.2.6]).

Lemma 3.11. *Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$ be such that $\Gamma \nvdash \Delta$. Then for each set S of cardinality $|\mathrm{Sent}_\sigma|$ there exists a strongly prime σ_S-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

Proof. Let κ and S be as in the previous proof. Again the σ_S-sentences can be arranged into a transfinite sequence of length κ:

$$\mathrm{Sent}_{\sigma_S} \ = \ \langle \Phi_\alpha : \alpha \in \kappa \rangle.$$

Now define $\langle (\Gamma_\alpha, \Delta_\alpha) : \alpha \in \kappa \rangle$ by transfinite recursion as follows.

- Suppose $\alpha = 0$. Then $(\Gamma_\alpha, \Delta_\alpha) := (\Gamma, \Delta)$.
- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \nvdash \Delta_\beta$ where Φ_β does not start with $\exists$. Then

$$(\Gamma_\alpha, \Delta_\alpha) \ := \ (\Gamma_\beta \cup \{\Phi_\beta\}, \Delta_\beta).$$

- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \nvdash \Delta_\beta$ where Φ_β has the form $\exists x\, \Psi$. Then choose a c in $\{\underline{s} \mid s \in S\} \setminus \mathrm{Const}\,(\Gamma_\beta \cup \{\Phi_\beta\} \cup \Delta)$, and take

$$(\Gamma_\alpha, \Delta_\alpha) \ := \ (\Gamma_\beta \cup \{\Phi_\beta, \Psi\,(x/c)\}, \Delta_\alpha).$$

[8] Note that, by construction, we have $\left|\mathrm{Const}\,(\Gamma_\beta \cup \{\Phi_\beta\}) \setminus \mathrm{Const}\,(\Gamma \cup \Delta)\right| < \kappa = |S|$.

- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \vdash \Delta_\beta$ where Φ_β does not start with $\forall$. Then

$$(\Gamma_\alpha, \Delta_\alpha) \ := \ (\Gamma_\beta, \Delta_\beta \cup \{\Phi_\beta\}).$$

- Suppose $\alpha = \beta + 1$ and $\Gamma_\beta \cup \{\Phi_\beta\} \vdash \Delta$ where Φ_β has the form $\forall x\, \Psi$. Then choose a c in $\{\underline{s} \mid s \in S\} \setminus \mathrm{Const}\,(\Gamma_\beta \cup \{\Phi_\beta\} \cup \Delta)$, and take

$$(\Gamma_\alpha, \Delta_\alpha) \ := \ \begin{cases} (\Gamma_\beta, \Delta_\beta \cup \{\Phi_\beta, \Psi\,(x/c)\}) & \text{if } \Gamma_\beta \nvdash \Delta \cup \{\Phi_\beta, \Psi\,(x/c)\} \\ (\Gamma_\beta, \Delta_\beta \cup \{\Phi_\beta\}) & \text{otherwise.} \end{cases}$$

- Suppose α is a limit ordinal. Then $\Gamma_\alpha := \bigcup_{\beta \in \alpha} \Gamma_\beta$ and $\Delta_\alpha := \bigcup_{\beta \in \alpha} \Delta_\beta$.

It is not hard to verify that $\Gamma' := \bigcup_{\alpha \in \kappa} \Gamma_\alpha$ has the desired properties. □

Although we need only prime σ-theories for our present purposes, strongly prime σ-theories will be employed heavily in Section 6.

For the rest of this section, $S^\star$ will denote a fixed set of cardinality $|\mathrm{Sent}_\sigma|$. Intuitively, it is a kind of 'constant universe'. Call $S \subseteq S^\star$ *admissible* iff $|S^\star \setminus S| = |S^\star|$.

Corollary 3.12. *Let $S \subseteq S^\star$ be admissible. Suppose $\Gamma \subseteq \mathrm{Sent}_{\sigma_S}$ and $\Delta \sqsubseteq \mathrm{Form}_{\sigma_S}$ are such that $\Gamma \nvdash \Delta$. Then there exist an admissible $S' \supseteq S$ and a prime $\sigma_{S'}$-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

Proof. Since $|S^\star \setminus S| = |\mathrm{Sent}_\sigma|$, there exists an admissible $S' \supseteq S$ such that

$$|S' \setminus S| \ = \ |\mathrm{Sent}_\sigma|.$$

Moreover, $|\mathrm{Sent}_\sigma| = |\mathrm{Sent}_{\sigma_S}|$ because $|S| \leqslant |\mathrm{Sent}_\sigma|$. The result now follows by Lemma 3.10 (by taking $\sigma := \sigma_S$ and $S := S' \setminus S$). □

Corollary 3.13. *Let $S \subseteq S^\star$ be admissible. Suppose $\Gamma \subseteq \mathrm{Sent}_{\sigma_S}$ and $\Delta \sqsubseteq \mathrm{Form}_{\sigma_S}$ are such that $\Gamma \nvdash \Delta$. Then there exists a strongly prime $\sigma_{S^\star}$-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*

Proof. This follows by Lemma 3.11 (by taking $\sigma := \sigma_S$ and $S := S^\star \setminus S$). □

Before proceeding further, a few observations concerning negation are worth making. Given a collection Γ of formulas, we write $\underline{\Gamma}$ for $\{\Phi \mid \neg\Phi \in \Gamma\}$.

Proposition 3.14. *Let $\Gamma \subseteq \mathrm{Sent}_\sigma$ be such that $\{\Phi \in \mathrm{Sent}_\sigma \mid \Gamma \vdash \Phi\} \subseteq \Gamma$ and $\underline{\Gamma} \neq \varnothing$.*

i. For every $\{\Phi_0, \ldots, \Phi_n\} \subseteq \mathrm{Sent}_\sigma$, if $\{\Phi_0, \ldots, \Phi_n\} \subseteq \underline{\Gamma}$, then $\Phi_0 \vee \cdots \vee \Phi_n \in \underline{\Gamma}$.

ii. For every $\Phi \in \mathrm{Sent}_\sigma$, if $\Phi \vdash \underline{\Gamma}$, then $\Phi \in \underline{\Gamma}$.

Proof. i. Using N, it can easily be verified that $\vdash \neg\Phi_0 \wedge \cdots \wedge \neg\Phi_n \to \neg\,(\Phi_0 \vee \cdots \vee \Phi_n)$, so the result follows.

ii. Suppose $\Phi \vdash \underline{\Gamma}$. Then, in view of (i), we have $\Phi \vdash \Psi$ for some $\Psi \in \underline{\Gamma}$. Thus $\vdash \Phi \to \Psi$ by Theorem 3.1, hence $\vdash \neg\Psi \to \neg\Phi$ by CR. Since $\neg\Psi \in \Gamma$, we get $\neg\Phi \in \Gamma$, i.e. $\Phi \in \underline{\Gamma}$. □

Finally, we are ready to adapt the canonical model method to QN_σ. Since there is no ambiguity, the subscript σ will generally be dropped. To this end, for every set S, take

$$\mathrm{Prime}_S \ := \ \text{the collection of all prime } \sigma_S\text{-theories.}$$

Associate with each $\Gamma \in \mathrm{Prime}_S$ the σ_S-structure $\mathfrak{A}_\Gamma^S$ with domain $\mathrm{Const}(\Gamma)$, such that all constant symbols of σ_S are interpreted as themselves, and for any atomic σ_S-sentence Φ,

$$\mathfrak{A}_\Gamma^S \models \Phi \quad :\Longleftrightarrow \quad \Phi \ \in \ \Gamma.$$

Denote by $\mathfrak{A}_\Gamma$ the σ-reduct of $\mathfrak{A}_\Gamma^S$. Clearly, every A_Γ-sentence has the form

$$\Phi\left(x_1/\underline{c_1}, \ldots, x_n/\underline{c_n}\right)$$

with $\{c_1, \ldots, c_n\} \subseteq \mathrm{Const}(\Gamma)$, so we shall identify it with the σ_S-sentence $\Phi(x_1/c_1, \ldots, x_n/c_n)$, provided that no confusion arises. Let the set of possible worlds be defined as

$$W^{\mathsf{QN}} \ := \ \bigcup\{\mathrm{Prime}_S \mid S \text{ is an admissible subset of } S^\star\}.$$

By the *canonical frame for* QN we mean $\mathcal{W}^{\mathsf{QN}} = \langle W^{\mathsf{QN}}, \leqslant^{\mathsf{QN}}, R^{\mathsf{QN}}\rangle$ where

$$\begin{aligned} \leqslant^{\mathsf{QN}} \ &:= \ \left\{(\Gamma_1, \Gamma_2) \in W^{\mathsf{QN}} \times W^{\mathsf{QN}} \mid \Gamma_1 \subseteq \Gamma_2\right\} \quad \text{and} \\ R^{\mathsf{QN}} \ &:= \ \left\{(\Gamma_1, \Gamma_2) \in W^{\mathsf{QN}} \times W^{\mathsf{QN}} \mid \underline{\Gamma_1} \cap \Gamma_2 = \varnothing,\ \mathrm{Const}(\Gamma_1) \subseteq \mathrm{Const}(\Gamma_2)\right\}. \end{aligned}$$

The *canonical model for* QN is $\mathcal{M}^{\mathsf{QN}} = \langle \mathcal{W}^{\mathsf{QN}}, \mathscr{A}^{\mathsf{QN}}\rangle$ where

$$\mathscr{A}^{\mathsf{QN}}(\Gamma) \ := \ \mathfrak{A}_\Gamma.$$

One readily verifies that $\mathcal{W}^{\mathsf{QN}}$ is a QN-frame, and $\mathcal{M}^{\mathsf{QN}}$ is a QN-model.

Lemma 3.15 (Canonical Model Lemma). *For any* $\Gamma \in W^{\mathsf{QN}}$ *and* A_Γ*-sentence* Φ*,*

$$\mathcal{M}^{\mathsf{QN}}, \Gamma \Vdash \Phi \quad \Longleftrightarrow \quad \Phi \ \in \ \Gamma.$$

Proof. By induction on the complexity of Φ.

In the case where Φ is atomic the result is immediate.

Suppose $\Phi = \neg\Psi$. Thus we need to show that

$$\neg\Psi \ \in \ \Gamma \quad \Longleftrightarrow \quad \text{for all } \ \Delta \in W^{\mathsf{QN}}, \ \ \underline{\Gamma} \cap \Delta = \varnothing \ \text{ implies } \ \Psi \notin \Delta$$

(because the right-hand side is equivalent to $\mathcal{M}^{\mathsf{QN}}, \Gamma \Vdash \neg\Psi$ by the inductive hypothesis). Trivially, the implication from left to right holds. Now for the converse, assume $\neg\Psi \notin \Gamma$. There are two possibilities.

i. Suppose $\underline{\Gamma}$ is non-empty. Therefore $\Psi \nvdash \underline{\Gamma}$ by Proposition 3.14. So by applying Corollary 3.12 we get $\Delta \in W^{\mathsf{QN}}$ such that $\Delta \nvdash \underline{\Gamma}$ (and hence $\underline{\Gamma} \cap \Delta = \varnothing$) while $\Psi \in \Delta$.

ii. Suppose $\underline{\Gamma}$ is empty. Then any $\Delta \in W^{\mathsf{QN}}$ with $\Psi \in \Delta$ will do the job.

Similarly to Došen, we did not require our prime theories to be non-trivial. So in (ii), one may take $\Delta := \mathrm{Sent}_\sigma$ where σ is the signature of Γ.[9]

In the other cases one can argue as in intuitionistic predicate logic. □

Naturally, this leads to:

[9] However, it can also be shown that $\{\Phi \in \mathrm{Sent}_\sigma \mid \Psi \nvdash \Phi\} \neq \varnothing$ for every $\Psi \in \mathrm{Sent}_\sigma$; thus Corollary 3.12 yields a non-trivial prime theory Δ containing Ψ.

Theorem 3.16 (Strong Completeness)**.** *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash \Delta \quad \Longleftrightarrow \quad \Gamma \vDash \Delta.$$

Proof. $\boxed{\Longrightarrow}$ This is Theorem 3.6.

$\boxed{\Longleftarrow}$ Assume $\Gamma \nvdash \Delta$. Fix an admissible $S \subseteq S^\star$ of cardinality $\aleph_0$ (hence $|S| = |\mathrm{Var}|$). Let λ be some one-to-one function from Var onto $\{\underline{s} \mid s \in S\}$. Now take

$$\Delta' \;:=\; \{\lambda\Psi \mid \Psi \in \Delta\}.$$

Using Proposition 3.8 we easily get $\Gamma \nvdash \Delta'$. By Corollary 3.12, there exists $\Gamma' \in W^{\mathsf{QN}}$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash \Delta'$. Clearly, λ can be viewed as a ground $A_{\Gamma'}$-substitution. So by Lemma 3.15, $\mathcal{M}^{\mathsf{QN}}, \Gamma' \Vdash \Phi$ for every $\Phi \in \Gamma$, while $\mathcal{M}^{\mathsf{QN}}, \Gamma' \nVdash \lambda\Psi$ for all $\Psi \in \Delta$. Consequently $\Gamma \nvDash \Delta$. □

It might be worth remarking that the 'propositional segment' of the completeness argument provided above differs, in certain respects, from the argument for N given in [3].

- Došen employed single-succedent derivability and consequence relations. This, in a sense, forced him to utilize Zorn's lemma, and hence the axiom of choice, even for the countable version of the canonical model lemma. This is not essential if multi-succedent derivability and consequence relations are used, as the above proof shows.
- Došen emphasized that the set of all propositional formulas should be treated as a prime theory in his canonical model construction. This is not necessary. Moreover, for logics in which at least one negated formula is derivable, we have $\underline{\Gamma} \neq \varnothing$ for every prime theory Γ; so (ii) from the proof of Lemma 3.15 becomes irrelevant.

On the other hand, many observations about N can be transferred to QN. For instance, as was noticed by Došen, the canonical frame for N is strictly condensed. The same holds for QN, of course. Thus we may limit ourselves to strictly condensed frames if needed.[10]

Similarly to [3], a quantified version QJ of Johansson's minimal logic J is obtained from QN by adding the two schemata

$$(\Phi \to \Psi) \to (\neg\Psi \to \neg\Phi) \quad \text{and} \quad \Phi \to \neg\neg\Phi$$

— or equivalently, the single scheme

$$(\Phi \to \neg\Psi) \to ((\Phi \to \neg\Psi) \to \neg\Phi).$$

As might be expected, one can prove the strong completeness of QJ with respect to an appropriate class of frames, which are intimately related to those used in the standard semantics of J; cf. [3, Sections 4–5]. Moreover, certain extensions of QJ may be treated in a similar way; cf. [3, Sections 6–7]. However, all of these derive the scheme

$$(\Phi \to \Psi) \to (\neg\Psi \to \neg\Phi),$$

hence make the modal rule CR redundant. We shall be mainly concerned with systems in which CR, though trivially admissible, is not derivable.

[10] Such frames are relatively easier to work with, and some authors start with condensed or strictly condensed frames when dealing with Došen-like semantics (cf. [14, 11]) — compare this to how partial orderings are often used instead of preorderings in the semantics of intuitionistic logic.

Finally, Došen frames may be viewed as bimodal Kripke frames, so N can be translated into a suitable classical bimodal logic as follows:

$$\begin{aligned}\tau(p) &:= \Box p;\\ \tau(\phi \circ \psi) &:= \tau(\phi) \circ \tau(\psi) \quad \text{where } \circ \in \{\wedge, \vee\};\\ \tau(\phi \to \psi) &:= \Box(\tau(\phi) \to \tau(\psi));\\ \tau(\neg\phi) &:= \blacksquare\neg\phi.\end{aligned}$$

Here $\Box$ and $\blacksquare$ are interpreted by means of $\leqslant$ and R respectively. The same holds for N^*, Hype, and certain other extensions of N.[11] This also carries over to the quantified setting, but the present paper takes a different route, and deals explicitly with intuitionistic implication.

4 A useful extension QN° of QN

While $\top$ is valid in all QN-models, there are QN-models in which $\bot$ (i.e. $\neg\top$) is satisfiable. We shall see how this rather undesirable feature can be avoided.

4.1 A Hilbert-type calculus

We define the logic QN° to be QN plus the following axiom schemata:

$\mathtt{N1}^\circ$. $\neg(\Phi \to \Phi) \to \Psi$;

$\mathtt{N2}^\circ$. $\neg\neg(\Phi \to \Phi)$.

More precisely, for a fixed signature σ, our predicate calculus for QN°_σ extends that for QN_σ by adding all axioms of the types $\mathtt{N1}^\circ$ and $\mathtt{N2}^\circ$. Obviously, the latter can be replaced by

$$\bot \to \Phi \quad \text{and} \quad \neg\bot$$

respectively; cf. the end of Subsection 3.1. Given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$, we write

$$\Gamma \vdash^\circ \Delta \quad :\Longleftrightarrow \quad \mathsf{QN}^\circ_\sigma \cup \Gamma \vdash \Delta.$$

The syntactical technique described in Section 3 is easily adapted to $\vdash^\circ$. From now on we shall assume that our prime theories are non-trivial.

4.2 A possible world semantics

If $\mathcal{W}$ is a frame, and $w \in W$, we let

$$R(w) := \{u \in W \mid wRu\}.$$

Call a frame $\mathcal{W}$ *serial* iff R is serial, i.e. $R(w) \neq \varnothing$ for each $w \in W$. Clearly, we have:

Proposition 4.1 (cf. [14, Section 2]). *A frame $\mathcal{W}$ is serial iff the propositional versions of $\mathtt{N1}^\circ$ and $\mathtt{N2}^\circ$ hold in every model for N based on $\mathcal{W}$.*

By QN°_σ*-models* are meant QN_σ-models based on serial frames, as one should expect. Then given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$, we write $\Gamma \vDash^\circ \Delta$ iff for any QN°_σ-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$, $w \in W$ and ground A_w-substitution λ,

$$\mathcal{M}, w \Vdash \Phi \quad \text{for all } \Phi \in \Gamma \quad \Longrightarrow \quad \mathcal{M}, w \Vdash \lambda\Psi \quad \text{for some } \Psi \in \Delta.$$

So $\vDash^\circ$ is simply the relativisation of $\vDash$ to the class of QN°_σ-models.

[11] In particular, see [15, Section 5] for a classical bimodal logic into which Hype is embedded.

4.3 Soundness and completeness

First, we quickly obtain:

Theorem 4.2. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash^\circ \Delta \quad \Longrightarrow \quad \Gamma \vDash^\circ \Delta.$$

Proof. As the argument for Theorem 3.6 shows, it suffices to prove that for any axiom Φ of the type $\mathtt{N1}^\circ$ or $\mathtt{N2}^\circ$ we have $\vDash^\circ \Phi$, i.e. Φ holds in every QN°_σ-model. So the desired result follows by Proposition 4.1. □

Now the *canonical frame for* QN°, denoted $\mathcal{W}^{\mathsf{QN}^\circ} = \langle W^{\mathsf{QN}^\circ}, \leqslant^{\mathsf{QN}^\circ}, R^{\mathsf{QN}^\circ} \rangle$, is defined exactly as that for QN, but with $\vdash$ replaced by $\vdash^\circ$ throughout.[12] Hence $\mathcal{W}^{\mathsf{QN}^\circ}$ is simply the subframe of $\mathcal{W}^{\mathsf{QN}}$ generated by the non-trivial worlds containing QN°_σ.

Proposition 4.3. $\mathcal{W}^{\mathsf{QN}^\circ}$ *is a serial frame.*

Proof. Let $\Gamma \in W^{\mathsf{QN}^\circ}$.

Claim: $\nvdash^\circ \underline{\Gamma}$, viz. $\mathsf{QN}^\circ \nvdash \underline{\Gamma}$.

For otherwise $\top \vdash^\circ \underline{\Gamma}$. Evidently, the analogue of Proposition 3.14 for $\vdash^\circ$ holds; therefore $\top$ is in $\underline{\Gamma}$, i.e. $\neg\top \in \Gamma$. In view of $\mathtt{N1}^\circ$, this implies the triviality of Γ, a contradiction.

So by applying Corollary 3.12 we get $\Delta \in W^{\mathsf{QN}^\circ}$ such that $\underline{\Gamma} \cap \Delta = \varnothing$. □

Naturally, by the *canonical model for* QN° we mean $\mathcal{M}^{\mathsf{QN}^\circ} = \langle \mathcal{W}^{\mathsf{QN}^\circ}, \mathscr{A}^{\mathsf{QN}^\circ} \rangle$ where $\mathscr{A}^{\mathsf{QN}^\circ}$ is the restriction of $\mathscr{A}^{\mathsf{QN}}$ to $\mathcal{W}^{\mathsf{QN}^\circ}$. Clearly, $\mathcal{M}^{\mathsf{QN}^\circ}$ is a QN°-model, and furthermore, the Canonical Model Lemma for QN implies that for QN°.

Theorem 4.4 (Strong Completeness)**.** *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash^\circ \Delta \quad \Longleftrightarrow \quad \Gamma \vDash^\circ \Delta.$$

Proof. This is perfectly analogous to the proof of Theorem 3.16. □

As might be expected, we obtain the following.

Proposition 4.5. *The* $\{\wedge, \vee, \rightarrow, -\}$*-fragment of* QN° *is precisely intuitionistic predicate logic.*

Proof. Let $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ be a QN°_σ-model. So in particular, R is serial. Then by choosing some $w' \in R(w)$ for each $w \in W$, we obtain

$$\mathcal{M}, w \Vdash \bot \quad \Longleftrightarrow \quad \mathcal{M}, w' \nVdash \top \quad \Longleftrightarrow \quad 0 \neq 0.$$

Therefore $-\Phi$ behaves semantically as the negation of Φ in intuitionistic predicate logic. Hence the result follows by the well-known completeness theorem for this logic (see, e.g., [6]). □

5 A quantified version QN^* of N^*

In [1], in the course of developing a framework for the study of logic programs with negation, a propositional logic N^* extending N was introduced. It turns out that N^* is, in effect, intimately connected with Došen's work in intuitionistic modal logic (see [2]) as well as Vakarelov's classification of negations (see [18]); cf. also [4]. We are going to bring quantifiers into the picture.

[12] As with QN, we do not explicitly mention σ since no ambiguity is likely to arise.

5.1 A Hilbert-type calculus

Similarly to [1], we define the logic QN^* to be QN° plus the axiom scheme

$\mathtt{N}^*$. $\neg(\Phi \wedge \Psi) \to \neg\Phi \vee \neg\Psi$.

Let $\vdash^*$ denote the derivability relation of QN^*. Notice that the converses of $\mathtt{N}$ and $\mathtt{N}^*$ are derivable already in QN_σ (even without $\mathtt{N}$):

- $\neg(\Phi \vee \Psi) \to \neg\Phi \wedge \neg\Psi$ can be obtained from D1, D2 and C3 using CR and MP;
- $\neg\Phi \vee \neg\Psi \to \neg(\Phi \wedge \Psi)$ can be obtained from C1, C2 and D3 using CR and MP.

Thus QN^*_σ proves De Morgan's laws.

5.2 A possible world semantics

Call a frame $\mathcal{W}$ *special* iff R is serial, and for each $w \in W$, $R(w)$ is directed with respect to $\leqslant$. The motivation for this definition comes from:

Proposition 5.1 (see [14]). *A frame $\mathcal{W}$ is special iff the propositional versions of* $\mathtt{N1}^\circ$, $\mathtt{N2}^\circ$ *and* $\mathtt{N}^*$ *hold in every model for* N *based on* $\mathcal{W}$.

By QN^*_σ-*models* are meant QN_σ-models based on special frames, as one may expect. Let $\vDash^*$ be the relativisation of $\vDash$ to the class of QN^*_σ-models.

5.3 Soundness and completeness

We quickly obtain:

Theorem 5.2. *For any $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$,*

$$\Gamma \vdash^* \Delta \quad \Longrightarrow \quad \Gamma \vDash^* \Delta.$$

Proof. The result follows by Proposition 5.1. □

Before proceeding further, let us make a useful observation about the behavior of $\neg$ in QN^*. Given $\Gamma \subseteq \mathrm{Sent}_\sigma$, we take Γ^* to be $\{\Phi \in \mathrm{Sent}_\sigma \mid \neg\Phi \notin \Gamma\}$; thus $\Gamma^* = \mathrm{Sent}_\sigma \setminus \underline{\Gamma}$.

Proposition 5.3 (cf. [14]). *Suppose $\Gamma \subseteq \mathrm{Sent}_\sigma$ has the following properties:*

- $\Gamma \neq \mathrm{Sent}_\sigma$;
- $\{\Phi \in \mathrm{Sent}_\sigma \mid \Gamma \vdash^* \Phi\} \subseteq \Gamma$;
- *for every $\Phi \vee \Psi \in \Gamma$ we have $\Phi \in \Gamma$ or $\Psi \in \Gamma$.*

Then Γ^* also has these properties.

Proof. In the propositional case, the desired result can be extracted from [14], or more precisely, from the proof of Proposition 2.11 therein. We provide a slightly more general argument.

Since $\vdash^* \neg\bot$, we have $\neg\bot \in \Gamma$, i.e. $\bot \notin \Gamma^*$. Thus $\Gamma^* \neq \mathrm{Sent}_\sigma$.

Assume $\Gamma^* \vdash^* \Phi$. Then $\vdash^* (\Psi_1 \wedge \cdots \wedge \Psi_n) \to \Phi$ for some $\{\Psi_1, \ldots, \Psi_n\} \subseteq \Gamma^*$.[13] In particular, $\vdash^* \neg\Phi \to \neg(\Psi_1 \wedge \cdots \wedge \Psi_n)$. At the same time, since $\neg\Psi_i \notin \Gamma$ for each $i \in \{1, \ldots, n\}$, we obtain $\neg\Psi_1 \vee \ldots \vee \neg\Psi_n \notin \Gamma$. Therefore $\neg(\Psi_1 \wedge \ldots \wedge \Psi_n) \notin \Gamma$ by $\mathtt{N}^*$, hence $\neg\Phi \notin \Gamma$, i.e. $\Phi \in \Gamma^*$.

Assume $\Phi \vee \Psi \in \Gamma^*$, i.e. $\neg(\Phi \vee \Psi) \notin \Gamma$. Then we have $\neg\Phi \wedge \neg\Psi \notin \Gamma$. So $\neg\Phi \notin \Gamma$ or $\neg\Psi \notin \Gamma$, i.e. $\Phi \in \Gamma^*$ or $\Psi \in \Gamma^*$. □

[13] Note that empty conjunction (which arises when $n = 0$) will traditionally be identified with $\top$.

Now the *canonical frame for* QN^*, denoted $\mathcal{W}^{\mathsf{QN}^*} = \langle W^{\mathsf{QN}^*}, \leqslant^{\mathsf{QN}^*}, R^{\mathsf{QN}^*} \rangle$, is the subframe of $\mathcal{W}^{\mathsf{QN}}$ generated by the non-trivial worlds containing QN^*_σ.

Proposition 5.4. *$\mathcal{W}^{\mathsf{QN}^*}$ is a special frame.*

Proof. Let $\Gamma \in W^{\mathsf{QN}^*}$. Evidently, we have $\Gamma^* \nvdash^* \underline{\Gamma}$, so there exists $\Delta \in W^{\mathsf{QN}^*}$ such that $\Gamma^* \subseteq \Delta$ and $\Delta \nvdash^* \underline{\Gamma}$. Obviously, $\underline{\Gamma} \cap \Delta = \varnothing$. Thus R^{QN^*} is serial.

Let $\{\Gamma, \Delta_1, \Delta_2\} \subseteq W^{\mathsf{QN}^*}$ be such that $\underline{\Gamma} \cap \Delta_1 = \varnothing$ and $\underline{\Gamma} \cap \Delta_2 = \varnothing$. Notice that Δ_1 and Δ are closed under conjunction, while $\underline{\Gamma}$ is closed under disjunction by Proposition 3.14.

<u>Claim</u>: $\Delta_1 \cup \Delta_2 \nvdash^* \underline{\Gamma}$, viz. $\mathsf{QN}^* \cup \Delta_1 \cup \Delta_2 \nvdash \underline{\Gamma}$.

For otherwise $\Psi_1 \wedge \Psi_2 \vdash^* \Phi$ for some $\Psi_1 \in \Delta_1$, $\Psi_2 \in \Delta_2$ and $\Phi \in \underline{\Gamma}$. Evidently, the analogue of Proposition 3.14 for $\vdash^*$ holds; therefore $\Psi_1 \wedge \Psi_2$ is in $\underline{\Gamma}$, i.e. $\neg\,(\Psi_1 \wedge \Psi_2) \in \Gamma$. Consequently $\neg\Psi_1 \vee \neg\Psi_2 \in \Gamma$ by $\mathtt{N}^*$, and thus $\neg\Psi_1 \in \Gamma$ or $\neg\Psi_2 \in \Gamma$, i.e. $\Psi_1 \in \underline{\Gamma}$ or $\Psi_2 \in \underline{\Gamma}$, a contradiction.

Since $\Delta_1 \cup \Delta_2 \nvdash^* \underline{\Gamma}$, there is $\Delta \in W^{\mathsf{QN}^*}$ such that $\Delta_1 \cup \Delta_2 \subseteq \Delta$ and $\Delta \nvdash^* \underline{\Gamma}$. Obviously, we have $\Delta_1 \subseteq \Delta$, $\Delta_2 \subseteq \Delta$ and $\underline{\Gamma} \cap \Delta = \varnothing$. ☐

Naturally, by the *canonical model for* QN^* we mean $\mathcal{M}^{\mathsf{QN}^*} = \langle \mathcal{W}^{\mathsf{QN}^*}, \mathscr{A}^{\mathsf{QN}^*} \rangle$ where $\mathscr{A}^{\mathsf{QN}^*}$ is the restriction of $\mathscr{A}^{\mathsf{QN}}$ to $\mathcal{W}^{\mathsf{QN}^*}$. Clearly, $\mathcal{M}^{\mathsf{QN}^*}$ is a QN^*-model, and furthermore, the Canonical Model Lemma for QN implies that for QN^*.

Theorem 5.5 (Strong Completeness)**.** *For any $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \sqsubseteq \mathrm{Form}_\sigma$,*

$$\Gamma \vdash^* \Delta \quad \Longleftrightarrow \quad \Gamma \vDash^* \Delta.$$

Proof. This is perfectly analogous to the proof of Theorem 3.16. ☐

It was shown in [14] that the $\{\wedge, \vee, \rightarrow, -\}$-fragment of N^* is precisely intuitionistic propositional logic. This generalises to QN^*, of course.

Proposition 5.6. *The $\{\wedge, \vee, \rightarrow, -\}$-fragment of QN^* is precisely intuitionistic predicate logic.*

Proof. The argument for Proposition 4.5 applies. ☐

6 A useful extension $\mathsf{QN}^\sharp$ of QN^*

Although QN^* and N^* share certain nice properties, it seems that QN^*, unlike N^*, does not have a Routley-style semantics.[14] So we are going to modify QN^* appropriately.

6.1 A Hilbert-type calculus

We define the logic $\mathsf{QN}^\sharp$ to be QN^* plus the following axiom schemata:

$\mathtt{N1}^\sharp$. $\forall x\, \neg\Phi \rightarrow \neg\exists x\, \Phi$;

$\mathtt{N2}^\sharp$. $\neg\forall x\, \Phi \rightarrow \exists x\, \neg\Phi$;

$\mathtt{CD}$. $\forall x\, (\Phi \vee \Psi) \rightarrow \Phi \vee \forall x\, \Psi$ for x not free in Φ.

[14] Roughly, for the Routley-style canonical model construction to work, we need Γ^*, as defined in Section 5, to be a prime theory, provided Γ is itself a prime theory. This is true for N^* (see [14]) but, in general, not for QN^*. Thus we are lead to considering quantified analogues of De Morgan's laws.

Let $\vdash^\sharp$ denote the derivability relation of $\mathsf{QN}^\sharp$. Notice that the converses of $\mathtt{N1}^\sharp$ and $\mathtt{N2}^\sharp$ are derivable already in QN (even without $\mathtt{N}$), as can be readily verified.

Intuitively, $\mathsf{QN}^\sharp$ has both modal and intuitionistic features. In this context we may think of $\mathtt{N1}^\sharp$ and $\mathtt{N2}^\sharp$ as playing the role of Barcan's formula. Further, $\mathtt{CD}$ may be viewed as an analogue to Barcan's formula in intuitionistic predicate logic with constant domains.

At the same time, it should be remarked that the 'constant domain condition' for R implies that for $\leqslant$, provided R is serial. More precisely, the following holds.

Proposition 6.1. *Let $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ be a QN°_σ-model such that for all $u, v \in W$,*

$$u\ R\ v \quad \Longrightarrow \quad A_u \ = \ A_v.$$

Then for any $u, v \in W$, $u \leqslant v$ implies $A_u = A_v$.

Proof. Assume $u \leqslant v$. Since R is serial, there exists $v' \in W$ such that vRv'. Remembering that $\leqslant \circ R \subseteq R \circ \leqslant^{-1}$, it follows that for some $u' \in W$ we have uRu' and $v' \leqslant u'$. Hence

$$A_u \ \subseteq \ A_v \ = \ A_{v'} \ \subseteq \ A_{u'} \ = \ A_u,$$

and therefore $A_u = A_v$ as desired. □

Naturally, we aim at developing an elegant constant domain semantics for $\mathsf{QN}^\sharp$. Henceforth we shall assume that σ is at most countable.[15]

6.2 A possible world semantics

Similarly to [1], call $\mathcal{W}$ a $*$-*frame* iff for each $w \in W$, $R(w)$ has a greatest element with respect to $\leqslant$. Clearly, $*$-frames are special. Next, observe that for every QN_σ-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ with $\mathcal{W}$ a $*$-frame,

$$\mathcal{M}, w \Vdash \neg\Psi \quad \Longleftrightarrow \quad \mathcal{M}, \widehat{w} \nVdash \Psi \ \text{ where } \widehat{w} \text{ is the greatest element of } R(w).$$

This leads to a Routley-style semantics for $\mathsf{QN}^\sharp$-extensions (cf. [16]). More precisely, similarly to [1], by a *Routley frame* we mean a triple $\mathcal{W} = \langle W, \leqslant, * \rangle$ where:

- W is a non-empty set;
- $\leqslant$ is a preordering on W;
- $*$ is an anti-monotone function from W to W.

Obviously, since $*$ is a binary relation on W, it may be viewed as an accessibility relation. And further, we readily verify that

$$\leqslant \circ * \ \subseteq \ * \circ \leqslant^{-1}.$$

Hence any Routley frame turns out to be a $*$-frame.[16] On the other hand, with each $*$-frame $\mathcal{W} = \langle W, \leqslant, R \rangle$ we can associate the Routley frame $\mathcal{W}^* = \langle W, \leqslant, * \rangle$ given by

$$w^* \ := \ \text{the greatest element of } R(w).$$

If $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ is a QN_σ-model with $\mathcal{W}$ a $*$-frame, we write $\mathcal{M}^*$ for $\langle \mathcal{W}^*, \mathscr{A} \rangle$.

[15] The point is that canonical model lemmas for constant domain semantics require analogues of [10, Theorem 14.2] and [6, Lemma 7.2.3], whose proofs rely on the countability of σ.

[16] Notice, however, that in [1, 14] all frames for N^* are assumed to be condensed — but then a Routley frame for N^* is not, in general, a frame for N^*.

Proposition 6.2. *For every* QN_σ*-model* $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ *with* $\mathcal{W}$ *a* $*$*-frame,* $w \in W$ *and* A_w*-sentence* Φ*,*

$$\mathcal{M}, w \Vdash \Phi \quad \Longleftrightarrow \quad \mathcal{M}^*, w \Vdash \Phi.$$

Proof. An easy induction on the complexity of Φ. □

Thus, in terms of validity, it makes no difference whether our semantics for $\mathsf{QN}^\sharp$ is based on $*$-frames or Routley frames. We choose the latter. Call $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ a $\mathsf{QN}^\sharp_\sigma$*-model* iff:

- $A_u = A_v$ for all $u, v \in W$;
- $\mathcal{W}$ is a Routley frame.

In other words, $\mathsf{QN}^\sharp_\sigma$-models are 'constant domain' models for QN_σ which are based on Routley frames. If $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ is a $\mathsf{QN}^\sharp_\sigma$-model, we take

$$A \; := \; \bigcap_{w \in W} A_w$$

— then $A = A_w$ for each $w \in W$, of course. Let $\vDash^\sharp$ denote the relativisation of $\vDash$ to the class of $\mathsf{QN}^\sharp_\sigma$-models.

6.3 Soundness and completeness

As usual, we quickly obtain:

Theorem 6.3. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash^\sharp \Delta \quad \Longrightarrow \quad \Gamma \vDash^\sharp \Delta.$$

Proof. Clearly, it suffices to prove that for any axiom Φ of the type $\mathtt{N1}^\sharp$, $\mathtt{N2}^\sharp$ or $\mathtt{CD}$ we get $\vDash^\sharp \Phi$. Now consider a $\mathsf{QN}^\sharp_\sigma$-model $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$.

Let $w \in W$, and assume $w \Vdash \forall x \, \neg\Phi$. So in particular, for each $a \in A$ we have $w^* \nVdash \Phi\,(x/\underline{a})$. Thus $w^* \nVdash \exists x \, \Phi$, i.e. $w \Vdash \neg\exists x \, \Phi$.

Let $w \in W$, and assume $w \Vdash \neg\forall x \, \Phi$, i.e. $w^* \nVdash \forall x \, \Phi$. There exist $u \in W$ and $a \in A$ such that $w^* \leqslant u$ and $u \nVdash \Phi\,(x/\underline{a})$. Evidently, $w^* \nVdash \Phi\,(x/\underline{a})$, i.e. $w \Vdash \neg\Phi\,(x/\underline{a})$. Hence $w \Vdash \exists x \, \neg\Phi$.

For $\mathtt{CD}$ one can argue as in intuitionistic predicate logic with constant domains. □

For the rest of this section, $S^\star$ will denote a fixed countable set; we shall write $\sigma_\star$ instead of $\sigma_{S^\star}$. Naturally, *strongly prime* $\sigma_\star$*-theories over* $\mathsf{QN}^\sharp_{\sigma_\star}$ are simply those that contain $\mathsf{QN}^\sharp_{\sigma_\star}$.

Next, we shall apply the canonical model construction to $\mathsf{QN}^\sharp_\sigma$ by extracting an appropriate constant domain submodel of $\mathcal{M}^{\mathsf{QN}^*_\sigma}$. The subscript σ will often be dropped. Define

$$W^{\mathsf{QN}^\sharp_\sigma} \; := \; \text{the collection of all strongly prime } \sigma_\star\text{-theories over } \mathsf{QN}^\sharp_{\sigma_\star}.$$

The following can be proved exactly as in intuitionistic predicate logic with constant domains — see [6, Lemma 7.2.3], which in turn is inspired by [10, Theorem 14.2] that deals with classical predicate modal logic.

Lemma 6.4. *Let* Γ *be a strongly prime* σ*-theory. Suppose* $\Phi \in \mathrm{Sent}_\sigma$ *and* $\Psi \in \mathrm{Form}_\sigma$ *are such that* $\Gamma \cup \{\Phi\} \nvdash \Psi$*. Then there exists a strongly prime* σ*-theory* $\Gamma' \supseteq \Gamma \cup \{\Phi\}$ *such that* $\Gamma' \nvdash \Psi$*.*

Recall that Γ^* denotes $\{\Phi \in \text{Sent} \mid \neg\Phi \notin \Gamma\}$. And $W^{\mathsf{QN}^\sharp}$ is closed under this operation:

Proposition 6.5. *$\Gamma^* \in W^{\mathsf{QN}^\sharp}$ for all $\Gamma \in W^{\mathsf{QN}^\sharp}$.*

Proof. Let Γ be a strongly prime $\sigma_\star$-theory over $\mathsf{QN}^\sharp_{\sigma_\star}$.

Replacing $\vdash^*$ by $\vdash^\sharp$ in the proof of Proposition 5.3, we see already that Γ^* does not coincide with Sent, is closed under $\vdash^\sharp$, and has the disjunction property.

Assume $\exists x\, \Phi \in \Gamma^*$, i.e. $\neg\exists x\, \Phi \notin \Gamma$. So we have $\forall x\, \neg\Phi \notin \Gamma$ by $\mathtt{N1}^\sharp$. Hence there is a constant symbol c such that $\neg\Phi\,(x/c) \notin \Gamma$, i.e. $\Phi\,(x/c) \in \Gamma^*$.

Assume $\Phi\,(x/c) \in \Gamma^*$ for all constant symbols c, i.e. $\neg\Phi\,(x/c) \notin \Gamma$ for each c. Hence we have $\exists x\, \neg\Phi \notin \Gamma$. So $\neg\forall x\, \Phi \notin \Gamma$ by $\mathtt{N2}^\sharp$, i.e. $\forall x\, \Phi \in \Gamma^*$. □

In particular, it follows that for every $\Gamma \in W^{\mathsf{QN}^\sharp}$,

$$\Gamma^* \;=\; \text{the greatest element of } \{\Delta \in W^{\mathsf{QN}^\sharp} \mid \underline{\Gamma} \cap \Delta = \varnothing\}$$

(with respect to inclusion). Consider the function $*^{\mathsf{QN}^\sharp}$ from $W^{\mathsf{QN}^\sharp}$ to $W^{\mathsf{QN}^\sharp}$ given by

$$*^{\mathsf{QN}^\sharp}(\Gamma) \;:=\; \Gamma^*.$$

By the *canonical frame for* $\mathsf{QN}^\sharp$ we mean $\mathcal{W}^{\mathsf{QN}^\sharp} = \langle W^{\mathsf{QN}^\sharp}, \leqslant^{\mathsf{QN}^\sharp}, *^{\mathsf{QN}^\sharp}\rangle$ where

$$\leqslant^{\mathsf{QN}^\sharp} \;:=\; \Big\{(\Gamma_1, \Gamma_2) \in W^{\mathsf{QN}^\sharp} \times W^{\mathsf{QN}^\sharp} \mid \Gamma_1 \subseteq \Gamma_2\Big\},$$

as usual. Clearly, $\mathcal{W}^{\mathsf{QN}^\sharp}$ is a Routley frame. Finally, by the *canonical model for* $\mathsf{QN}^\sharp$ we mean

$$\mathcal{M}^{\mathsf{QN}^\sharp} \;=\; \langle \mathcal{W}^{\mathsf{QN}^\sharp}, \mathscr{A}^{\mathsf{QN}^\sharp}\rangle$$

where $\mathscr{A}^{\mathsf{QN}^\sharp}$ is the restriction of $\mathscr{A}^{\mathsf{QN}^*}$ to $\mathcal{W}^{\mathsf{QN}^\sharp}$. Note that $\mathcal{M}^{\mathsf{QN}^\sharp}$ is a $\mathsf{QN}^\sharp$-model with domain $\text{Const}_{\sigma_\star}$.[17] For convenience we write C instead of $\text{Const}_{\sigma_\star}$ below.

Lemma 6.6 (Canonical Model Lemma)**.** *For any $\Gamma \in W^{\mathsf{QN}^\sharp}$ and C-sentence Φ,*

$$\mathcal{M}^{\mathsf{QN}^\sharp}, \Gamma \Vdash \Phi \quad\Longleftrightarrow\quad \Phi \;\in\; \Gamma.$$

Proof. By induction on the complexity of Φ.

In the case where Φ is atomic the result is immediate.

Suppose $\Phi = \neg\Psi$. Then we have

$$\neg\Psi \;\in\; \Gamma \quad\Longleftrightarrow\quad \Psi \;\notin\; \Gamma^* \quad\Longleftrightarrow\quad \mathcal{M}^{\mathsf{QN}^\sharp}, \Gamma^* \nVdash \Psi \quad\Longleftrightarrow\quad \mathcal{M}^{\mathsf{QN}^\sharp}, \Gamma \Vdash \neg\Psi$$

(here the second equivalence is justified by the inductive hypothesis).

In the other cases one can argue as in intuitionistic predicate logic with constant domains; these do not involve $*^{\mathsf{QN}^\sharp}$. In particular, Lemma 6.4 is used to show that:

$$\mathcal{M}^{\mathsf{QN}^\sharp}, \Gamma \Vdash \Psi \to \Theta \quad\Longrightarrow\quad \Psi \to \Theta \;\in\; \Gamma.$$

The crucial point here is that applying Lemma 6.4 does not require adding new constants. □

[17]Of course, $\text{Const}\,(\Gamma) = \text{Const}_{\sigma_\star}$ for any strongly prime $\sigma^\star$-theory Γ over $\mathsf{QN}^\sharp_{\sigma_\star}$.

This leads us to:

Theorem 6.7 (Strong Completeness)**.** *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$,

$$\Gamma \vdash^\sharp \Delta \quad \Longleftrightarrow \quad \Gamma \vDash^\sharp \Delta.$$

Proof. $\boxed{\Longrightarrow}$ This is Theorem 6.3.

$\boxed{\Longleftarrow}$ Assume $\Gamma \nvdash^\sharp \Delta$. Fix an admissible $S \subseteq S^\star$ of cardinality $\aleph_0$; let λ be some one-to-one function from Var onto $\{\underline{s} \mid s \in S\}$, and take

$$\Delta' \;:=\; \{\lambda\Psi \mid \Psi \in \Delta\}.$$

Then $\Gamma \nvdash^\sharp \Delta'$. By Corollary 3.13, there is $\Gamma' \in W^{\mathsf{QN}^\sharp}$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash^\sharp \Delta'$. Evidently, λ can be viewed as a ground C-substitution. By Lemma 6.6, $\mathcal{M}^{\mathsf{QN}^\sharp}, \Gamma' \Vdash \Phi$ for all $\Phi \in \Gamma$, while $\mathcal{M}^{\mathsf{QN}^\sharp}, \Gamma' \nVdash \lambda\Psi$ for all $\Psi \in \Delta$. Consequently $\Gamma \nvDash^\sharp \Delta$. ☐

Finally, the analogue of Proposition 5.6 holds.

Proposition 6.8. *The* $\{\wedge, \vee, \rightarrow, -\}$*-fragment of* $\mathsf{QN}^\sharp$ *is precisely intuitionistic predicate logic with constant domains.*

Proof. Let $\mathcal{M} = \langle \mathcal{W}, \mathscr{A} \rangle$ be a $\mathsf{QN}^\sharp_\sigma$-model. Then for all $w \in W$,

$$\mathcal{M}, w \Vdash \bot \quad \Longleftrightarrow \quad \mathcal{M}, w^* \nVdash \top \quad \Longleftrightarrow \quad 0 \neq 0.$$

Hence the result follows. ☐

7 Concerning Leitgeb's quantified Hype

In [12], a certain system of 'hyperintensional' logic is advocated.[18] Denote by Hype and QHype its propositional and quantified versions respectively. S. P. Odintsov has recently observed that, in effect, Hype coincides with the logic obtained from N^* by adding the laws of double negation introduction and elimination. This leads to a Routley-style semantics for Hype which is simpler than the one suggested by Leitgeb. See [15] for further discussion.

Still, the application to semantic paradoxes given in [12] requires QHype. Note that Leitgeb provides only a rough sketch of the weak completeness proof for QHype, which refers to [9] and [17] extensively (filling the gaps is then left to the reader). I shall give a self-contained proof of the strong completeness of QHype with respect to a suitable Routley-style semantics.

7.1 A Hilbert-type calculus and its variations

We define the logic $\mathsf{QN}^\bullet$ to be QN plus the following axiom schemata:

`N1`$^\bullet$. $\Phi \rightarrow \neg\neg\Phi$;

`N2`$^\bullet$. $\neg\neg\Phi \rightarrow \Phi$.

Let $\vdash^\bullet$ denote the derivability relation of $\mathsf{QN}^\bullet$. We shall see shortly that $\mathsf{QN}^\bullet$ extends $\mathsf{QN}^\sharp$ and eventually coincides with Leitgeb's QHype.

Proposition 7.1. `N` *is derivable from* `I1`–`I2`, `C1`–`C2`, `D3` *and* `N1`$^\bullet$ *using* `MP` *and* `CR`.

[18] Whether this system should really be called 'hyperintensional' is a matter of dispute; cf. [15].

Proof. We reason as follows:

1.	$\neg\Phi \wedge \neg\Psi \to \neg\Phi$	C1
2.	$\neg\neg\Phi \to \neg(\neg\Phi \wedge \neg\Psi)$	from 1 using CR
3.	$\Phi \to \neg(\neg\Phi \wedge \neg\Psi)$	from $\mathtt{N1}^\bullet$, 2
4.	$\neg\Phi \wedge \neg\Psi \to \neg\Psi$	C2
5.	$\neg\neg\Psi \to \neg(\neg\Phi \wedge \neg\Psi)$	from 4 using CR
6.	$\Psi \to \neg(\neg\Phi \wedge \neg\Psi)$	from $\mathtt{N1}^\bullet$, 5
7.	$\Phi \vee \Psi \to \neg(\neg\Phi \wedge \neg\Psi)$	from 3, 6, D3
8.	$\neg\neg(\neg\Phi \wedge \neg\Psi) \to \neg(\Phi \vee \Psi)$	from 7 using CR
9.	$(\neg\Phi \wedge \neg\Psi) \to \neg(\Phi \vee \Psi)$	from $\mathtt{N1}^\bullet$, 8.

This line of reasoning implicitly involves I1–I2 and MP, of course. □

Thus N turns out to be redundant in $\mathsf{QN}^\bullet$.

Proposition 7.2. *$\mathtt{N1}^\circ$, $\mathtt{N2}^\circ$ and $\mathtt{N}^*$ are derivable in $\mathsf{QN}^\bullet$.*

Proof. For $\mathtt{N1}^\circ$, $\mathtt{N2}^\circ$ and $\mathtt{N}^*$ we argue as follows:

1.	$\top$	positive intuitionistic logic
2.	$\neg\Psi \to \top$	positive intuitionistic logic
3.	$\neg\top \to \neg\neg\Psi$	from 2 using CR
4.	$\neg\top \to \Psi$	from 3, $\mathtt{N2}^\bullet$
1.	$\top$	positive intuitionistic logic
2.	$\top \to \neg\neg\top$	$\mathtt{N1}^\bullet$
3.	$\neg\neg\top$	from 1, 2
1.	$\neg\Phi \to \neg\Phi \vee \neg\Psi$	D1
2.	$\neg(\neg\Phi \vee \neg\Psi) \to \neg\neg\Phi$	from 1 using CR
3.	$\neg(\neg\Phi \vee \neg\Psi) \to \Phi$	from 2, $\mathtt{N2}^\bullet$
4.	$\neg\Psi \to \neg\Phi \vee \neg\Psi$	D2
5.	$\neg(\neg\Phi \vee \neg\Psi) \to \neg\neg\Psi$	from 4 using CR
6.	$\neg(\neg\Phi \vee \neg\Psi) \to \Psi$	from 5, $\mathtt{N2}^\bullet$
7.	$\neg(\neg\Phi \vee \neg\Psi) \to \Phi \wedge \Psi$	from 3, 6, C3
8.	$\neg(\Phi \wedge \Psi) \to \neg\neg(\neg\Phi \vee \neg\Psi)$	from 7 using CR
9.	$\neg(\Phi \wedge \Psi) \to \neg\Phi \vee \neg\Psi$	from 8, $\mathtt{N2}^\bullet$.

(Remember that any two formulas of the form $\Theta \to \Theta$ are interchangeable over QN.) □

Consequently $\mathsf{QN}^\bullet$ extends QN^*.

Proposition 7.3. *$\mathtt{N1}^\sharp$ and $\mathtt{N2}^\sharp$ are derivable in $\mathsf{QN}^\bullet$.*

Proof. For $\texttt{N1}^\sharp$ and $\texttt{N2}^\sharp$ we may reason as follows:

1.	$\forall x\, \neg\Phi \to \neg\Phi$	$\texttt{Q1}$
2.	$\neg\neg\Phi \to \neg\forall x\, \neg\Phi$	from 1 using $\texttt{CR}$
3.	$\Phi \to \neg\forall x\, \neg\Phi$	from $\texttt{N1}^\bullet$, 2
4.	$\exists x\, \Phi \to \neg\forall x\, \neg\Phi$	from 3 using $\texttt{BR2}$
5.	$\neg\neg\forall x\, \neg\Phi \to \neg\exists x\, \Phi$	from 4 using $\texttt{CR}$
6.	$\forall x\, \neg\Phi \to \neg\exists x\, \Phi$	from $\texttt{N1}^\bullet$, 5
1.	$\neg\Phi \to \exists x\, \neg\Phi$	$\texttt{Q2}$
2.	$\neg\exists x\, \neg\Phi \to \neg\neg\Phi$	from 1 using $\texttt{CR}$
3.	$\neg\exists x\, \neg\Phi \to \Phi$	from 2, $\texttt{N2}^\bullet$
4.	$\neg\exists x\, \neg\Phi \to \forall x\, \Phi$	from 3 using $\texttt{BR1}$
5.	$\neg\forall x\, \Phi \to \neg\neg\exists x\, \neg\Phi$	from 4 using $\texttt{CR}$
6.	$\neg\forall x\, \Phi \to \exists x\, \neg\Phi$	from 5, $\texttt{N2}^\bullet$.

(Clearly, $\texttt{N1}^\sharp$–$\texttt{N2}^\sharp$ can also be derived from their converses, once $\texttt{N1}^\bullet$–$\texttt{N2}^\bullet$ are available.) □

This immediately implies:

Corollary 7.4. QHype *(as presented in [12]) and* $\mathsf{QN}^\bullet$ *coincide.*

Leitgeb's axiomatisation is not 'minimal', and includes a number of redundant schemata (in particular, it assumes not only $\texttt{N1}^\bullet$–$\texttt{N2}^\bullet$ but also all de Morgan's laws and $\texttt{N2}^\sharp$); still, it leads to the same logic. From now on we shall use QHype and $\mathsf{QN}^\bullet$ interchangeably.

Yet another simple fact about QHype is worth mentioning here:

Proposition 7.5 (see [12])**.** $\texttt{CD}$ *is derivable in* QHype.

Hence QHype ($\mathsf{QN}^\bullet$) extends $\mathsf{QN}^\sharp$ as well.

7.2 A Routley-style semantics

Call a Routley frame $\mathcal{W} = \langle W, \leqslant, * \rangle$ *involutive* iff $w^{**} = w$ for each $w \in W$. Define $\mathsf{QN}^\bullet_\sigma$*-models* to be constant domain models for QN_σ which are based on involutive Routley frames, i.e. these are precisely $\mathsf{QN}^\sharp_\sigma$-models whose underlying frames are involutive. Let $\vDash^\bullet$ be the relativisation of $\vDash$ to the class of $\mathsf{QN}^\bullet_\sigma$-models.

7.3 Soundness and completeness

We immediately get:

Theorem 7.6. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \sqsubseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash^\bullet \Delta \quad \Longrightarrow \quad \Gamma \vDash^\bullet \Delta.$$

Proof. Clearly, for any axiom Φ of the type $\texttt{N1}^\bullet$ or $\texttt{N2}^\bullet$ we have $\vDash^\bullet \Phi$. So the result follows. □

Now the *canonical frame for* $\mathsf{QN}^\bullet$, denoted $\mathcal{W}^{\mathsf{QN}^\bullet} = \langle W^{\mathsf{QN}^\bullet}, \leqslant^{\mathsf{QN}^\bullet}, *^{\mathsf{QN}^\bullet} \rangle$, is the subframe of $\mathcal{W}^{\mathsf{QN}^\sharp}$ generated by the worlds containing $\mathsf{QN}^\bullet$.

Proposition 7.7. $\mathcal{W}^{\mathsf{QN}^\bullet}$ *is involutive.*

Proof. Let $\Gamma \in W^{\mathsf{QN}^\bullet}$. Obviously, for any $\Phi \in \text{Sent}$,

$$\Phi \in \Gamma^{**} \iff \neg\Phi \notin \Gamma^{*} \iff \neg\neg\Phi \in \Gamma \iff \Phi \in \Gamma.$$

Hence Γ^{**} coincides with Γ. □

Naturally, by the *canonical model for* $\mathsf{QN}^\bullet$ we mean $\mathcal{M}^{\mathsf{QN}^\bullet} = \langle \mathcal{W}^{\mathsf{QN}^\bullet}, \mathscr{A}^{\mathsf{QN}^\bullet} \rangle$ where $\mathscr{A}^{\mathsf{QN}^\bullet}$ is the restriction of $\mathscr{A}^{\mathsf{QN}^\sharp}$ to $\mathcal{W}^{\mathsf{QN}^\bullet}$. Clearly, $\mathcal{M}^{\mathsf{QN}^\bullet}$ is a $\mathsf{QN}^\bullet$-model, and moreover, the Canonical Model Lemma for $\mathsf{QN}^\sharp$ implies that for $\mathsf{QN}^\bullet$.

Theorem 7.8 (Strong Completeness). *For any* $\Gamma \subseteq \text{Sent}_\sigma$ *and* $\Delta \sqsubseteq \text{Form}_\sigma$*,*

$$\Gamma \vdash^\bullet \Delta \iff \Gamma \vDash^\bullet \Delta.$$

Proof. This is perfectly analogous to the proof of Theorem 6.7. □

In a nutshell, QHype has a nice Routley-style semantics, with respect to which it is strongly complete. The proposed semantics looks much simpler than the one provided in [12].

Not surprisingly, the analogue of Proposition 6.8 holds.

Proposition 7.9 (see [12]). *The* $\{\wedge, \vee, \to, -\}$*-fragment of* $\mathsf{QN}^\bullet$ *is precisely intuitionistic predicate logic with constant domains.*

Proof. The argument for Proposition 6.8 applies. □

8 On the constructive properties of ∨ and ∃

In [12], Leitgeb mistakenly claimed that Hype has the disjunction property, but his argument is flawed; cf. [15, 5]. In this section we shall briefly discuss related issues.

8.1 Disjunction

It was observed in [5, Proposition 6] that N^* — and hence QN^* — does not have the disjunction property. More precisely, $\neg p \vee \neg(-q \wedge (p \to q))$ belongs to N^*, but neither of its disjuncts does. To give a simpler counterexample, consider the following scheme:

`WEM`. $\neg\Phi \vee \neg-\Phi$.

Here '`WEM`' stands for 'weak excluded middle'.

Proposition 8.1 (cf. [15]). `WEM` *is derivable in* QN^*.

Proof. We argue as follows:

1.	$\Phi \wedge -\Phi \to \bot$	positive intuitionistic logic
2.	$\neg\bot \to \neg(\Phi \wedge -\Phi)$	from 1 using `CR`
3.	$\neg(\Phi \wedge -\Phi)$	from `N2`°, 2
4.	$\neg\Phi \vee \neg-\Phi$	from 3, `N`*.

(Notice that the argument involves no quantifier axioms or rules.) □

Call a QN^*-extension *subclassical* iff it is a subset of classical predicate logic.

Corollary 8.2. *No subclassical* QN^**-extension has the disjunction property.*

Proof. Let L be a subclassical QN^*-extension. Obviously, there is a formula Φ such that neither $\neg\Phi$ nor $\neg{-}\Phi$ is in L. However, $\neg\Phi \vee \neg{-}\Phi$ belongs to L. $\square$

Using the soundness theorems, it is easy to verify semantically that QN^*, $\mathsf{QN}^\sharp$ and $\mathsf{QN}^\bullet$ are subclassical. So neither of them has the disjunction property. The same applies to their propositional versions, of course.

8.2 Existential quantifier

The existential property is a bit more tricky. We assume, for simplicity, that σ contains exactly two constant symbols, 0 and 1. Now consider the following scheme:

`WCP`. $\neg{-}(\Phi(x/0) \wedge \Phi(x/1)) \vee \exists x\, \neg\Phi$.

Here '`WCP`' stands for 'weak choice principle'.

Proposition 8.3. *`WCP` is derivable in* QN^*.

Proof. We shall write $\Phi(0)$ and $\Phi(1)$ for $\Phi(x/0)$ and $\Phi(x/1)$ respectively. Take

$$\Phi^\star \ := \ \neg{-}(\Phi(0) \wedge \Phi(1)) \vee \exists x\, \neg\Phi.$$

We reason as follows:

1.	$\neg\Phi(0) \vee \neg\Phi(1) \to \exists x\, \neg\Phi$	predicate intuitionistic logic
2.	$\neg(\Phi(0) \wedge \Phi(1)) \to \exists x\, \neg\Phi$	from `N`*, 1
3.	$\neg(\Phi(0) \wedge \Phi(1)) \vee \neg{-}(\Phi(0) \wedge \Phi(1)) \to \Phi^\star$	2, positive intuitionistic logic
4.	$\Phi^\star$	from `WEM`, 3.

(Intuitively, if not $\neg{-}(\Phi(0) \wedge \Phi(1))$, then $\neg\Phi(0)$ or $\neg\Phi(1)$, and therefore $\exists x\, \neg\Phi$.) $\square$

Corollary 8.4. *No subclassical* QN^**-extension has the existential property.*

Proof. Let L be a subclassical QN^*-extension. Obviously, there is a formula $\Phi(x)$ (with exactly one free variable) such that

$$\neg{-}(\Phi(0) \wedge \Phi(1)) \vee \neg\Phi(0) \quad \text{and} \quad \neg{-}(\Phi(0) \wedge \Phi(1)) \vee \neg\Phi(1)$$

are not in L. On the other hand, intuitionistic predicate logic proves

$$\Psi \vee \exists x\, \Theta \to \exists x\, (\Psi \vee \Theta)$$

for x not free in Φ. Consequently, $\exists x\, (\neg{-}(\Phi(0) \wedge \Phi(1)) \vee \neg\Phi(x))$ belongs to L. $\square$

In particular, neither of QN^*, $\mathsf{QN}^\sharp$ and $\mathsf{QN}^\bullet$ has the existential property.

Contraclassical logics, i.e. those which are not subclassical, are somewhat more complicated. Still, it seems unlikely that a reasonable contraclassical QN^*-extension will have the disjunction property or the existential property. On the other hand, simple semantical arguments — which look very much like in intuitionistic predicate logic — show that QN and QN° do have both the disjunction property and the existential property.

9 Final comments

The technique developed in this article can be applied to other predicate logics. Here is a brief description of some directions that might profitably be explored.

- Naturally, since N plays a key role in studying propositional intuitionistic modal logics, it would be interesting to investigate predicate versions of such logics.
- It may be reasonable to develop predicate versions of the logics presented in [11].
- It will be useful to introduce and study sequent systems for QN and its extensions — this should help us analyse some metamathematical properties.
- One may consider various results on intuitionistic predicate logic (Harrop's theorems, the Curry–Howard correspondence, etc.) and try to extend them to certain QN-extensions in a suitable way.
- We could also try to employ QN^*-extensions to provide a suitable framework for studying first-order logic programs with negation; compare this to [1].

On the linguistic side, if we want to study negation as used in natural language, and focus our attention on its modal aspects, this should be done in a predicate setting. Still, all these things fall beyond the scope of the present article, and are the subject of future work.

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