Modal bilattice logic and its extensions*

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Abstract

In this paper we consider the lattices of extensions of three logics: (1) modal bilattice logic; (2) full Belnap–Dunn bimodal logic; (3) classical bimodal logic. We shall prove that these lattices are isomorphic to each other. Furthermore, the isomorphisms constructed will preserve various nice properties — such as tabularity, pretabularity, decidability or Craig's interpolation property.

Keywords many-valued modal logic · strong negation · first-degree entailment · algebraic logic

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1 Introduction

This paper deals with modal logics based on FDE, also known as the *Belnap–Dunn useful four-valued logic* (see [2, 3, 5]). Four truth values are used in defining these:

1. T, which intuitively stands for 'true';
2. F, which intuitively stands for 'false';
3. N, which intuitively stands for 'neither true nor false';
4. B, which intuitively stands for 'both true and false'.

One such logic, called *modal bilattice logic*, was introduced in [8, 7]; denote it by MBL.[1] Notice that T, F, N and B are expressible as terms in the language of MBL.

Another important FDE-based modal logic is BK, which was introduced in [13] and has been studied in [11, 10, 9]. Though its Kripke semantics makes use of all the four truth values, only T and F are expressible as terms in the language of BK. Let FBK denote the logic obtained from BK by expanding its original language to include constant symbols for N and B; we shall call it *full Belnap–Dunn modal logic*. It is worth noting that the Kripke semantics of BK and FBK use two-valued (or classical) accessibility relations, while that of MBL employs four-valued ones. Furthermore, $\Box$ behaves differently in these logics. In particular, BK and FBK are normal (i.e. closed under monotonicity rules), but MBL is not.

Adequate algebraic semantics for MBL, BK and FBK have been described in [7, 11, 9]. These semantics associate with each $L \in \{\mathsf{MBL}, \mathsf{BK}, \mathsf{FBK}\}$ a suitable variety of algebras in the language of L — which we shall call *L-algebras*. Given an L, denote by $\mathcal{E}L$ the lattice of all L-extensions (defined appropriately). Then the 'adequateness' of the variety of L-algebras ensures that there exists a natural isomorphism between the lattice of all its subvarieties and $\mathcal{E}L$.

We shall denote classical bimodal logic by K^2, and FBK's bimodal version by FBK^2.[2] In Section 3, it will be shown that there is a one-one correspondence between MBL-algebras and FBK^2-algebras; this will allow us to prove algebraically that $\mathcal{E}\mathsf{MBL}$ and $\mathcal{E}\mathsf{FBK}^2$ are isomorphic as lattices. Moreover, in Section 4, using a technique similar to that applied in [9] we shall prove that $\mathcal{E}\mathsf{FBK}^2$ and $\mathcal{E}\mathsf{K}^2$ are isomorphic as lattices — this is quite surprising because FBK is, in a sense, four-valued, while K is two-valued.

From the proofs of the foregoing results we can extract explicit isomorphisms between $\mathcal{E}\mathsf{MBL}$, $\mathcal{E}\mathsf{FBK}^2$ and $\mathcal{E}\mathsf{K}^2$ (which are induced by appropriate computable formula translations). It turns out that these isomorphisms preserve various nice properties — such as tabularity, pretabularity, decidability or Craig's interpolation property. In this way they not only preserve lattice structure, but also much of what may be called 'metamathematical structure'.

2 Preliminaries

2.1 Syntactic conventions

In this paper we shall deal with three different propositional languages:

$$\mathcal{L}_\circ \ := \ \{\wedge, \vee, \to, \bot, \top, \Box_+, \Box_-\};$$

[1] The non-modal base of MBL is the bilattice logic of [1].

[2] Of course, the adequate algebraic semantics for FBK can be easily modified to obtain an adequate algebraic semantics for FBK^2.

$$\begin{aligned}\mathcal{L}_* &:= \{\wedge, \vee, \rightarrow, \sim, \bot, \top, \mathtt{n}, \mathtt{b}, \Box_+, \Box_-, \Diamond_+, \Diamond_-\};\\ \mathcal{L}_\star &:= \{\wedge, \vee, \otimes, \oplus, \rightarrow, \sim, \bot, \top, \mathtt{n}, \mathtt{b}, \Box\}.\end{aligned}$$

In particular, $\mathcal{L}_\circ$ will be identified with the language of the least (normal) bimodal logic K^2. For convenience, we shall use the following abbreviations.

ABBREVIATION	DEFINITION	NAME
$\neg\phi$	$\phi \rightarrow \bot$	weak negation
$\phi \Rightarrow \psi$	$(\phi \rightarrow \psi) \wedge (\sim\psi \rightarrow \sim\phi)$	strong implication
$\phi \leftrightarrow \psi$	$(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$	weak equivalence
$\phi \Leftrightarrow \psi$	$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$	strong equivalence

The symbols $\top$, $\bot$, $\mathtt{n}$ and $\mathtt{b}$ will be read as 'true', 'false', 'undefined' and 'overdefined' respectively.[3] Let Prop be the set of all propositional variables. Finally, for each $\mathcal{L} \in \{\mathcal{L}_\circ, \mathcal{L}_*, \mathcal{L}_\star\}$, denote by $\mathrm{Form}_\mathcal{L}$ the set of all $\mathcal{L}$-formulas.

2.2 Classical bimodal logic

Traditionally, we call an $\mathcal{L}_\circ$-algebra $\mathfrak{A} = \langle A; \wedge, \vee, \rightarrow, \bot, \top, \Box_+, \Box_-\rangle$ a *bimodal algebra* iff it satisfies the following conditions:

- its $\{\wedge, \vee, \rightarrow, \bot, \top\}$-reduct is a Boolean algebra;
- $\Box_+\top = \top$, and $\Box_+ (a \wedge b) = \Box_- a \wedge \Box_- b$ for any $a, b \in A$;
- $\Box_-\top = \top$, and $\Box_- (a \wedge b) = \Box_+ a \wedge \Box_+ b$ for any $a, b \in A$.

Let $\mathcal{V}_\circ$ be the class of all bimodal algebras. Obviously, $\mathcal{V}_\circ$ is a variety, i.e. it is axiomatizable by identities. Denote by $\mathcal{SV}_\circ$ the collection of all subvarieties of $\mathcal{V}_\circ$, and by $\mathcal{E}\mathsf{K}^2$ the collection of all bimodal logics.[4]

Next, we write $\mathfrak{A} \Vdash_\circ \phi$ iff $\phi = \top$ holds in $\mathfrak{A}$, i.e. belongs to the equational theory of $\mathfrak{A}$. For each class $\mathcal{K}$ of bimodal algebras and each set Γ of $\mathcal{L}_\circ$-formulas, define

$$\begin{aligned}\mathbf{L}_\circ(\mathcal{K}) &:= \{\phi \in \mathrm{Form}_{\mathcal{L}_\circ} \mid \mathfrak{A} \Vdash_\circ \phi \text{ for all } \mathfrak{A} \in \mathcal{K}\},\\ \mathbf{V}_\circ(\Gamma) &:= \{\mathfrak{A} \in \mathcal{V}_\circ \mid \mathfrak{A} \Vdash_\circ \phi \text{ for all } \phi \in \Gamma\}.\end{aligned}$$

Evidently, $\mathcal{SV}_\circ$ and $\mathcal{E}\mathsf{K}^2$ may be ordered by inclusion; so they may be viewed as lattices.

FOLKLORE 2.1.
$\mathbf{L}_\circ$ *and* $\mathbf{V}_\circ$ *induce mutually inverse dual isomorphisms between* $\mathcal{SV}_\circ$ *and* $\mathcal{E}\mathsf{K}^2$.

We now turn to similar results for FBK^2 and MBL.

2.3 Full Belnap–Dunn bimodal logic

At this point it is important to distinguish the *Belnap–Dunn modal logic*, denoted BK, from the *full Belnap–Dunn modal logic*, denoted $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$. The former was introduced in [13], while the latter was studied in [9]. We may assume that their languages are

$$\{\wedge, \vee, \rightarrow, \sim, \bot, \top, \Box, \Diamond\} \quad \text{and} \quad \{\wedge, \vee, \rightarrow, \sim, \bot, \top, \mathtt{n}, \mathtt{b}, \Box, \Diamond\}$$

[3]Thus our notation is closer to that of [9] than to that of [7].

[4]Here all bimodal logics are assumed to be normal.

respectively.[5] It was shown in [9] that BK and $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$ are, in fact, very different when it comes to studying their extensions. Henceforth we shall write FBK instead of $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$. Notice that, since $\Diamond$ can be expressed as $\sim\Box\sim$, $\{\Box, \Diamond\}$ will be viewed as one modality.

Let FBK^2 be FBK's bimodal version. The deductive system for FBK and the corresponding algebraic semantics are readily adapted to FBK^2; cf. [9]. Besides appropriate axioms and modus ponens, the system for FBK^2 includes four monotonicity rules, viz.

$$\frac{\phi \to \psi}{\heartsuit\phi \to \heartsuit\psi} \quad (\mathtt{M}\heartsuit)$$

where $\heartsuit \in \{\Box_+, \Box_-, \Diamond_+, \Diamond_-\}$. Further — by an FBK^2-*extension* we mean a set of $\mathcal{L}_*$-formulas that contains FBK^2 and is closed under modus ponens, the four monotonicity rules and substitutions. Denote by $\mathcal{E}\mathsf{FBK}^2$ the collection of all FBK^2-extensions.

Given $\mathfrak{A} \in \mathcal{V}_\circ$, by the *full twist-structure over* $\mathfrak{A}$ *in* $\mathcal{L}_*$ we mean the $\mathcal{L}_*$-algebra

$$\mathfrak{A}_* \;:=\; \langle A \times A; \wedge, \vee, \to, \sim, \bot, \top, \mathtt{n}, \mathtt{b}, \Box_+, \Diamond_+, \Box_-, \Diamond_- \rangle$$

whose operations are defined as follows:

$$\begin{aligned}
(a,b) \wedge (c,d) &:= (a \wedge c, b \vee d);\\
(a,b) \vee (c,d) &:= (a \vee c, b \wedge d);\\
(a,b) \to (c,d) &:= (\neg a \vee c, a \wedge d);\\
\sim(a,b) &:= (b,a);\\
\bot &:= (\bot, \top);\\
\top &:= (\top, \bot);\\
\mathtt{n} &:= (\bot, \bot);\\
\mathtt{b} &:= (\top, \top);\\
\Box_+(a,b) &:= (\Box_+ a, \neg\Box_+\neg b);\\
\Diamond_+(a,b) &:= (\neg\Box_+\neg a, \Box_+ b);\\
\Box_-(a,b) &:= (\Box_- a, \neg\Box_-\neg b);\\
\Diamond_-(a,b) &:= (\neg\Box_-\neg a, \Box_- b).^6
\end{aligned}$$

An $\mathcal{L}_*$-algebra is called an FBK^2-*algebra* iff it is isomorphic to $\mathfrak{A}_*$ for some bimodal algebra $\mathfrak{A}$. Let $\mathcal{V}_*$ be the class of all FBK^2-algebras.

THEOREM 2.2 (see [9, 11]).
$\mathcal{V}_*$ is a variety.

Denote by $\mathcal{S}\mathcal{V}_*$ the collection of all subvarieties of $\mathcal{V}_*$.

The *semantical consequence relation* $\vDash_*$ *for* FBK^2 can be described as follows: $\Gamma \vDash_* \phi$ iff for any $\mathfrak{M} \in \mathcal{V}_*$ and valuation v in $\mathfrak{M}$,

$$v(\psi \to \bot) \;=\; v(\bot) \quad \text{for all } \psi \in \Gamma \quad \Longrightarrow \quad v(\psi \to \bot) \;=\; v(\bot)$$

— this is equivalent to the condition that for any $\mathfrak{A} \in \mathcal{V}_\circ$ and valuation v in $\mathfrak{A}_*$,

$$\pi_1(v(\psi)) = \top^{\mathfrak{A}} \quad \text{for all } \psi \in \Gamma \quad \Longrightarrow \quad \pi_1(v(\phi)) = \top^{\mathfrak{A}}$$

where π_1 denotes the 1^{st} projection function from $A \times A$ onto A. Then we have:

[5] Originally, $\top$ was not treated as primitive in BK and $\mathsf{BK}^{\mathtt{b}}_{\mathtt{n}}$, but it can be defined as $\sim\bot$.
[6] We use $\neg a$ as shorthand for $a \to \bot$, of course.

THEOREM 2.3 (see [9, 13]).
The global derivability relation for FBK^2 *coincides with its semantical consequence relation.*[7]

Next, we write $\mathfrak{M} \Vdash_* \phi$ iff $\phi \to \bot = \bot$ holds in $\mathfrak{M}$. The operations $\mathbf{L}_*$ and $\mathbf{V}_*$ are defined exactly as $\mathbf{L}_\circ$ and $\mathbf{V}_\circ$, but with $\circ$ replaced by $*$.

THEOREM 2.4 (see [9, 11]).
$\mathbf{L}_*$ *and* $\mathbf{V}_*$ *induce mutually inverse dual isomorphisms between* $\mathcal{SV}_*$ *and* $\mathcal{E}\mathsf{FBK}^2$.

2.4 Modal bilattice logic

The deductive system for MBL and the corresponding algebraic semantics were provided in [7]. Besides appropriate axioms and modus ponens, the system for MBL includes a weak monotonicity rule, namely

$$\frac{\phi \Rightarrow \psi}{\Box\phi \Rightarrow \Box\psi} \quad (\mathtt{WM}_\Box).$$

However, since MBL is known to be non-normal, the classical rule

$$\frac{\phi \to \psi}{\Box\phi \to \Box\psi} \quad (\mathtt{M}_\Box)$$

is not admissible in MBL. So by an MBL-*extension* we mean a set of $\mathcal{L}_\star$-formulas that contains MBL and is closed under modus ponens, the weak monotonicity rule (that uses strong implication) and substitutions. Denote by $\mathcal{E}\mathsf{MBL}$ the collection of all MBL-extensions.

Given $\mathfrak{A} \in \mathcal{V}_\circ$, by the *full twist-structure over* $\mathfrak{A}$ *in* $\mathcal{L}_\star$ we mean the $\mathcal{L}_\star$-algebra

$$\mathfrak{A}_\star \ := \ \langle A \times A; \wedge, \vee, \otimes, \oplus, \to, \sim, \bot, \top, \mathtt{n}, \mathtt{b}, \Box \rangle$$

whose operations are defined by:

$$\begin{aligned}
(a,b) \wedge (c,d) &:= (a \wedge c, b \vee d);\\
(a,b) \vee (c,d) &:= (a \vee c, b \wedge d);\\
(a,b) \otimes (c,d) &:= (a \wedge c, b \wedge d);\\
(a,b) \oplus (c,d) &:= (a \vee c, b \vee d);\\
(a,b) \to (c,d) &:= (\neg a \vee c, a \wedge d);\\
\sim(a,b) &:= (b,a);\\
\bot &:= (\bot, \top);\\
\top &:= (\top, \bot);\\
\mathtt{n} &:= (\bot, \bot);\\
\mathtt{b} &:= (\top, \top);\\
\Box(a,b) &:= (\Box_+ a \wedge \Box_- \neg b, \neg\Box_+ \neg b).
\end{aligned}$$

An $\mathcal{L}_\star$-algebra is called *a modal bilattice*, or an MBL-*algebra*, iff it is isomorphic to $\mathfrak{A}_\star$ for some bimodal algebra $\mathfrak{A}$. Let $\mathcal{V}_\star$ be the class of all modal bilattices.

[7]Here the adjective 'global' indicates that we can use all rules, and not just modus ponens.

THEOREM 2.5 (see [7]).
$\mathcal{V}_\star$ is a variety.

Denote by $\mathcal{SV}_\star$ the collection of all subvarieties of $\mathcal{V}_\star$.

The *semantical consequence relation* $\vDash_\star$ *for* MBL is described exactly as that for FBK^2, but with $*$ replaced by $\star$.

THEOREM 2.6 (see [7]).
The global derivability relation for MBL *coincides with its semantical consequence relation.*

Next, we write $\mathfrak{M} \Vdash_\star \phi$ iff $\phi \to \bot = \bot$ holds in $\mathfrak{M}$. Now $\mathbf{L}_\star$ and $\mathbf{V}_\star$ are defined exactly as $\mathbf{L}_*$ and $\mathbf{V}_*$, but with $*$ replaced by $\star$.

THEOREM 2.7.
$\mathbf{L}_\star$ and $\mathbf{V}_\star$ induce mutually inverse dual isomorphisms between $\mathcal{SV}_\star$ and $\mathcal{E}$MBL.

Proof. First of all, it is easy to verify that $\mathbf{L}_\star$ maps each class of bimodal lattices to some MBL-extension. Note also that for any $\mathfrak{M} \in \mathcal{V}_\star$ and $\phi, \psi \in \mathrm{Form}_{\mathcal{L}_\star}$,

$$\phi = \psi \text{ holds in } \mathfrak{M} \quad \Longleftrightarrow \quad (\phi \Leftrightarrow \psi) \to \bot = \bot \text{ holds in } \mathfrak{M}. \tag{$*$}$$

Hence for every $V \in \mathcal{SV}_\star$ we have $\mathbf{V}_\star(\mathbf{L}_\star(V)) = V$. Moreover, for each $L \in \mathcal{E}\mathsf{MBL}$, using its Lindenbaum–Tarski algebra, it can be checked that $\mathbf{L}_\star(\mathbf{V}_\star(L)) = L$.[8] Finally, both $\mathbf{L}_\star$ and $\mathbf{V}_\star$ are obviously order-reversing. ☐

3 From MBL to FBK^2 and back again

We begin by showing that from an algebraic perspective, MBL and FBK^2 are two variants of essentially the same logic.

LEMMA 3.1.
Let $\mathfrak{A}$ be a bimodal algebra. Then:

i. the operations $\oplus$, $\otimes$, $\Box$ of $\mathfrak{A}_\star$ are definable by terms in $\mathfrak{A}_*$;

ii. the operations $\Box_+$, $\Box_-$, $\Diamond_+$, $\Diamond_-$ of $\mathfrak{A}_*$ are definable by terms in $\mathfrak{A}_\star$.[9]

Proof. Notice that we can work with each of the two coordinates separately, because in $\mathfrak{A}_*$ and $\mathfrak{A}_\star$ we have

$$(a_1, a_2) \;=\; (a_1, \bot) \wedge (\top, a_2) \;=\; ((a_1,\ ?\ ) \vee \mathtt{n}) \wedge ((\ ?\ , a_2) \vee \mathtt{b}).$$

where the question marks may be replaced by arbitrary elements of A. In what follows for each $i \in \{1, 2\}$ the notation $(a_1, a_2) =_i (a'_1, a'_2)$ will mean that $a_i = a'_i$.

$\boxed{\text{i}}$ In the case of $\oplus$, observe that

$$(a, b) \oplus (c, d) \;=_1\; (a, b) \vee (c, d),$$
$$(a, b) \oplus (c, d) \;=_2\; (a, b) \wedge (c, d).$$

[8] Compare this with the proof of Proposition 5.1 in [11].

[9] Obviously, the $\{\wedge, \vee, \to, \sim, \bot, \top, \mathtt{n}, \mathtt{b}\}$-reducts of $\mathfrak{A}_*$ and $\mathfrak{A}_\star$ coincide.

So $x \oplus y$ can be expressed as $((x \vee y) \vee \mathtt{n}) \wedge ((x \wedge y) \vee \mathtt{b})$. Similarly for $\otimes$. It remains to define $\Box$ in $\mathfrak{A}_*$. This can be done by using

$$\begin{aligned}
\Box\,(a,b) &=_1 (\Box_+ a \wedge \Box_- \neg b, \neg\Box_+ \neg b)\\
&=_1 (\Box_+ a, \neg\Box_+ \neg b) \wedge (\Box_- \neg b, \neg\Box_- \neg b)\\
&=_1 \Box_+\,(a,b) \wedge \Box_-\,(\neg b, b)\\
&=_1 \Box_+\,(a,b) \wedge \Box_- \neg\,(b,a)\\
&=_1 \Box_+\,(a,b) \wedge \Box_- \neg\!\sim(a,b),\\
\Box\,(a,b) &=_2 \Box_+\,(a,b)
\end{aligned}$$

— hence the $\mathcal{L}_*$-term $((\Box_+ x \wedge \Box_-\,({\sim}x \to \bot)) \vee \mathtt{n}) \wedge (\Box_+ x \vee \mathtt{b})$ defines $\Box$ in $\mathfrak{A}_*$.

$\boxed{\text{ii}}$ Since $\Diamond_+$ and $\Diamond_-$ can be expressed as ${\sim}\Box_+{\sim}$ and ${\sim}\Box_-{\sim}$ respectively, we need to consider only $\Box_+$ and $\Box_-$. In the case of $\Box_+$, observe that

$$\begin{aligned}
\Box_+\,(a,b) &=_1 (\Box_+ a, \bot)\\
&=_1 (\Box_+ a \wedge \Box_- \neg\bot, \neg\Box_+ \neg\bot)\\
&=_1 \Box\,(a,\bot)\\
&=_1 \Box\,((a,b) \vee \mathtt{n}),\\
\Box_+\,(a,b) &=_2 \Box\,(a,b).
\end{aligned}$$

For $\Box_-$ we may use

$$\begin{aligned}
\Box_-\,(a,b) &=_1 (\Box_- a, \neg\Box_+ a)\\
&=_1 (\Box_+ \top \wedge \Box_- \neg\neg a, \neg\Box_+ \neg\neg a)\\
&=_1 \Box\,(\top, \neg a)\\
&=_1 \Box\,((a,\neg a) \vee \mathtt{b})\\
&=_1 \Box\,({\sim}\,(\neg a, a) \vee \mathtt{b})\\
&=_1 \Box\,({\sim}\neg\,(a,b) \vee \mathtt{b}),\\
\Box_-\,(a,b) &=_2 (\Box_- \neg b, \neg\Box_- \neg b)\\
&=_2 {\sim}\,(\neg\Box_- \neg b, \Box_- \neg b)\\
&=_2 {\sim}\neg\,(\Box_- \neg b, \neg\Box_+ \neg b)\\
&=_2 {\sim}\neg\,(\Box_+ \top \wedge \Box_- \neg b, \neg\Box_+ \neg b)\\
&=_2 {\sim}\neg\Box\,(\top, b)\\
&=_2 {\sim}\neg\Box\,((\neg b, b) \vee \mathtt{b})\\
&=_2 {\sim}\neg\Box\,(\neg\,(b,a) \vee \mathtt{b})\\
&=_2 {\sim}\neg\Box\,(\neg{\sim}\,(a,b) \vee \mathtt{b}).
\end{aligned}$$

(Here the argument for the second coordinate is a modification of that for the first.) $\Box$

For each bimodal algebra $\mathfrak{A}$ and each function f from Prop to $A \times A$, denote by $f^{\mathfrak{A}_*}$ and $f^{\mathfrak{A}_\star}$ the valuations determined by f in $\mathfrak{A}_*$ and $\mathfrak{A}_\star$ respectively.

Proposition 3.2.
There are linear-time computable translations

$$\tau : \mathrm{Form}_{\mathcal{L}_\star} \to \mathrm{Form}_{\mathcal{L}_*} \quad \textit{and} \quad \rho : \mathrm{Form}_{\mathcal{L}_*} \to \mathrm{Form}_{\mathcal{L}_\star}$$

such that for any bimodal algebra $\mathfrak{A}$ *and function* f *from* Prop *to* $A \times A$ *the following hold:*

i. $f^{\mathfrak{A}_\star}(\phi) = f^{\mathfrak{A}_*}(\tau(\phi))$ *for every* $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$*;*

ii. $f^{\mathfrak{A}_*}(\phi) = f^{\mathfrak{A}_\star}(\rho(\phi))$ *for every* $\phi \in \mathrm{Form}_{\mathcal{L}_*}$*.*

Proof. To define τ and ρ, we can use the terms constructed in the proof of Proposition 3.1. $\square$

Remember, both $\Vdash_*$ and $\Vdash_\star$ were defined by means of the identity $x \to \bot = \bot$.

Proposition 3.3.
Let $\mathfrak{A}$ *be a bimodal algebra. Then:*

i. for every $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$*,*

$$\mathfrak{A}_\star \Vdash_\star \phi \quad \Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \tau(\phi);$$

ii. for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$*,*

$$\mathfrak{A}_* \Vdash_* \phi \quad \Longleftrightarrow \quad \mathfrak{A}_\star \Vdash_\star \rho(\phi).$$

Proof. This follows by (i) and (ii) of Proposition 3.2. $\square$

Next, for each MBL-extension L, define

$$\dot{\tau}(L) \ := \ \text{the least } \mathsf{FBK}^2\text{-extension containing } \tau[L].$$[10]

Similarly with FBK^2 and ρ in place of MBL and τ.

Corollary 3.4.
For any $L \in \mathcal{E}\mathsf{MBL}$ *and* $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$*,*

$$\phi \ \in \ L \quad \Longleftrightarrow \quad \tau(\phi) \ \in \ \dot{\tau}(L).$$

Similarly with FBK^2*,* $*$ *and* ρ *in place of* MBL*,* $\star$ *and* τ*.*

Proof. We shall consider only the case with τ. The argument for ρ is perfectly analogous, and therefore we omit it.

Let $L \in \mathcal{E}\mathsf{MBL}$. By Proposition 3.3, for each bimodal algebra $\mathfrak{A}$,

$$\begin{aligned} \mathfrak{A}_\star \Vdash_\star \phi \quad \text{for all } \phi \in L \quad &\Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \tau(\phi) \quad \text{for all } \phi \in L \\ &\Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \psi \quad \text{for all } \psi \in \tau[L] \\ &\Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \psi \quad \text{for all } \psi \in \dot{\tau}(L), \end{aligned}$$

and therefore $\mathfrak{A}_\star \in \mathbf{V}_\star(L)$ iff $\mathfrak{A}_* \in \mathbf{V}_*(\dot{\tau}(L))$. So for every $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$,

$$\begin{aligned} \phi \ \in \ L \quad &\Longleftrightarrow \quad \phi \ \in \ \mathbf{L}_\star(\mathbf{V}_\star(L)) \\ &\Longleftrightarrow \quad \mathfrak{A}_\star \Vdash_\star \phi \quad \text{for all } \mathfrak{A}_\star \in \mathbf{V}_\star(L) \\ &\Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \tau(\phi) \quad \text{for all } \mathfrak{A}_* \in \mathbf{V}_*(\dot{\tau}(L)) \\ &\Longleftrightarrow \quad \tau(\phi) \ \in \ \mathbf{L}_\star(\mathbf{V}_\star(\dot{\tau}(L))) \\ &\Longleftrightarrow \quad \tau(\phi) \ \in \ \dot{\tau}(L). \end{aligned}$$

(Here the first and last equivalences are guaranteed by Theorems 2.4 and 2.7 respectively.) $\square$

[10] Here $\tau[L]$ is the image of L under τ, i.e. $\{\tau(\phi) \mid \phi \in L\}$.

Theorem 3.5.
$\dot{\tau}$ and $\dot{\rho}$ are mutually inverse isomorphisms between $\mathcal{E}\mathsf{MBL}$ and $\mathcal{E}\mathsf{FBK}^2$.

Proof. By Proposition 3.2, if $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$, then $\phi = \rho(\tau(\phi))$ holds in all $\mathfrak{A}_\star \in \mathcal{V}_\star$, and therefore $\phi \Leftrightarrow \rho(\tau(\phi)) \in \mathsf{MBL}$. Consequently, for any $L \in \mathcal{E}\mathsf{MBL}$ and $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$,

$$\begin{aligned} \phi \in L \quad &\overset{3.4}{\Longleftrightarrow} \quad \tau(\phi) \in \dot{\tau}(L) \\ &\overset{3.4}{\Longleftrightarrow} \quad \rho(\tau(\phi)) \in \dot{\rho}(\dot{\tau}(L)) \\ &\Longleftrightarrow \quad \phi \in \dot{\rho}(\dot{\tau}(L)). \end{aligned}$$

Thus for each $L \in \mathcal{E}\mathsf{MBL}$ we have $L = \dot{\rho}(\dot{\tau}(L))$. Similarly, $L = \dot{\rho}(\dot{\tau}(L))$ for every $L \in \mathcal{E}\mathsf{FBK}^2$. Finally, both $\dot{\tau}$ and $\dot{\rho}$ are obviously order-preserving. $\square$

It turns out that $\dot{\tau}$ and $\dot{\rho}$ preserve many nice properties. We shall only give a few examples, which will concern tabularity, decidability and interpolation.

As usual, $L \in \mathcal{E}\mathsf{MBL}$ will be called *tabular* iff $L = \mathbf{L}_\star(\{\mathfrak{M}\})$ for some $\mathfrak{M} \in \mathcal{V}_\star$. Similarly with FBK^2 and $*$ in place of MBL and $\star$.

Corollary 3.6.
For every $L \in \mathcal{E}\mathsf{MBL}$,

$$L \text{ is tabular} \quad \Longleftrightarrow \quad \dot{\tau}(L) \text{ is tabular.}$$

Proof. We shall consider only the right-to-left implication. The left-to-right implication can be proved similarly, and therefore we omit it.

Suppose L is tabular; this means that $L = \mathbf{L}_\star(\{\mathfrak{A}_\star\})$ for some bimodal algebra $\mathfrak{A}$. Then for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$,

$$\begin{aligned} \phi \in \dot{\tau}(L) \quad &\overset{3.4}{\Longleftrightarrow} \quad \rho(\phi) \in \dot{\rho}(\dot{\tau}(L)) \\ &\overset{3.5}{\Longleftrightarrow} \quad \rho(\phi) \in L \\ &\Longleftrightarrow \quad \mathfrak{A}_\star \Vdash_\star \rho(\phi) \\ &\overset{3.3}{\Longleftrightarrow} \quad \mathfrak{A}_* \Vdash_* \phi. \end{aligned}$$

Thus $\dot{\tau}(L) = \mathbf{L}_*(\{\mathfrak{A}_*\})$, and therefore $\dot{\tau}(L)$ is tabular as well. $\square$

Further, we shall call $L \in \mathcal{E}\mathsf{MBL}$ *pretabular* iff for every $L' \in \mathcal{E}\mathsf{MBL}$,

$$L \subsetneq L' \quad \Longrightarrow \quad L' \text{ is tabular.}$$

Similarly with FBK^2 in place of MBL.

Corollary 3.7.
For every $L \in \mathcal{E}\mathsf{MBL}$,

$$L \text{ is pretabular} \quad \Longleftrightarrow \quad \dot{\tau}(L) \text{ is pretabular.}$$

Proof. This follows from Theorem 3.5 and Corollary 3.6. $\square$

Concerning decidable logics:

Corollary 3.8.
For every $L \in \mathcal{E}\mathsf{MBL}$,

$$L \text{ is decidable} \iff \dot{\tau}(L) \text{ is decidable.}$$

*Here 'decidable' can be replaced by '*co-NP*-complete', '*PSPACE*-complete', etc.*

Proof. Remember, both τ and ρ are linear-time computable. By Corollary 3.4, τ computably reduces L to $\dot{\tau}(L)$, while ρ computably reduces $\dot{\tau}(L)$ to $\dot{\rho}(\dot{\tau}(L))$ — which coincides with L by Theorem 3.5. Hence L and $\dot{\tau}(L)$ are computably equivalent. □

For each formula ϕ, denote by $\mathrm{var}(\phi)$ the collection of all propositional variables that occur in ϕ. Note that neither τ nor ρ adds or removes variables:

- $\mathrm{var}(\phi) = \mathrm{var}(\tau(\phi))$ for every $\phi \in \mathrm{Form}_{\mathcal{L}_\star}$;
- $\mathrm{var}(\phi) = \mathrm{var}(\rho(\phi))$ for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$.

As in classical modal logic, we say that $L \in \mathcal{E}\mathsf{MBL}$ has *Craig's interpolation property* — or *CIP* for short — iff for any $\phi \to \psi \in L$ there exists $\chi \in \mathrm{Form}_{\mathcal{L}_\star}$ such that

$$\{\phi \to \chi, \chi \to \psi\} \subseteq L \quad \text{and} \quad \mathrm{var}(\chi) \subseteq \mathrm{var}(\phi) \cap \mathrm{var}(\psi).$$

Similarly with FBK^2 and $*$ in place of MBL and $\star$.

Corollary 3.9.
For every $L \in \mathcal{E}\mathsf{MBL}$,

$$L \text{ has CIP} \iff \dot{\tau}(L) \text{ has CIP.}$$

Proof. We shall again consider only the right-to-left implication. Notice that both τ and ρ distribute over $\to$ (and also over $\wedge$, $\vee$ and $\sim$).

Assume L has CIP. Let $\phi \to \psi$ be in $\dot{\tau}(L)$. By Corollary 3.4, $\rho(\phi) \to \rho(\psi)$ is in $\dot{\rho}(\dot{\tau}(L))$ — which coincides with L by Theorem 3.5. So there exists $\chi \in \mathrm{Form}_{\mathcal{L}_\star}$ such that

$$\{\rho(\phi) \to \chi, \chi \to \rho(\psi)\} \subseteq L \quad \text{and} \quad \mathrm{var}(\chi) \subseteq \mathrm{var}(\phi) \cap \mathrm{var}(\psi).$$

Applying Corollary 3.4 to $\rho(\phi) \to \chi$ and $\chi \to \rho(\psi)$, we get

$$\tau(\rho(\phi)) \to \tau(\chi) \in \dot{\tau}(L) \quad \text{and} \quad \tau(\chi) \to \tau(\rho(\psi)) \in \dot{\tau}(L).$$

Finally, since both $\tau(\rho(\phi)) \Leftrightarrow \phi$ and $\tau(\rho(\psi)) \Leftrightarrow \psi$ are in FBK^2 (see the proof of Theorem 3.5), we have $\phi \to \tau(\chi) \in \dot{\tau}(L)$ and $\tau(\chi) \to \psi \in \dot{\tau}(L)$. Therefore $\tau(\chi)$ is an 'interpolant' for $\phi \to \psi$ in L. □

In this sense the mappings $\dot{\tau}$ and $\dot{\rho}$ not only preserve lattice structure but also much of what may be called 'metamathematical structure'.

4 From FBK^2 to K^2 and back again

The relationship between FBK^2 and K^2 is more subtle than that between MBL and FBK^2; still, results analogous to those in the previous section can be obtained in this case as well. To ease the exposition, we shall use a special representation for $\mathcal{L}_*$-formulas.

We say that an $\mathcal{L}^*$-formula ϕ is a *negation normal form* — or an *nnf* for short — iff each occurrence of $\sim$ in ϕ immediately precedes some atomic subformula (i.e. a propositional variable or a constant symbol). It is easy to show the following.[11]

PROPOSITION 4.1.
For every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$ there exists a nnf ϕ' such that $\phi \Leftrightarrow \phi' \in \mathsf{FBK}^2$.

Proof. For each $\mathcal{L}_*$-formula ϕ, define ϕ' recursively as follows:

- if $\phi \in \mathrm{Prop} \cup \{\bot, \top, \mathtt{n}, \mathtt{b}\}$, then $\phi' := \phi$;
- if $\phi = \sim\psi$ where $\psi \in \mathrm{Prop} \cup \{\bot, \top, \mathtt{n}, \mathtt{b}\}$, then $\phi' := \phi$;
- if $\phi = \psi \odot \chi$ where $\odot \in \{\wedge, \vee, \to\}$, then $\phi' := \psi' \odot \chi'$;
- if $\phi = \sim(\psi \wedge \chi)$, then $\phi' := (\sim\psi)' \vee (\sim\chi)'$;
- if $\phi = \sim(\psi \vee \chi)$, then $\phi' := (\sim\psi)' \wedge (\sim\chi)'$;
- if $\phi = \sim(\psi \to \chi)$, then $\phi' := \neg\neg\psi' \wedge (\sim\chi)'$;
- if $\phi = \heartsuit\psi$ where $\heartsuit \in \{\Box_+, \Box_-, \Diamond_+, \Diamond_-\}$, then $\phi' := \heartsuit\psi'$;
- if $\phi = \sim\Box_+\psi$, then $\phi' := \Diamond_+(\sim\psi)'$;
- if $\phi = \sim\Box_-\psi$, then $\phi' := \Diamond_-(\sim\psi)'$;
- if $\phi = \sim\Diamond_+\psi$, then $\phi' := \Box_+(\sim\psi)'$;
- if $\phi = \sim\Diamond_-\psi$, then $\phi' := \Box_-(\sim\psi)'$;
- if $\phi = \sim\sim\psi$, then $\phi' := \psi'$.

It is not hard to verify that ϕ' has the desired property. $\Box$

For each $\mathcal{L}_*$-formula ϕ, denote by ϕ' the nnf constructed in the proof above. Notice that the mapping $\phi \mapsto \phi'$ is polynomial-time computable.

Before proceeding, let us make one useful observation:

PROPOSITION 4.2.
Let $\mathfrak{A}$ be a bimodal algebra. Then for every $\phi \in \mathrm{Form}_{\mathcal{L}_\circ}$,

$$\mathfrak{A} \Vdash_\circ \phi \quad \Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \phi.$$

Proof. By an easy induction on ϕ. $\Box$

COROLLARY 4.3.
For every $L \in \mathcal{E}\mathsf{K}^2$ there exists $L' \in \mathcal{E}\mathsf{FBK}^2$ such that $L' \cap \mathrm{Form}_{\mathcal{L}_\circ} = L$.

[11] Compare this with [9, Proposition 2.1] and [12, Proposition 8.1.1] — which are about BK and Nelson's logics respectively. It should be noted that in these two results $\leftrightarrow$ is used instead of $\Leftrightarrow$. In fact, the 'strong' version of the nnf theorem holds also for BK, but fails for Nelson's logics.

Proof. Let L be a K^2-extension. Take

$$L' \;:=\; \{\phi \in \mathrm{Form}_{\mathcal{L}_*} \mid \mathfrak{A}_* \Vdash_* \phi \text{ for all } \mathfrak{A} \in \mathbf{V}_\circ(L)\}.$$

One easily sees that L' is an FBK^2-extension. Moreover, for every $\phi \in \mathrm{Form}_{\mathcal{L}_\circ}$,

$$\begin{aligned}
\phi \;\in\; L' \quad &\iff \quad \mathfrak{A}_* \Vdash_* \phi \text{ for all } \mathfrak{A} \in \mathbf{V}_\circ(L) \\
&\overset{4.2}{\iff} \quad \mathfrak{A} \Vdash_\circ \phi \text{ for all } \mathfrak{A} \in \mathbf{V}_\circ(L) \\
&\iff \quad \phi \;\in\; \mathbf{L}_\circ(\mathbf{V}_\circ(L)) \\
&\overset{2.1}{\iff} \quad \phi \;\in\; L.
\end{aligned}$$

Thus the $\mathcal{L}_\circ$-fragment of L' coincides with L. □

For convenience, we shall assume that the propositional variables are indexed by natural numbers, i.e.

$$\mathrm{Prop} \;=\; \{p_0, p_1, p_2, \ldots\}.$$

Now consider $\lambda : \mathrm{Form}_{\mathcal{L}_*} \to \mathrm{Form}_{\mathcal{L}_\circ}$ given as follows.

- If $\phi = \phi'$, then $\lambda(\phi)$ is defined recursively:
  - if $\phi = p_i$, then $\lambda(\phi) := p_{2i}$;
  - if $\phi = \bot$ or $\phi = \mathtt{n}$, then $\lambda(\phi) = \bot$;
  - if $\phi = \top$ or $\phi = \mathtt{b}$, then $\lambda(\phi) = \top$;
  - if $\phi = {\sim}p_i$, then $\lambda(\phi) := p_{2i+1}$;
  - if $\phi = {\sim}\top$ or $\phi = {\sim}\mathtt{n}$, then $\lambda(\phi) = \bot$;
  - if $\phi = {\sim}\bot$ or $\phi = {\sim}\mathtt{b}$, then $\lambda(\phi) = \top$;
  - if $\phi = \psi \odot \chi$ where $\odot \in \{\vee, \wedge, \to\}$, then $\lambda(\phi) := \lambda(\psi) \odot \lambda(\chi)$;
  - if $\phi = \heartsuit\psi$ where $\heartsuit \in \{\Box_+, \Box_-\}$, then $\lambda(\phi) := \heartsuit\lambda(\psi)$;
  - if $\phi = \Diamond_+\psi$, then $\lambda(\phi) := \neg\Box_+\neg\lambda(\psi)$;
  - if $\phi = \Diamond_-\psi$, then $\lambda(\phi) := \neg\Box_-\neg\lambda(\psi)$.
- Otherwise we let $\lambda(\phi)$ be $\lambda(\phi')$.

Clearly, λ is polynomial-time computable.

Proposition 4.4.
Let $\mathfrak{A}$ be a bimodal algebra. Then for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$,

$$\mathfrak{A}_* \Vdash_* \phi \quad \iff \quad \mathfrak{A}_* \vDash_* \lambda(\phi).$$

Proof. By Proposition 4.1 and the definition of λ, without loss of generality, we may assume that ϕ is an nnf, i.e. $\phi = \phi'$.

$\boxed{\Longrightarrow}$ For each valuation v in $\mathfrak{A}_*$, consider the valuation $v^\flat$ in $\mathfrak{A}_*$ such that

$$v^\flat(p_{2i}) \;:=\; v(p_i) \quad \text{and} \quad v^\flat(p_{2i+1}) \;:=\; {\sim}v(p_i).$$

It is easy to show that for every nnf ϕ,

$$\pi_1(v(\phi)) \;=\; \pi_1(v^\flat(\lambda(\phi))),$$

and therefore $\mathfrak{A}_* \nVdash_* \phi$ implies $\mathfrak{A}_* \nVdash_* \lambda(\phi)$.

$\boxed{\Longleftarrow}$ For each valuation v in $\mathfrak{A}_*$, consider the valuation $v^\sharp$ in $\mathfrak{A}_*$ such that

$$\pi_1\left(v^\sharp(p_i)\right) \;:=\; \pi_1(v(p_{2i})) \quad \text{and} \quad \pi_2\left(v^\sharp(p_i)\right) \;:=\; \pi_1(v(p_{2i+1})).$$

It is easy to show that for every nnf ϕ,

$$\pi_1(v(\lambda(\phi))) \;=\; \pi_1(v^\sharp(\phi)),$$

and therefore $\mathfrak{D}^{\bowtie} \nvDash \lambda(\phi)$ implies $\mathfrak{D}^{\bowtie} \nvDash \phi$. □

Next, for each FBK^2-extension L, define

$$\dot{\lambda}(L) \;:=\; \{\lambda(\phi) \mid \phi \in L\}.$$

As we shall shortly see, $\dot{\lambda}(L)$ coincides with the $\mathcal{L}_\circ$-fragment of L.

Corollary 4.5.
Let L be an FBK^2-extension. Then for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$,

$$\phi \;\in\; L \quad \Longleftrightarrow \quad \lambda(\phi) \;\in\; L.$$

Proof. This follows easily from Proposition 4.4 and Theorem 2.4:

$$\begin{aligned}
\phi \;\in\; L \quad & \overset{2.4}{\Longleftrightarrow} \quad \phi \;\in\; \mathbf{L}_*(\mathbf{V}_*(L)) \\
& \Longleftrightarrow \quad \mathfrak{A}_* \Vdash_* \phi \text{ for all } \mathfrak{A}_* \in \mathbf{V}_*(L) \\
& \overset{4.4}{\Longleftrightarrow} \quad \mathfrak{A}_* \Vdash_* \lambda(\phi) \text{ for all } \mathfrak{A}_* \in \mathbf{V}_*(L) \\
& \Longleftrightarrow \quad \lambda(\phi) \;\in\; \mathbf{L}_*(\mathbf{V}_*(L)) \\
& \overset{2.4}{\Longleftrightarrow} \quad \lambda(\phi) \;\in\; L
\end{aligned}$$

(where ϕ is an arbitrary $\mathcal{L}_*$-formula). □

Going in the opposite direction, define $\iota : \mathrm{Form}_{\mathcal{L}_\circ} \to \mathrm{Form}_{\mathcal{L}_*}$ recursively as follows:

- if $\phi = p_{2i}$, then $\iota(\phi) := p_i$;
- if $\phi = p_{2i+1}$, then $\iota(\phi) := {\sim}p_i$;
- if $\phi = \bot$ or $\phi = \top$, then $\iota(\phi) := \phi$;
- if $\phi = \psi \odot \chi$ where $\odot \in \{\vee, \wedge, \to\}$, then $\iota(\phi) := \iota(\psi) \odot \iota(\chi)$;
- if $\phi = \heartsuit\psi$ where $\heartsuit \in \{\Box_+, \Box_-\}$, then $\iota(\phi) := \heartsuit\lambda(\psi)$.

In other words, $\iota(\phi)$ is obtained from ϕ by substituting p_i for p_{2i} and ${\sim}p_i$ for p_{2i+1}. Obviously, $\lambda(\iota(\phi)) = \phi$ for each $\mathcal{L}_\circ$-formula ϕ.

Corollary 4.6.
For every $L \in \mathcal{E}\mathsf{FBK}^2$ we have $\dot{\lambda}(L) = L \cap \mathrm{Form}_{\mathcal{L}_\circ}$.

Proof. Let L be an FBK^2-extension. By Corollary 4.5, we have $\dot{\lambda}(L) \subseteq L \cap \mathrm{Form}_{\mathcal{L}_\circ}$. On the other hand, for every $\phi \in \mathrm{Form}_{\mathcal{L}_\circ}$, if $\phi \in L$, then $\iota(\phi) \in L$ (because L is closed under substitutions), hence $\lambda(\iota(\phi)) \in \dot{\lambda}(L)$, i.e. $\phi \in \dot{\lambda}(L)$; thus $L \cap \mathrm{Form}_{\mathcal{L}_\circ} \subseteq \dot{\lambda}(L)$. □

Theorem 4.7.
$\dot{\lambda}$ is an isomorphism from $\mathcal{E}\mathsf{FBK}^2$ onto $\mathcal{E}\mathsf{K}^2$.

Proof. By Corollaries 4.6 and 4.3, $\dot{\lambda}$ is surjective. It remains to show that $\dot{\lambda}$ is an embedding of $\mathcal{E}\mathsf{FBK}^2$ into $\mathcal{E}\mathsf{K}^2$.

Let L_1 and L_2 be FBK^2-extensions. Observe that for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$,

$$\begin{aligned} \phi \in L_1 \setminus L_2 &\overset{4.5}{\iff} \lambda(\phi) \in L_1 \setminus L_2 \\ &\iff \lambda(\phi) \in (L_1 \cap \mathrm{Form}_{\mathcal{L}_\circ}) \setminus (L_2 \cap \mathrm{Form}_{\mathcal{L}_\circ}) \\ &\overset{4.6}{\iff} \lambda(\phi) \in \dot{\lambda}(L_1) \setminus \dot{\lambda}(L_2). \end{aligned}$$

Hence $L_1 \nsubseteq L_2$ iff $\dot{\lambda}(L_1) \nsubseteq \dot{\lambda}(L_2)$. From this it follows that $\dot{\lambda}$ is injective, and moreover, both $\dot{\lambda}$ and $\dot{\lambda}^{-1}$ are order-preserving. □

Now we turn to transfer results.

Corollary 4.8.
For every $L \in \mathcal{E}\mathsf{FBK}^2$,

$$L \text{ is tabular} \iff \dot{\lambda}(L) \text{ is tabular.}$$

Proof. $\boxed{\Longrightarrow}$ Suppose L is tabular; this means that $L = \mathbf{L}_*(\{\mathfrak{A}_*\})$ for some bimodal algebra $\mathfrak{A}$. Then for every $\phi \in \mathrm{Form}_{\mathcal{L}_\circ}$,

$$\begin{aligned} \phi \in \dot{\lambda}(L) &\overset{4.6}{\iff} \phi \in L \\ &\iff \mathfrak{A}_* \Vdash_* \phi \\ &\overset{4.2}{\iff} \mathfrak{A} \Vdash_\circ \phi. \end{aligned}$$

Thus $\dot{\lambda}(L) = \mathbf{L}_\circ(\{\mathfrak{A}\})$, and therefore $\dot{\lambda}(L)$ is tabular as well.

$\boxed{\Longleftarrow}$ Conversely, suppose that $\dot{\lambda}(L) = \mathbf{L}_\circ(\{\mathfrak{A}\})$ for some bimodal algebra $\mathfrak{A}$. Then for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$,

$$\begin{aligned} \phi \in L &\overset{4.5}{\iff} \lambda(\phi) \in L \cap \mathrm{Form}_{\mathcal{L}_\circ} \\ &\overset{4.6}{\iff} \lambda(\phi) \in \dot{\lambda}(L) \\ &\iff \mathfrak{A} \Vdash_\circ \lambda(\phi) \\ &\overset{4.2}{\iff} \mathfrak{A}_* \Vdash_* \lambda(\phi) \\ &\overset{4.4}{\iff} \mathfrak{A}_* \Vdash_* \phi. \end{aligned}$$

Thus L coincides with $\mathbf{L}_*(\{\mathfrak{A}_*\})$. □

Corollary 4.9.
For every $L \in \mathcal{E}\mathsf{FBK}^2$,

$$L \text{ is pretabular} \iff \dot{\lambda}(L) \text{ is pretabular.}$$

Proof. This follows from Theorem 4.7 and Corollary 4.8. $\square$

Corollary 4.10.
For every $L \in \mathcal{E}\mathsf{FBK}^2$,

$$L \text{ is decidable} \iff \dot{\lambda}(L) \text{ is decidable.}$$

Here 'decidable' can be replaced by '$\mathsf{co\text{-}NP}$-complete', 'PSPACE-complete', etc.

Proof. Remember, λ is polynomial-time computable. By Corollaries 4.5 and 4.6, for every $\phi \in \mathrm{Form}_{\mathcal{L}_*}$,

$$\phi \in L \iff \lambda(\phi) \in \dot{\lambda}(L).$$

Thus λ computably reduces L to $\dot{\lambda}(L)$. Furthermore, Corollary 4.6 ensures that for any $\phi \in \mathrm{Form}_{\mathcal{L}_\circ}$,

$$\phi \in \dot{\lambda}(L) \iff \phi \in L.$$

Thus the identity function on $\mathrm{Form}_{\mathcal{L}_\circ}$ computably reduces $\dot{\lambda}(L)$ to L. Hence L and $\dot{\lambda}(L)$ are computably equivalent. $\square$

Corollary 4.11.
For every $L \in \mathcal{E}\mathsf{FBK}^2$,

$$L \text{ has CIP} \iff \dot{\lambda}(L) \text{ has CIP.}$$

Proof. Consider an arbitrary $\mathcal{L}_*$-formula ϕ. Observe that for any $\mathfrak{A} \in \mathcal{V}_\circ$ and valuation v in $\mathfrak{A}_*$,

$$\pi_1(v(\iota(\lambda(\phi)))) = \pi_1(v(\phi)),$$

and therefore $\iota(\lambda(\phi)) \leftrightarrow \phi \in \mathsf{FBK}^2$. We will need this observation in what follows.

$\boxed{\Longrightarrow}$ Suppose L has CIP. Let $\phi \to \psi$ be in $\dot{\lambda}(L)$. Then $\phi \to \psi$ is in L by Corollary 4.5. So there exists $\chi \in \mathrm{Form}_{\mathcal{L}_*}$ such that

$$\{\phi \to \chi, \chi \to \psi\} \subseteq L \quad \text{and} \quad \mathrm{var}(\chi) \subseteq \mathrm{var}(\phi) \cap \mathrm{var}(\psi).$$

By Proposition 4.1, without loss of generality, we may assume that $\chi = \chi'$. Our goal is to turn χ into a suitable $\mathcal{L}_\circ$-formula. Then, applying Corollary 4.5 to $\phi \to \chi$ and $\chi \to \psi$, we get

$$\lambda(\phi) \to \lambda(\chi) \in L \quad \text{and} \quad \lambda(\chi) \to \lambda(\psi) \in L.$$

Clearly, no propositional variable with odd index occurs in $\lambda(\phi)$ or $\lambda(\psi)$, but such variables may occur in $\lambda(\chi)$. Take

$$\theta := \text{ the result of replacing each } p_{2i+1} \text{ in } \lambda(\chi) \text{ by } \top.$$

Obviously, $\lambda(\phi) \to \theta \in L$ and $\theta \to \lambda(\psi) \in L$. Moreover,

$$\iota(\lambda(\phi)) \to \iota(\theta) \in L \quad \text{and} \quad \iota(\theta) \to \iota(\lambda(\psi)) \in L.$$

Since both $\iota(\lambda(\phi)) \leftrightarrow \phi$ and $\iota(\lambda(\psi)) \leftrightarrow \psi$ are in FBK^2 (by the remark above), we have $\phi \to \iota(\theta) \in L$ and $\iota(\theta) \to \psi \in L$. By construction, $\iota(\theta)$ is an $\mathcal{L}_\circ$-formula. Thus by Proposition 4.6,

$$\phi \to \iota(\theta) \in \dot{\lambda}(L) \quad \text{and} \quad \phi \to \iota(\theta) \in \dot{\lambda}(L).$$

Furthermore, one easily verifies that $\mathrm{var}(\iota(\theta)) \subseteq \mathrm{var}(\phi) \cap \mathrm{var}(\psi)$.

$\boxed{\Longleftarrow}$ Suppose $\dot{\lambda}(L)$ has CIP. Let $\phi \to \psi$ be in L. Hence $\lambda(\phi) \to \lambda(\psi)$ is in $\dot{\lambda}(L)$. So there exists $\chi \in \mathrm{Form}_{\mathcal{L}_\circ}$ such that

$$\{\lambda(\phi) \to \chi, \chi \to \lambda(\psi)\} \;\subseteq\; \dot{\lambda}(L) \quad \text{and} \quad \mathrm{var}(\chi) \;\subseteq\; \mathrm{var}(\lambda(\phi)) \cap \mathrm{var}(\lambda(\psi)).$$

Since $\dot{\lambda}(L) \subseteq L$ by Corollary 4.5, and L is closed under substitutions, we get

$$\iota(\lambda(\phi)) \to \iota(\chi) \;\in\; L \quad \text{and} \quad \iota(\chi) \to \iota(\lambda(\psi)) \;\in\; L.$$

By the remark above, this implies that $\phi \to \iota(\chi) \in L$ and $\iota(\chi) \to \psi \in L$. Furthermore, one easily checks that $\mathrm{var}(\iota(\chi)) \subseteq \mathrm{var}(\phi) \cap \mathrm{var}(\psi)$. □

Here it is worth discussing a more delicate version of CIP for FBK^2-extensions. Given a nnf ϕ and a propositional variable p, call an occurrence of p in ϕ *positive* iff it is not inside the scope of $\sim$, and *negative* otherwise. For each nnf ϕ, take

$$\mathrm{var}^+(\phi) \;:=\; \{p \in \mathrm{Prop} \mid p \text{ occurs positively in } \phi\}$$
$$\text{and} \quad \mathrm{var}^-(\phi) \;:=\; \mathrm{var}(\phi) \setminus \mathrm{var}^+(\phi).$$

We say that $L \in \mathcal{E}\mathsf{FBK}^2$ has *the strong interpolation property* — or *SIP* for short — iff for any nnf $\phi \to \psi$ in L there exists a nnf χ such that

$$\{\phi \to \chi, \chi \to \psi\} \;\subseteq\; L, \quad \mathrm{var}^+(\chi) \;\subseteq\; \mathrm{var}^+(\phi) \cap \mathrm{var}^+(\psi)$$
$$\text{and} \quad \mathrm{var}^-(\chi) \;\subseteq\; \mathrm{var}^-(\phi) \cap \mathrm{var}^-(\psi).$$

It turns out that SIP and CIP are equivalent for FBK^2-extensions.

Proposition 4.12.
For every $L \in \mathcal{E}\mathsf{FBK}^2$,

$$L \text{ has SIP} \quad \Longleftrightarrow \quad L \text{ has CIP.}$$

Proof. $\boxed{\Longrightarrow}$ Trivial (by Proposition 4.1).

$\boxed{\Longleftarrow}$ Assume L has CIP; then $\dot{\lambda}(L)$ has CIP by Corollary 4.11. Let $\phi \to \psi$ be a nnf in L. Hence $\lambda(\phi) \to \lambda(\psi)$ is in $\dot{\lambda}(L)$. So there exists $\chi \in \mathrm{Form}_{\mathcal{L}_\circ}$ such that

$$\{\lambda(\phi) \to \chi, \chi \to \lambda(\psi)\} \;\subseteq\; \dot{\lambda}(L) \quad \text{and} \quad \mathrm{var}(\chi) \;\subseteq\; \mathrm{var}(\lambda(\phi)) \cap \mathrm{var}(\lambda(\psi)).$$

Exactly as in the proof of the right-to-left implication in Corollary 4.11, from this we get $\phi \to \iota(\chi) \in L$ and $\iota(\chi) \to \psi \in L$. Furthermore, one easily verifies that

$$\mathrm{var}^+(\iota(\chi)) \;\subseteq\; \mathrm{var}^+(\phi) \cap \mathrm{var}^+(\psi) \quad \text{and} \quad \mathrm{var}^+(\iota(\chi)) \;\subseteq\; \mathrm{var}^+(\phi) \cap \mathrm{var}^+(\psi).$$

Therefore $\iota(\chi)$ is a 'strong interpolant' for $\phi \to \psi$ in L. □

Finally, the relationship between MBL and K^2 is completely analogous to that between FBK^2 and K^2; the corresponding results can be easily obtained by combining τ and λ.[12]

[12] One may try to get similar results in a different way, using a pair (ν_1, ν_2) of translations from MBL into K^2, suggested in [7, Section 5]. This requires a separate study.

5 Final comments

Lattices of logics are complicated objects. Usually their study requires algebraic semantics, and not Kripke-style semantics. Sometimes we are interested only in logics with certain nice properties; then it may be helpful to pass from one lattice to another by means of an isomorphism that preserves (many of) the desired properties. As has been shown above, such isomorphisms exist between the lattices of extensions of MBL, FBK^2 and K^2. It seems that almost every natural property (at least if it is expressible in terms of algebra) should be preserved under these isomorphisms.[13] On the other hand, one may try to provide a counter-example to this claim, which would also be useful and give us a better understanding of the situation.

The situation with MBL, FBK^2 and K^2 is similar to that with FBK and K; cf. [9]. Therefore it seems plausible that for any reasonable FDE-based modal logic L, if all the four truth values are expressible in its language, then there exists an isomorphism from the lattice of extensions of L onto the lattice of extensions of K or K^2 that preserves various nice properties. Of course, this claim is somewhat vague. Still, it may be interesting to study other FDE-based modal logics (see [14, 4]) from this point of view.

[13] For example, it is well-known that for bimodal logics, CIP can be expressed in terms of algebra: L has CIP iff $\mathbf{V}_\circ(L)$ is superamalgamable; see [6] for details.

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