On the decision problem
for quantified probability logics

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Abstract

Let $\mathsf{QPL}^{\mathrm{e}}$ expand the quantifier-free 'polynomial' probability logic of [4] by adding quantifiers over arbitrary events; it can be viewed as a one-sorted elementary language for reasoning about probability spaces. We prove that the Σ_2-fragment of the $\mathsf{QPL}^{\mathrm{e}}$-theory of finite spaces is hereditarily undecidable. By earlier observations, this implies that Π_2 is the maximal decidable prefix fragment of $\mathsf{QPL}^{\mathrm{e}}$. Moreover, we obtain similar results for two natural one-sorted logics of probability that emerge from [1].

Keywords: probability logic, decidability, prefix fragments, elementary theories

1 Introduction

The decision problem for (classical) first-order logic can be stated as: given a signature σ, let Val_σ be the set of all valid first-order σ-sentences; we want to know which *prefix fragments* of Val_σ are decidable. Here each prefix fragment is denoted by a suitable element of

$$\{\Sigma_n, \Pi_n \mid n \in \mathbb{N}\}.$$

Recall that a first-order σ-formula is Σ_n iff it has the form

$$\underbrace{\exists \vec{x}_1 \, \forall \vec{x}_2 \, \ldots \, \vec{x}_n}_{n-1 \text{ alternations}} \Psi$$

where $\vec{x}_1, \ldots, \vec{x}_n$ are tuples of variables and Ψ contains no quantifiers; the definition of Π_n results by interchanging $\exists$ and $\forall$.[1] By using dummy quantifiers, we may order the prefix fragments in the natural way:

$$\Sigma_0 \longrightarrow \Sigma_1 \longrightarrow \Sigma_2 \qquad \cdots$$
$$\Pi_0 \longrightarrow \Pi_1 \longrightarrow \Pi_2 \qquad \cdots$$

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[1] So in particular, both Σ_0 and Π_0 correspond to the quantifier-free formulas.

Thus the above problem can be restated as: given a signature σ, find the minimal undecidable — or alternatively, maximal decidable — prefix fragments of Val_σ. Considerable effort has been devoted to this problem, leading to a complete solution; see [2] for more information.

Of course, similar problems arise for other quantified logics. Roughly, if $\mathscr{L}$ is a language with one sort of variables and two standard quantifiers, i.e. $\forall$ and $\exists$, then the decision problem for $\mathscr{L}$ is stated in a way similar to that for first-order language.[2] In this article, we shall be concerned with the decision problem for quantified probability logics.

In particular, we shall consider a one-sorted elementary language for reasoning about probability spaces, denoted by QPL^e, which can be viewed as obtained from the well-known 'polynomial' language from [4, Section 5] by adding quantifiers over arbitrary events. As was proved in [13] (cf. [11]), the validity problem for Π_2-QPL^e-sentences is decidable, but that for Π_3-QPL^e-sentences is not. We are going to improve this result by replacing Π_3 by Σ_2 — which yields the desired solution of the decision problem for QPL^e.

Furthermore, using the technique developed for QPL^e, we shall solve the decision problems for two other languages, denoted by $\mathscr{L}_1$ and $\mathscr{L}_2$, which are natural one-sorted fragments of the 'first-order' languages studied in [1]. Both of these include quantifiers over elements of a given domain; however, $\mathscr{L}_1$ uses probability distributions on the domain, while $\mathscr{L}_2$ utilizes external distributions on an additional set of possible worlds, each of which can be identified with a first-order structure. It should also be remarked that unlike QPL^e, these depend on the choice of a signature ς.[3] Briefly, we aim to show that for each $i \in \{1, 2\}$ and any sufficiently rich ς, the minimal undecidable prefix fragment of $\mathscr{L}_i(\varsigma)$ is Σ_2 — so the maximal decidable one is Π_2. Here 'sufficiently rich' agrees with the richness conditions of the complexity results in [1].

In effect, we shall prove more than what is mentioned above. For instance, the Σ_2-fragment of the QPL^e-theory of finite probability spaces turns out to be hereditarily undecidable; similarly for $\mathscr{L}_1$ and $\mathscr{L}_2$. Also $\mathscr{L}_1$ and $\mathscr{L}_2$ have the finite model property for Π_2-sentences.

2 Quantifying over events

By a *probability space*, or simply a *space*, we mean a pair $\langle \mathscr{A}, \mathsf{P} \rangle$ where:

- $\mathscr{A}$ is a sigma-algebra, i.e. a Boolean algebra in which every countable set of elements has a supremum (and hence an infimum);

- P is a *probability measure* on $\mathscr{A}$, i.e. a function from $\mathscr{A}$ to $[0, 1]$ such that for every sequence $A_0, A_1, \ldots$ of pairwise disjoint elements of $\mathscr{A}$,

$$\mathsf{P}\left(\bigvee_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mathsf{P}(A_n),$$

and $\mathsf{P}(\top) = 1$ where $\top$ denotes the greatest element of $\mathscr{A}$.

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[2]For convenience, $\mathscr{L}$ will occasionally be identified with the collection of all valid $\mathscr{L}$-sentences.

[3]Following [1], we shall focus on predicate and constant symbols.

Elements of $\mathscr{A}$ are called *events*, which are measurable with respect to P. Note, in passing, that the countable additivity of P implies $\mathsf{P}(\bot) = 0$ where $\bot$ is the least element of $\mathscr{A}$. We shall write $\mathcal{K}_{\mathrm{all}}$ for the class of all spaces.

The formal language QPL$^{\mathrm{e}}$ includes:

- *Boolean variables* $X, Y, Z, \ldots$, which are intended to range over events;

- the symbols $\top, \bot, \wedge, \vee$ and $\neg$ of the language of Boolean algebras;

- a special symbol μ, which will be interpreted as a probability measure;

- the symbols $0, 1, +, -, \cdot$ and $\leqslant$ of the language of ordered fields;

- the quantifier symbols $\forall$ and $\exists$.

The *Boolean terms* are build up from $\bot, \top$ and the Boolean variables by use of $\wedge, \vee$ and $\neg$:

- if ϕ_1 and ϕ_2 are Boolean terms, so are $(\phi_1 \wedge \phi_2)$ and $(\phi_1 \vee \phi_2)$;

- if ϕ is a Boolean term, so is $\neg\phi$.[4]

Naturally, they represent Boolean combinations of events. By a μ-*term* we mean an expression of the form $\mu(\phi)$ where ϕ is a Boolean term. Intuitively, $\mu(\phi)$ is the probability of ϕ.

We shall use $\wedge, \vee$ and $\neg$ to denote the Boolean operations as well as the ordinary logical connectives. Since their Boolean versions will not occur outside the scope of μ, the interpretations of $\wedge, \vee$ and $\neg$ will always be clear from the context. By a *basic* QPL$^{\mathrm{e}}$-*formula* we mean an expression of the form

$$f(\mu(\phi_1), \ldots, \mu(\phi_m)) \leqslant g(\mu(\phi_{m+1}), \ldots, \mu(\phi_{m+n}))$$

where f and g are polynomials with integer coefficients, and $\phi_1, \ldots, \phi_{m+n}$ are Boolean terms. The QPL$^{\mathrm{e}}$-*formulas* are built up from the basic QPL$^{\mathrm{e}}$-formulas in the obvious way, as in classical first-order logic. We treat $t_1 = t_2$ as an abbreviation for $t_1 \leqslant t_2 \wedge t_2 \leqslant t_1$ and also adopt other standard abbreviations. Traditionally, a QPL$^{\mathrm{e}}$-*sentence* is a QPL$^{\mathrm{e}}$-formula with no free variable occurrences. Denote by $\mathrm{Sent}^{\mathrm{e}}$ the collection of all QPL$^{\mathrm{e}}$-sentences.

The satisfiability relation $\Vdash$ for QPL$^{\mathrm{e}}$ can be defined in the customary way, treating the logical connective symbols and quantifiers classically. For example, consider

$$\Theta(X) := \mu(X) \neq 0 \wedge \forall Y (\mu(X \wedge Y) \neq 0 \rightarrow \mu(X \wedge \neg Y) = 0).$$

Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be a probability space. Then for every $A \in \mathscr{A}$: $\mathscr{P} \Vdash \Theta(A)$ iff A is an atom of $\mathscr{A}$ modulo events of measure zero.

Remark 2.1. The quantifier-free fragment of QPL$^{\mathrm{e}}$ is essentially the 'polynomial' logic studied in [4, Section 5], provided that we treat variables as constant symbols.

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[4]Normal conventions for dropping brackets are in force. For instance, $\phi_1 \wedge \phi_2$ stands for $(\phi_1 \wedge \phi_2)$.

Next, a QPL^e-formula $\Phi\left(X_1, \ldots, X_n\right)$ is called *valid* if $\mathscr{P} \Vdash \Phi\left(A_1, \ldots, A_n\right)$ for every space $\mathscr{P}$ and all $A_1, \ldots, A_n$ in $\mathscr{A}$. Using Tarski's algorithm from [16], it is rather easy to obtain:

Proposition 2.2 (see [4])

The validity problem for quantifier-free QPL^e*-formulas is decidable.*[5]

Denote by Val^e the collection of all valid QPL^e-sentences. Given a class $\mathcal{K}$ of spaces, define the QPL^e-*theory of* $\mathcal{K}$ to be

$$\mathrm{Th}\left(\mathcal{K}\right) \;:=\; \{\Phi \in \mathrm{Sent}^e \mid \mathscr{P} \Vdash \Phi \text{ for every } \mathscr{P} \in \mathcal{K}\}.$$

Thus Val^e coincides with $\mathrm{Th}\left(\mathcal{K}_{\mathrm{all}}\right)$. In this paper, however, we shall deal mainly with finite probability spaces. For each $n \in \mathbb{N}$, let

$$\mathcal{K}_n \;:=\; \text{the class of all spaces with at most } n \text{ events.}$$

We write $\mathcal{K}_{\mathrm{fin}}$ for the union of $\{\mathcal{K}_n \mid n \in \mathbb{N}\}$, i.e. the class of all finite spaces. An argument similar to that for Proposition 2.2 leads to the following.

Proposition 2.3 (see [13])

$\mathrm{Th}\left(\mathcal{K}_n\right)$ *is decidable, uniformly in* n.

Finally, the notions of Σ_n- and Π_n-formula in QPL^e are analogous to those in first-order logic, as described in the Introduction. Sometimes we say that Φ is *existential* instead of 'Φ is Σ_1'. Given $\Gamma \subseteq \mathrm{Sent}^e$, take

$$\Sigma_n\text{-}\Gamma \;:=\; \{\Phi \in \Gamma \mid \Phi \text{ is } \Sigma_n\}.$$

Similarly for Π_n-Γ. Observe that the validity problem for Σ_n-QPL^e-formulas (which may contain free variables) is equivalent to that for Π_{n+1}-QPL^e-sentences, and thus corresponds to Π_{n+1}-Val^e. So Proposition 2.2 establishes the decidability of Π_1-Val^e. This can be improved:

Theorem 2.4 (see [13])

Π_2-$\mathrm{Th}\left(\mathcal{K}_{\mathrm{fin}}\right)$ *is decidable. Moreover,*

$$\Pi_2\text{-}\mathrm{Th}\left(\mathcal{K}_{\mathrm{fin}}\right) \;=\; \Pi_2\text{-}\mathrm{Th}\left(\mathcal{K}_{\mathrm{all}}\right).$$

On the other hand, as was also shown in [13], Π_3-Val^e is undecidable. But what about Σ_2? We are going to prove that Σ_2-Val^e and $\mathrm{Sent}^e \setminus \Sigma_2$-$\mathrm{Th}\left(\mathcal{K}_{\mathrm{fin}}\right)$ are computably inseparable. This implies, in particular, the undecidability of Σ_2-Val^e.

Remark 2.5. Our work may be compared to the study of prefix fragments of elementary theories in [7]; cf. also [10].

See [14] and [15] for more results about QPL^e.

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[5]In fact, [4] contains a deeper computational complexity result, stated in terms of polynomial-time reducibility; see also [8] for an adequate infinitary calculus, and [6] for related finitary calculi and further complexity results.

3 Probabilities on the domain

We are about to describe a natural one-sorted fragment $\mathscr{L}_1$ of Halpern's 'first-order' logic of probability of type 1. In brief, $\mathscr{L}_1$ is obtained from the language of [1, Section 2] by:

    i. excluding quantifiers over reals — but keeping those over elements of the domain, which are more significant;

    ii. requiring that μ can only be applied to quantifier-free first-order formulas.

In particular, (ii) implies that no nested occurrences of μ are allowed. Notice that the lower bound arguments provided in [1] agree with (ii).

Consider a (first-order) signature ς. For technical reasons, we shall restrict ourselves to signatures containing no function symbols; cf. the richness conditions in [1]. By an $\mathscr{L}_1(\varsigma)$-*structure* we mean a triple $\langle D, \pi, \mathsf{p} \rangle$ where:

- D is a non-empty set;

- π is a ς-structure, as defined in first-order logic, with domain D;

- p is a *discrete probability distribution* on D, i.e. a function from D to $[0, 1]$ such that

$$|\{d \in D \mid \mathsf{p}(d) \neq 0\}| \leqslant \aleph_0 \quad \text{and} \quad \sum_{d \in D} \mathsf{p}(d) = 1,$$

which generates the probability measure P on the powerset of D as follows:

$$\mathsf{P}(A) := \sum_{d \in A} \mathsf{p}(d).$$

Note, in passing, that given p as above and a non-zero $k \in \mathbb{N}$, we can define a discrete distribution p^k on D^k by

$$\mathsf{p}^k(d_1, \ldots, d_k) := \mathsf{p}(d_1) \cdot \ldots \cdot \mathsf{p}(d_k),$$

which generates the measure P^k on the powerset of D^k, of course. Evidently, if $A \subseteq D^k$, and A' is obtained from A by permuting some of the coordinates, then $\mathsf{P}^k(A')$ coincides with $\mathsf{P}^k(A)$.

Remark 3.1. We assume that if the equality symbol belongs to ς, then it has arity 2 and is always interpreted as the equality relation on the corresponding domain.

Let var be a countable set of *individual variables*; denote by $\mathrm{Form}_\varsigma^\circ$ the set of all quantifier-free first-order ς-formulas. Then the $\mathscr{L}_1(\varsigma)$-formulas are built up exactly as in $\mathsf{QPL}^{\mathrm{e}}$, except that:

- instead of Boolean variables, we use individual variables;

- instead of μ-terms, we employ expressions of the form $\mu_{\vec{x}}(\phi)$ where $\vec{x}$ is a non-empty tuple of individual variables, and ϕ belongs to $\mathrm{Form}_\varsigma^\circ$.

Next, consider an $\mathscr{L}_1(\varsigma)$-structure $\mathcal{M} = \langle D, \pi, \mathsf{p} \rangle$. Hence the variables are intended to range over elements of D. Given a function ζ from var to D, we interpret each $\mu_{(x_1,\ldots,x_k)}(\phi)$ as

$$\mathsf{P}^k \left(\left\{ (d_1, \ldots, d_k) \in D^k \mid \phi \text{ is true in } \pi \text{ under } \zeta^{\vec{x}}_{\vec{d}} \right\} \right)$$

where $\zeta^{\vec{x}}_{\vec{d}}$ is the function from var to D such that

$$\zeta^{\vec{x}}_{\vec{d}}(u) = \begin{cases} d_i & \text{if } u = x_i \text{ with } i \in \{1, \ldots, k\} \\ \zeta(u) & \text{otherwise.} \end{cases}$$

Now the satisfiability relation for $\mathscr{L}_1(\varsigma)$ can be defined in the natural way, like that for QPL$^{\mathrm{e}}$. We shall ambiguously denote it by $\Vdash$, since this should cause no confusion.

We can proceed as in QPL. The following notations will be used.

Notation	Meaning
$\mathscr{K}^1_{\mathrm{all}}(\varsigma)$	the class of all $\mathscr{L}_1(\varsigma)$-structures
$\mathscr{K}^1_n(\varsigma)$	the class of all $\mathscr{L}_1(\varsigma)$-structures with at most n elements
$\mathscr{K}^1_{\mathrm{fin}}(\varsigma)$	the class of all finite $\mathscr{L}_1(\varsigma)$-structures
$\mathrm{Sent}^1_\varsigma$	the collection of all $\mathscr{L}_1(\varsigma)$-sentences
Val^1_ς	the collection of all valid $\mathscr{L}_1(\varsigma)$-sentences
$\mathrm{Th}(\mathscr{K})$	the $\mathscr{L}_1(\varsigma)$-theory of $\mathscr{K}$

Here $n \in \mathbb{N}$ and $\mathscr{K} \subseteq \mathscr{K}^1_{\mathrm{all}}(\varsigma)$. As one would expect, we have:

Proposition 3.2 (see [1])

$\mathrm{Th}(\mathscr{K}^1_n(\varsigma))$ *is decidable, uniformly in* n.

We are going to prove that for every sufficiently rich ς, the sets

$$\Sigma_2\text{-Val}^1_\varsigma \quad \text{and} \quad \mathrm{Sent}^1_\varsigma \setminus \Sigma_2\text{-Th}\left(\mathscr{K}^1_{\mathrm{fin}}(\varsigma)\right)$$

are computably inseparable $-$ this implies, among other things, the undecidability of $\Sigma_2\text{-Val}^1_\varsigma$. We shall also obtain the analogue of Theorem 2.4 for $\mathscr{L}_1$.

4 Probabilities on possible worlds

We now turn to a natural one-sorted fragment $\mathscr{L}_2$ of Halpern's 'first-order' logic of probability of type 2. It is obtained from the language of [1, Section 3] in a way similar to that of $\mathscr{L}_1$.

Consider a signature ς. By an $\mathscr{L}_2(\varsigma)$-*structure* we mean a quadruple $\langle D, \Omega, \pi, \mathsf{p} \rangle$ where:

- D and Ω are non-empty sets;

- π is a function from Ω to the class of all ς-structures with domain D;

- p is a discrete probability distribution on Ω — rather than on D.

In this context, elements of Ω are called *possible worlds*.

The $\mathscr{L}_2(\varsigma)$-formulas are built up exactly as in $\mathscr{L}_1(\varsigma)$, except that the subscripts $\vec{x}$ are dropped: we use μ instead of $\mu_{\vec{x}}$ throughout. Next, consider an $\mathscr{L}_2(\varsigma)$-structure $\mathcal{M} = \langle D, \Omega, \pi, \mathsf{p} \rangle$. Given a function ζ from var to D, we interpret each $\mu(\phi)$ as

$$\mathsf{P}\left(\{\omega \in \Omega \mid \phi \text{ is true in } \pi(\omega) \text{ under } \zeta\}\right)$$

where P is the probability measure (on the powerset of Ω) generated by p. Then the satisfiability relation for $\mathscr{L}_2(\varsigma)$ can be defined in the obvious way. We shall denote it by $\Vdash$ as well.

Remark 4.1. Z. Ognjanović and his colleagues have developed suitable infinitary calculi for some languages similar to $\mathscr{L}_2$; see [9] for details.

The notation for $\mathscr{L}_2$ is perfectly similar to that for $\mathscr{L}_1$, but with 1 replaced by 2, and one additional modification: we write $\mathscr{K}_n^2(\varsigma)$ for the class of all $\mathscr{L}_2(\varsigma)$-structures with at most n elements and at most n worlds.

Proposition 4.2 (see [1])

$\mathrm{Th}\left(\mathscr{K}_n^2(\varsigma)\right)$ *is decidable, uniformly in n.*

Our theorems for $\mathscr{L}_2$ will be analogous to those for $\mathscr{L}_1$.

See [1] and [5] for more results about Halpern's 'first-order' logics of probability.

5 Finite spaces

Given a signature σ, denote by Sent_σ the set of all (first-order) σ-sentences. Following [7], we call $\Gamma \subseteq \mathrm{Sent}_\sigma$ *hereditarily undecidable* if for every $\Delta \subseteq \mathrm{Sent}_\sigma$,

$$\mathrm{Val}_\sigma \cap \Gamma \ \subseteq \ \Delta \ \subseteq \ \Gamma \quad \Longrightarrow \quad \Delta \ \text{ is undecidable}$$

(or equivalently, $\mathrm{Val}_\sigma \cap \Gamma$ and $\mathrm{Sent}_\sigma \setminus \Gamma$ are computably inseparable). This notion can be readily adapted to $\mathsf{QPL}^{\mathrm{e}}$, $\mathscr{L}_1$ and $\mathscr{L}_2$.

In particular, let σ be $\langle R^2 \rangle$ where R is a binary predicate symbol. By a *simple graph* we mean a σ-structure $\mathfrak{G}$ such that $R^{\mathfrak{G}}$, i.e. the interpretation of R in $\mathfrak{G}$, is irreflexive and symmetric. We take $\mathbf{G}_{\mathrm{fin}}$ to be the class of all finite simple graphs.

Theorem 5.1 (see [7])

$\Sigma_2\text{-}\mathrm{Th}\left(\mathbf{G}_{\mathrm{fin}}\right)$ *is hereditarily undecidable.*[6]

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[6]This strengthens I. A. Lavrov's earlier result; see [3, Theorem 3.3.3].

We want to have the same for the Σ_2-fragment of the QPL$^{\mathrm{e}}$-theory of finite probability spaces. A key role in our argument will be played by:

Proposition 5.2

There are an existential QPL$^{\mathrm{e}}$*-formula* $\Phi_{\mathrm{all}}(X)$ *and a quantifier-free* QPL$^{\mathrm{e}}$*-formula* $\Phi_R(X_1, X_2)$ *such that for every* $\mathfrak{G} \in \mathbf{G}_{\mathrm{fin}}$ *one can find* $\mathscr{P} \in \mathcal{K}_{\mathrm{fin}}$ *satisfying the following conditions:*

a. $H := \{A \in \mathscr{A} \mid \mathscr{P} \Vdash \Phi_{\mathrm{all}}(A)\}$ *is non-empty;*

b. *the* σ*-structure* $\mathfrak{H}$ *with domain* H *in which* R *is interpreted by* $\Phi_R(X_1, X_2)$*, i.e.*

$$R^{\mathfrak{H}} \;=\; \{(A_1, A_2) \in H^2 \mid \mathscr{P} \Vdash \Phi_R(A_1, A_2)\},$$

is isomorphic to $\mathfrak{G}$.

Proof. Consider an arbitrary finite simple graph $\mathfrak{G}$. For convenience, we may assume that

$$G \;=\; \{1, \dots, N\}$$

where N is the cardinality of G.[7] Further, the relation $R^{\mathfrak{G}}$, being irreflexive and symmetric, can be identified with

$$R(\mathfrak{G}) \;:=\; \{\{i, j\} \mid (i, j) \in R^{\mathfrak{A}}\}.$$

Let K be the cardinality of $R(\mathfrak{G})$ and ν be a one-one function from $R(\mathfrak{G})$ onto $\{1, \dots, K\}$. In this way each $\{i, j\} \in R(\mathfrak{G})$ is coded by $\nu(\{i, j\})$. Given $n \in G$, define

$$I_n \;:=\; \{\nu(\{m, n\}) \mid \{m, n\} \in R(\mathfrak{G})\},$$

i.e. I_n is the set of all codes of edges that are incident on n.

We begin by describing a suitable finite probability space $\mathscr{P}$ — and then show how the desired quantifier-free QPL$^{\mathrm{e}}$-formulas emerge. Take Ω to be a set of cardinality $N + K + 3$, say

$$\Omega \;:=\; \{\gamma_1, \dots, \gamma_N\} \cup \{\delta_1, \dots, \delta_K\} \cup \{\lambda_1, \lambda_2, \lambda_*\},$$

and define $\mathsf{p} : \Omega \to [0, 1]$ by

$$\mathsf{p}(\gamma_n) := r - \sum_{k \in I_n} \frac{r}{3^k}, \quad \mathsf{p}(\delta_k) := \frac{r}{3^k},$$

$$\mathsf{p}(\lambda_1) := \frac{r}{2}, \quad \mathsf{p}(\lambda_2) := \frac{1}{3} - \frac{r}{2} \quad \text{and} \quad \mathsf{p}(\lambda_*) := \frac{2}{3} - \sum_{n=1}^{N} \mathsf{p}(\gamma_n) - \sum_{k=1}^{K} \mathsf{p}(\delta_k)$$

where r equals $1/(6N + 3)$, just to make everything work. Observe that

$$\sum_{n=1}^{N} \mathsf{p}(\gamma_n) + \sum_{k=1}^{K} \mathsf{p}(\delta_k) \;=\; \sum_{n=1}^{N} \left(r - \sum_{k \in I_n} \frac{r}{3^k} \right) + \sum_{k=1}^{K} \frac{r}{3^k}$$

$$<\; N \cdot r + \frac{r}{2} \;=\; \left(N + \frac{1}{2} \right) \cdot r$$

$$=\; \frac{2N + 1}{2} \cdot r \;=\; \frac{2N + 1}{2 \cdot (6N + 3)} \;=\; \frac{1}{2 \cdot 3} \;=\; \frac{1}{6}$$

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[7] Since G is non-empty, we have $N \neq 0$.

(hence, in particular, $0 \leqslant \mathsf{p}(\lambda_*) \leqslant 1$). Moreover, we have $\sum_{\omega \in \Omega} \mathsf{p}(\omega) = 1$. Let $\mathscr{P} = \langle \mathscr{A}, \mathsf{P} \rangle$ be the corresponding discrete space — so $\mathscr{A}$ is the power set of Ω, and $\mathsf{P} : \mathscr{A} \to [0,1]$ is given by

$$\mathsf{P}(A) := \sum_{\omega \in A} \mathsf{p}(\omega).$$

Then $\{\lambda_1\}$ and $\{\lambda_2\}$ can be distinguished from the other events in $\mathscr{A}$ by using the quantifier-free QPL^e-formula

$$\Phi_\circ(U, V) := \mu(U \wedge V) = 0 \wedge 3\mu(U \vee V) = 1 \wedge 0 < \mu(U) < \mu(V).$$

To put it formally:

Claim 1

Let $\mathfrak{G}$ and $\mathscr{P}$ be as above. Then for any $A_1, A_2 \in \mathscr{A}$,

$$\mathscr{P} \Vdash \Phi_\circ(A_1, A_2) \quad \Longleftrightarrow \quad A_1 = \{\lambda_1\} \quad \text{and} \quad A_2 = \{\lambda_2\}.$$

Proof. $\boxed{\Longleftarrow}$ Obviously, $\mathsf{P}(\{\lambda_1\} \cap \{\lambda_2\}) = 0$, $\mathsf{P}(\{\lambda_1\} \cup \{\lambda_2\}) = 1/3$ and $\mathsf{P}(\{\lambda_1\}) > 0$. Moreover, since $r \leqslant 1/9$, we also have

$$\mathsf{p}(\lambda_1) \leqslant \frac{1}{9 \cdot 2} = \frac{1}{18} \quad \text{and} \quad \mathsf{p}(\lambda_2) \geqslant \frac{1}{3} - \frac{1}{18} = \frac{5}{18}.$$

These imply $\mathsf{P}(\{\lambda_1\}) < \mathsf{P}(\{\lambda_2\})$.

$\boxed{\Longrightarrow}$ Suppose that $\mathsf{P}(A_1 \cap A_2) = 0$, $\mathsf{P}(A_1 \cup A_2) = 1/3$ and $0 < \mathsf{P}(A_1) < \mathsf{P}(A_2)$. Clearly, if $\{\lambda_1, \lambda_2\} \subseteq A_1 \cup A_2$, then $A_1 = \{\lambda_1\}$ and $A_2 = \{\lambda_2\}$. Thus it suffices to show that both λ_1 and λ_2 belong to $A_1 \cup A_2$. First, note that λ_* cannot be in $A_1 \cup A_2$, since

$$\mathsf{p}(\lambda_*) > \frac{2}{3} - \frac{1}{6} = \frac{1}{2},$$

Now if $\lambda_2 \notin A_1 \cup A_2$, then

$$\mathsf{P}(A_1 \cup A_2) \leqslant \mathsf{P}(\Omega \setminus \{\lambda_2, \lambda_*\}) = 1 - \mathsf{p}(\lambda_2) - \mathsf{p}(\lambda_*) < 1 - \frac{5}{18} - \frac{1}{2} = \frac{2}{9},$$

which is a contradiction. Hence $\lambda_2 \in A_1 \cup A_2$, and therefore

$$\mathsf{P}((A_1 \cup A_2) \setminus \{\lambda_2\}) = \mathsf{P}(A_1 \cup A_2) - \mathsf{p}(\lambda_2) = \frac{r}{2}.$$

Further, for each $n \in \{1, \ldots, N\}$, γ_n cannot be in $A_1 \cup A_2$, since $\mathsf{p}(\gamma_n) > r - r/2 = r/2$. So if λ_1 is not in $A_1 \cup A_2$, then

$$\mathsf{P}((A_1 \cup A_2) \setminus \{\lambda_2\}) \leqslant \mathsf{P}(\{\delta_1, \ldots, \delta_K\}) = \sum_{k=1}^{K} \frac{r}{3^k} < \frac{r}{2},$$

which is a contradiction. Hence $\lambda_1 \in A_1 \cup A_2$. $\qquad\square$

Next, with each $n \in G$ we associate the event

$$\llbracket n \rrbracket := \{\gamma_n\} \cup \{\delta_k \mid k \in I_n\}.$$

Denote by $\llbracket \Omega \rrbracket$ the collection of all these events. It turns out that $\llbracket \Omega \rrbracket$ can be defined in $\mathscr{P}$ by the existential QPL$^{\mathrm{e}}$-formula

$$\Phi_{\mathrm{all}}(X) := \exists U \, \exists V \, (\Phi_{\circ}(U, V) \wedge \mu(X) = 2\mu(U)),$$

which has the same meaning as $\mu(X) = r$, of course.

Claim 2

Let $\mathfrak{G}$ and $\mathscr{P}$ be as above. Then for every $A \in \mathscr{A}$,

$$\mathscr{P} \Vdash \Phi_{\mathrm{all}}(A) \quad \Longleftrightarrow \quad A \in \llbracket \Omega \rrbracket.$$

Proof. $\boxed{\Longleftarrow}$ By the construction of $\llbracket \Omega \rrbracket$.

$\boxed{\Longrightarrow}$ Suppose that $\mathsf{P}(A) = r$. Remember, we have

$$r < \frac{1}{9}, \quad \mathsf{p}(\lambda_2) \geqslant \frac{5}{18} \quad \text{and} \quad \mathsf{p}(\lambda_*) > \frac{1}{2}.$$

So neither λ_2 nor λ_* can be in A. Moreover, if $\lambda_1 \in A$, then

$$\mathsf{P}(A \setminus \{\lambda_1\}) = \mathsf{P}(A) - \mathsf{p}(\lambda_1) = \frac{r}{2},$$

which implies that $\gamma_1, \ldots, \gamma_N$ do not belong to A (since $\mathsf{p}(\gamma_n) > r/2$ for each $n \in \{1, \ldots, N\}$), and therefore

$$\mathsf{P}(A \setminus \{\lambda_1\}) \leqslant \mathsf{P}(\{\delta_1, \ldots, \delta_K\}) < \frac{r}{2},$$

which is a contradiction. Hence λ_1 cannot be in A as well. To summarize, we have

$$A \subseteq \{\gamma_1, \ldots, \gamma_N\} \cup \{\delta_1, \ldots, \delta_K\}.$$

Clearly, there exists one and only one $n \in \{1, \ldots, N\}$ such that $\gamma_n \in A$. This gives us

$$A \setminus \{\gamma_n\} \subseteq \{\delta_1, \ldots, \delta_K\} \quad \text{and} \quad \mathsf{P}(A \setminus \{\gamma_n\}) = \sum_{k \in I_n} \frac{r}{3^k}.$$

It is easy to verify that $A \setminus \{\gamma_n\}$ equals $\{\delta_k \mid k \in I_n\}$. Thus A coincides with $\llbracket n \rrbracket$. $\qquad \square$

Finally, observe that for all $i, j \in G$,

$$\begin{aligned}
(i, j) \in R^{\mathfrak{G}} &\iff \{i, j\} \in R(\mathfrak{G}) \\
&\iff i \neq j \text{ and } I_i \cap I_j \neq \varnothing \\
&\iff i \neq j \text{ and } \llbracket i \rrbracket \cap \llbracket j \rrbracket \neq \varnothing \\
&\iff 0 < \mathsf{P}(\llbracket i \rrbracket \cap \llbracket j \rrbracket) < \mathsf{P}(\llbracket i \rrbracket \cup \llbracket j \rrbracket).
\end{aligned}$$

So we can take

$$\Phi_R(X_1, X_2) := 0 < \mu(X_1 \wedge X_2) < \mu(X_1 \vee X_2).$$

Then the corresponding σ-structure $\mathfrak{H}$ (as in the statement of the proposition) turns out to be isomorphic to $\mathfrak{G}$ via the mapping $n \mapsto \llbracket n \rrbracket$. $\qquad \square$

Now we are ready to obtain the desired result.

Theorem 5.3

Σ_2-Th $(\mathcal{K}_{\mathrm{fin}})$ *is hereditarily undecidable.*

Proof. For convenience, we shall assume that the logical connective symbols in classical first-order logic are the same as those in QPL (namely $\wedge$, $\vee$, $\neg$) and associate with each individual variable x a distinguished Boolean variable X.

Let $\Phi_{\mathrm{all}}(X)$ and $\Phi_R(X_1, X_2)$ be as in the statement of Proposition 5.2. Given a σ-sentence Φ, define $\tau(\Phi)$ recursively:

$$
\begin{aligned}
\tau(R(x,y)) &:= \Phi_R(X, Y); \\
\tau(\Psi \wedge \Theta) &:= \tau(\Psi) \wedge \tau(\Theta); \\
\tau(\Psi \vee \Theta) &:= \tau(\Psi) \vee \tau(\Theta); \\
\tau(\neg\Psi) &:= \neg\tau(\Psi); \\
\tau(\forall x\, \Psi) &:= \forall X\, (\Phi_{\mathrm{all}}(X) \to \tau(\Psi)); \\
\tau(\exists x\, \Psi) &:= \exists X\, (\Phi_{\mathrm{all}}(X) \wedge \tau(\Psi)).
\end{aligned}
$$

To avoid clashes of variables, we require that Φ contain no occurrences of the individual variables whose uppercase versions are bound in $\Phi_{\mathrm{all}}(X)$ or $\Phi_R(X_1, X_2)$. Take

$$
\tau^*(\Phi) := \Phi_{\mathrm{spec}} \to \tau(\Phi)
$$

where Φ_{spec} denotes the 'specification' sentence

$$
\exists X\, \Phi_{\mathrm{all}}(X) \wedge \forall X\, \neg\Phi_R(X, X) \wedge \forall X_1\, \forall X_2\, (\Phi_R(X_1, X_2) \to \Phi_R(X_2, X_1)).
$$

Observe that for every σ-sentence Φ:

$$
\begin{aligned}
\Phi \in \mathrm{Val}_\sigma &\implies \tau^*(\Phi) \in \mathrm{Val}^{\mathrm{e}}; \\
\Phi \in \mathrm{Th}(\mathbf{G}_{\mathrm{fin}}) &\iff \tau^*(\Phi) \in \mathrm{Th}(\mathcal{K}_{\mathrm{fin}}).
\end{aligned}
$$

And if Φ is in Σ_2, then we can effectively convert $\tau^*(\Phi)$ to an equivalent Σ_2-QPL$^{\mathrm{e}}$-sentence $\rho(\Phi)$. In this way ρ maps the Σ_2-σ-sentences into the Σ_2-QPL$^{\mathrm{e}}$-sentences.

Now suppose Δ is such that Σ_2-Val$^{\mathrm{e}} \subseteq \Delta \subseteq \Sigma_2$-Th $(\mathcal{K}_{\mathrm{fin}})$. Clearly, we have

$$
\Sigma_2\text{-Val}_\sigma \subseteq \rho^{-1}[\Delta] \subseteq \Sigma_2\text{-Th}(\mathbf{G}_{\mathrm{fin}}).
$$

Therefore $\rho^{-1}[\Delta]$ cannot be decidable. Evidently, $\rho^{-1}[\Delta]$ is reducible to Δ via ρ. It follows that Δ must be undecidable, as desired. $\qquad\square$

It is known that Σ_2-Th $(\mathbf{G}_{\mathrm{fin}})$ is Π_1^0-complete; see [7, Theorem 4.2] and [12, Corollary 3.8]. This quickly leads us to:

Corollary 5.4

Σ_2-Th $(\mathcal{K}_{\mathrm{fin}})$ *is* Π_1^0-*complete.*

Proof. Let ρ be as in the proof of Theorem 5.3. Obviously, ρ reduces Σ_2-Th $(\mathbf{G}_{\mathrm{fin}})$ to Σ_2-Th $(\mathcal{K}_{\mathrm{fin}})$; hence Σ_2-Th $(\mathcal{K}_{\mathrm{fin}})$ is Π_1^0-hard. On the other hand, Σ_2-Th $(\mathcal{K}_{\mathrm{fin}})$ is trivially reducible to Th $(\mathcal{K}_{\mathrm{fin}})$, which is Π_1^0-bounded by Proposition 2.3. $\qquad\square$

At the same time, Theorem 5.3 implies the undecidability of the validity problem for Σ_2-QPL$^{\mathrm{e}}$-sentences. In view of Theorem 2.4, this means that Σ_2 is the minimal undecidable prefix fragment of QPL$^{\mathrm{e}}$, which completes the analysis of the decision problem for QPL$^{\mathrm{e}}$.

6 Finite structures of type 1

Next, we wish to adapt the technique developed in the previous section to deal with $\mathscr{L}_1$. Instead of Proposition 5.2, we shall use:

Proposition 6.1

Let ς be $\langle = \rangle$. Then there are existential $\mathscr{L}_1(\varsigma)$-formulas

$$\Phi_{\mathrm{all}}(x), \quad \Phi_R^1(x_1, x_2) \quad \text{and} \quad \Phi_R^0(x_1, x_2)$$

such that for every $\mathfrak{G} \in \mathbf{G}_{\mathrm{fin}}$ one can find $\mathcal{M} \in \mathscr{K}_{\mathrm{fin}}^1(\varsigma)$ satisfying the following conditions:

a. *$H := \{d \in D \mid \mathcal{M} \Vdash \Phi_{\mathrm{all}}(d)\}$ is non-empty;*

b. *for all $d_1, d_2 \in H$ we have*

$$\mathcal{M} \Vdash \Phi_R^1(d_1, d_2) \quad \Longleftrightarrow \quad \mathcal{M} \nVdash \Phi_R^0(d_1, d_2);$$

c. *the σ-structure $\mathfrak{H}$ with domain H in which R is interpreted by $\Phi_R^1(x_1, x_2)$, i.e.*

$$R^{\mathfrak{H}} = \{(d_1, d_2) \in H^2 \mid \mathcal{M} \Vdash \Phi_R^1(d_1, d_2)\},$$

is isomorphic to $\mathfrak{G}$.

Proof. Since ς is fixed by the context, we shall write $\mathscr{L}_1$ instead of $\mathscr{L}_1(\varsigma)$.

Consider an arbitrary finite simple graph $\mathfrak{G}$. As before, we may assume that $G = \{1, \ldots, N\}$ where N is the cardinality of G, and also identify the relation $R^{\mathfrak{A}}$ with

$$R(\mathfrak{G}) := \{\{i, j\} \mid (i, j) \in R^{\mathfrak{A}}\}.$$

Denote by G_2 the collection of all subsets of G containing exactly two elements. For each $S \in G_2$, let ε_S tell us whether or not S belongs to $R(\mathfrak{G})$:

$$\varepsilon_S := \begin{cases} 1 & \text{if } S \in R(\mathfrak{G}) \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to describe a suitable finite $\mathscr{L}_1$-structure $\mathcal{M}$. Take

$$\Omega \;:=\; \{\gamma_n \mid n \in G\} \cup \{\delta_S \mid S \in G_2\} \cup \{\lambda_1, \lambda_2, \lambda_*\},$$

and define $\mathsf{p} : \Omega \to [0,1]$ by

$$\mathsf{p}(\gamma_n) \;:=\; r - \frac{r}{3^n}, \quad \mathsf{p}(\delta_{\{i,j\}}) \;:=\; \frac{r}{3^i} + \frac{r}{3^j} + \varepsilon_{\{i,j\}} \cdot r,$$

$$\mathsf{p}(\lambda_1) \;:=\; \frac{r}{2}, \quad \mathsf{p}(\lambda_2) \;:=\; \frac{1}{3} - \frac{r}{2} \quad \text{and} \quad \mathsf{p}(\lambda_*) \;:=\; \frac{2}{3} - \sum_{n=1}^{N} \mathsf{p}(\gamma_n) - \sum_{S \in G_2} \mathsf{p}(\delta_S)$$

where r equals $1/(6N^2 + 3)$. Observe that

$$
\begin{aligned}
\sum_{n=1}^{N} \mathsf{p}(\gamma_n) + \sum_{S \in G_2} \mathsf{p}(\delta_S) \;&=\; \sum_{n=1}^{N} \left(r - \frac{r}{3^n} \right) + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\frac{r}{3^i} + \frac{r}{3^j} + \varepsilon_{\{i,j\}} \cdot r \right) \\
&<\; N \cdot r + \frac{N \cdot (N-1)}{2} \cdot \left(\frac{r}{2} + r \right) \\
&=\; N \cdot r + N \cdot (N-1) \cdot \frac{3r}{4} \;<\; N \cdot r + N \cdot (N-1) \cdot r \\
&=\; N^2 \cdot r \;=\; \frac{N^2}{6N^2 + 3} \;<\; \frac{1}{6}.
\end{aligned}
$$

Trivially, we have $\sum_{\omega \in \Omega} \mathsf{p}(\omega) = 1$. Let $\mathcal{M}$ be the $\mathscr{L}_1$-structure $\langle D, \pi, \mathsf{p} \rangle$ where D coincides with Ω, and π is the (unique) ς-structure with domain D. Similarly to before, we can distinguish λ_1 and λ_2 from the other elements of $D = \Omega$ by using the quantifier-free $\mathscr{L}_1$-formula

$$\Phi_\circ(u, v) \;:=\; \mu_z(z = u \wedge z = v) = 0 \wedge 3\mu_z(z = u \vee z = v) = 1 \wedge$$
$$0 < \mu_z(z = u) < \mu_z(z = v).$$

To be precise, it is easy to show that for all $d_1, d_2 \in D$,

$$\mathcal{M} \Vdash \Phi_\circ(d_1, d_2) \quad \Longleftrightarrow \quad d_1 = \lambda_1 \quad \text{and} \quad d_2 = \lambda_2;$$

cf. the proof of Claim 1 (in the proof of Proposition 5.2).[8]

Intuitively, we are going to identify each $n \in G$ with γ_n. Note that $\{\gamma_1, \ldots, \gamma_N\}$ can be defined in $\mathcal{M}$ by the existential $\mathscr{L}_1$-formula

$$\Phi_{\text{all}}(x) \;:=\; \exists u \, \exists v \, (\Phi_\circ(u, v) \wedge \mu_z(z = u) < \mu_z(z = x) < 2\mu_z(z = u)),$$

which has the same meaning as $r/2 < \mu_z(z = x) < r$.

Next, observe that for every $\{i, j\} \in G_2$ there exists a unique $\omega \in \Omega$ satisfying

$$\mathsf{p}(\omega) \;=\; \frac{r}{3^i} + \frac{r}{3^j} + r \quad \text{or} \quad \mathsf{p}(\omega) \;=\; \frac{r}{3^i} + \frac{r}{3^j}. \tag{$\dagger$}$$

The existence of ω is obvious, since we can substitute $\delta_{\{i,j\}}$ for ω. As for the uniqueness, suppose that ω satisfies ($\dagger$). So in particular, either $r < \mathsf{p}(\omega) < 3r/2$ or $\mathsf{p}(\omega) < r/2$.

---

[8] Again, since $r \leqslant \frac{1}{9}$, we have $\mathsf{p}(\lambda_1) \leqslant \frac{1}{18}$, $\mathsf{p}(\lambda_2) \geqslant \frac{5}{18}$ and $\mathsf{p}(\lambda_*) > \frac{1}{2}$.

- If $\omega \in \{\gamma_1, \ldots, \gamma_N\}$, then $r/2 < \mathsf{p}(\omega) < r$ — a contradiction.

- If $\omega = \lambda_1$, then $\mathsf{p}(\omega) = r/2$ — a contradiction.

- If $\omega = \lambda_2$, then $\mathsf{p}(\omega) \geqslant 5/18$, which contradicts $\mathsf{p}(\omega) < 3r/2 \leqslant 3/18$.

- If $\omega = \lambda_*$, then $\mathsf{p}(\omega) > 1/2$, which contradicts $\mathsf{p}(\omega) < 3/18$ again.

- Assume ω is $\delta_{\{i',j'\}}$ where $\{i',j'\} \in G_2$. Clearly, $\{i',j'\} \in R(\mathfrak{G})$ iff $\mathsf{p}(\omega) > r$. Hence

$$\frac{r}{3^{i'}} + \frac{r}{3^{j'}} \;=\; \frac{r}{3^i} + \frac{r}{3^j},$$

which implies $\{i',j'\} = \{i,j\}$.

Thus the only element of Ω that satisfies (†) is $\delta_{\{i,j\}}$. In addition, we have

$$
\begin{aligned}
\varepsilon_{\{i,j\}} \;=\; \varepsilon \quad &\Longleftrightarrow \quad \mathsf{p}\left(\delta_{\{i,j\}}\right) = \frac{r}{3^i} + \frac{r}{3^j} + \varepsilon \cdot r \\
&\Longleftrightarrow \quad \mathsf{p}(\gamma_i) + \mathsf{p}(\gamma_j) + \mathsf{p}\left(\delta_{\{i,j\}}\right) = (2 + \varepsilon) \cdot r \\
&\Longleftrightarrow \quad \mathsf{p}(\gamma_i) + \mathsf{p}(\gamma_j) + \mathsf{p}\left(\delta_{\{i,j\}}\right) = (4 + 2\varepsilon) \cdot \mathsf{p}(\lambda_1)
\end{aligned}
$$

where ε is either 0 or 1. Finally, take

$$
\begin{aligned}
\Phi_R^\varepsilon(x_1, x_2) \;:=\; \exists u\, \exists v\, \exists y\, (\Phi_\circ(u,v) \wedge \mu_z\,(z = x_1 \wedge z = x_2) = 0\, \wedge \\
\mu_z\,(z = x_1 \vee z = x_2 \vee z = y) = (4 + 2\varepsilon)\, \mu_z\,(z = u)).
\end{aligned}
$$

It follows that $\Phi_{\mathrm{all}}(x)$, $\Phi_R^1(x_1, x_2)$ and $\Phi_R^0(x_1, x_2)$ do the job. $\qquad\square$

Naturally, since we have two formulas for R, the corresponding translation should be slightly more subtle than that of Theorem 5.3; cf. also [7, Lemma 3.1].

Theorem 6.2

Suppose that ς contains the equality symbol. Then Σ_2-$\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^1(\varsigma)\right)$ is hereditarily undecidable.

Proof. Remember, we treat $\to$ as defined, rather than primitive.

Let $\Phi_{\mathrm{all}}(x)$, $\Phi_R^1(x_1, x_2)$ and $\Phi_R^0(x_1, x_2)$ be as in the statement of Proposition 6.1. Now, given a σ-sentence Φ, define $\tau(\Phi)$ as follows:

$$\tau(R(x,y)) \;:=\; \neg\Phi_R^0(x,y);$$

$$
\begin{aligned}
\tau(\neg\neg\Psi) &:= \tau(\Psi); & \tau(\neg R(x,y)) &:= \neg\Phi_R^1(x,y); \\
\tau(\neg(\Psi \wedge \Theta)) &:= \tau(\neg\Psi \vee \neg\Theta); & \tau(\Psi \wedge \Theta) &:= \tau(\Psi) \wedge \tau(\Theta); \\
\tau(\neg(\Psi \vee \Theta)) &:= \tau(\neg\Psi \wedge \neg\Theta); & \tau(\Psi \vee \Theta) &:= \tau(\Psi) \vee \tau(\Theta); \\
\tau(\neg\forall x\,\Psi) &:= \tau(\exists x\,\neg\Psi); & \tau(\forall x\,\Psi) &:= \forall x\,(\Phi_{\mathrm{all}}(x) \to \tau(\Psi)); \\
\tau(\neg\exists x\,\Psi) &:= \tau(\forall x\,\neg\Psi); & \tau(\exists x\,\Psi) &:= \exists x\,(\Phi_{\mathrm{all}}(x) \wedge \tau(\Psi))
\end{aligned}
$$

(avoiding clashes of variables, of course). Clearly, if Φ is quantifier-free, then $\tau\left(\Phi\right)$ is equivalent to a Π_1-$\mathscr{L}_1\left(\varsigma\right)$-formula. Take

$$\tau^*\left(\Phi\right) \; := \; \Phi_{\text{spec}} \rightarrow \tau\left(\Phi\right)$$

where Φ_{spec} denotes the 'specification' sentence

$$\exists x\,\Phi_{\text{all}}\left(x\right) \wedge \forall x_1\,\forall x_2\,\left(\Phi_R^1\left(x_1,x_2\right) \leftrightarrow \neg\Phi_R^0\left(x_1,x_2\right)\right) \wedge$$
$$\forall x\,\neg\Phi_R^1\left(x,x\right) \wedge \forall x_1\,\forall x_2\,\left(\Phi_R^1\left(x_1,x_2\right) \rightarrow \Phi_R^1\left(x_2,x_1\right)\right).$$

Proceed as in the proof of Theorem 5.3. $\qquad\square$

In addition, we have:

Corollary 6.3

Suppose that ς contains the equality symbol. Then Σ_2-$\mathrm{Th}\left(\mathscr{K}_{\text{fin}}^1\left(\varsigma\right)\right)$ is Π_1^0-complete.

Proof. The argument is like that for Corollary 5.4 — except that we use Proposition 3.2 instead of Proposition 2.3. $\qquad\square$

Evidently, the condition 'ς contains the equality symbol' can be replaced by 'ς contains at least one predicate symbol of arity greater than or equal to 2'. On the other hand, as was proved in [1] (by using a representation result from [5]), if ς consists only of unary predicate symbols, then the validity problem for $\mathscr{L}_1\left(\varsigma\right)$-sentences — and even those in the full language of [1, Section 2] over ς — is decidable.

Consequently, for every sufficiently rich ς, the validity problem for Σ_2-$\mathscr{L}_1\left(\varsigma\right)$-sentences is undecidable. It will be shown in Section 8 that this cannot be improved.

7 Finite structures of type 2

In the case of $\mathscr{L}_2$, we shall need two analogues of Proposition 5.2. This might be expected because of the richness conditions in [1].

Proposition 7.1

Let ς be $\langle=;c\rangle$ where c is a constant symbol. Then there are existential $\mathscr{L}_2\left(\varsigma\right)$-formulas

$$\Phi_{\text{all}}\left(x\right), \quad \Phi_R^1\left(x_1,x_2\right) \quad \textit{and} \quad \Phi_R^0\left(x_1,x_2\right)$$

such that for every $\mathfrak{G} \in \mathbf{G}_{\text{fin}}$ one can find $\mathcal{M} \in \mathscr{K}_{\text{fin}}^2\left(\varsigma\right)$ satisfying the following conditions:

a. $H := \{d \in D \mid \mathcal{M} \Vdash \Phi_{\text{all}}\left(d\right)\}$ is non-empty;

b. for all $d_1, d_2 \in H$ we have

$$\mathcal{M} \Vdash \Phi_R^0\left(d_1,d_2\right) \quad \Longleftrightarrow \quad \mathcal{M} \nVdash \Phi_R^1\left(d_1,d_2\right);$$

15

c. the σ-structure $\mathfrak{H}$ with domain H in which R is interpreted by $\Phi_R^1(x_1, x_2)$, i.e.

$$R^{\mathfrak{H}} \; = \; \left\{ (d_1, d_2) \in H^2 \mid \mathcal{M} \Vdash \Phi_R^1(d_1, d_2) \right\},$$

is isomorphic to $\mathfrak{G}$.

Proof. The argument is like that for Proposition 6.1, except that:

- we let $\mathcal{M}$ be the $\mathscr{L}_1$-structure $\langle D, \Omega, \pi, \mathsf{p} \rangle$ where D coincides with Ω, and for each $\omega \in \Omega$, $\pi(\omega)$ is the ς-structure with domain D in which c is interpreted as ω;

- we first replace μ_z by μ and then z by c throughout — so, for example, $\mu_z(z = x_1 \wedge z = x_2)$ turns into $\mu(c = x_1 \wedge c = x_2)$.[9]

In fact, a similar trick was used in [1] to transfer certain complexity results. $\square$

It gives us two formulas for R, as in Proposition 6.1.

Theorem 7.2

Suppose that ς extends $\langle =; c \rangle$. Then Σ_2-$\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^2(\varsigma)\right)$ is hereditarily undecidable.

Proof. This is similar to Theorem 6.2. $\square$

In addition, we have:

Corollary 7.3

Suppose that ς extends $\langle =; c \rangle$. Then Σ_2-$\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^2(\varsigma)\right)$ is Π_1^0-complete.

Proof. The argument is like that for Corollary 5.4 — except that we use Proposition 4.2 instead of Proposition 2.3. $\square$

Here is another analogue of Proposition 5.2:

Proposition 7.4

Let ς be $\langle P^1 \rangle$ where P is a unary predicate symbol, and take

$$\Phi_R(x_1, x_2) \; := \; 0 < \mu(P(x_1) \wedge P(x_2)) < \mu(P(x_1) \vee P(x_2)).$$

Then for every $\mathfrak{G} \in \mathbf{G}_{\mathrm{fin}}$ one can find $\mathcal{M} \in \mathscr{K}_{\mathrm{fin}}^2(\varsigma)$ such that the σ-structure $\mathfrak{H}$ with domain D (the first component of $\mathcal{M}$) in which R is interpreted by $\Phi_R(x_1, x_2)$, i.e.

$$R^{\mathfrak{H}} \; = \; \left\{ (d_1, d_2) \in D^2 \mid \mathcal{M} \Vdash \Phi_R(d_1, d_2) \right\},$$

is isomorphic to $\mathfrak{G}$.

---

[9]It is crucial that c is non-rigid, i.e. it does not have to designate the same object in all possible worlds.

Proof. We are going to apply a simplified version of the argument for Proposition 5.2.

Consider an arbitrary finite simple graph $\mathfrak{G}$. Let Ω, p and $[\![\cdot]\!]$ be as in the proof of Proposition 5.2. For each $\omega \in \Omega$, define $\pi(\omega)$ to be the ς-structure with domain G in which P is interpreted as $\{n \in G \mid \omega \in [\![n]\!]\}$. So by construction, for every $n \in G$,

$$\{\omega \in \Omega \mid \pi(\omega) \vDash P(n)\} \;=\; [\![n]\!].$$

Let $\mathcal{M}$ be the $\mathscr{L}_2(\varsigma)$-structure $\langle D, \Omega, \pi, \mathsf{p}\rangle$ where D coincides with G. Then for all $i, j \in G$,

$$
\begin{aligned}
(i,j) \;\in\; R^{\mathfrak{G}} \quad &\Longleftrightarrow \quad 0 \;<\; \mathsf{P}\left([\![i]\!] \cap [\![j]\!]\right) \;<\; \mathsf{P}\left([\![i]\!] \cup [\![j]\!]\right) \\
&\Longleftrightarrow \quad 0 \;<\; \mathsf{P}\left(\{\omega \in \Omega \mid \pi(\omega) \vDash P(i) \wedge P(j)\}\right) \\
&\qquad\qquad <\; \mathsf{P}\left(\{\omega \in \Omega \mid \pi(\omega) \vDash P(i) \vee P(j)\}\right) \\
&\Longleftrightarrow \quad \mathcal{M} \Vdash \Phi_R(i,j),
\end{aligned}
$$

as desired. $\qquad\qquad\qquad\square$

Naturally, the corresponding translation is going to be simpler.

Theorem 7.5

Suppose that ς extends $\langle P^1 \rangle$. Then Σ_2-Th $(\mathscr{K}^2_{\mathrm{fin}}(\varsigma))$ is hereditarily undecidable.

Proof. Let $\Phi_R(x_1, x_2)$ be as in the statement of Proposition 7.4. Given a σ-sentence Φ, define $\rho(\Phi)$ to be the result of replacing each $R(x, y)$ in Φ by $\Phi_R(x, y)$. Take

$$\tau^*(\Phi) \;:=\; \Phi_{\mathrm{spec}} \to \tau(\Phi)$$

where Φ_{spec} denotes the 'specification' sentence

$$\forall x \, \neg\Phi_R(x, x) \wedge \forall x_1 \, \forall x_2 \, (\Phi_R(x_1, x_2) \to \Phi_R(x_2, x_1)).$$

Then one can proceed as in the proof of Theorem 5.3. $\qquad\qquad\qquad\square$

Of course, we also have:

Corollary 7.6

Suppose that ς extends $\langle P^1 \rangle$. Then Σ_2-Th $(\mathscr{K}^2_{\mathrm{fin}}(\varsigma))$ is Π^0_1-complete.

Evidently, the condition 'ς extends $\langle =, c \rangle$ or $\langle P^1 \rangle$' can be replaced by 'ς does not coincide with $\langle = \rangle$'. On the other hand, it is rather easy to prove that the validity problem for $\mathscr{L}_2(\langle = \rangle)$-sentences is decidable.

Consequently, for every sufficiently rich ς, the validity problem for Σ_2-$\mathscr{L}_2(\varsigma)$-sentences is undecidable. It will soon be shown that this cannot be improved.

8 Concerning Π_2-fragments

We aim to prove the analogues of Theorem 2.4 for $\mathscr{L}_1$ and $\mathscr{L}_2$. Technically, it is easier to start with $\mathscr{L}_2$, and then deduce the desired result for $\mathscr{L}_1$ from that for $\mathscr{L}_2$.

Assume ς contains at least one constant symbol. For convenience, we shall write $\mathrm{Sent}^{\circ}_{\varsigma}$ for the collection of all quantifier-free σ-sentences. Given a ς-structure $\mathfrak{A}$, let

$$\mathrm{Th}^{\circ}\left(\mathfrak{A}\right) \;:=\; \left\{\Phi \in \mathrm{Sent}^{\circ}_{\varsigma} \mid \mathfrak{A} \vDash \Phi\right\}.$$

Now consider an arbitrary $\mathscr{L}_2\left(\varsigma\right)$-structure $\mathcal{M} = \langle D, \Omega, \pi, \mathsf{p}\rangle$. We are going to describe a special $\mathscr{L}_2\left(\varsigma\right)$-structure

$$\mathcal{M}_{\circ} = \langle D_{\circ}, \Omega_{\circ}, \pi_{\circ}, \mathsf{p}_{\circ}\rangle$$

which satisfies the same quantifier-free $\mathscr{L}_2\left(\varsigma\right)$-sentences. For this purpose, define the equivalence relation $\equiv$ on Ω as follows:

$$\omega_1 \equiv \omega_2 \quad :\Longleftrightarrow \quad \mathrm{Th}^{\circ}\left(\pi\left(\omega_1\right)\right) \;=\; \mathrm{Th}^{\circ}\left(\pi\left(\omega_2\right)\right).$$

For each $\omega \in \Omega$, denote by $[\omega]$ the equivalence class of ω under $\equiv$. Take $\Omega_{\circ}$ to be the set of all such classes. The discrete distribution $\mathsf{p}_{\circ}$ on $\Omega_{\circ}$ is then given by

$$\mathsf{p}_{\circ}\left([\omega]\right) \;:=\; \sum_{\omega' \in [\omega]} \mathsf{p}\left(\omega'\right).$$

Finally, let $D_{\circ}$ consist of all constant symbols in ς, and for each $\omega \in \Omega$, define $\pi_{\circ}\left([\omega]\right)$ to be the ς-structure with domain $D_{\circ}$ such that:

$$\begin{aligned}
c^{\pi_{\circ}([\omega])} &\;:=\; c, \\
P^{\pi_{\circ}([\omega])} &\;:=\; \{(t_1, \ldots, t_n) \in D_{\circ}^n \mid \pi\left(\omega\right) \vDash P\left(t_1, \ldots, t_n\right)\}.^{10}
\end{aligned}$$

This completes the description of $\mathcal{M}_{\circ}$. And it is straightforward to verify that for every quantifier-free $\mathscr{L}_2\left(\varsigma\right)$-sentence Φ,

$$\mathcal{M} \Vdash \Phi \quad \Longleftrightarrow \quad \mathcal{M}_{\circ} \Vdash \Phi. \tag{$\star$}$$

From $(\star)$ we can get the finite model property for $\mathscr{L}_2\left(\varsigma\right)$-sentences without quantifiers and, with the help of Proposition 4.2, the corresponding decidability result:

Proposition 8.1

The quantifier-free fragments of $\mathrm{Th}\left(\mathscr{K}^2_{\mathrm{all}}\left(\varsigma\right)\right)$ and $\mathrm{Th}\left(\mathscr{K}^2_{\mathrm{fin}}\left(\varsigma\right)\right)$ coincide. Moreover, the fragment in question is decidable.

Proof. Clearly, we may assume ς is finite — provided that our decision procedure will be uniform in ς, of course. Take N to be the number of atomic ς-sentences. Then for any $\mathscr{L}_2\left(\varsigma\right)$-structure $\mathcal{M}$ we have

$$|D_{\circ}| \;\leqslant\; N \quad \text{and} \quad |\Omega_{\circ}| \;\leqslant\; 2^N;$$

---

[10] Here c is a constant symbol, P is a predicate symbol different from $=$, and n is the arity of P.

so in particular, $\mathcal{M}_\circ$ is finite.

Consider an arbitrary quantifier-free $\mathscr{L}_2(\varsigma)$-sentence Φ. Obviously, if $\Phi \in \mathrm{Th}\left(\mathscr{K}_{\mathrm{all}}^2(\varsigma)\right)$, then $\Phi \in \mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^2(\varsigma)\right)$. For the converse, suppose that Φ does not belong to $\mathrm{Th}\left(\mathscr{K}_{\mathrm{all}}^2(\varsigma)\right)$, i.e. $\mathcal{M} \nVdash \Phi$ for some $\mathscr{L}_2(\varsigma)$-structure $\mathcal{M}$. Then $\mathcal{M}_\circ \nVdash \Phi$ by $(\star)$.

Thus the quantifier-free theories of $\mathscr{K}_{\mathrm{all}}^2(\varsigma)$ and $\mathscr{K}_{\mathrm{fin}}^2(\varsigma)$ coincide with that of $\mathscr{K}_{2^N}^2(\varsigma)$, which is decidable by Proposition 4.2.[11] $\qquad\square$

Further, by modifying the above argument we can get the following.

Theorem 8.2

Π_2-$\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^2(\varsigma)\right)$ *is decidable. Moreover,*

$$\Pi_2\text{-}\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^2(\varsigma)\right) \;=\; \Pi_2\text{-}\mathrm{Th}\left(\mathscr{K}_{\mathrm{all}}^2(\varsigma)\right).$$

Proof. Again, we may assume ς is finite.

Consider an arbitrary Π_2-$\mathscr{L}^2(\varsigma)$-sentence Φ. By definition, Φ has the form

$$\forall x_0 \ldots \forall x_n \exists y_0 \ldots \exists y_k \, \Psi(x_0, \ldots, x_n, y_1, \ldots, y_0)$$

where Ψ is quantifier-free. Let ς^* be the signature obtained from ς by adding n new constant symbols $c_0, \ldots, c_n$. Denote by C the collection of all constant symbols in ς^*. Take N to be the number of atomic ς^*-sentences.

Next, we call a $\mathscr{L}^2(\varsigma^*)$-structure $\mathcal{M}$ *special* iff each $c \in \{c_1, \ldots, c_n\}$ is interpreted rigidly, i.e. for all $\omega_1, \omega_2 \in \Omega$,

$$c^{\pi(\omega_1)} \;=\; c^{\pi(\omega_2)}.$$

Denote by $\mathscr{K}_\bullet$ the class of all such structures. Observe that

$$\Phi \;\in\; \mathrm{Th}\left(\mathscr{K}_{\mathrm{all}}^2(\varsigma)\right) \quad\Longleftrightarrow\quad \exists \vec{y}\,\Psi(\mathbf{c}, \vec{y}) \;\in\; \mathrm{Th}\left(\mathscr{K}_\bullet\right)$$

where $\Psi(\mathbf{c}, \vec{y})$ abbreviates $\Psi(c_1, \ldots, c_n, y_1, \ldots, y_k)$. And this remains true if we restrict ourselves to finite structures on both sides. So it suffices to show that

$$\exists \vec{y}\,\Psi(\mathbf{c}, \vec{y}) \;\in\; \mathrm{Th}\left(\mathscr{K}_\bullet\right) \quad\Longleftrightarrow\quad \exists \vec{y}\,\Psi(\mathbf{c}, \vec{y}) \;\in\; \mathrm{Th}\left(\mathscr{K}_\bullet \cap \mathscr{K}_{\mathrm{fin}}^2(\varsigma)\right).$$

The implication from left to right is obvious. For the other direction, suppose that $\exists \vec{y}\,\Psi(\mathbf{c}, \vec{y})$ does not belong to $\mathrm{Th}\left(\mathscr{K}_\bullet\right)$, i.e. $\mathcal{M} \nVdash \exists \vec{y}\,\Psi(\mathbf{c}, \vec{y})$ for some special $\mathscr{L}^2(\varsigma^*)$-structure $\mathcal{M}$. Hence we also have $\mathcal{M} \nVdash \bigvee_{\vec{t} \in \mathrm{C}^k} \Psi(\mathbf{c}, \vec{t})$. Then $\mathcal{M}_\circ \nVdash \bigvee_{\vec{t} \in \mathrm{C}^k} \Psi(\mathbf{c}, \vec{t})$ by $(\star)$. Since $D_\circ$ equals C, this is equivalent to $\mathcal{M}_\circ \nVdash \exists \vec{y}\,\Psi(\mathbf{c}, \vec{y})$. Note that $\mathcal{M}_\circ$ is special and has size at most 2^N.

It follows that the Π_2-theories of $\mathscr{K}_{\mathrm{all}}^2(\varsigma)$ and $\mathscr{K}_{\mathrm{fin}}^2(\varsigma)$ coincide with that of $\mathscr{K}_{2^N}^2(\varsigma)$, which is decidable by Proposition 4.2. $\qquad\square$

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[11]In effect, it is easy to reduce the quantifier-free theory of $\mathscr{K}_{2^N}^2(\varsigma)$ to the first-order theory of the ordered field of reals; cf. Proposition 2.2.

Then, instead of proving the analogous result for $\mathscr{L}_1$ directly, one can utilize a special translation of $\mathscr{L}_1$ into $\mathscr{L}_2$ described in [1, Section 6].

Given a signature ς, let ς' be obtained from ς by adding countably many new constant symbols $c_0, c_1, c_2, \ldots$

Theorem 8.3 (see [1])

Suppose that ς contains the equality symbol. Then there exists a computable translation τ^{12} of $\mathscr{L}_1(\varsigma)$ into $\mathscr{L}_2(\varsigma')$ such that for all $\mathscr{L}_1(\varsigma)$-sentences Φ and $\mathtt{s} \in \{\mathrm{all}, \mathrm{fin}\}$,

$$\Phi \ \in \ \mathrm{Th}\left(\mathscr{K}_{\mathtt{s}}^1(\varsigma)\right) \quad \Longleftrightarrow \quad \tau^{12}(\Phi) \ \in \ \mathrm{Th}\left(\mathscr{K}_{\mathtt{s}}^2(\varsigma')\right).$$

Moreover, for each $n \in \mathbb{N}$, if Φ is in Π_{n+2}, so is $\tau^{12}(\Phi)$.

Combining Theorems 8.2 and 8.3 leads us to:

Theorem 8.4

Π_2-$\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^1(\varsigma)\right)$ *is decidable. Moreover,*

$$\Pi_2\text{-}\mathrm{Th}\left(\mathscr{K}_{\mathrm{fin}}^1(\varsigma)\right) \ = \ \Pi_2\text{-}\mathrm{Th}\left(\mathscr{K}_{\mathrm{all}}^1(\varsigma)\right).$$

Consequently, the undecidability results of Sections 6 and 7 turn out to be precise. So the decision problems for $\mathscr{L}_1$ and $\mathscr{L}_2$ are solved.

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