An ‘elementary’ perspective on reasoning about probability spaces

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This is a pre-print version of the article
published in *Logic Journal of the IGPL*.

DOI: 10.1093/jigpal/jzae042

Abstract

This paper is concerned with a two-sorted probabilistic language, denoted by QPL, which contains quantifiers over events and over reals, and can be viewed as an elementary language for reasoning about probability spaces. The fragment of QPL containing only quantifiers over reals is a variant of the well-known ‘polynomial’ language from [3, Section 6]. We shall prove that the QPL-theory of the Lebesgue measure on $[0, 1]$ is decidable, and moreover, all atomless spaces have the same QPL-theory. Also we shall introduce the notion of elementary invariant for QPL, and use it to translate the semantics for QPL into the setting of elementary analysis. This will allow us to obtain further decidability results as well as to provide exact complexity upper bounds for a range of interesting undecidable theories.

Keywords: probability logic, quantification over events, quantification over reals, decidability, complexity, elementary invariants, elementary theories

Contents

1 Introduction

We shall be concerned with the two-sorted probabilistic language QPL proposed in [16]. It can be obtained by combining the elementary language of Boolean algebras and that of (ordered) fields in a natural way. Its fragment containing only quantifiers over reals (but not over events) is a variant of the well-known 'polynomial' language from [3, Section 6]. In general, the present work may be seen as a study in elementary theories of various classes of probability spaces; cf. [2], [6], [12], [17] and [18].

Several important complexity results about QPL have been obtained in [16]. In particular, we know that for any class $\mathcal{K}$ of probability spaces, if $\mathcal{K}$ contains all infinite discrete spaces, then its QPL-theory is at least as complex as complete second-order arithmetic. The QPL-theory of finite spaces is much simpler but still undecidable — it turns out to be Π^0_1-complete. Furthermore, these results continue to hold if we exclude quantifiers over reals. On the other hand, for each positive integer n, the QPL-theory of spaces with exactly n elements is easily shown to be decidable. One may wonder whether there are more interesting examples of decidable probabilistic theories. We shall answer this question affirmatively by proving that every atomless space has the same QPL-theory as the Lebesgue measure on $[0, 1]$, and this theory is decidable. To derive the latter, Tarski's decidability result about the ordered field of reals (see [18]) will be utilised.

Another interesting problem arising in quantified probability logic concerns complexity upper bounds: probability spaces cannot be directly encoded in the language of higher-order arithmetic, which makes it difficult to provide complexity upper bounds for many natural undecidable probabilistic theories. To overcome this difficulty, we shall introduce the notion of elementary invariant for QPL, and use it to translate the semantics for QPL into the setting of elementary analysis — or that of second-order arithmetic. In particular, this will allow us to prove that for each 'analytical' class of probability spaces its QPL-theory is reducible to complete second-order arithmetic, which solves one of the main problems of [16]. At the same time, the translation mentioned above can be used to derive some further decidability results.

Certainly many probabilistic languages have been proposed and studied during the last several decades. They are interesting in their own right. Still, QPL is quite different from other probability logics with quantifiers, e.g. those in [1] and [7]. In particular, in Halpern's languages of type 1 we have:

i. quantifiers over 'elementary events', but not over measurable sets of elementary events, i.e. not over arbitrary events;

ii. formulas like $\mu(\{x \mid \Phi(x, \vec{y})\}) \geqslant 1/2$ where Φ is a first-order formula in a given signature, and the individual variables x, $\vec{y}$ are intended to range over elementary events;

iii. formulas containing nested occurrences of μ.

There are pros and cons to dealing with such languages. For instance, while (ii) is a useful feature, it also leads to well-known measurability problems — since a projection of a measurable set is not necessarily measurable; see [16, Section 5] for further discussion. This paper aims at presenting an 'elementary' perspective on the subject, where arbitrary events are treated as a basic sort of object (compare [12]).

Note that, in general, probability logics with quantifiers tend to be highly undecidable. But the situation is different with quantifier-free languages, or even those containing only quantifiers over reals. In particular, the reader may consult [3], [4] and [8] for some nice complexity results — dealing with classes like NP, $\exists\mathbb{R}$ and PSPACE — and related axiomatizations. One natural direction of future research is to obtain similar results for decidable fragments of QPL.

2 Quantified probability logic

By a *probability space*, or simply a *space*, we mean a pair $\langle \mathscr{A}, \mathsf{P} \rangle$ where:

- $\mathscr{A}$ is a σ-algebra, i.e. a Boolean algebra in which every countable set of elements has a supremum (and hence an infimum);
- P is a *probability measure* on $\mathscr{A}$, i.e. a function from $\mathscr{A}$ to $[0, 1]$ such that for any countable set S of pairwise disjoint elements of $\mathscr{A}$,

  $$\mathsf{P}\left(\bigvee S\right) \;=\; \sum\nolimits_{A \in S} \mathsf{P}\,(A),$$

  and also $\mathsf{P}\,(\top) = 1$ where $\top$ denotes the greatest element of $\mathscr{A}$.

Each element of $\mathscr{A}$ is called an *event*, which is measurable with respect to P.

Since the very definition of a space involves two different sorts of object, our formal language QPL will include two disjoint countable sets of variables:

$$\mathrm{Var} \;:=\; \{X, Y, Z, \dots\} \quad \text{and} \quad \mathrm{var} \;:=\; \{x, y, z, \dots\}.$$

Elements of Var are intended to range over events, and called *Boolean variables*, while those of var are intended to range over reals, and called *field variables*. This in turn suggests considering two sets of function symbols:

$$\{\bot, \top, \wedge, \vee, \neg\} \quad \text{and} \quad \{0, 1, +, \,\cdot\,, -\},$$

viz. the symbols of the language of Boolean algebras and those of the language of fields. In addition to these, we will need a special symbol μ to denote a probability measure.

The *Boolean terms* are build up from $\bot$, $\top$ and the Boolean variables as follows:

- if ϕ is a Boolean term, so is $\neg\phi$;
- if ϕ_1 and ϕ_2 are Boolean terms, so are $\phi_1 \wedge \phi_1$ and $\phi_1 \vee \phi_2$.

Naturally, they represent Boolean combinations of events. By a *μ-term* we mean an expression of the form $\mu(\phi)$ where ϕ is a Boolean term. The *field terms* are built up from 0, 1, the field variables and the μ-terms as follows:

- if t is a field term, so is $-t$;
- if t_1 and t_2 are field terms, so are $t_1 \cdot t_2$ and $t_1 + t_2$.

Roughly, each field term can be put in the form

$$f(x_1, \ldots, x_m, \mu(\phi_1), \ldots, \mu(\phi_n))$$

where f is a polynomial with integer coefficients, x_1, …, x_m are field variables, and ϕ_1, …, ϕ_n are Boolean terms. Now by a *basic* QPL*-formula* we mean an expression of the form

$$t_1 < t_2 \quad \text{or} \quad t_1 = t_2 \quad \text{or} \quad t_1 \leqslant t_2$$

where t_1 and t_2 are field terms.

We shall use $\wedge$, $\vee$ and $\neg$ to denote not only the Boolean operations but also the ordinary logical connectives. Since their Boolean versions will not occur outside the scope of μ, the interpretations of $\wedge$, $\vee$ and $\neg$ will always be clear from the context. Take the quantifier symbols to be $\forall$ and $\exists$. Then the QPL*-formulas* are built up from the basic QPL-formulas using the logical connective symbols and quantifiers — binding either Boolean or field variables — in the customary way. We abbreviate $\neg\Phi \vee \Psi$ to $\Phi \to \Psi$ and $(\Phi \to \Psi) \wedge (\Psi \to \Phi)$ to $\Phi \leftrightarrow \Psi$.

For each QPL-formula Φ, define $\mathrm{Free}(\Phi)$ to be the set of all variables (of any of the two sorts) that occur free in Φ. We also let

$$\mathrm{FV}(\Phi) \;:=\; \mathrm{Free}(\Phi) \cap \mathrm{Var} \quad \text{and} \quad \mathrm{Fv}(\Phi) \;:=\; \mathrm{Free}(\Phi) \cap \mathrm{var}.$$

Call Φ a QPL*-sentence* if $\mathrm{Free}(\Phi) = \varnothing$. Denote by Sent the collection of all QPL-sentences.

The satisfiability relation $\Vdash$ for QPL can be defined in the obvious way, and it behaves like one would expect. In more detail, let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a space. By a *Boolean valuation in $\mathscr{P}$* we mean a partial function from Var to $\mathscr{A}$. By a *field valuation* is meant a partial function from var to $\mathbb{R}$. To sum up, we deal with pairs $\langle \zeta, \iota \rangle$ where $\zeta : \subseteq \mathrm{Var} \to \mathscr{A}$ and $\iota : \subseteq \mathrm{var} \to \mathbb{R}$. Then

$$\mathscr{P} \Vdash \Phi[\zeta, \iota]$$

is defined by induction on Φ, provided that $\mathrm{Free}(\Phi) \subseteq \mathrm{dom}\,\zeta \cup \mathrm{dom}\,\iota$, as in classical two-sorted first-order logic. Clearly, it does not matter what values $\langle \zeta, \iota \rangle$ assigns to $(\mathrm{Var} \cup \mathrm{var}) \setminus \mathrm{Free}(\Phi)$. If $\Phi \in \mathrm{Sent}$, we often omit $[\zeta, \iota]$ and write $\mathscr{P} \Vdash \Phi$ instead of $\mathscr{P} \Vdash \Phi[\zeta, \iota]$. For example, consider

$$\Phi \;:=\; \forall x\, (0 \leqslant x \leqslant 1 \to \exists X\, x = \mu(X)).$$

Then $\mathscr{P} \Vdash \Phi$ iff for every $r \in [0, 1]$ there exists $A \in \mathscr{A}$ such that $\mathsf{P}(A) = r$. We say that Φ_1 and Φ_2 are *semantically equivalent* if for any space $\mathscr{P}$, Boolean valuation ζ in $\mathscr{P}$ and field valuation ι,

$$\mathscr{P} \Vdash \Phi_1[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{P} \Vdash \Phi_2[\zeta, \iota].$$

— intuitively, this means that Φ_1 and Φ_2 are interchangeable.

Obviously, $\mathscr{A}$ may vary from space to space, while $\mathbb{R}$ remains unchanged. If we expand QPL to $\underline{\mathsf{QPL}}$ by adding a new constant symbol $\underline{r}$ for each $r \in \mathbb{R}$, then ζ can be used instead of $\langle \zeta, \iota \rangle$; thus we may limit ourselves to the case where $\mathrm{Free}\,(\Phi) \subseteq \mathrm{Var}$. However, the expanded language $\underline{\mathsf{QPL}}$ will be uncountable, namely of cardinality $2^{\aleph_0}$.

Remark 2.1. The fragment of QPL containing only quantifiers over reals (but not over events) can be viewed as the 'polynomial' logic described earlier in [3, Section 6], where Boolean variables are treated as constant symbols and called *propositional variables* — so Boolean terms become *propositional formulas.* A study of quantification over propositional formulas in probability logic has been carried out in [13].

Let $\mathcal{K}$ be a class of probability spaces. By the QPL-*theory of* $\mathcal{K}$, written $\mathrm{Th}\,(\mathcal{K})$, we mean

$$\{\Phi \in \mathrm{Sent} \mid \mathscr{P} \Vdash \Phi \text{ for all } \mathscr{P} \in \mathcal{K}\}.$$

The set of all sentences in $\mathrm{Th}\,(\mathcal{K})$ that do not contain quantifiers over reals is denoted by $\mathrm{Th}^{\mathrm{e}}\,(\mathcal{K})$. We shall often write $\mathrm{Th}\,(\mathscr{P})$ and $\mathrm{Th}^{\mathrm{e}}\,(\mathscr{P})$ instead of $\mathrm{Th}\,(\{\mathscr{P}\})$ and $\mathrm{Th}^{\mathrm{e}}\,(\{\mathscr{P}\})$ respectively. Two probability spaces are called *elementarily equivalent* if their QPL-theories coincide (in other words, they are indistinguishable by means of QPL-sentences).

In general, QPL-theories tend to have very high degrees of undecidability. In particular, using an alternative description of the analytical hierarchy given in [14, Section 3] (see [15] for further results in this area), we can obtain:

Theorem 2.2 (see [16])

Let $\mathcal{K}$ be a class of spaces. Suppose that $\mathcal{K}$ contains all infinite discrete spaces. Then $\mathrm{Th}^{\mathrm{e}}\,(\mathcal{K})$ is at least as complex as complete second-order arithmetic, i.e. the latter is reducible to $\mathrm{Th}^{\mathrm{e}}\,(\mathcal{K})$.

Still, some interesting examples of decidable QPL-theories will be discovered later.

3 Passing to quotients

Two spaces $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1 \rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2 \rangle$ are called *isomorphic* if there exists $f : \mathscr{A}_1 \to \mathscr{A}_2$ such that:

- f is an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$, thought of as Boolean algebras;
- $\mathsf{P}_1\,(A) = \mathsf{P}_2\,(f\,(A))$ for all $A \in \mathscr{A}_1$.

Naturally, one can easily show that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are isomorphic via f, then for any QPL-formula Φ, Boolean valuation ζ in $\mathscr{P}_1$ and field valuation ι,

$$\mathscr{P}_1 \Vdash \Phi\,[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{P}_2 \Vdash \Phi\,[\zeta \circ f, \iota],$$

which implies $\mathrm{Th}(\mathscr{P}_1) = \mathrm{Th}(\mathscr{P}_2)$.

The notion of isomorphism may be relaxed in the following way. Consider the QPL-formula

$$X \approx Y \;:=\; \mu((X \wedge \neg Y) \vee (Y \wedge \neg X)) = 0.$$

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a space. Evidently, $X \approx Y$ defines (in $\mathscr{P}$) a very natural equivalence relation on $\mathscr{A}$, namely

$$\mathscr{E} \;:=\; \{(A_1, A_2) \in \mathscr{A} \times \mathscr{A} \mid \mathscr{P} \Vdash A_1 \approx A_2\}.$$

In particular, $\mathscr{E}$ turns out to be a congruence relation on $\mathscr{A}$. For each $A \in \mathscr{A}$, denote by $[A]_{\approx}$ the equivalence class of A under $\mathscr{E}$, i.e.

$$[A]_{\approx} \;:=\; \{B \in \mathscr{A} \mid (A, B) \in \mathscr{E}\}.$$

Now take $\mathscr{A}_{\approx}$ to be the collection of all such classes — or rather, the quotient Boolean algebra of $\mathscr{A}$ modulo $\mathscr{E}$ — and define $\mathsf{P}_{\approx} : \mathscr{A}_{\approx} \to [0,1]$ by

$$\mathsf{P}_{\approx}([A]_{\approx}) \;:=\; \mathsf{P}(A)$$

(note that $\mathsf{P}(A) = \mathsf{P}(B)$ for all $B \in [A]_{\approx}$). It is easy to check that $\mathscr{P}_{\approx} = \langle \mathscr{A}_{\approx}, \mathsf{P}_{\approx} \rangle$ is a probability space, called the *quotient space of $\mathscr{P}$ modulo events of measure zero.* Moreover, one can show that for any QPL-formula Φ, Boolean valuation ζ in $\mathscr{P}$ and field valuation ι,

$$\mathscr{P} \Vdash \Phi[\zeta, \iota] \quad\Longleftrightarrow\quad \mathscr{P}_{\approx} \Vdash \Phi[\zeta_{\approx}, \iota]$$

where $\zeta_{\approx}$ is the Boolean valuation given by

$$\zeta_{\approx}(X) \;:=\; [\zeta(X)]_{\approx}.$$

Finally, we call $\mathscr{P}_1$ and $\mathscr{P}_2$ *weakly isomorphic* if $(\mathscr{P}_1)_{\approx}$ and $(\mathscr{P}_2)_{\approx}$ are isomorphic. Consequently, if two spaces are weakly isomorphic, then they are elementarily equivalent, i.e. have the same QPL-theory.

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a space. Now if $\mathscr{P} \Vdash A_1 \approx A_2$, i.e. the symmetric difference of A_1 and A_2 has measure zero, then A_1 and A_2 are indistinguishable in $\mathscr{P}$ by QPL-formulas (with parameters). So definability in $\mathscr{P}$ reduces to definability in $\mathscr{P}_{\approx}$. For instance, consider

$$X \preccurlyeq Y \;:=\; X \wedge Y \approx X.$$

Obviously, $\mathscr{P} \Vdash A_1 \preccurlyeq A_2$ iff $[A_1]_{\approx}$ is less than or equal to $[A_2]_{\approx}$ in the Boolean algebra $\mathscr{A}_{\approx}$. Next, consider

$$\mathrm{At}(X) \;:=\; \mu(X) \neq 0 \wedge \forall Y\, (\mu(Y) \neq 0 \wedge Y \preccurlyeq X \to Y \approx X).$$

Clearly, $\mathscr{P} \Vdash \mathrm{At}(A)$ iff $[A]_{\approx}$ is an atom of $\mathscr{A}_{\approx}$.[1] We call $\mathscr{P}$ *atomless* if $\mathscr{P} \Vdash \neg\exists X\, \mathrm{At}(X)$, i.e. $\mathscr{A}_{\approx}$ is an atomless Boolean algebra. There are alternative definitions, of course, but the one given here is in the spirit of Boolean algebras; cf. [5].

[1] Recall that $A \in \mathscr{A}$ is an *atom* of $\mathscr{A}$ if A is minimal in $\mathscr{A} \setminus \{\bot\}$ where $\bot$ denotes the least element of $\mathscr{A}$; see [5], for example.

Remark 3.1. On the other hand, there are probability logics in which we cannot safely pass from each structure to the corresponding quotient structure modulo events of measure, without altering its theory. In particular, this applies to the languages studied in [1]. In effect, the Π^2_1-hardness arguments provided in [1] work over 'generalized discrete spaces', which may contain uncountably many elements of measure zero, but fail over ordinary discrete spaces, whose underlying sets are at most countable.

4 The standard atomless space

The most standard choice of atomless probability space is

$$\mathscr{L} \;:=\; \langle \mathscr{S}, \mathsf{L} \rangle$$

where $\mathscr{S}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ (i.e. of the closed unit interval of real numbers), and L is the corresponding measure.

Intuitively, $\mathscr{L}$ behaves as smooth as one may expect in the present context. So the next result looks rather natural.

Theorem 4.1

$\mathrm{Th}\,(\mathscr{L})$ *is decidable.*

We shall reduce the problem of testing membership in $\mathrm{Th}\,(\mathscr{L})$ to that of testing membership in the first-order theory of the ordered field $\mathfrak{R}$ of reals, which is well-known to be decidable (see [17] for details). Our argument will be based on a number of lemmas and propositions.

For each finite set S of Boolean variables, by a *basic conjunction over* S we mean an expression of the form

$$X_1^{\varepsilon_1} \wedge \ldots \wedge X_n^{\varepsilon_n}$$

where $\{\varepsilon_1, \ldots, \varepsilon_n\} \subseteq \{0,1\}$, X_1, …, X_n are pairwise distinct elements of S, and n equals $|S|$ (so every element of S occurs in this conjunction).[2] In case S is empty, we assume that $\perp$ is the only basic conjunction over S. Moreover, we shall identify two basic conjunctions if they differ only in the order of the conjuncts. Call a QPL-formula *normal* if it has the form

$$\mathrm{Q}_1 v_1 \,\ldots \mathrm{Q}_n v_n \,\Psi$$

where $\{\mathrm{Q}_1, \ldots, \mathrm{Q}_n\} \subseteq \{\forall, \exists\}$, v_1, …, v_n are pairwise distinct elements of $\mathrm{Free}\,(\Psi)$, Ψ is quantifier-free, and every μ-term occurring in Ψ has the form $\mu\,(\psi)$ with ψ a basic conjunction over $\mathrm{FV}\,(\Psi)$. Without loss of generality, we can restrict attention to such QPL-formulas:

Lemma 4.2

Every QPL*-formula is semantically equivalent to a normal* QPL*-formula.*

[2]We shall sometimes write ϕ^1 instead of ϕ and ϕ^0 instead of $\neg\phi$. Similarly for QPL-formulas and formulas in the first-order language of $\mathfrak{R}$.

Proof. Let Φ be a QPL-formula. As in classical first-order logic, Φ can be put in the form

$$\mathrm{Q}_1 v_1 \ \ldots \mathrm{Q}_n v_n \, \Theta$$

where $\{\mathrm{Q}_1, \ldots, \mathrm{Q}_n\} \subseteq \{\forall, \exists\}$, $v_1, \ldots, v_n$ are pairwise distinct elements of $\mathrm{Free}\,(\Theta)$ and Θ is quantifier-free. It remains to show that Θ is semantically equivalent to a quantifier-free QPL-formula Ψ such that $\mathrm{Free}\,(\Psi) = \mathrm{Free}\,(\Theta)$, and every μ-term occurring in Ψ has the form $\mu\,(\psi)$ with ψ a basic conjunction over $\mathrm{FV}\,(\Psi)$. To this end, for each Boolean term ϕ whose variables lie in $\mathrm{FV}\,(\Theta)$, we define

$$C\,(\phi) \;:=\; \begin{array}{c}\text{the set of all basic conjunctions over } \mathrm{FV}\,(\Theta) \text{ that} \\ \text{imply } \phi \text{ in classical propositional logic.}\end{array}$$

Take Ψ to be the result of replacing each μ-term $\mu\,(\phi)$ in Θ by $\sum_{\psi \in C(\phi)} \mu\,(\psi)$. It is easy to see that Ψ has the required properties. $\square$

Let S be a set of Boolean variables, $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1 \rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2 \rangle$ be probability spaces, ζ_1 and ζ_2 be functions from S to $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. We say that ζ_1 and ζ_2 are *similar* if for every basic conjunction ϕ over S,

$$\mathsf{P}_1\,(\zeta_1\,(\phi)) \;=\; \mathsf{P}_2\,(\zeta_2\,(\phi)).$$

Note that if ζ_1 and ζ_2 are similar, then we also have $\mathsf{P}_1\,(\zeta_1\,(\phi)) = \mathsf{P}_2\,(\zeta_2\,(\phi))$ for all Boolean terms ϕ whose variables lie in S.

Lemma 4.3

Let S be a finite set of Boolean variables, ζ and ξ be functions from S to $\mathscr{S}$, X be a Boolean variable that lies outside S. Suppose ζ and ξ are similar (with respect to $\mathscr{L}$). Then for every $A \in \mathscr{S}$ there exists $B \in \mathscr{S}$ such that ζ^X_A and ξ^X_B are similar.[3]

Proof. Consider an arbitrary $A \in \mathscr{S}$. A simple fact from analysis tells us that for any $B \in \mathscr{S}$,

$$\{\mathsf{L}\,(C) \mid C \in \mathscr{S} \text{ and } C \subseteq B\} \;=\; [0, \mathsf{L}\,(B)]. \tag{†}$$

Using (†), for each basic conjunction ϕ over S we can choose $B_\phi \in \mathscr{S}$ such that $B_\phi \subseteq \xi\,(\phi)$ and $\mathsf{L}\,(B_\phi) = \mathsf{L}\left(\zeta^X_A\,(\phi \wedge X)\right)$, and then take

$$B \;:=\; \bigcup \{B_\phi \mid \phi \text{ is a basic conjunction over } S\}\,.$$

It is easy to see that ξ^X_B is similar to ζ^X_A. $\square$

Using Lemma 4.3, we show that similar Boolean valuations in $\mathscr{L}$ are indistinguishable by QPL-formulas in the following sense:

Proposition 4.4

Let Φ be a QPL-formula, ζ and ξ be functions from $\mathrm{FV}\,(\Phi)$ to $\mathscr{S}$. Suppose that ζ and ξ are similar (with respect to $\mathscr{L}$). Then for every function ι from $\mathrm{Fv}\,(\Phi)$ to $\mathbb{R}$,

$$\mathscr{L} \Vdash \Phi\,[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{L} \Vdash \Phi\,[\xi, \iota].$$

[3] Here ζ^X_A denotes $\zeta \cup \{(X, A)\}$.

Proof. By induction on the complexity of Φ. By Lemma 4.2, we may assume Φ is normal. Note that all subformulas of Φ are also normal.

In the case where Φ is quantifier-free the result is immediate.

Suppose $\Phi = \exists X\, \Psi$. Assume $\mathscr{L} \Vdash \Phi\,[\zeta, \iota]$, i.e. there exists $A \in \mathscr{S}$ such that $\mathscr{L} \Vdash \Psi\,\left[\zeta_A^X, \iota\right]$. By Lemma 4.3, since ζ and ξ are similar, one can find $B \in \mathscr{S}$ such that ζ_A^X and ξ_B^X are similar as well. Thus $\mathscr{L} \Vdash \Psi\,\left[\xi_B^X, \iota\right]$ by the inductive hypothesis. It follows that $\mathscr{L} \Vdash \Phi\,[\xi, \iota]$. The converse implication holds by symmetry.

The case where $\Phi = \forall X\, \Psi$ can be handled similarly.

The cases where Φ is $\exists x\, \Psi$ or $\forall x\, \Psi$ are trivial. □

Call a Boolean valuation ζ in $\mathscr{L}$ (i.e. a partial function from Var to $\mathscr{S}$) *compact* if for each basic conjunction ϕ over $\operatorname{dom}\zeta$, $\zeta\,(\phi)$ is a subinterval of $[0,1]$.

Lemma 4.5

Let S be a finite set of Boolean variables, ζ be a compact function from S to $\mathscr{S}$, and X be a Boolean variable that lies outside S. Then for every $A \in \mathscr{S}$ there exists $B \in \mathscr{S}$ such that ζ_B^X is compact and similar to ζ_A^X.

Proof. Consider an arbitrary $A \in \mathscr{S}$. For each basic conjunction ϕ over S, denote the infimum of $\zeta\,(\phi)$ in $\mathbb{R}$ — i.e. the left endpoint of the interval $\zeta\,(\phi)$ — by a_ϕ, and set

$$B_\phi \;:=\; \zeta\,(\phi) \cap \left[0, a_\phi + \mathsf{L}\left(\zeta_A^X\,(\phi \wedge X)\right)\right]$$

— so in particular, we have $\mathsf{L}\,(B_\phi) = \mathsf{L}\left(\zeta_A^X\,(\phi \wedge X)\right)$. Take

$$B \;:=\; \bigcup \{B_\phi \mid \phi \text{ is a basic conjunction over } S\}.$$

It is easy to see that ζ_B^X has the required properties. □

Corollary 4.6

Let S be a finite set of Boolean variables and ζ be a function from S to $\mathscr{S}$. Then there exists a compact function ξ from S to $\mathscr{L}$ such that ζ and ξ are similar.

Proof. This can easily be shown using Lemmas 4.3 and 4.5, by induction on $|S|$. □

Next we describe an alternative semantics for normal QPL-formulas over $\mathscr{L}$, which uses only compact Boolean valuations. For any normal QPL-formula Φ, compact function ζ from $\mathrm{FV}\,(\Phi)$ to $\mathscr{S}$ and function ι from $\mathrm{Fv}\,(\Phi)$ to $\mathbb{R}$, define $\mathscr{L} \rhd \Phi\,[\zeta, \iota]$ recursively:

- in the case where Φ is quantifier-free, set $\mathscr{L} \rhd \Phi\,[\zeta, \iota]$ iff $\mathscr{L} \Vdash \Phi\,[\zeta, \iota]$;
- in the case where $\Phi = \exists X\, \Psi$ we set

$$\mathscr{L} \rhd \Phi\,[\zeta, \iota] \quad :\Longleftrightarrow \quad \begin{array}{c}\text{there exists } A \in \mathscr{S} \text{ such that} \\ \zeta_A^X \text{ is compact and } \mathscr{L} \rhd \Psi\,\left[\zeta_A^X, \iota\right].\end{array}$$

- in the case where $\Phi = \forall X\ \Psi$ we set

$$\mathscr{L} \rhd \Phi[\zeta, \iota] \quad :\Longleftrightarrow \quad \begin{array}{c}\text{for all } A \in \mathscr{S}, \text{if } \zeta_A^X \text{ is compact,}\\ \text{then } \mathscr{L} \rhd \Psi\left[\zeta_A^X, \iota\right].\end{array}$$

- the cases where Φ is $\exists x\ \Psi$ or $\forall x\ \Psi$ are treated in the standard way.

This semantics turns out to be equivalent to the usual one:

Proposition 4.7

Let Φ be a normal QPL*-formula. Then for any compact function ζ from $\mathrm{FV}(\Phi)$ to $\mathscr{S}$ and function ι from $\mathrm{Fv}(\Phi)$ to $\mathbb{R}$,*

$$\mathscr{L} \Vdash \Phi[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{L} \rhd \Phi[\zeta, \iota].$$

Proof. By induction on the complexity of Φ.

In the case where Φ is quantifier-free the result is obvious.

Suppose $\Phi = \exists X\ \Psi$. Assume $\mathscr{L} \Vdash \Phi[\zeta, \iota]$, i.e. there exists $A \in \mathscr{S}$ such that $\mathscr{L} \Vdash \Psi\left[\zeta_A^X, \iota\right]$. By Lemma 4.5, one can find $B \in \mathscr{S}$ such that ζ_B^X is compact and similar to ζ_A^X. Hence $\mathscr{L} \Vdash \Psi\left[\zeta_B^X, \iota\right]$ by Proposition 4.4, and thus $\mathscr{L} \rhd \Psi\left[\zeta_B^X, \iota\right]$ by the inductive hypothesis. It follows that $\mathscr{L} \rhd \Phi[\zeta, \iota]$. The converse is much easier: if $\mathscr{L} \rhd \Phi[\zeta, \iota]$, then there exists $A \in \mathscr{S}$ such that ζ_A^X is compact and $\mathscr{L} \rhd \Psi\left[\zeta_A^X, \iota\right]$; hence $\mathscr{L} \Vdash \Psi\left[\zeta_A^X, \iota\right]$ by the inductive hypothesis, which gives $\mathscr{L} \Vdash \Phi[\zeta, \iota]$.

The case where $\Phi = \forall X\ \Psi$ can be handled similarly.

The cases where Ψ is $\exists x\ \Psi$ or $\forall x\ \Psi$ are trivial. □

Finally, $\{\Phi \in \mathrm{Sent} \mid \Phi \text{ is normal and } \mathscr{L} \rhd \Phi\}$ can be easily reduced to the first-order theory of $\mathfrak{R}$. Consider a normal QPL-sentence Φ. Given a Boolean variable X occurring in Φ, define

$$\Phi_X \ := \ \text{the subformula of } \Phi \text{ beginning with } \mathrm{Q}X \text{ for some } \mathrm{Q} \in \{\forall, \exists\}.$$

(The normality of Φ guarantees that Φ_X exists and is unique.) For our reduction, we shall represent X as a union of finitely many intervals. To this end, set

$$\ell(X) \ := \ 2^{|\mathrm{FV}(\Phi_X)|}.$$

The intervals for X will be coded by the $\ell(X)$-tuple

$$\underline{X} \ := \ \langle\langle a_{X,i}, b_{X,i}, c_{X,i}, d_{X,i}\rangle\rangle_{i=1}^{\ell(X)}$$

of quadruples of fresh field variables, which must be different from the field variables occurring in Φ. The intuition is that each $\langle a_{X,i}, b_{X,i}, c_{X,i}, d_{X,i}\rangle$ corresponds to

$$A_{X,i} \ := \ \begin{cases} [a_{X,i}, b_{X,i}) & \text{if } c_{X,i} = 1 \text{ and } d_{X,i} \neq 1 \\ (a_{X,i}, b_{X,i}] & \text{if } c_{X,i} \neq 1 \text{ and } d_{X,i} = 1 \\ [a_{X,i}, b_{X,i}] & \text{if } c_{X,i} = 1 \text{ and } d_{X,i} = 1 \\ (a_{X,i}, b_{X,i}) & \text{otherwise.} \end{cases}$$

Now let S be a finite set of Boolean variables. We use $\underline{S}$ to abbreviate $\langle \underline{X} \rangle_{X \in S}$. Roughly speaking, $\underline{S}$ determines the function ζ from S to $\mathscr{S}$ given by

$$\zeta(X) := \bigcup_{i=1}^{\ell(X)} A_{X,i}.$$

For each function δ from S to $\{0,1\}$ we introduce the first-order formulas

$$\begin{aligned}
\mathtt{E}_\delta(x, \underline{S}) &:= 0 \leqslant x \leqslant 1 \wedge \\
&\bigwedge\nolimits_{X \in S} \left(\bigvee\nolimits_{i=1}^{\ell(X)} (a_{X,i} < x < b_{X,i} \vee (x = a_{X,i} \wedge c_{X,i} = 1) \vee (x = b_{X,i} \wedge d_{X,i} = 1)) \right)^{\delta(X)}, \\
\mathtt{I}_\delta(a, b, \underline{S}) &:= a \leqslant b \wedge \\
&\qquad \forall x \left((a < x < b \to \mathtt{E}_\delta(x, \underline{S})) \wedge (\mathtt{E}_\delta(x, \underline{S}) \to a \leqslant x \leqslant b) \right).
\end{aligned}$$

Intuitively, if $\underline{S}$ determines ζ, then

$$\begin{aligned}
\mathtt{E}_\delta(x, \underline{S}) &\text{ says } `x \text{ belongs to } \zeta\left(\bigwedge\nolimits^{\delta} S\right)\text{'}, \\
\mathtt{I}_\delta(a, b, \underline{S}) &\text{ says } `\zeta\left(\bigwedge\nolimits^{\delta} S\right) \text{ is a subinterval of } [0,1] \text{ with endpoints } a \text{ and } b\text{'}
\end{aligned}$$

where $\bigwedge^{\delta} S$ denotes $\bigwedge_{X \in S} X^{\delta(X)}$. Hence the condition '$\zeta$ is compact' can be expressed by

$$\mathtt{C}(\underline{S}) := \bigwedge\nolimits_{\delta \in \{0,1\}^S} \exists a \, \exists b \, \mathtt{I}_\delta(a, b, \underline{S})$$

where $\{0,1\}^S$ is the set of all functions from S to $\{0,1\}$. Also, for each $X \in S$, to make sure that the field variables in $\underline{X}$ behave as expected, we need

$$\mathtt{P}(\underline{X}) := \bigwedge\nolimits_{n=1}^{\ell(X)} 0 \leqslant a_{X,n} \leqslant b_{X,n} \leqslant 1.$$

Using these first-order formulas, Φ can be translated into $\tau(\Phi)$ as follows.

- Letting Ψ be the quantifier-free part of Φ, define

  $$\Psi^* := \exists a_1 \, \exists b_1 \, \ldots \, \exists a_{2^{|S|}} \, \exists b_{2^{|S|}} \left(\bigwedge\nolimits_{i=1}^{2^{|S|}} \mathtt{I}_{\delta_i}(a_i, b_i, \underline{S}) \wedge \underline{\Psi} \right)$$

  where $S = \mathrm{FV}(\Psi)$, $\{\delta_1, \ldots, \delta_{2^{|S|}}\} = \{0,1\}^S$, and $\underline{\Psi}$ is the result of replacing each μ-term $\mu\left(\bigwedge^{\delta_i} S\right)$ in Φ by $(b_i - a_i)$.

- Next, for each subformula Θ of Φ beginning with a quantifier, we recursively define

  $$\Theta^* := \begin{cases} \exists \underline{X} \left(\mathtt{P}(\underline{X}) \wedge \mathtt{C}(\underline{S}) \wedge \Omega^* \right) & \text{if } \Theta = \exists X \, \Omega \\ \forall \underline{X} \left(\mathtt{P}(\underline{X}) \wedge \mathtt{C}(\underline{S}) \to \Omega^* \right) & \text{if } \Theta = \forall X \, \Omega \\ \exists x \, \Omega^* & \text{if } \Theta = \exists x \, \Omega \\ \forall x \, \Omega^* & \text{if } \Theta = \forall x \, \Omega \end{cases}$$

  where $S = \mathrm{FV}(\Psi)$. Now set $\tau(\Phi)$ to be Φ^*.

It is easy to show the following.

Proposition 4.8

For every normal QPL*-sentence* Φ*,*

$$\mathscr{L} \rhd \Phi \quad \Longleftrightarrow \quad \tau(\Phi) \ \textit{ is true in } \ \mathfrak{R}.$$

Proof. Observe that if S is a finite set of Boolean variables, and ζ is a compact function from S to $\mathscr{S}$, then for every $X \in S$, $\zeta(X)$ can be represented as

$$\bigcup \left\{ \zeta \left(\bigwedge^{\delta} S \right) \mid \delta \in \{0,1\}^{S} \text{ and } \delta(X) = 1 \right\},$$

which is a union of $2^{|S|-1}$ intervals. Hence the function ℓ works nicely. The rest of the argument is straightforward, by the definition of $\rhd$. $\square$

Proof of Theorem 4.1. Clearly, Lemma 4.2 can be effectivised, i.e. there exists a computable function that, given any QPL-formula Φ, finds a normal QPL-formula Φ' semantically equivalent to Φ. So for every QPL-sentence Φ,

$$\begin{aligned} \mathscr{L} \Vdash \Phi \quad & \Longleftrightarrow \quad \mathscr{L} \Vdash \Phi' \\ & \overset{4.7}{\Longleftrightarrow} \quad \mathscr{L} \rhd \Phi' \\ & \overset{4.8}{\Longleftrightarrow} \quad \tau(\Phi') \ \text{ is true in } \ \mathfrak{R}. \end{aligned}$$

Hence the QPL-theory of $\mathscr{L}$ is reducible to the first-order theory of $\mathfrak{R}$. $\square$

Since $\mathrm{Th}(\mathscr{L})$ includes the first-order theory of $\mathfrak{R}$, Theorem 4.1 may be viewed as an expansion of Tarski's famous result. In subsequent sections, the technique described above will be developed further, with the argument of Theorem 4.1 serving as a framework for various generalizations.

5 Arbitrary atomless spaces

There is an analogue of (†) from the proof of Lemma 4.3:

Proposition 5.1 (see [10])

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ *be an atomless space. Then for every* $A \in \mathscr{A}$*,*

$$\{\mathsf{P}(B) \mid B \in \mathscr{A} \textit{ and } B \leqslant A\} \; = \; [0, \mathsf{P}(A)]$$

where $\leqslant$ *denotes the ordering relation in* $\mathscr{A}$*.*

Since [10] was written in French, we provide an alternative proof in the Appendix; our argument is somewhat more direct and uses transfinite recursion instead of Zorn's lemma.

As for Lemma 4.3 itself, it can be strengthened as follows.

Lemma 5.2

Let $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1 \rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2 \rangle$ be atomless spaces, S be a finite set of Boolean variables, ζ and ξ be functions from S to $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively, X be a Boolean variable that lies outside S. Suppose ζ and ξ are similar. Then for every $A \in \mathscr{S}$ there exists $B \in \mathscr{S}$ such that ζ_A^X and ξ_B^X are similar.

Proof. Consider an arbitrary $A \in \mathscr{S}$. Now using Proposition 5.1, for each basic conjunction ϕ over S, choose $B_\phi \in \mathscr{A}_2$ such that $B_\phi \leqslant \xi(\phi)$ and $\mathsf{P}_2(B_\phi) = \mathsf{P}_1\left(\zeta_A^X(\phi \wedge X)\right)$, and then take

$$B := \bigvee \{B_\phi \mid \phi \text{ is a basic conjunction over } \mathrm{FV}(\Phi)\}.$$

It is easy to see that ξ_B^X is similar to ζ_A^X. □

This in turn leads to a stronger version of Proposition 4.4:

Proposition 5.3

Let $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1 \rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2 \rangle$ be atomless spaces, Φ be a QPL*-formula, ζ and ξ be functions from $\mathrm{FV}(\Phi)$ to $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Suppose ζ and ξ are similar. Then for every function ι from $\mathrm{Fv}(\Phi)$ to $\mathbb{R}$,*

$$\mathscr{P}_1 \Vdash \Phi[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{P}_2 \Vdash \Phi[\xi, \iota].$$

Proof. The argument is essentially the same as that for Proposition 4.4, by induction on the complexity of Φ. By Lemma 4.2, we may assume Φ is normal.

In the case where Φ is quantifier-free the result is immediate.

Suppose $\Phi = \exists X\, \Psi$. Assume $\mathscr{P}_1 \Vdash \Phi[\zeta, \iota]$, i.e. there exists $A \in \mathscr{A}_1$ such that $\mathscr{P}_1 \Vdash \Psi\left[\zeta_A^X, \iota\right]$. By Lemma 5.2, since ζ and ξ are similar, one can find $B \in \mathscr{A}_2$ such that ζ_A^X and ξ_B^X are similar too. Thus $\mathscr{P}_2 \Vdash \Psi\left[\xi_B^X, \iota\right]$ by the inductive hypothesis. It follows that $\mathscr{P}_2 \Vdash \Phi[\xi, \iota]$. The converse implication holds by symmetry.

The case where $\Phi = \forall X\, \Psi$ can be handled similarly.

The cases where Φ is $\exists x\, \Psi$ or $\forall x\, \Psi$ are trivial. □

Theorem 5.4

Let $\mathscr{P}$ be an atomless space. Then $\mathrm{Th}(\mathscr{P})$ coincides with $\mathrm{Th}(\mathscr{L})$, and hence is decidable.

Proof. Proposition 5.3 implies $\mathrm{Th}(\mathscr{P}) = \mathrm{Th}(\mathscr{L})$, and $\mathrm{Th}(\mathscr{L})$ is decidable by Theorem 4.1. □

Thus all atomless probability spaces are elementarily equivalent to each other, i.e. have the same QPL-theory, namely $\mathrm{Th}(\mathscr{L})$. This suggests that a nice elementary classification of spaces should be based on the notion of atom (cf. [5]).

6 Elementary invariants

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a space. Take

$$\mathrm{D}_{\mathscr{P}} \;:=\; \text{the collection of all atoms of } \mathscr{A}_{\approx}.$$

Observe that $\mathrm{D}_{\mathscr{P}}$ is at most countable.[4] Denote $\bigvee \mathrm{D}_{\mathscr{P}}$ by $\mathrm{d}_{\mathscr{P}}$. By the *elementary invariant of* $\mathscr{P}$ we mean the function $\sharp_{\mathscr{P}}$ from $(0,1]$ to $\mathbb{N}$ given by

$$\sharp_{\mathscr{P}}(r) \;:=\; |\{A \in \mathrm{D}_{\mathscr{P}} \mid \mathsf{P}_{\approx}(A) = r\}|.$$

Note that since $\{r \in \mathbb{R}_+ \mid \sharp_{\mathscr{P}}(r) \neq 0\}$ is at most countable, $\sharp_{\mathscr{P}}$ may be encoded as a subset of $\mathbb{N}$ — this will play an important role in Section 7. We call $\mathscr{P}$ *atomic* if $\mathscr{A}_{\approx}$ is an atomic Boolean algebra (cf. [5]), i.e.

$$\mathscr{P} \Vdash \forall X\,(\mu(X) \neq 0 \to \exists Y\,(\mathrm{At}(Y) \wedge Y \preccurlyeq X)).$$

Evidently, we have

$$\begin{aligned}
\mathscr{P} \text{ is atomic} \quad &\iff \quad \mathsf{P}_{\approx}(\neg \mathrm{d}_{\mathscr{P}}) \;=\; 0\\
&\iff \quad \mathrm{d}_{\mathscr{P}} \;=\; [\top]_{\approx}\\
&\iff \quad \sum\nolimits_{r \in (0,1]} r \cdot \sharp_{\mathscr{P}}(r) \;=\; 1\\
&\iff \quad \mathscr{P}_{\approx} \text{ is discrete.}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathscr{P} \text{ is atomless} \quad &\iff \quad \mathsf{P}_{\approx}(\mathrm{d}_{\mathscr{P}}) \;=\; 0\\
&\iff \quad \mathrm{d}_{\mathscr{P}} \;=\; [\bot]_{\approx}\\
&\iff \quad \sharp_{\mathscr{P}} \text{ is the zero function.}
\end{aligned}$$

In general, modulo events of measure zero, each probability space can be represented as a convex combination of a discrete space and an atomless space. In addition to $\sharp_{\mathscr{P}}$, we shall make use of the function $\lambda_{\mathscr{P}}$ from $\mathscr{A} \times (0,1]$ to $\mathbb{N}$ given by

$$\lambda_{\mathscr{P}}(A, r) \;:=\; |\{B \in \mathrm{D}_{\mathscr{P}} \mid B \leqslant [A]_{\approx} \text{ and } \mathsf{P}_{\approx}(B) = r\}|.$$

Obviously, $\lambda_{\mathscr{P}}(\top, r) = \sharp_{\mathscr{P}}(r)$ for all $r \in (0,1]$. Thus $\lambda_{\mathscr{P}}$ extends $\sharp_{\mathscr{P}}$.

The following modification of Proposition 5.1 will turn out to be helpful.

Proposition 6.1

Let $\mathscr{P} = \langle \Omega, \mathscr{A}, \mathsf{P} \rangle$ *be a space. Then for every* $A \in \mathscr{A}$,

$$[A]_{\approx} \;\leqslant \neg \mathrm{d}_{\mathscr{P}} \quad \Longrightarrow \quad \{\mathsf{P}(B) \mid B \in \mathscr{A} \textit{ and } B \leqslant A\} \;=\; [0, \mathsf{P}(A)].$$

[4] See (2) in the proof of Proposition 5.1 provided in Appendix.

Proof. Let $A \in \mathscr{A}$ be such that $[A]_{\approx} \leqslant \neg \mathrm{d}_{\mathscr{P}}$. If $\mathsf{P}(A) = 0$, then the result is trivial. Suppose A has positive measure. Take $\mathscr{A}_A$ to be $\{B \in \mathscr{A} \mid B \leqslant A\}$, and define $\mathsf{P}_A : \mathscr{A}_A \to [0,1]$ by

$$\mathsf{P}_A(B) \;:=\; \frac{\mathsf{P}(B)}{\mathsf{P}(A)}.$$

Obviously, $\mathscr{P}_A = \langle \mathscr{A}_A, \mathsf{P}_A \rangle$ is an atomless space. Therefore

$$\begin{aligned}\{\mathsf{P}(B) \mid B \in \mathscr{A} \text{ and } B \leqslant A\} \;&=\; \{\mathsf{P}_A(B) \cdot \mathsf{P}(A) \mid B \in \mathscr{A}_A\} \\ &\overset{5.1}{=}\; [0, \mathsf{P}(A)].\end{aligned}$$

□

Let S be a set of Boolean variables, $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1 \rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2 \rangle$ be probability spaces, ζ_1 and ζ_2 be functions from S to $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. We say that ζ_1 and ζ_2 are *congruent* if they are similar, and moreover, for any basic conjunction ϕ over S and $r \in (0,1]$,

$$\lambda_{\mathscr{P}_1}(\zeta_1(\phi), r) \;=\; \lambda_{\mathscr{P}_2}(\zeta_2(\phi), r).$$

So if ζ_1 and ζ_2 are congruent, then the elementary invariants of $\mathscr{P}_1$ and $\mathscr{P}_2$ coincide. Moreover, if $\mathscr{P}_1$ and $\mathscr{P}_2$ are atomless, then ζ_1 and ζ_2 are congruent iff they are similar.

Now Lemma 5.2 can be generalised as follows.

Lemma 6.2

Let $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1 \rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2 \rangle$ be spaces, S be a finite set of Boolean variables, ζ and ξ be functions from S to $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively, X be a Boolean variable not in S. Suppose that ζ and ξ are congruent. Then for every $A \in \mathscr{A}_1$ there exists $B \in \mathscr{A}_2$ such that ζ^X_A and ξ^X_B are congruent.

Proof. Consider an arbitrary $A \in \mathscr{A}_1$. For convenience, fix $D_1 \in \mathscr{A}_1$ and $D_2 \in \mathscr{A}_2$ such that

$$[D_1]_{\approx} \;=\; \mathrm{d}_{\mathscr{P}_1} \quad \text{and} \quad [D_2]_{\approx} \;=\; \mathrm{d}_{\mathscr{P}_2}.$$

Intuitively, D_1 and D_2 correspond to the atomic parts of $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively (modulo events of measure zero). Let ϕ be a basic conjunction over S. Then since ζ and ξ are congruent, we can choose $B^+_\phi \in \mathscr{A}_2$ such that:

- $B^+_\phi \leqslant \xi(\phi) \wedge D_2$;
- $\lambda_{\mathscr{P}_2}\left(B^+_\phi, r\right) = \lambda_{\mathscr{P}_1}\left(\zeta^X_A(\phi \wedge X) \wedge D_1, r\right)$ for all $r \in (0,1]$.

Observe that $\mathsf{P}_1(\zeta(\phi) \wedge \neg D_1) = \mathsf{P}_2(\xi(\phi) \wedge \neg D_2)$, because $\mathsf{P}_1(\zeta(\phi) \wedge D_1) = \mathsf{P}_2(\xi(\phi) \wedge D_2)$ and $\mathsf{P}_1(\zeta(\phi)) = \mathsf{P}_2(\xi(\phi))$. So using Proposition 6.1 we can choose $B^-_\phi \in \mathscr{A}_2$ such that:

- $B^-_\phi \leqslant \xi(\phi) \wedge \neg D_2$;
- $\mathsf{P}_2\left(B^-_\phi\right) = \mathsf{P}_1\left(\zeta^X_A(\phi \wedge X) \wedge \neg D_1\right)$.

Denote $B_\phi^+ \vee B_\phi^-$ by B_ϕ. Putting the B_ϕ's together, we get

$$B \ := \ \bigvee \{B_\phi \mid \phi \text{ is a basic conjunction over } S\}\,.$$

It is straightforward to check that ξ_B^X is congruent to ζ_A^X. □

Corollary 6.3

Let $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1\rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2\rangle$ be spaces, S be a finite set of Boolean variables and ζ be a function from S to $\mathscr{A}_1$. Suppose that $\sharp_{\mathscr{P}_1}$ coincides with $\sharp_{\mathscr{P}_2}$. Then there exists a function ξ from S to $\mathscr{A}_2$ such that ζ and ξ are congruent.

Proof. This can easily be shown using Lemma 6.2, by induction on $|S|$. □

We are ready for the most general version of Proposition 5.3:

Proposition 6.4

Let $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1\rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2\rangle$ be spaces, Φ be a QPL-formula, ζ and ξ be functions from $\mathrm{FV}(\Phi)$ to $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Suppose ζ and ξ are congruent. Then for every function ι from $\mathrm{Fv}(\Phi)$ to $\mathbb{R}$,

$$\mathscr{P}_1 \Vdash \Phi\,[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{P}_2 \Vdash \Phi\,[\xi, \iota].$$

Proof. The argument is like that for Proposition 5.3 except that we replace 'similar' by 'congruent', and use Lemma 6.2 instead of Lemma 5.2. □

The use of the term 'elementary invariant' is justified by:

Theorem 6.5

For any two spaces $\mathscr{P}_1 = \langle \mathscr{A}_1, \mathsf{P}_1\rangle$ and $\mathscr{P}_2 = \langle \mathscr{A}_2, \mathsf{P}_2\rangle$,

$$\mathrm{Th}\,(\mathscr{P}_1) \ = \ \mathrm{Th}\,(\mathscr{P}_2) \quad \Longleftrightarrow \quad \mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_1) \ = \ \mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_2) \quad \Longleftrightarrow \quad \sharp_{\mathscr{P}_1} \ = \ \sharp_{\mathscr{P}_2}.$$

In particular, two spaces are elementarily equivalent iff their elementary invariants coincide.

Proof. Obviously, $\mathrm{Th}\,(\mathscr{P}_1) = \mathrm{Th}\,(\mathscr{P}_2)$ implies $\mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_1) = \mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_2)$.

Suppose to the contrary that $\mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_1) = \mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_2)$ but $\sharp_{\mathscr{P}_1} \neq \sharp_{\mathscr{P}_2}$. The latter means we have $\sharp_{\mathscr{P}_1}(r) \neq \sharp_{\mathscr{P}_2}(r)$ for some $r \in (0,1]$. Without loss of generality we may assume $\sharp_{\mathscr{P}_1}(r) > \sharp_{\mathscr{P}_2}(r)$. Clearly, there are rational numbers p and q such that $p < r < q$, and for every $s \in (p,q)$,

$$s \ \neq \ r \quad \Longrightarrow \quad \sharp_{\mathscr{P}_1}(s) \ = \ \sharp_{\mathscr{P}_2}(s) \ = \ 0.$$

Take N to be $\sharp_{\mathscr{P}_1}(r)$, and consider the sentence

$$\Phi \ := \ \exists X_1 \ldots \exists X_N \left(\bigwedge_{i=1}^{N} (\mathrm{At}\,(X_i) \wedge p < \mu\,(X_i) < q) \wedge \bigwedge_{i=1}^{N-1} \bigwedge_{j=i+1}^{N} \neg\,(X_i \approx X_j)\right).$$

Evidently, $\mathscr{P}_1 \Vdash \Phi$ and $\mathscr{P}_2 \nVdash \Phi$. Hence $\mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_1) \neq \mathrm{Th}^{\mathrm{e}}\,(\mathscr{P}_2)$, which is a contradiction.

Finally, if $\sharp_{\mathscr{P}_1} = \sharp_{\mathscr{P}_2}$, then Proposition 6.4 guarantees that for every QPL-sentence Φ,

$$\mathscr{P}_1 \Vdash \Phi \quad \Longleftrightarrow \quad \mathscr{P}_2 \Vdash \Phi,$$

and thus $\mathrm{Th}(\mathscr{P}_1) = \mathrm{Th}(\mathscr{P}_2)$. □

Remark 6.6. Analogous formulations arise in the metamathematics of Boolean algebras; cf. [5]. Although the two lines of research are related, they are in a sense incomparable, since QPL deals with measures on Boolean algebras of a special kind.

7 Translation

Next we want to adapt the alternative semantics over $\mathscr{L}$ provided in Section 4 to deal with spaces other than $\mathscr{L}$, and translate the new version into the setting of elementary analysis.

We start by showing that each space has the same elementary invariant as some space of a special kind, which will be easier to work with. Given $r \in (0,1]$, set

$$\mathscr{S}_r \;:=\; \{A \in \mathscr{S} \mid A \subseteq [0,r]\}.$$

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a probability space. For convenience, we define

$$\begin{aligned} \mathrm{h}(\mathscr{P}) \;&:=\; \mathsf{P}_{\approx}(\mathrm{d}_{\mathscr{P}}) \\ &=\; \textstyle\sum_{r \in (0,1]} r \cdot \sharp_{\mathscr{P}}(r). \end{aligned}$$

So $\mathrm{h}(\mathscr{P})$ is the measure of the atomic part of $\mathscr{P}$. Call $\mathscr{P}$ *special* if there are an initial segment I of $\mathbb{N}_+$ (i.e. of the positive integers) and a function f from I to $(0,1]$ such that:

- $\mathscr{A}$ is the least σ-algebra containing $\{\{i\} \mid i \in I\} \cup \mathscr{S}_{1-\mathrm{h}(\mathscr{P})}$;
- for any $S^+ \subseteq I$ and $S^- \in \mathscr{S}_{1-\mathrm{h}(\mathscr{P})}$,

  $$\mathsf{P}\left(S^+ \cup S^-\right) \;=\; \sum_{i \in S^+} f(i) + \mathsf{L}\left(S^-\right)$$

  where the empty sum is identified with zero.

Obviously, $\mathscr{L}$ is special, with $I = f = \varnothing$. In general, every special space is a convex combination of a discrete spaces and $\mathscr{L}$. Whenever $\mathscr{P}$ is special, I and f are uniquely determined by $\mathscr{P}$:

- I equals $\{1, \ldots, |\mathrm{D}_{\mathscr{P}}|\}$ if $\mathrm{D}_{\mathscr{P}}$ is finite, and $\mathbb{N}_+$ otherwise;
- f is the function from I to $(0,1]$ that maps each $i \in I$ to $\mathsf{P}(\{i\})$.

Moreover, the collection of all atoms of $\mathscr{A}$ then coincides with $\{\{i\} \mid i \in I\}$, and hence

$$\mathrm{h}(\mathscr{P}) \;=\; \sum_{i \in I} f(i).$$

On the other hand, for every partial function f from $\mathbb{N}_+$ to $(0,1]$, if $\mathrm{dom}\, f$ is an initial segment of $\mathbb{N}_+$ and $\sum_{i \in I} f(i) \leqslant 1$, then f determines a unique special space.

Proposition 7.1

For every space there exists a special space with the same elementary invariant.[5]

Proof. Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a space. Take

$$I := \begin{cases} \{1, \dots, |\mathrm{D}_{\mathscr{P}}|\} & \text{if } \mathrm{D}_{\mathscr{P}} \text{ is finite} \\ \mathbb{N}_+ & \text{otherwise.} \end{cases}$$

So there exists a one-one function g from I onto $\mathrm{D}_{\mathscr{P}}$. Define f to be $g \circ \mathsf{P}_{\approx}$ — i.e. f is the function from I to $(0, 1]$ given by

$$f(i) := \mathsf{P}_{\approx}(g(i)).$$

Since $\operatorname{range} f \subseteq (0, 1]$ and $\sum_{i \in I} f(i) \leqslant 1$, we conclude that f determines a (unique) special space. It is easy to see that this space has the same elementary invariant as $\mathscr{P}$. $\square$

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a special space; take f to be the corresponding partial function from $\mathbb{N}_+$ to $(0, 1]$ and I to be $\operatorname{dom} f$. For convenience, given $A \in \mathscr{A}$, we define

$$A^{\circ} := A \cap I \quad \text{and} \quad A^{\bullet} := A \cap [0, 1 - \mathrm{h}(\mathscr{P})].$$

Intuitively, A° is the atomic (discrete) part of A and $A^{\bullet}$ is its atomless part. So we have

$$\mathsf{P}(A^{\circ}) = \sum_{i \in A^{\circ}} f(i) \quad \text{and} \quad \mathsf{P}(A^{\bullet}) = \mathsf{L}(A^{\bullet}).$$

Call a Boolean valuation ζ in $\mathscr{P}$ *compact* if for each basic conjunction ϕ over $\operatorname{dom} \zeta$, $(\zeta(\phi))^{\bullet}$ is a subinterval of $[0, 1]$. Clearly, this extends the definition of compact Boolean valuation in $\mathscr{L}$ given in Section 4. We can then generalize Lemma 4.5 as follows.

Lemma 7.2

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a special probability space, S be a finite set of Boolean variables, ζ be a compact function from S to $\mathscr{A}$, and X be a Boolean variable that lies outside S. Then for every $A \in \mathscr{A}$ there exists $B \in \mathscr{A}$ such that ζ_B^X is compact and congruent to ζ_A^X.

Proof. Consider an arbitrary $A \in \mathscr{A}$. For each basic conjunction ϕ over S, denote the infimum of $(\zeta(\phi))^{\bullet}$ in $\mathbb{R}$ by a_{ϕ}, and set

$$C_{\phi} := (\zeta(\phi))^{\bullet} \cap \left[0, a_{\phi} + \mathsf{L}\left(\left(\zeta_A^X(\phi \wedge X)\right)^{\bullet}\right)\right]$$

— so in particular, $C_{\phi} \in \mathscr{S}_{1 - \mathrm{h}(\mathscr{P})}$ and $\mathsf{L}(C_{\phi}) = \mathsf{L}\left(\left(\zeta_A^X(\phi \wedge X)\right)^{\bullet}\right)$. Take

$$B := A^{\circ} \cup \bigcup \{C_{\phi} \mid \phi \text{ is a basic conjunction over } S\}.$$

It is easy to see that ζ_B^X has the required properties. $\square$

[5]Furthermore, this special space will be unique if we require that the corresponding f be non-increasing.

Corollary 7.3

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a special space, S be a finite set of Boolean variables and ζ be a function from S to $\mathscr{A}$. Then there exists a compact function ξ from S to $\mathscr{A}$ such that ζ and ξ are congruent.

Proof. This can easily be shown using Lemmas 6.2 and 7.2, by induction on $|S|$. $\square$

Fix a special probability space $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$. For any normal QPL-formula Φ, compact function ζ from $\mathrm{FV}(\Phi)$ to $\mathscr{A}$ and function ι from $\mathrm{Fv}(\Phi)$ to $\mathbb{R}$, the definition of $\mathscr{P} \rhd \Phi[\zeta, \iota]$ is like that of $\mathscr{L} \rhd \Phi[\zeta, \iota]$ except that we use the extended notion of compact Boolean valuation. Again, it turns out that the alternative semantics is equivalent to the usual one:

Proposition 7.4

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a special space, and Φ be a normal QPL-formula. Then for any compact function ζ from $\mathrm{FV}(\Phi)$ to $\mathscr{A}$ and function ι from $\mathrm{Fv}(\Phi)$ to $\mathbb{R}$,

$$\mathscr{P} \Vdash \Phi[\zeta, \iota] \quad \Longleftrightarrow \quad \mathscr{P} \rhd \Phi[\zeta, \iota].$$

Proof. The argument is like that for Proposition 4.7 except that we replace 'similar' by 'congruent', and use Lemma 7.2 instead of Lemma 4.5 and Proposition 6.4 instead of Proposition 4.4. $\square$

Finally, we are ready to adapt the translation τ described in Section 4 to deal with spaces other than $\mathscr{L}$. The output language will be that of elementary analysis, which contains not only quantifiers over reals but also quantifiers over natural numbers. This language has the same expressive power as that of second-order arithmetic, and furthermore, sequences of reals and the concept of limit are representable in it; see, e.g., [9, Chapter 16] and [11, Chapter II]. We shall write $\mathfrak{R}^2$ for the standard model of elementary analysis.

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a special space. So $\mathscr{P}$ is determined by a suitable f; take I to be $\operatorname{dom} f$. To work with the atomic part of $\mathscr{P}$, we introduce the list

$$\overline{\mathfrak{p}} \ := \ \langle \mathfrak{p}_i \rangle_{i \in I}$$

of fresh field variables. Intuitively, we associate with each $\mathfrak{p}_i$ the real number $f(i)$. If $I = \mathbb{N}_+$, and one wants to avoid using infinitely many variables, then $\overline{\mathfrak{p}}$ may be replaced by a function variable ranging over functions from $\mathbb{N}_+$ to $\mathbb{R}$. Now consider a normal QPL-sentence Φ. The notation of Section 4 will be adopted to handle the atomless part of our coding. As for its atomic part, given a Boolean variable X occurring in Φ, we set

$$\overline{X} \ := \ \langle e_{X,i} \rangle_{i \in I}$$

where the $e_{X,i}$'s are fresh field variables. The intuition is that each $e_{X,i}$ corresponds to

$$\begin{cases} \{i\} & \text{if } e_{X,i} = 1 \\ \varnothing & \text{otherwise.} \end{cases}$$

Now let S be a finite set of variables. We use $\overline{S}$ to abbreviate $\langle \overline{S} \rangle_{X \in S}$. Roughly speaking, $\overline{S}$ and $\underline{S}$ determine the function ζ from S to $\mathscr{A}$ given by

$$\zeta\left(X\right) \;:=\; \bigcup_{i=1}^{\ell(X)} A_{X,i} \cup \{i \in I \mid e_{X,i} = 1\}.$$

— cf. Section 4. For each function δ from S to $\{0,1\}$ we introduce the formulas

$$\begin{aligned}
\mathtt{E}^{\bullet}_{\delta}\left(x, \underline{S}, \overline{\mathfrak{p}}\right) \;&:=\; \text{the result of replacing } 0 \leqslant x \leqslant 1 \text{ in } \mathtt{E}_{\delta}\left(x, \underline{S}\right) \text{ by } 0 \leqslant x \leqslant 1 - \textstyle\sum_{i \in I} \mathfrak{p}_i,\\
\mathtt{I}^{\bullet}_{\delta}\left(a, b, \underline{S}, \overline{\mathfrak{p}}\right) \;&:=\; \text{the result of replacing } \mathtt{E}_{\delta}\left(x, \underline{S}\right) \text{ in } \mathtt{I}_{\delta}\left(a, b, \underline{S}\right) \text{ by } \mathtt{E}^{\bullet}_{\delta}\left(x, \underline{S}, \overline{\mathfrak{p}}\right).
\end{aligned}$$

Intuitively, if $\underline{S}$ and $\overline{S}$ determine ζ, then

$$\begin{aligned}
\mathtt{E}^{\bullet}_{\delta}\left(x, \underline{S}, \overline{\mathfrak{p}}\right) \;&\text{says}\; \text{`}x \text{ belongs to } \left(\zeta\left(\textstyle\bigwedge^{\delta} S\right)\right)^{\bullet}\text{'},\\
\mathtt{I}^{\bullet}_{\delta}\left(a, b, \underline{S}, \overline{\mathfrak{p}}\right) \;&\text{says}\; \text{`}\left(\zeta\left(\textstyle\bigwedge^{\delta} S\right)\right)^{\bullet} \text{ is a subinterval of } [0,1] \text{ with endpoints } a \text{ and } b\text{'}.
\end{aligned}$$

Therefore the condition 'ζ is compact' can be expressed by

$$\mathtt{C}^{\bullet}\left(\underline{S}\right) \;:=\; \textstyle\bigwedge_{\delta \in \{0,1\}^S} \exists a\, \exists b\, \mathtt{I}^{\bullet}_{\delta}\left(a, b, \underline{S}\right).$$

In addition, we introduce the term

$$\mathtt{t}_{\delta}\left(\overline{S}, \overline{\mathfrak{p}}\right) \;:=\; \sum_{i \in I} \Big(\mathfrak{p}_i \cdot \textstyle\prod_{X \in S} \left(\delta\left(X\right) \cdot e_{X,i} + \left(1 - \delta\left(X\right)\right) \cdot \left(1 - e_{X,i}\right)\right)\Big).$$

It is easy to verify that

$$\mathtt{t}_{\delta}\left(\overline{S}, \overline{\mathfrak{p}}\right) \;\text{ represents }\; \mathsf{P}\left(\left(\zeta\left(\textstyle\bigwedge^{\delta} S\right)\right)^{\circ}\right),$$

provided that the $e_{X,i}$'s take their values in $\{0,1\}$. Moreover, for each $X \in S$, to ensure that the field variables in $\underline{X}$ and $\overline{X}$ behave as expected, we need

$$\begin{aligned}
\mathtt{Q}\left(\underline{X}, \overline{X}, \overline{\mathfrak{p}}\right) \;:=\; &\textstyle\bigwedge_{n=1}^{\ell(X)} 0 \leqslant a_{X,n} \leqslant b_{X,n} \leqslant 1 - \displaystyle\sum_{i \in I} \mathfrak{p}_i \wedge\\
&\bigwedge_{i \in I} \left(e_{X,i} = 0 \vee e_{X,i} = 1\right).
\end{aligned}$$

And of course, $\overline{\mathfrak{p}}$ must be such that

$$\mathtt{A}\left(\overline{\mathfrak{p}}\right) \;:=\; \textstyle\bigwedge_{i \in I} 0 < \mathfrak{p}_i \leqslant 1 \wedge \displaystyle\sum_{i \in I} \mathfrak{p}_i \leqslant 1.$$

We shall call a list $\langle r_i \rangle_{i \in I}$ of reals *acceptable* if it satisfies $\mathtt{A}\left(\overline{\mathfrak{p}}\right)$ in $\mathfrak{R}^2$. Obviously, $\overline{f} = \langle f\left(i\right) \rangle_{i \in I}$ is acceptable. Using these first-order formulas, Φ can be translated into $\tau_I\left(\Phi\right)$ as follows.[6]

[6]In our translation, I is used explicitly, while f plays the role of an external parameter.

- Letting Ψ be the quantifier-free part of Φ, define

$$\Psi_I^* \ := \ \exists a_1 \, \exists b_1 \, \ldots \, \exists a_{2^{|S|}} \, \exists b_{2^{|S|}} \left(\bigwedge_{i=1}^{2^{|S|}} \mathtt{I}^\bullet_{\delta_i} \left(a_i, b_i, \underline{S}, \overline{\mathfrak{p}}\right) \wedge \overline{\underline{\Psi}} \right)$$

where $S = \mathrm{FV}(\Psi)$, $\{\delta_1, \ldots, \delta_{2^{|S|}}\} = \{0,1\}^S$, and $\overline{\underline{\Psi}}$ is the result of replacing each μ-term $\mu\left(\bigwedge^{\delta_i} S\right)$ by $(b_i - a_i) + \mathtt{t}_{\delta_i}\left(\overline{S}, \overline{\mathfrak{p}}\right)$.

- Next, for each subformula Θ of Φ beginning with a quantifier, we recursively define

$$\Theta_I^* \ := \ \begin{cases} \exists \underline{X} \, \exists \overline{X} \left(\mathtt{Q}\left(\underline{X}, \overline{X}, \overline{\mathfrak{p}}\right) \wedge \mathtt{C}^\bullet\left(\underline{S}\right) \wedge \Omega_I^* \right) & \text{if } \Theta = \exists X \, \Omega \\ \forall \underline{X} \, \forall \overline{X} \left(\mathtt{Q}\left(\underline{X}, \overline{X}, \overline{\mathfrak{p}}\right) \wedge \mathtt{C}^\bullet\left(\underline{S}\right) \to \Omega_I^* \right) & \text{if } \Theta = \forall X \, \Omega \\ \exists x \, \Omega_I^* & \text{if } \Theta = \exists x \, \Omega \\ \forall x \, \Omega_I^* & \text{if } \Theta = \forall x \, \Omega \end{cases}$$

where $S = \mathrm{FV}(\Psi)$. Now set $\tau_I(\Phi)$ to be Φ_I^*.

Evidently, the variables that occur free in $\tau_I(\Phi)$ are the $\mathfrak{p}_i$'s. It is straightforward to verify that the following generalization of Proposition 4.8 holds.

Proposition 7.5

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a special space, f and I be as above. Then for every normal QPL*-sentence Φ,*

$$\mathscr{P} \rhd \Phi \quad \Longleftrightarrow \quad \tau_I(\Phi)\left[\overline{\mathfrak{p}}/\overline{f}\right] \text{ is true in } \mathfrak{R}^2$$

where $\tau_I(\Phi)\left[\overline{\mathfrak{p}}/\overline{f}\right]$ is obtained from $\tau_I(\Phi)$ by substituting $\overline{f}$ for $\overline{\mathfrak{p}}$.

Proof. The argument is analogous to that for Proposition 4.8, via the extended definition of $\rhd$. $\square$

In the next section we shall apply Proposition 7.5 to derive further decidability results and also to solve one of the main problems of [16].

8 Some further applications

For each $N \in \mathbb{N}$, take

$$\mathcal{K}_N \ := \ \{\mathscr{P} \mid \mathscr{P} \text{ is a probability space and } |\mathrm{D}_{\mathscr{P}}| = N\}.$$

Thus $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ belongs to $\mathcal{K}_N$ iff $\mathscr{A}_\approx$ contains exactly N atoms.

Theorem 8.1

$\mathrm{Th}(\mathcal{K}_N)$ *is decidable, for any $N \in \mathbb{N}$.*

Proof. Take I to be $\{1, \ldots, N\}$. Given a QPL-formula Φ, define Φ' as in the proof of Theorem 4.1. Now for every QPL-sentence Φ,

$$\begin{aligned}
\Phi \in \mathrm{Th}(\mathcal{K}_N) &\iff \Phi' \in \mathrm{Th}(\mathcal{K}_N) \\
&\overset{7.1,\ 6.5}{\iff} \mathscr{P} \Vdash \Phi' \text{ for all special } \mathscr{P} \in \mathcal{K}_N \\
&\overset{7.4}{\iff} \mathscr{P} \rhd \Phi' \text{ for all special } \mathscr{P} \in \mathcal{K}_N \\
&\overset{7.5}{\iff} \tau_I(\Phi') \text{ is true in } \mathfrak{R}^2 \text{ for all acceptable values of } \overline{\mathfrak{p}} \\
&\iff \forall \overline{\mathfrak{p}}\, (\mathtt{A}(\overline{\mathfrak{p}}) \to \tau_I(\Phi')) \text{ is true in } \mathfrak{R}^2.
\end{aligned}$$

Clearly, since I is finite, we can treat $\forall \overline{\mathfrak{p}}\, (\mathtt{A}(\overline{\mathfrak{p}}) \to \tau_I(\Phi'))$ as a sentence in the first-order language of $\mathfrak{R}$. Hence the QPL-theory of $\mathcal{K}_N$ is reducible to the first-order theory of $\mathfrak{R}$. □

Here is another easy application. We call a space $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ *algebraic* iff for every $A \in \mathrm{D}_{\mathscr{P}}$, $\mathsf{P}_{\approx}(A)$ is an algebraic real number.

Proposition 8.2

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be an algebraic space in $\mathcal{K}_N$, where $N \in \mathbb{N}$. Then $\mathrm{Th}(\mathscr{P})$ is decidable.

Proof. By Proposition 7.1 and Theorem 6.5, we may assume $\mathscr{P}$ is special. So $\mathscr{P}$ is determined by a suitable f; take I to be $\operatorname{dom} f$, which coincides with $\{1, \ldots, N\}$.

Observe that for each $i \in I$, since $\mathsf{P}(f(i))$ is an algebraic real number, there are a polynomial $g_i(x)$ with integer coefficients and rational numbers p_i, q_i such that for every $r \in \mathbb{R}$,

$$r = \mathsf{P}(f(i)) \iff g_i(r) = 0 \quad \text{and} \quad p_i < r < q_i.$$

Notice that each $p_i < \mathfrak{p}_i < q_i$ can be treated as a formula in the first-order language of $\mathfrak{R}$, because all rational numbers are definable in $\mathfrak{R}$. So the formula

$$\mathtt{B}(\overline{\mathfrak{p}}) := \bigwedge\nolimits_{i \in I} (g_i(\mathfrak{p}_i) = 0 \wedge p_i < \mathfrak{p}_i < q_i)$$

defines $\overline{f}$ in $\mathfrak{R}$. Then by an argument similar to that for Theorem 8.1, we have that for every QPL-sentence Φ,

$$\mathscr{P} \Vdash \Phi \iff \forall \overline{\mathfrak{p}}\, (\mathtt{B}(\overline{\mathfrak{p}}) \to \tau_I(\Phi')) \text{ is true in } \mathfrak{R}.$$

Hence the QPL-theory of $\mathscr{P}$ is decidable. □

Remark 8.3. On the other hand, as was proved in [16, Section 2], it is not hard to build an infinite discrete space $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ such that:

- for every $A \in \mathrm{D}_{\mathscr{P}}$, $\mathsf{P}_{\approx}(A)$ is a rational number;
- complete second-order arithmetic is reducible to $\mathrm{Th}^{\mathrm{e}}(\mathscr{P})$.

So in Proposition 8.2, the condition that $|\mathrm{D}_{\mathscr{P}}| < \infty$ cannot, in general, be weakened.

As was shown in [16, Section 2], there are many QPL-theories that are at least as complex as complete second-order arithmetic — or equivalently, the theory of $\mathfrak{R}^2$. We are going to prove that this upper bound is precise for a rich variety of classes of spaces. Given a space $\mathscr{P}$, define

$$[\mathscr{P}] \;:=\; \{\mathscr{P}' \mid \sharp_{\mathscr{P}'} = \sharp_{\mathscr{P}} \text{ and } \mathscr{P}' \text{ is special}\}.$$

By Proposition 7.1, $[\mathscr{P}]$ is non-empty, and therefore $\mathrm{Th}\,(\mathscr{P}) = \mathrm{Th}\,([\mathscr{P}])$ by Theorem 6.5. We also set

$$[\![\mathscr{P}]\!] \;:=\; \{f \mid f \text{ determines some } \mathscr{P}' \in [\mathscr{P}]\}\,.$$

Elements of $[\![\mathscr{P}]\!]$ may be thought of as codes for $\mathscr{P}$. We extend $[\cdot]$ and $[\![\cdot]\!]$ to classes of probability spaces as follows:

$$[\mathcal{K}] \;:=\; \bigcup_{\mathscr{P}\in\mathcal{K}} [\mathscr{P}] \quad \text{and} \quad [\![\mathcal{K}]\!] \;:=\; \bigcup_{\mathscr{P}\in\mathcal{K}} [\![\mathscr{P}]\!].$$

Notice that $[\mathcal{K}]$ has the same QPL-theory as $\mathcal{K}$:

$$\mathrm{Th}\,(\mathcal{K}) \;=\; \bigcap_{\mathscr{P}\in\mathcal{K}} \mathrm{Th}\,(\mathscr{P}) \;=\; \bigcap_{\mathscr{P}\in\mathcal{K}} \mathrm{Th}\,([\mathscr{P}]) \;=\; \mathrm{Th}\,([\mathcal{K}]).$$

We say that a class $\mathcal{K}$ of spaces is *analytical* if $[\![\mathcal{K}]\!]$ is definable in $\mathfrak{R}^2$ (as a set of partial functions from $\mathbb{N}_+$ to $(0,1]$).

Theorem 8.4

Let $\mathcal{K}$ be an analytical class of spaces. Then $\mathrm{Th}\,(\mathcal{K})$ is reducible to complete second-order arithmetic.

Proof. The translation described in Section 7 can easily be modified in such a way that I becomes a set variable ranging over initial segments of $\mathbb{N}_+$, and every list of field variables indexed by I — such as $\overline{\mathfrak{p}}$ or an $\overline{X}$ — is treated as a function variable ranging over functions from I to $\mathbb{R}$. For the modified version we shall write $\tau\,(I,\Phi)$ instead of $\tau_I\,(\Phi)$.

Take $\Theta\,(F)$ to be a formula that defines $[\![\mathcal{K}]\!]$ in $\mathfrak{R}^2$, where F is a function variable ranging over partial functions from $\mathbb{N}_+$ to $\mathbb{R}$. Given a QPL-formula Φ, define Φ' as in the proof of Theorem 4.1. Then for every QPL-sentence Φ,

$$\begin{aligned}
\Phi \;\in\; \mathrm{Th}\,(\mathcal{K}) \quad &\iff \quad \Phi' \;\in\; \mathrm{Th}\,(\mathcal{K}) \\
&\iff \quad \Phi' \;\in\; \mathrm{Th}\,([\mathcal{K}]) \\
&\overset{7.4}{\iff} \quad \mathscr{P} \rhd \Phi' \;\text{ for all } \mathscr{P} \in [\mathcal{K}] \\
&\overset{7.5}{\iff} \quad \tau\,(I,\Phi')\,[\overline{\mathfrak{p}}/f, I/\mathrm{dom}\,f] \;\text{ is true in }\; \mathfrak{R}^2 \;\text{ for all }\; f \in [\![\mathcal{K}]\!] \\
&\iff \quad \forall I\,\forall\overline{\mathfrak{p}}\,(\Theta\,(\overline{\mathfrak{p}}) \to \tau\,(I,\Phi')) \;\text{ is true in }\; \mathfrak{R}^2.
\end{aligned}$$

Therefore $\mathrm{Th}\,(\mathcal{K})$ is reducible to the theory of $\mathfrak{R}^2$, which is equivalent to complete second-order arithmetic. □

Remark 8.5. The analogue of Theorem 8.4 for analytical classes of discrete spaces can be proved in a much more direct way, since discrete spaces — unlike arbitrary ones — can be easily encoded in the language of second-order arithmetic; cf. the proof of [16, Lemma 2.2].

Combining this with a complexity result from [16, Section 2], we get:

Theorem 8.6

Let $\mathcal{K}$ be an analytical class of spaces. Suppose that $\mathcal{K}$ contains all infinite discrete spaces. Then both $\mathrm{Th}^{\mathrm{e}}(\mathcal{K})$ *and* $\mathrm{Th}(\mathcal{K})$ *are equivalent to complete second-order arithmetic.*

Proof. Immediate from Theorems 2.2 and 8.4. □

In particular, Theorem 8.6 applies to the class of all spaces and to the class of all infinite spaces, which solves one of the main problems of [16].

Remark 8.7. One may try to prove Theorem 8.4 in a somewhat different way, by identifying each space with its elementary invariant (which is justified by Theorem 6.5) and directly translating $\Vdash$ into the language of $\mathfrak{R}^2$, without using the alternative semantics. But even if we succeed in doing this, quantifiers over $\mathscr{S}$ (and its subalgebras) will be translated as set quantifiers; thus we shall not be able to obtain decidability results like Theorem 8.1 and Proposition 8.2. Therefore the notion of compact valuation and the relation $\rhd$ are essential here.

Acknowledgments. This work was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2022-265).

References

[1] M. Abadi and J. Y. Halpern, “Decidability and expressiveness for first-order logics of probability,” *Information and Computation* 112(1), 1–36, 1994.

[2] Yu. L. Ershov, I. A. Lavrov, A. D. Taimanov, M. A. Taitslin. Elementary theories, *Russian Mathematical Surveys* 20(4), 35–105, 1965.

[3] R. Fagin, J. Y. Halpern, N. Megiddo. A logic for reasoning about probabilities. *Information and Computation* 87(1–2), 78–128, 1990.

[4] D. Ibeling, T. Icard, K. Mierzewski, M. Mossé. Probing the quantitative–qualitative divide in probabilistic reasoning. *Annals of Pure and Applied Logic*, 103339, 2024. In Press.

[5] S. Koppelberg. General theory of Boolean algebras. In J. D. Monk (ed., with R. Bonnet), *Handbook of Boolean Algebras*, Vol. 1, pp. 1–311. North-Holland, 1989.

[6] A. Nies. Undecidable fragments of elementary theories. *Algebra Universalis* 35(1), 8–33, 1996.

[7] Z. Ognjanović, A. Ilić-Stepić. Logics with probability operators. In Z. Ognjanović (ed.), *Probabilistic Extensions of Various Logical Systems*, pp. 1–35. Springer, 2020.

[8] A. Perović, Z. Ognjanović, M. Rašković, Z. Marković. A probabilistic logic with polynomial weight formulas. In S. Hartmann, G. Kern-Isberner (eds.), *Foundations of Information and Knowledge Systems 2008*, LNCS 4932, pp. 239–252. Springer, 2008.

[9] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill Book Company, 1967.

[10] W. Sierpiński. Sur les fonctions d'ensemble additives et continues. *Fundamenta Mathematicae* 3(1), 240–246, 1922. (In French.)

[11] S. G. Simpson. *Subsystems of Second Order Arithmetic.* Second edition. Cambridge University Press, 2009.

[12] R. M. Solovay, R. D. Arthan, J. Harrison. Some new results on decidability for elementary algebra and geometry. *Annals of Pure and Applied Logic* 163(12), 1765–1802, 2012.

[13] S. O. Speranski. Complexity for probability logic with quantifiers over propositions. *Journal of Logic and Computation* 23(5), 1035–1055, 2013.

[14] S. O. Speranski. A note on definability in fragments of arithmetic with free unary predicates. *Archive for Mathematical Logic* 52 (5–6), 507–516, 2013.

[15] S. O. Speranski. Some new results in monadic second-order arithmetic. *Computability* 4(2), 159–174, 2015.

[16] S. O. Speranski. Quantifying over events in probability logic: an introduction. *Mathematical Structures in Computer Science* 27(8), 1581–1600, 2017.

[17] A. Tarski. *A Decision Method for Elementary Algebra and Geometry.* University of California Press, 1951.

[18] A. Tarski, A. Mostowski, R. Robinson. *Undecidable Theories.* North-Holland, 1953.

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Appendix

Proof of Proposition 5.1. If $\mathsf{P}\,(A) = 0$, then the result is trivial. Suppose A has positive measure, i.e. $\mathsf{P}\,(A) > 0$. Take $\mathscr{A}_A$ to be $\{B \in \mathscr{A} \mid B \leqslant A\}$, and define $\mathsf{P}_A : \mathscr{A}_A \to [0,1]$ by

$$\mathsf{P}_A\,(B) \;:=\; \frac{\mathsf{P}\,(B)}{\mathsf{P}\,(A)}.$$

Evidently, $\mathscr{P}_A = \langle \mathscr{A}_A, \mathsf{P}_A \rangle$ is also an atomless space, and it suffices to show that

$$\{\mathsf{P}_A(B) \mid B \in \mathscr{A}_A\} \;=\; [0,1].$$

To sum up, the original formulation of Proposition 5.1 is equivalent to the following one: for each atomless space $\mathscr{P}$,

$$\{\mathsf{P}(A) \mid A \in \mathscr{A}\} \;=\; [0,1].$$

Before proving this, let us make two observations:

1. if $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ is an atomless space, and $A \in \mathscr{A}$ has positive measure, then for every $n \in \mathbb{N}$ there exists $B \in \mathscr{A}$ such that $B \subseteq A$ and $0 < \mathsf{P}(B) \leqslant 1/2^n$;
2. if S is an uncountable set, and f is a function from S to $\mathbb{R}_+$ (i.e. to the set of all positive real numbers), then there exists a countable $U \subseteq S$ such that $\sum_{s \in U} f(s) = \infty$.

Here (1) can be easily proved by induction on n, and (2) is a basic fact from analysis.

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be an atomless space. Consider an arbitrary $r \in [0,1]$. We aim to find $A \in \mathscr{A}$ such that $\mathsf{P}(A) = r$. The cases where r is 0 or 1 are trivial. Suppose $0 < r < 1$. Then we define $\rho : \mathrm{Ord} \to \mathscr{A}$ by transfinite recursion as follows.[7]

- If α is at most countable, and $\mathsf{P}\left(\bigvee_{\beta<\alpha} \rho(\beta)\right) = r$, then set $\rho(\alpha) := \bot$.
- If α is at most countable, and $\mathsf{P}\left(\bigvee_{\beta<\alpha} \rho(\beta)\right) < r$, then define $\rho(\alpha)$ to be some $B \in \mathscr{A}$ such that
  $$B \wedge \bigvee_{\beta<\alpha} \rho(\beta) \;=\; \varnothing \quad \text{and} \quad 0 < \mathsf{P}(B) \;\leqslant\; r - \mathsf{P}\left(\bigvee_{\beta<\alpha} \rho(\beta)\right),$$
  whose existence is guaranteed by (1).
- If α is uncountable, then set $\rho(\alpha) := \bot$.

Notice that for each finite or countable ordinal α, the members of $\langle \rho(\beta) \rangle_{\beta<\alpha}$ are pairwise disjoint, and their sum is less than or equal to r. Now take

$$S \;:=\; \{\alpha \in \mathrm{Ord} \mid \rho(\alpha) \neq \bot\}.$$

Obviously, we have $\mathsf{P}(\rho(\alpha)) > 0$ for all $\alpha \in S$. So by (2), S must be at most countable (since one may define the function f from S to $\mathbb{R}_+$ by $f(\alpha) = \mathsf{P}(\rho(\alpha))$. Moreover, S is an initial segment of Ord, and hence an ordinal. Finally, we observe that

$$\mathsf{P}\left(\bigvee_{\alpha \in S} \rho(\alpha)\right) \;=\; r,$$

for otherwise $\rho(S)$ would not be equal to $\bot$, i.e. S would belong to S. $\square$

[7] Here Ord denotes the class of all ordinals.