# On the complexity of first-order logics of probability

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**Abstract**

This article is concerned with Halpern's first-order logics of probability, which we denote by $\mathscr{L}_1$ and $\mathscr{L}_2$: the first of these deals with probability distributions on the domain, while the second employs distributions on external sets of possible worlds. The proofs of the complexity lower bound results for $\mathscr{L}_1$ and $\mathscr{L}_2$ given in [1] relied heavily on using polynomials. We shall obtain the same lower bounds for small fragments of $\mathscr{L}_1$ and $\mathscr{L}_2$ in which neither addition nor multiplication is allowed. Further, it will be studied what happens if we exclude field variables, and hence quantifiers over reals; the upper bound proofs here will utilize suitable analogues of the (downward) Löwenheim–Skolem theorem.

*Keywords:* probability logic, quantification, undecidability, complexity, higher-order arithmetic

## 1 Introduction

Briefly stated, a probabilistic logical system is any formal language whose semantics makes use of probability measures. Languages of this kind play an important role in logic and its applications to computer science and artificial intelligence. For more information, see, e.g., [9].

Many quantified probability logics may be viewed as modifications of two 'first-order' logics of probability studied in [3] and [1]. The latter include quantifiers over elements of a given domain, and also quantifiers over reals. However, the first of them employs probability distributions on the domain, while the second deals with distributions on additional sets of possible worlds. We denote them by $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Naturally, they depend on the choice of a signature, like classical first-order logic.[1] On the other hand, $\mathscr{L}_1$ and $\mathscr{L}_2$ may be treated as expansions of the well-known language studied in [2, Section 6]; see [12] for another approach to expanding that language.

The main complexity results for $\mathscr{L}_1$ and $\mathscr{L}_2$ were obtained in [1]. Briefly, for each $i \in \{1, 2\}$, if $\varsigma$ is a sufficiently rich signature, then the validity problem for $\mathscr{L}_i(\varsigma)$ is $\Pi^2_1$-complete; and if only at

[1]Following [1], we shall focus on predicate and constant symbols.

most countable structures are considered, then the corresponding problem is equivalent to that of recognizing true sentences in second-order arithmetic.[2] The lower bound arguments in [1] relied heavily on using polynomials of probabilities of first-order formulas, and hence, in particular, they fail if we limit ourselves to linear terms. In this paper, by providing new and more advanced arguments we are going to derive the same lower bounds for small fragments of $\mathscr{L}_1$ and $\mathscr{L}_2$ in which neither addition nor multiplication is allowed. In these fragments:

- each basic formula is either an equality between two probabilities or an inequality between some probability and a field variable;
- the probability symbol $\mu$ can only be applied to quantifier-free first-order formulas;
- all occurrences of first-order formulas must be in the scope of $\mu$.

Further, let $\mathscr{L}_1^\natural$ and $\mathscr{L}_2^\natural$ be obtained from $\mathscr{L}_1$ and $\mathscr{L}_2$ by excluding field variables, and hence quantifiers over reals.[3] We are going to analyze the complexity of the small fragments of $\mathscr{L}_1^\natural$ and $\mathscr{L}_2^\natural$ as well. The upper bound proofs here will utilize suitable analogues of the (downward) Löwenheim–Skolem theorem. Note that the present work may be compared — or contrasted — with [6], which is concerned with 'quantitative' propositional probability logics like those in [2] and their 'qualitative' fragments, whose complexity is defined in terms of polynomial-time reducibility.

The rest of the article is organized as follows. In Sections 2 and 3, the syntax and semantics of $\mathscr{L}_1$ and $\mathscr{L}_2$ are described, and some relevant results of [1] are stated. Section 4 contains some material on higher-order arithmetic. In Sections 5 and 6, we show that the earlier hardness results for Halpern's logics can be strengthened in a crucial way — by excluding multiplication and addition. Finally, Section 7 provides suitable analogues of the Löwenheim–Skolem theorem for $\mathscr{L}_1^\natural$ and $\mathscr{L}_2^\natural$, which imply that the corresponding validity problems belong to $\Pi_1^1$, as expected.

## 2 Probabilities on the domain

The purpose of this section is to recall the definition of Halpern's 'first-order' logic of probability of type 1, which we denote by $\mathscr{L}_1$, and some related complexity results.

Consider a (first-order) signature $\varsigma$. We shall restrict ourselves to signatures without equality, and containing no function symbols. Then, following [1], by an $\mathscr{L}_1(\varsigma)$-*structure* we mean a triple $\langle D, \pi, \mathsf{p} \rangle$ where:

- $D$ is a non-empty set;
- $\pi$ is a $\varsigma$-structure, as defined in first-order logic, with domain $D$;

[2] At the same time, for practically any reasonable class of probability spaces, its elementary theory can be reduced to complete second-order arithmetic; see [17] for details.

[3] See [19] for a study of their prefix fragments in terms of hereditary undecidability.

- $\mathsf{p}$ is a *discrete probability distribution* on $D$, i.e. a function from $D$ to $[0,1]$ such that

$$|\{d \in D \mid \mathsf{p}(d) \neq 0\}| \leqslant \aleph_0 \quad \text{and} \quad \sum_{d \in D} \mathsf{p}(d) = 1,$$

  which generates the probability measure $\mathsf{P}$ on the powerset of $D$ as follows:

$$\mathsf{P}(A) := \sum_{d \in A} \mathsf{p}(d).$$

Note, in passing, that given $\mathsf{p}$ as above and a non-zero $k \in \mathbb{N}$, we can define a discrete distribution $\mathsf{p}^k$ on $D^k$ by

$$\mathsf{p}^k(d_1, \ldots, d_k) := \mathsf{p}(d_1) \cdot \ldots \cdot \mathsf{p}(d_k),$$

which generates the measure $\mathsf{P}^k$ on the powerset of $D^k$, of course. Evidently, if $A \subseteq D^k$, and $A'$ is obtained from $A$ by permuting some of the coordinates, then $\mathsf{P}^k(A')$ coincides with $\mathsf{P}^k(A)$.

As for the syntax of $\mathscr{L}_1$, the corresponding alphabet includes two disjoint countable sets

$$\mathrm{Var} := \{x, y, z, \ldots\} \quad \text{and} \quad \mathbf{Var} := \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots\},$$

whose elements are called *individual variables* and *field variables* respectively. Of course, the latter are intended to range over reals. In addition, we have:

- the logical symbols $\top$, $\bot$, $\wedge$, $\vee$ and $\neg$;
- the quantifier symbols $\forall$ and $\exists$;
- the symbols $0$, $1$, $+$, $-$, $\cdot$, $=$ and $\leqslant$ of the language of ordered fields;
- a special symbol $\mu$, which will be interpreted using probability measures.[4]

Given a signature $\varsigma$, let $\mu\text{-Form}^1_\varsigma$ and $\mu\text{-Term}^1_\varsigma$ be the sets defined simultaneously by the following conditions:

1. $\mu\text{-Form}^1_\varsigma$ contains all atomic fist-order $\varsigma$-formulas, including $\top$ and $\bot$;
2. $\mu\text{-Term}^1_\varsigma$ contains $0$ and $1$;
3. $\mu\text{-Term}^1_\varsigma$ contains all field variables;
4. $\mu\text{-Form}^1_\varsigma$ is closed under $\wedge$, $\vee$ and $\neg$;
5. $\mu\text{-Form}^1_\varsigma$ is closed under $\mathrm{Q}x$, for all $\mathrm{Q} \in \{\forall, \exists\}$ and $x \in \mathrm{Var}$;
6. $\mu\text{-Form}^1_\varsigma$ is closed under $\mathrm{Q}\mathfrak{a}$, for all $\mathrm{Q} \in \{\forall, \exists\}$ and $\mathfrak{a} \in \mathbf{Var}$;
7. if $\phi$ belongs to $\mu\text{-Form}^1_\varsigma$, and $\vec{x}$ is a non-empty tuple of elements of $\mathrm{Var}$, then $\mu_{\vec{x}}(\phi)$ belongs to $\mu\text{-Term}^1_\varsigma$;

[4]Here $\top$ and $\bot$ stand for 'truth' and 'falsity' respectively.

8. $\mu$-$\mathrm{Term}^1_\varsigma$ is closed under $+$, $-$ and $\cdot$;

9. if $t_1$ and $t_2$ belong to $\mu$-$\mathrm{Term}^1_\varsigma$, then $t_1 = t_2$ and $t_1 \leqslant t_2$ belong to $\mu$-$\mathrm{Form}^1_\varsigma$.

Elements of these sets are called $\mathscr{L}_1(\varsigma)$*-formulas* and $\mathscr{L}_1(\varsigma)$*-terms* respectively. By the *depth* of an $\mathscr{L}_1(\varsigma)$-formula $\phi$, denoted by $\mathrm{dp}(\phi)$, we mean the largest number of nested occurrences of $\mu$ in $\phi$; similarly for $\mathscr{L}_1(\varsigma)$-terms. An $\mathscr{L}_1(\varsigma)$-formula $\phi$ is:

- *basic* if $\phi$ has the form $t_1 = t_2$ or $t_1 \leqslant t_2$ where $t_1$ and $t_2$ are $\mathscr{L}_1(\varsigma)$-terms;
- *regular* if the depth of $\phi$ is positive, and each occurrence of a first-order $\varsigma$-formula in $\phi$ is in the scope of exactly $\mathrm{dp}(\phi)$ occurrences of $\mu$.

For instance, if $\varsigma$ contains a binary predicate symbol $R$, then:

i. $\mu_x(R(\mathfrak{a}, x)) \leqslant 1$ and $\mu_{\mathfrak{a}}(\bot) = 0$ are not $\mathscr{L}_1(\varsigma)$-formulas;

ii. $0 \leqslant \mathfrak{a} \leqslant 1 \to \exists x(\mu_y(R(x,y)) = \mathfrak{a})$ is a regular $\mathscr{L}_1(\varsigma)$-formula of depth $1$;

iii. $\mu_x(R(x,x) \wedge \mu_y(R(x,y)) \leqslant \mathfrak{a}) = 1$ is a non-regular $\mathscr{L}_1(\varsigma)$-formula of depth $2$.

Finally, an $\mathscr{L}_1(\varsigma)$*-sentence* is an $\mathscr{L}_1(\varsigma)$-formula with no free variable occurrences — provided that $\mu_{(x_1,\ldots,x_k)}$ binds all occurrences of $x_1, \ldots, x_k$ in its scope.

Now consider an $\mathscr{L}_1(\varsigma)$-structure $\mathcal{M} = \langle D, \pi, \mathsf{p}\rangle$. Hence the individual variables are intended to range over $D$. By a *valuation in* $\mathcal{M}$ we mean a pair $\langle \zeta, \gamma\rangle$ where $\zeta$ and $\gamma$ are functions from $\mathrm{Var}$ and $\mathbf{Var}$ to $D$ and $\mathbb{R}$ respectively. Then

$$\mathcal{M} \Vdash \phi[\zeta, \gamma]$$

read as '$\phi$ is true in $\mathcal{M}$ under $\langle \zeta, \gamma\rangle$', can be defined by induction on the depth of $\phi$. Of course, in case $\phi$ is an atomic first-order $\varsigma$-formula, we employ the $\varsigma$-structure $\pi$, viz.

$$\mathcal{M} \Vdash \phi[\zeta, \gamma] \iff \pi \vDash \phi[\zeta].$$

Assuming $\mathrm{dp}(\phi) > 0$, the idea is that given an arbitrary valuation $\langle \eta, \delta\rangle$ in $\mathcal{M}$, we interpret each $\mu_{(x_1,\ldots,x_k)}(\psi)$ with $\mathrm{dp}(\psi) < \mathrm{dp}(\phi)$ as

$$\mathsf{P}^k\left(\left\{(d_1, \ldots, d_k) \in D^k \mid \mathcal{M} \Vdash \psi\left[\eta^{\vec{x}}_{\vec{d}}, \delta\right]\right\}\right)$$

where $\eta^{\vec{x}}_{\vec{d}}$ is the function from $\mathrm{Var}$ to $D$ such that

$$\eta^{\vec{x}}_{\vec{d}}(u) = \begin{cases} d_i & \text{if } u = x_i \text{ with } i \in \{1, \ldots, k\} \\ \eta(u) & \text{otherwise.} \end{cases}$$

For example, take

$$\phi(x, \mathfrak{a}) := \mathfrak{a} + \mathfrak{a} = 1 \wedge \mu_y(R(x,y)) = \mathfrak{a}.$$

Then $\mathcal{M} \Vdash \phi[\zeta, \gamma]$ iff both $\gamma(\mathfrak{a})$ and $\mathsf{P}(\{d \in D \mid \pi \models R(\zeta(x), d)\})$ are equal to $1/2$. See [3] and [1] for details. We call an $\mathscr{L}_1(\varsigma)$-sentence *valid* if it is true in all $\mathscr{L}_1(\varsigma)$-structures. In fact, neither the lower bound arguments in [1] nor those in the present article require iterations of $\mu$. Thus we shall deal mainly with $\mathscr{L}_1(\varsigma)$-formulas of depth 1.

**Theorem 2.1** (see [1])

*Let $\varsigma$ be $\langle R^2 \rangle$ where $R$ is a binary predicate symbol. Then the validity problem for $\mathscr{L}_1(\varsigma)$-sentences is $\Pi^2_1$-complete. However, if we limit ourselves to at most countable domains, the corresponding problem becomes $\Pi^1_\infty$-complete.*[5]

**Remark 2.2.** In the case of at most countable domains the upper bound argument — which gives us a translation into second-order arithmetic — is straightforward; cf. [11, § 16.2]. But the general upper bound argument, though similar in spirit, requires an analogue of the Löwenheim–Skolem theorem, which ensures that every satisfiable $\mathscr{L}_1(\varsigma)$-sentence is true in some $\mathscr{L}_1(\varsigma)$-structure $\mathcal{M}$ with $|D| \leqslant 2^{\aleph_0}$; see the proof of Theorem 5.2 in [1].

Let $\mathscr{L}_1^\natural$ be the sublanguage of $\mathscr{L}_1$ obtained by excluding field variables, and hence quantifiers over reals. So the definitions of $\mathscr{L}_1^\natural(\varsigma)$-formula and $\mathscr{L}_1^\natural(\varsigma)$-term are like those for $\mathscr{L}_1$ except that items 3 and 6 are removed. The following result is not explicitly stated in [1], but it can be easily extracted from the proof of Theorem 5.5 there.

**Theorem 2.3** (see [1])

*Let $\varsigma$ be as before. Then the validity problem for $\mathscr{L}_1^\natural(\varsigma)$-sentences is $\Pi^1_1$-hard, even if we confine ourselves to at most countable domains.*

This lower bound turns out to be precise. Naturally, in the case of at most countable domains a direct upper bound argument applies. In the general case we need to show that, in addition, every satisfiable $\mathscr{L}_1^\natural(\varsigma)$-sentence is true in some $\mathscr{L}_1(\varsigma)$-structure $\mathcal{M}$ with $|D| \leqslant \aleph_0$ — this will be done in Section 7. So the corresponding problems are both $\Pi^1_1$-complete.

## 3 Probabilities on possible worlds

We now turn to Halpern's 'first-order' logic of probability of type 2, denoted by $\mathscr{L}_2$. See [10] for a similar logic with somewhat more general semantics.

Consider a signature $\varsigma$. An $\mathscr{L}_2(\varsigma)$-*structure* is a quadruple $\langle D, \Omega, \pi, \mathsf{p} \rangle$ where:

- $D$ and $\Omega$ are non-empty disjoint sets;
- $\pi$ is a function from $\Omega$ to the class of all $\varsigma$-structures with domain $D$;
- $\mathsf{p}$ is a discrete probability distribution on $\Omega$ — rather than on $D$.

[5] See Section 4 for the notion of $\Pi^2_1$-completeness and the like.

In this context, elements of $\Omega$ are viewed as *possible worlds*; by the *universe* we mean the union of $D$ and $\Omega$. In effect, the corresponding semantics limits the use of iterations of $\mu$ in $\mathscr{L}_2$, and a more flexible approach allows $\mathsf{p}$ to depend on the choice of a possible world; cf. [10]. Still, this difference turns out to be somewhat irrelevant to our development because our lower bound arguments will not require nested occurrences of $\mu$.

The sets $\mu\text{-Form}^2_\varsigma$ and $\mu\text{-Term}^2_\varsigma$, whose elements are called $\mathscr{L}_2(\varsigma)$*-formulas* and $\mathscr{L}_2(\varsigma)$*-terms*, are built up exactly as $\mu\text{-Form}^1_\varsigma$ and $\mu\text{-Term}^1_\varsigma$, except that the subscripts $\vec{x}$ are dropped: we use $\mu$ instead of $\mu_{\vec{x}}$ throughout. The related syntactic notions are defined exactly as in $\mathscr{L}_1$.

Next, consider an $\mathscr{L}_2(\varsigma)$-structure $\mathcal{M} = \langle D, \Omega, \pi, \mathsf{p}\rangle$. As before, by a *valuation in* $\mathcal{M}$ we mean a pair $\langle \zeta, \gamma\rangle$ where $\zeta$ and $\gamma$ are functions from $\mathrm{Var}$ and $\mathbf{Var}$ to $D$ and $\mathbb{R}$ respectively. Then

$$\mathcal{M}, \omega \Vdash \phi\,[\zeta, \gamma]$$

read as '$\phi$ is true at $\omega$ in $\mathcal{M}$ under $\langle \zeta, \gamma\rangle$', can be defined by induction on the depth of $\phi$. Naturally, in case $\phi$ is an atomic first-order $\varsigma$-formula, we employ

$$\mathcal{M}, \omega \Vdash \phi\,[\zeta, \gamma] \quad \Longleftrightarrow \quad \pi(\omega) \vDash \phi\,[\zeta].$$

Assuming $\mathrm{dp}(\phi) > 0$, the idea is that given an arbitrary valuation $\langle \eta, \delta\rangle$ in $\mathcal{M}$, we interpret each $\mu(\psi)$ with $\mathrm{dp}(\psi) < \mathrm{dp}(\phi)$ as

$$\mathsf{P}\left(\{\omega \in \Omega \mid \mathcal{M}, \omega \Vdash \psi\,[\eta, \delta]\}\right)$$

where $\mathsf{P}$ is the probability measure on the powerset of $\Omega$ generated by $\mathsf{p}$. Notice that whenever $\phi$ is regular, we have

$$\mathcal{M}, \omega \Vdash \phi\,[\zeta, \gamma] \quad \Longleftrightarrow \quad \mathcal{M}, \omega' \Vdash \phi\,[\zeta, \gamma] \ \text{ for each } \omega' \in \Omega,$$

and therefore $\mathcal{M}, \omega \Vdash \phi\,[\zeta, \gamma]$ may be abbreviated $\mathcal{M} \Vdash \phi\,[\zeta, \gamma]$. For example, let $\varsigma$ contain a unary predicate symbol $U$, and take

$$\phi(x, \mathfrak{a}) \ := \ \mathfrak{a} + \mathfrak{a} \leqslant \mu(U(x)).$$

Then $\mathcal{M} \Vdash \phi\,[\zeta, \gamma]$ iff $\mathsf{P}\left(\{\omega \in \Omega \mid \pi(\omega) \vDash U(\zeta(x))\}\right)$ is greater than or equal to $2\gamma(\mathfrak{a})$. Finally, an $\mathscr{L}_2(\varsigma)$-sentence is *valid* if it is true at every world of every $\mathscr{L}_2(\varsigma)$-structure.[6]

The complexity results for $\mathscr{L}_2$ are analogous to those for $\mathscr{L}_1$. Actually, the situation is worse: the lower bound proofs for $\mathscr{L}_2$ require only a unary predicate symbol, not a binary one.

**Theorem 3.1** (see [1])

*Let $\varsigma$ be $\langle U^1\rangle$ where $U$ is a unary predicate symbol. Then the validity problem for $\mathscr{L}_2(\varsigma)$-sentences is $\Pi^2_1$-complete. However, if we limit ourselves to at most countable universes, the corresponding problem becomes $\Pi^1_\infty$-complete.*

[6] Note, in passing, that since the foregoing semantics utilizes the same distribution $\mathsf{p}$ at all worlds, some iterations of $\mu$ become redundant. For instance, both $\mu(\mu(U(x)) \neq 0) \neq 0$ and $\mu(\mu(U(x)) \neq 0) = 1$ have the same meaning as $\mu(U(x)) \neq 0$. However, we shall deal mainly with $\mathscr{L}_2(\varsigma)$-formulas of depth 1.

Let $\mathscr{L}_2^\natural$ be the sublanguage of $\mathscr{L}_2$ obtained by excluding field variables. The argument for the following result is similar to that for Theorem 6.1.

**Theorem 3.2** (see [1])

*Let $\varsigma$ be as before. Then the validity problem for $\mathscr{L}_2^\natural(\varsigma)$-sentences is $\Pi_1^1$-hard, even if we confine ourselves to at most countable universes.*

Again, this lower bound turns out to be precise; a suitable analogue of the Löwenheim–Skolem theorem (needed in the general case) will be proved in Section 7.

**Remark 3.3.** Z. Ognjanović and his colleagues have developed suitable infinitary calculi for some languages similar to $\mathscr{L}_2^\sharp$; see [10] and [9, Chapters 1–2] for more information and background.[7]

# 4 Concerning higher-order arithmetic

In general, we shall assume familiarity with basic notions and methods of higher-order arithmetic; see, e.g., [11]. Therefore only a brief summary of some special results will be given below; cf. also [13] (which generalizes [4]), [14] and [16].

Recall that in second-order arithmetic, in addition to *individual variables* $x$, $y$, $z$, …, which are intended to range over $\mathbb{N}$, we have *$k$-ary set variables*

$$X^k, \ Y^k, \ Z^k, \ \ldots,$$

intended to range over the powerset of $\mathbb{N}^k$, for each positive natural number $k$. Hence the atomic second-order formulas additionally include all expressions of the form

$$X^k(t_1, \ldots, t_k)$$

where $t_1$, …, $t_k$ are terms. In what follows we shall write $X$ instead of $X^1$.

Let $\mathfrak{N}$ be the standard model of arithmetic presented in the signature $\sigma := \langle 0, \mathsf{s}, +, \cdot; = \rangle$. We write $\sigma_\mathsf{s}$ for the much smaller signature $\langle 0, \mathsf{s}; = \rangle$ and $\mathfrak{N}_\mathsf{s}$ for the $\sigma_\mathsf{s}$-reduct of $\mathfrak{N}$. Take

$$\sigma_\mathsf{s}^\sharp := \langle 0, \mathsf{s}; =, Y^2 \rangle$$

where $Y^2$ is treated as a binary predicate symbol. For each $S \subseteq \mathbb{N}^2$, denote by $\langle \mathfrak{N}_\mathsf{s}, S \rangle$ the $\sigma_\mathsf{s}^\sharp$-expansion of $\mathfrak{N}_\mathsf{s}$ in which $Y^2$ is interpreted as $S$.

We start with a relatively simple but important observation:

**Lemma 4.1** (see [18, Section 5])

[7] Intuitively, the reason why quantifiers over reals are avoided here is that probability logics with both quantifiers over reals and some other sort of quantifiers are usually at least as complex as complete second-order arithmetic, but reasonable infinitary calculi can only handle $\Pi_1^1$-sets; see [8] for details.

*There exist first-order $\sigma_{\mathsf{s}}^{\sharp}$-formulas $\Psi_{+}(x,y,z)$, $\Psi_{\cdot}(x,y,z)$ and a first-order $\sigma_{\mathsf{s}}^{\sharp}$-sentence $\Delta$ such that for every $S \subseteq \mathbb{N}^2$,*

$$\langle \mathfrak{N}_{\mathsf{s}}, S \rangle \vDash \Delta \quad \Longleftrightarrow \quad \begin{array}{c} \Psi_{+}(x,y,z) \text{ and } \Psi_{\cdot}(x,y,z) \text{ define} \\ \text{addition and multiplication respectively in } \langle \mathfrak{N}_{\mathsf{s}}, S \rangle. \end{array}$$

It leads to a number of nice definability and complexity results. We are going to mention only three of them, which will be utilized in Sections 5 and 6; see [18, Section 5] for more on this.

Recall that a second-order $\sigma$-formula is in $\Pi_1^1$ if it has the form $\forall \vec{X}\, \Psi$ where $\vec{X}$ is a tuple of set variables, and $\Psi$ contains no set quantifiers. Denote

$$\begin{aligned} \mathrm{Th}^2(\mathfrak{N}) &:= \text{ the full second-order theory of } \mathfrak{N}, \\ \Pi_1^1\text{-}\mathrm{Th}^2(\mathfrak{N}) &:= \text{ the } \Pi_1^1\text{-fragment of } \mathrm{Th}^2(\mathfrak{N}). \end{aligned}$$

Assume some Gödel numbering of the language of second-order arithmetic has been chosen, and hence its formulas may be identified with natural numbers. Then $S \subseteq \mathbb{N}$ is $\Pi_1^1$-*hard* if $\Pi_1^1$-$\mathrm{Th}^2(\mathfrak{N})$ (viewed as a subset of $\mathbb{N}$) is computably reducible to $S$, and $\Pi_n^1$-*complete* if the converse also holds; see [11] for discussion and alternative definitions.[8]

**Corollary 4.2**

*Let $\mathrm{S}_1^1$ denote the collection of all second-order $\sigma_{\mathsf{s}}$-sentences of the form $\forall Y^2\, \Psi$ where $Y^2$ is a binary set variable, and $\Psi$ contains no set quantifiers. Then $\{\Phi \in \mathrm{S}_1^1 \mid \mathfrak{N} \vDash \Phi\}$ is $\Pi_1^1$-complete.*

Further, as in [1], we say that $S \subseteq \mathbb{N}$ is $\Pi_\infty^1$-*hard* if $\mathrm{Th}^2(\mathfrak{N})$ is computably reducible to $S$, and $\Pi_\infty^1$-*complete* if the converse also holds.

**Corollary 4.3**

*Let $\mathrm{S}_\infty^1$ denote the collection of all second-order $\sigma_{\mathsf{s}}$-sentences of the form $\forall Y^2\, \Psi$ where $Y^2$ is a binary set variable, and $\Psi$ contains only unary set quantifiers. Then $\{\Phi \in \mathrm{S}_\infty^1 \mid \mathfrak{N} \vDash \Phi\}$ is $\Pi_\infty^1$-complete.*

In third-order arithmetic we also have *class variables*

$$\mathcal{X},\ \mathcal{Y},\ \mathcal{Z},\ \ldots,$$

intended to range over the powerset of the powerset of $\mathbb{N}$. It would be more accurate to call these *unary class variables*, but we shall not deal with class variables of greater arities. Hence the atomic third-order formulas additionally include all expressions of the form $\mathcal{X}(X)$.

A third-order $\sigma$-formula is in $\Pi_1^2$ if it has the form $\forall \vec{\mathcal{X}}\, \Psi$ where $\vec{\mathcal{X}}$ is a tuple of class variables, and $\Psi$ contains no class quantifiers. The definitions of $\Pi_1^2$-hardness and $\Pi_1^2$-completeness are like those of $\Pi_1^1$-hardness and $\Pi_1^1$-completeness.

[8] Here $S$ is *computably reducible* to $T$ if there exists a computable $f : \mathbb{N} \to \mathbb{N}$ (which can be thought of as an input transformation) such that for every $n \in \mathbb{N}$,

$$n \in S \quad \Longleftrightarrow \quad f(n) \in T.$$

**Corollary 4.4**

*Let* $\mathrm{S}_1^2$ *denote the collection of all third-order* $\sigma_{\mathsf{s}}$*-sentences of the form*

$$\forall \mathcal{X}\, \forall Y^2\, \Psi$$

*where* $\mathcal{X}$ *is a class variable,* $Y^2$ *is a binary set variable, and* $\Psi$ *contains only unary set quantifiers and no class quantifiers. Then* $\{\Phi \in \mathrm{S}_1^2 \mid \mathfrak{N} \vDash \Phi\}$ *is* $\Pi_1^2$*-complete.*

**Remark 4.5.** For our purposes, we shall use slightly modified versions of the last two corollaries. Namely, consider the arithmetical formula

$$\overline{\mathrm{Cof}}\,(X) \;:=\; \forall x\, \exists y\, (x < y \wedge \neg X\,(y)),$$

which says that $X$ is not co-finite, i.e. the complement of $X$ is not finite.[9] Unary set quantifiers are relativized to $\overline{\mathrm{Cof}}\,(X)$ in the obvious way:

$$\forall^{\star} X\, \Phi \;:=\; \forall X\, \big(\overline{\mathrm{Cof}}\,(X) \to \Phi\big) \quad \text{and} \quad \exists^{\star} X\, \Phi \;:=\; \exists X\, \big(\overline{\mathrm{Cof}}\,(X) \wedge \Phi\big).$$

Then since co-finite (as well as finite) subsets of $\mathbb{N}$ can be coded by natural numbers, it is straightforward to obtain the analogues of Corollaries 4.3 and 4.4 in which all unary set quantifiers are to be relativized to $\overline{\mathrm{Cof}}\,(X)$.

# 5 The case of structures of type 2

It is more convenient to start with $\mathscr{L}_2$, and then adapt the corresponding arguments to $\mathscr{L}_1$, though this is not crucial. We write $\mathrm{Form}_{\varsigma}^{\circ}$ for the set of all quantifier-free first-order $\varsigma$-formulas.

Call a regular $\mathscr{L}_2\,(\varsigma)$-formula *flat* if each of its basic subformulas has the form

$$\mu\,(\phi) \;=\; \mu\,(\psi) \quad \text{or} \quad \mu\,(\phi) \;\leqslant\; \mathfrak{a}$$

where $\phi$ and $\psi$ belong to $\mathrm{Form}_{\varsigma}^{\circ}$, and $\mathfrak{a}$ is a field variable. Naturally, we shall write $\mu\,(\phi) = 0$ and $\mu\,(\phi) = 1$ instead of $\mu\,(\phi) = \mu\,(\top)$ and $\mu\,(\phi) = \mu\,(\bot)$ respectively. Obviously, '$\mu\,(\phi) \;\leqslant\; \mathfrak{a}$' must be omitted in the case of $\mathscr{L}_2^{\natural}$.

Now Corollary 4.2 can be utilized to get:

**Theorem 5.1**

*Let* $\varsigma$ *be* $\langle U^1 \rangle$ *where* $U$ *is a unary predicate symbol. Then the validity problem for flat* $\mathscr{L}_2^{\natural}\,(\varsigma)$*-sentences is* $\Pi_1^1$*-hard, even if we confine ourselves to at most countable universes.*

*Proof.* The basic idea is the following. Imagine a family $\langle E_{ij} : i, j \in \mathbb{N} \rangle$ of pairwise disjoint events (in some probability space) with positive measures. Take

$$E_i \;:=\; \bigcup\nolimits_{j \in \mathbb{N}} E_{ij} \quad \text{and} \quad E_j^{*} \;:=\; \bigcup\nolimits_{i \in \mathbb{N}} E_{ij}.$$

[9] Obviously, the standard ordering relation on $\mathbb{N}$ is first-order definable in $\mathfrak{N}$. Moreover, it can be defined in $\mathfrak{N}_{\mathsf{s}}$ by the monadic second-order $\sigma_{\mathsf{s}}$-formula $\forall X\, (\forall u\, (X\,(u) \to X\,(\mathsf{s}\,(u))) \wedge X\,(x) \to X\,(y))$.

Intuitively, $E_i$ and $E_j^*$ can be viewed as the $i$th row and the $j$th column respectively; of course, $E_{ij}$ equals $E_i \cap E_j^*$. Let us interpret natural numbers as rows, and assume that the successor function on the rows is definable. Then, provided that we can switch from every row to the corresponding column, a given subset $S$ of $\mathbb{N}^2$ may be encoded as the event

$$E_S \;:=\; \bigcup_{(i,j)\in S} E_{ij}$$

– because $(i,j)$ belongs to $S$ iff $E_{ij}$ belongs to $E_S$, i.e. $E_i \cap E_j^* \cap E_S$ has a non-zero measure. To realize this idea within the flat fragment of $\mathscr{L}_2^{\natural}$, some additional machinery will be needed.

Consider an arbitrary $\mathscr{L}_2(\varsigma)$-structure $\mathcal{M} = \langle D, \Omega, \pi, \mathsf{p}\rangle$. With each $d \in D$, associate the corresponding event

$$[\![d]\!] \;:=\; \{\omega \in \Omega \mid \pi(\omega) \vDash U(d)\}.$$

Denote by $\mathscr{D}$ the collection of all these events. If $x$ is a variable, let us write $[x]$ for $U(x)$. When it comes to explaining the meaning of formulas, we may also use expressions like $[\![x]\!]$, which depend on the choice of valuation. For instance, $\mu([x]) \neq 0$ means that $[\![x]\!]$ has a non-zero measure: given a valuation $\langle \zeta, \gamma\rangle$ in $\mathcal{M}$, this formula holds iff $\mathsf{P}([\![\zeta(x)]\!])$ is not zero. Then the (flat) formula

$$x \approx y \;:=\; \mu(([x] \wedge \neg[y]) \vee ([y] \wedge \neg[x])) = 0$$

says 'the symmetric difference of $[\![x]\!]$ and $[\![y]\!]$ has measure zero'. For expository purposes, assume that $\mathsf{p}(\omega) > 0$ for all $\omega \in \Omega$. While this restriction is not necessary, it will make some descriptions below simpler. Thus $x \approx y$ means that $[\![x]\!]$ equals $[\![y]\!]$. So

$$x \preccurlyeq y \;:=\; \mu([x] \wedge \neg[y]) = 0$$

says '$[\![x]\!]$ is a subset of $[\![y]\!]$'. For convenience, take

$$\underline{\mathscr{D}} \;:=\; \text{the closure of } \mathscr{D} \text{ under finite intersection and complementation.}$$

Naturally, it can be viewed as a Boolean algebra. Observe that the formula

$$\mathrm{At}(x) \;:=\; \mu([x]) \neq 0 \wedge \forall y\, (\mu([x] \wedge [y]) \neq 0 \to \mu([x] \wedge [y]) = \mu([x]))$$

holds iff $[\![x]\!]$ is an atom of $\underline{\mathscr{D}}$, i.e. a minimal non-empty event in $\underline{\mathscr{D}}$. We shall also need the following formulas:

$$\begin{aligned}
\mathrm{Disj}_2(x,y) \;&:=\; \mu([x] \wedge [y]) = 0;\\
\mathrm{Disj}_3(x,y,z) \;&:=\; \mu([x] \wedge [y]) = \mu([x] \wedge [z]) = \mu([y] \wedge [z]) = 0;\\
\mathrm{DEq}_2(x,y) \;&:=\; \mathrm{Disj}_2(x,y) \wedge \mu([x]) = \mu([y]);\\
\mathrm{DEq}_3(x,y,z) \;&:=\; \mathrm{Disj}_3(x,y,z) \wedge \mu([x]) = \mu([y]) = \mu([z]);\\
\mathrm{Step}_2(x,y) \;&:=\; \exists y_1\, \exists y_2\, (\mathrm{DEq}_2(y_1,y_2) \wedge \mu([x]) = \mu([y_1] \vee [y_2]) \wedge \mu([y]) = \mu([y_1]));\\
\mathrm{Step}_3(x,y) \;&:=\; \exists y_1\, \exists y_2\, \exists y_3\, (\mathrm{DEq}_3(y_1,y_2,y_3) \;\wedge\\
&\qquad\qquad \mu([x]) = \mu([y_1] \vee [y_2] \vee [y_3]) \wedge \mu([y]) = \mu([y_1])).
\end{aligned}$$

Their meanings are clear. In effect, the purpose of $\mathrm{Step}_2(x, y)$ is to guarantee that the measure of $[\![y]\!]$ is two times smaller than that of $[\![x]\!]$ — but we have to do it in a special way to stay within the flat fragment. Similarly for $\mathrm{Step}_3(x, y)$. For technical reasons, suppose that $\mathcal{M}$ satisfies

$$\mathrm{Tech} \;:=\; \forall u\, (\mathrm{At}(u) \to \exists v\, (\mathrm{At}(v) \wedge \mathrm{Step}_2(u, v)) \wedge \exists v\, (\mathrm{At}(v) \wedge \mathrm{Step}_3(u, v))).$$

With Tech in mind, the formula

$$\mathrm{Ind}_2(x) \;:=\; \forall u\, (\mathrm{At}(u) \wedge u \preccurlyeq x \to \exists v\, (\mathrm{At}(v) \wedge v \preccurlyeq x \wedge \mathrm{Step}_2(u, v)))$$

holds iff for every atom $[\![u]\!]$ (of $\mathscr{D}$) below $[\![x]\!]$ there exists an atom $[\![v]\!]$ below $[\![x]\!]$ whose measure is two times smaller than that of $[\![u]\!]$. Then

$$\begin{aligned}\mathrm{Seq}_2(u, x) \;:=\;& \mathrm{At}(u) \wedge u \preccurlyeq x \wedge \mathrm{Ind}_2(x) \;\wedge \\ & \forall v_1\, \forall v_2\, (\mathrm{At}(v_1) \wedge \mathrm{At}(v_2) \wedge v_1 \preccurlyeq x \wedge v_2 \preccurlyeq x \to \neg \mathrm{DEq}_2(v_1, v_2)) \wedge \\ & \forall v\, (\mathrm{At}(v) \wedge v \preccurlyeq x \wedge \mu([v]) \neq \mu([u]) \to \exists w\, (\mathrm{At}(w) \wedge w \preccurlyeq x \wedge \mathrm{Step}_2(w, v)))\end{aligned}$$

means that $[\![u]\!]$ is an atom, and $[\![x]\!]$ is a minimal event above $[\![u]\!]$ satisfying $\mathrm{Ind}_2(x)$. Similarly, we can obtain $\mathrm{Ind}_3(x)$ and $\mathrm{Seq}_3(u, x)$ using $\mathrm{Step}_3(x, y)$, or $\mathrm{Ind}_6(x)$ and $\mathrm{Seq}_6(u, x)$ via

$$\mathrm{Step}_6(x, y) \;:=\; \exists z\, (\mathrm{Step}_2(x, z) \wedge \mathrm{Step}_3(z, y)).$$

Finally, we need the formula

$$\begin{aligned}\mathrm{Base}(x_a, x_b, x_c) \;:=\;& \mathrm{Disj}_3(x_a, x_b, x_c) \wedge \\ & \mu([x_a] \vee [x_b] \vee [x_c]) = 1 \wedge \mu([x_a] \vee [x_b]) = \mu([x_c]) \wedge \mu([x_a]) = \mu([x_b]) \wedge \\ & \exists u\, (\mathrm{At}(u) \wedge \mathrm{Step}_2(x_a, u) \wedge u \preccurlyeq x_a) \wedge \mathrm{Ind}_2(x_a) \wedge \\ & \exists u\, (\mathrm{At}(u) \wedge \mathrm{Step}_2(x_b, u) \wedge u \preccurlyeq x_b) \wedge \mathrm{Ind}_2(x_b) \wedge \\ & \exists u\, (\mathrm{At}(u) \wedge \mathrm{Step}_3(x_c, u) \wedge u \preccurlyeq x_c) \wedge \mathrm{Ind}_2(x_c) \wedge \mathrm{Ind}_3(x_c)\,.\end{aligned}$$

It guarantees that:

- $[\![x_a]\!]$, $[\![x_b]\!]$ and $[\![x_c]\!]$ are pairwise disjoint;
- the measures of $[\![x_a]\!]$, $[\![x_b]\!]$ and $[\![x_c]\!]$ are equal to $1/4$, $1/4$ and $1/2$;
- there exist sequences $\langle a_i : i \in \mathbb{N} \rangle$ and $\langle b_i : i \in \mathbb{N} \rangle$ of elements of $D$ such that

  $$[\![x_a]\!] \;=\; \bigcup\nolimits_{i \in \mathbb{N}} [\![a_i]\!] \quad \text{and} \quad [\![x_b]\!] \;=\; \bigcup\nolimits_{i \in \mathbb{N}} [\![b_i]\!],$$

  and further, each $[\![a_i]\!]$ and $[\![b_i]\!]$ is an atom of measure $1/2^{i+3}$;
- there exists a family $\langle c_{ij} : i, j \in \mathbb{N} \rangle$ of elements of $D$ such that

  $$[\![x_c]\!] \;=\; \bigcup\nolimits_{i,j \in \mathbb{N}} [\![c_{ij}]\!],$$

  and further, each $[\![c_{ij}]\!]$ is an atom of measure $1/\left(2^{i+1} \cdot 3^{j+1}\right)$.

Clearly, in case $\mathrm{Base}\,(x_a, x_b, x_c)$ holds, every atom has the form $[\![a_i]\!]$ or $[\![b_j]\!]$ or $[\![c_{ij}]\!]$, since

$$2 \cdot \sum_{i\in\mathbb{N}} \frac{1}{2^{i+3}} + \sum_{i,j\in\mathbb{N}} \frac{1}{2^{i+1}\cdot 3^{j+1}} \;=\; \frac{1}{2} \cdot \sum_{i\in\mathbb{N}} \frac{1}{2^{i+1}} + \sum_{i\in\mathbb{N}} \frac{1}{2^{i+1}} \cdot \sum_{j\in\mathbb{N}} \frac{1}{3^{j+1}} \;=\; \frac{1}{2} + \frac{1}{2} \;=\; 1.$$

Moreover, each of the $[\![c_{ij}]\!]$'s is uniquely determined by its measure. In particular, $[\![c_{00}]\!]$ can be captured by

$$\mathrm{Start}\,(x) \;:=\; \mathrm{At}\,(x) \wedge \exists y\, (\mu\,([y]) = \mu\,(\neg[y]) \wedge \mathrm{Step}_3\,(y, x)).$$

In fact, the atoms below $[\![x_a]\!]$ and $[\![x_b]\!]$ play supporting roles. For instance, $\mathrm{Step}_3\,([\![c_{ij}]\!], [\![c_{ij+1}]\!])$ can be justified by finding $S \subseteq \mathbb{N}$ such that

$$\frac{1}{2^{i+1}\cdot 3^{j+2}} \;=\; \sum_{k\in S} \frac{1}{2^{k+3}}$$

and extending $\mathscr{D}$ to contain both $\bigcup_{k\in S}[\![a_k]\!]$ and $\bigcup_{k\in S}[\![b_k]\!]$. However, we shall be mainly concerned with $[\![x_c]\!]$, which will conveniently be viewed as an infinite matrix: for any $i, j \in \mathbb{N}$,

$$C_i \;:=\; \bigcup\{[\![c_{ij}]\!] \mid j \in \mathbb{N}\} \quad \text{and} \quad C^*_j \;:=\; \bigcup\{[\![c_{ij}]\!] \mid i \in \mathbb{N}\}$$

correspond to the $i$th row and $j$th column respectively; the diagonal of this matrix is

$$E \;:=\; \bigcup\{[\![c_{ii}]\!] \mid i \in \mathbb{N}\}.$$

To make sure that all the rows, the columns and the diagonal belong to $\mathscr{D}$, one can add

$$\begin{aligned}\mathrm{Aux} \;:=\; & \exists u\, \exists y\, (\mathrm{Start}\,(u) \wedge \mathrm{Seq}_2\,(u, y) \wedge \forall v\, (\mathrm{At}\,(v) \wedge v \preccurlyeq y \to \exists z\, \mathrm{Seq}_3\,(v, z))) \wedge \\ & \exists u\, \exists y\, (\mathrm{Start}\,(u) \wedge \mathrm{Seq}_3\,(u, y) \wedge \forall v\, (\mathrm{At}\,(v) \wedge v \preccurlyeq y \to \exists z\, \mathrm{Seq}_2\,(v, z))) \wedge \\ & \exists u\, \exists y\, (\mathrm{Start}\,(u) \wedge \mathrm{Seq}_6\,(u, y)).\end{aligned}$$

which guarantees, in particular, that for some $c_0, c_1, \ldots$ and $c^*_0, c^*_1, \ldots$,

$$[\![c_0]\!] = C_0, \;\; [\![c_1]\!] = C_1, \;\; \ldots \quad \text{and} \quad [\![c^*_0]\!] = C^*_0, \;\; [\![c^*_1]\!] = C^*_1, \;\; \ldots$$

Thus we are going to deal with $\mathscr{L}_2\,(\varsigma)$-structures that satisfy the sentence

$$\mathrm{Req} \;:=\; \mathrm{Tech} \wedge \exists x_a\, \exists x_b\, \exists x_c\, \mathrm{Base}\,(x_a, x_b, x_c) \wedge \mathrm{Aux}.$$

It is straightforward to check that such structures do exist; we shall call them *admissible*. Further, for every $S \subseteq \mathbb{N}^2$ there exists an admissible $\mathcal{M}$ such that

$$\bigcup_{(i,j)\in S} [\![c_{ij}]\!] \;\in\; \mathscr{D}.$$

This will allow us to interpret a free binary predicate on the natural numbers.

Now consider the following formulas:

$$\begin{aligned}\mathrm{Row}^0\,(x) \;&:=\; \exists u\, (\mathrm{Start}\,(u) \wedge \mathrm{Seq}_3\,(u, x));\\ \mathrm{Col}^0\,(x) \;&:=\; \exists u\, (\mathrm{Start}\,(u) \wedge \mathrm{Seq}_2\,(u, x));\\ \mathrm{Row}\,(x) \;&:=\; \exists y\, \exists u\, \big(\mathrm{Col}^0\,(y) \wedge \mathrm{At}\,(u) \wedge u \preccurlyeq y \wedge \mathrm{Seq}_3\,(u, x)\big);\\ \mathrm{Col}\,(x) \;&:=\; \exists y\, \exists u\, \big(\mathrm{Row}^0\,(y) \wedge \mathrm{At}\,(u) \wedge u \preccurlyeq y \wedge \mathrm{Seq}_2\,(u, x)\big);\\ \mathrm{Diag}\,(x) \;&:=\; \exists u\, (\mathrm{Start}\,(u) \wedge \mathrm{Seq}_6\,(u, x));\\ \mathrm{Match}\,(x, y) \;&:=\; \exists z\, (\mathrm{Diag}\,(z) \wedge \mu\,([x] \wedge [y] \wedge [z]) \neq 0).\end{aligned}$$

Their meanings are clear. Note that $\mathrm{Match}\,(x, y)$ can be used to switch from rows to columns, and vice versa: if $\mathcal{M}$ is an admissible $\mathscr{L}_2\,(\varsigma)$-structure, then for any $i, j \in \mathbb{N}$,

$$\mathcal{M} \Vdash \mathrm{Match}\left(c_i, c_j^*\right) \quad \iff \quad i \;=\; j.$$

Let us think of natural numbers as rows. Hence the successor function is captured by $\mathrm{Step}_2\,(x, y)$. To interpret a binary set variable, we introduce

$$\Gamma\,(x, y, z) \;:=\; \exists y^*\,(\mathrm{Col}\,(y^*) \wedge \mathrm{Match}\,(y, y^*) \wedge \mu\,([x] \wedge [y^*] \wedge [z]) \neq 0).$$

To see how it works, observe that for every $S \subseteq \mathbb{N}^2$,

$$S \;=\; \left\{(i, j) \in \mathbb{N}^2 \mid \mathcal{M} \Vdash \Gamma\,(c_i, c_j, s)\right\},$$

provided that $\mathcal{M}$ is admissible, $\bigcup_{(i,j)\in S}[\![c_{ij}]\!]$ belongs to $\mathscr{D}$ and equals $[\![s]\!]$. Thus elements of $\mathscr{D}$ may be treated as binary relations on $\mathbb{N}$.

We are ready to show the $\Pi^1_1$-hardness of the validity problem for flat $\mathscr{L}_2^\natural\,(\varsigma)$-sentences. Let $\Phi$ be a $\sigma_{\mathsf{s}}$-sentence in $\mathrm{S}^1_1$; so it has the form $\forall Y^2\,\Psi$ where $\Psi$ contains no set variables. Without loss of generality, we may assume that:

- each atomic subformula of $\Psi$ has the form

$$x = y \quad \text{or} \quad x = 0 \quad \text{or} \quad \mathsf{s}\,(x) = y \quad \text{or} \quad Y^2\,(x, y);$$

- $\vee$ and $\exists$ do not occur in $\Psi$, although $\wedge$, $\neg$ and $\forall$ may occur in it.

For convenience, the set variable $Y^2$ will also be treated as distinguished individual variable. Now define $\tau\,(\Psi)$ recursively:

$$\begin{aligned}
\tau\,(x = y) \;&:=\; \mu\,([x]) = \mu\,([y]);\\
\tau\,(x = 0) \;&:=\; \mathrm{Row}^0\,(x);\\
\tau\,(\mathsf{s}\,(x) = y) \;&:=\; \mathrm{Step}_2\,(x, y);\\
\tau\left(Y^2\,(x, y)\right) \;&:=\; \Gamma\left(x, y, Y^2\right);\\
\tau\,(\Theta \wedge \Xi) \;&:=\; \tau\,(\Theta) \wedge \tau\,(\Xi);\\
\tau\,(\neg\Theta) \;&:=\; \neg\tau\,(\Theta);\\
\tau\,(\forall x\,\Theta) \;&:=\; \forall x\,(\mathrm{Row}\,(x) \to \tau\,(\Theta)).
\end{aligned}$$

By construction, $\tau\,(\Psi)$ is always flat. And it is straightforward to verify that

$$\mathfrak{N} \vDash \Phi \quad \iff \quad \mathrm{Req} \to \forall Y^2\,\tau\,(\Psi) \;\text{ is valid.}$$

Finally, apply Corollary 4.2. □

If we allow quantifiers over reals, then Corollary 4.3 can be used to obtain:

**Theorem 5.2**

*Let $\varsigma$ be as before. Then the validity problem for flat $\mathscr{L}_2(\varsigma)$-sentences is $\Pi^1_\infty$-hard, even if we confine ourselves to at most countable universes.*

*Proof.* We shall employ the notation of the proof of Theorem 5.1. Again, $\Gamma(x, y, z)$ will be used to interpret a free binary set variable, which is intuitively bounded by the outermost universal quantifier. As for unary set variables (each of which may be bounded by $\forall$ or $\exists$), they will be handled by means of field variables.

For technical reasons, $\mathrm{Req}$ has to be extended slightly. Namely, we need to add the condition that ensures that the collection $\mathscr{D}$ contains all finite unions of rows. To this end, take

$$\mathrm{Rows}(x) \;:=\; \forall u\,(\mathrm{At}(u) \wedge u \preccurlyeq x \to \exists v\,(\mathrm{Row}(v) \wedge u \preccurlyeq v \preccurlyeq x)).$$

Thus $\mathrm{Rows}(x)$ means that $[\![x]\!]$ is a union of rows. Obviously, the formula

$$\mathrm{Join}(x, y, z) \;:=\; \mu\,((([x] \vee [y]) \wedge \neg[z]) \vee ([z] \wedge \neg([x] \vee [y]))) = 0.$$

says '$[\![z]\!]$ is the union of $[\![x]\!]$ and $[\![y]\!]$'. Hence the sentence

$$\mathrm{Aux}' \;:=\; \forall x\,\forall y\,(\mathrm{Rows}(x) \wedge \mathrm{Rows}(y) \to \exists z\,\mathrm{Join}(x, y, z))$$

does the job. Now let $\mathrm{Req}'$ denote $\mathrm{Req} \wedge \mathrm{Aux}'$. In what follows $\mathscr{L}_2(\varsigma)$-structures satisfying $\mathrm{Req}'$ will be called *acceptable*.

As is well known, every $\varepsilon \in [0, 1/2)$ can be uniquely represented as

$$\varepsilon \;=\; \sum_{i=0}^{\infty} \frac{\varepsilon_i}{2^{i+2}}$$

where each $\varepsilon_i$ is either 0 or 1, and the sequence $\varepsilon_0, \varepsilon_1, \ldots$ contains infinitely many 0's. It is easy to verify that for all $k \in \mathbb{N}$,

$$\varepsilon_k \;=\; 1 \quad\iff\quad \sum_{i=0}^{k-1} \frac{\varepsilon_i}{2^{i+2}} + \frac{1}{2^{k+2}} \;\leqslant\; \varepsilon.$$[10]

For our purposes, we can view $\varepsilon$ as the set $\{i \in \mathbb{N} \mid \varepsilon_i = 1\}$, which gives us all subsets of $\mathbb{N}$ whose complements are not finite. To make this idea work, we need the formula

$$\begin{aligned}\mathrm{Init}(x) \;:=\; \mathrm{Rows}(x) \wedge \forall v\,\big(\mathrm{Row}(v) \wedge v \preccurlyeq x \wedge \neg\mathrm{Row}^0(v) \to \\ \exists w\,(\mathrm{Row}(w) \wedge w \preccurlyeq x \wedge \mathrm{Step}_2(w, v))\big),\end{aligned}$$

which says that $[\![x]\!]$ is an initial segment of the rows (thought of as natural numbers). Hence if $[\![u]\!]$ and $[\![v]\!]$ are rows, then

$$\mathrm{Geq}(u, v) \;:=\; \forall x\,(\mathrm{Init}(x) \wedge v \preccurlyeq x \to u \preccurlyeq x)$$

[10] Here we identify the empty sum (in case $k = 0$) with 0.

means that the probability of $[\![u]\!]$ is more than or equal to that of $[\![v]\!]$. So

$$\mathrm{Upper}\,(u,x) \;:=\; \mathrm{Init}\,(x) \wedge u \preccurlyeq x \wedge \forall y\,(\mathrm{Init}\,(y) \wedge u \preccurlyeq y \to x \preccurlyeq y).$$

holds iff $[\![x]\!]$ is the union of all rows that are at least as probable as $[\![u]\!]$. Notice that if $\mu\,([u]) \neq 0$, then the corresponding union contains only finitely many rows, and hence belongs to $\mathscr{D}$ by $\mathrm{Aux}'$. Next, let

$$\begin{aligned}\mathrm{Approx}\,(x,\mathfrak{a},u) \;:=\;& \forall v\,(\mathrm{Row}\,(v) \wedge v \preccurlyeq x \to \mathrm{Geq}\,(v,u)) \wedge \\ & \exists v\,\big(\mathrm{Row}^0\,(v) \wedge (v \preccurlyeq x \leftrightarrow \mu\,([v]) \leqslant \mathfrak{a})\big) \wedge \\ & \forall v\,(\mathrm{Row}\,(v) \wedge \mathrm{Geq}\,(v,u) \to \\ & \qquad (v \preccurlyeq x \leftrightarrow \exists y\,(\mathrm{Upper}\,(v,y) \wedge \mu\,(([x] \wedge [y]) \vee [v]) \leqslant \mathfrak{a}))).\end{aligned}$$

It is not hard to check that for any $k \in \mathbb{N}$, $S \subseteq \mathbb{N}$ and $\varepsilon \in [0, 1/2)$,

$$\mathcal{M} \Vdash \mathrm{Approx}\,(s,\varepsilon,c_k) \quad\Longleftrightarrow\quad S \;=\; \{i \in \mathbb{N} \mid i \leqslant k \text{ and } \varepsilon_i = 1\},$$

provided that $\mathcal{M}$ is acceptable, $\bigcup\{C^i \mid i \in S\}$ belongs to $\mathscr{D}$ and coincides with $[\![s]\!]$. Furthermore, the sentence

$$\mathrm{Req}' \to \forall\mathfrak{a}\,\forall u\,(0 \leqslant \mathfrak{a} < 1/2 \wedge \mathrm{Row}\,(u) \to \exists x\,\mathrm{Approx}\,(x,\mathfrak{a},u))$$

is valid.[11] Consequently, for the formula

$$\Sigma\,(u,\mathfrak{a}) \;:=\; \exists x\,(\mathrm{Approx}\,(x,\mathfrak{a},u) \wedge u \preccurlyeq x)$$

we have $\mathcal{M} \Vdash \Sigma\,(c_k,\varepsilon)$ iff $\varepsilon_k = 1$. Thus $\varepsilon$ plays the role of $\bigcup\{C_i \mid \varepsilon_i = 1\}$, even though the latter does not necessarily belong to $\mathscr{D}$.

We are ready to show the $\Pi^1_\infty$-hardness of the validity problem for flat $\mathscr{L}_2\,(\varsigma)$-sentences. Let $\Phi$ be a $\sigma_s$-sentence in $\mathsf{S}^1_\infty$. It has the form $\forall Y^2\,\Psi$ with $\Psi$ containing no set variables. By Remark 4.5, we may assume that

$$\text{all unary set quantifiers in } \Psi \text{ are relativized to } \overline{\mathrm{Cof}}\,(X).$$

Next, with each unary set variable $X$, associate a distinguished field variable $\mathfrak{a}$. We then extend the definition of $\tau\,(\Psi)$ given in the proof of Theorem 5.1 by adding two more cases:

$$\begin{aligned}\tau\,(X\,(x)) \;&:=\; \Sigma\,(x,\mathfrak{a});\\ \tau\,(\forall^\star X\,\Theta) \;&:=\; \forall\mathfrak{a}\,(0 \leqslant \mathfrak{a} < 1/2 \to \tau\,(\Theta)).\end{aligned}$$

By construction, $\tau\,(\Phi)$ is a flat $\mathscr{L}_2\,(\varsigma)$-sentence. And it is straightforward to check that

$$\mathfrak{N} \vDash \Phi \quad\Longleftrightarrow\quad \mathrm{Req}' \to \forall Y^2\,\tau\,(\Phi) \;\text{ is valid.}$$

Finally, apply the corresponding analogue of Corollary 4.3. □

[11] Here $0 \leqslant \mathfrak{a} < 1/2$ can be understood as $\mu\,(\bot) \leqslant \mathfrak{a} \wedge \exists x\,(\mu\,([x]) = \mu\,(\neg[x]) \wedge \neg\mu\,([x]) \leqslant \mathfrak{a})$.

In effect, Corollary 4.4 allows us to get a bit more:

**Theorem 5.3**

*Let $\varsigma$ be as before. Then the validity problem for flat $\mathscr{L}_2(\varsigma)$-sentences is $\Pi^2_1$-hard.*

*Proof.* Notice that if $\mathcal{M}$ is an admissible $\mathscr{L}_2(\varsigma)$-structure, then the corresponding matrix $[\![x_c]\!]$ can be captured by

$$\mathrm{Matrix}(x) \;:=\; \exists u\,(\mathrm{Start}(u) \wedge u \preccurlyeq x \wedge \mathrm{Ind}_2(x) \wedge \mathrm{Ind}_3(x) \wedge \mu([x]) = \mu(\neg[x])).$$

Now consider the formula

$$\mathrm{Spec}(\mathfrak{a}) \;:=\; \exists x\,(\mu([x]) = \mathfrak{a} \wedge \exists y\,(\mathrm{Matrix}(y) \wedge \mathrm{Disj}_2(x, y))).$$

Naturally, we are going to use it to translate $X \in \mathcal{X}$, which will lead us to the desired $\Pi^2_1$-hardness result. Call $S \subseteq \mathbb{N}$ *suitable* if its complement is not finite and $\sum_{i \in S} 1/2^{i+2} \notin \mathbb{Q}$. Roughly, we want $\sum_{i \in S} 1/2^{i+2}$ to be irrational here because $\mathrm{Req}'$ implies that certain real numbers, which can be assumed to be rational, satisfy $\mathrm{Spec}(\mathfrak{a})$. Take

$$\mathbb{S} \;:=\; \text{the collection of all suitable subsets of } \mathbb{N}.$$

Observe that for any $S \subseteq \mathbb{N}^2$ and $\mathscr{S} \subseteq \mathbb{S}$ there exists an acceptable $\mathscr{L}_2(\varsigma)$-structure $\mathcal{M}$ such that

$$\bigcup\nolimits_{(i,j) \in S} [\![c_{ij}]\!] \;\in\; \mathscr{D} \quad \text{and} \quad \{T \in \mathbb{S} \mid \mathcal{M} \Vdash \mathrm{Spec}(\epsilon_T)\} \;=\; \mathscr{S}$$

where $\epsilon_T$ denotes $\sum_{i \in T} 1/2^{i+2}$. This will allow us to interpret a free unary predicate on $\mathbb{S}$.

Let $\Phi$ be a $\sigma_s$-sentence in $\mathsf{S}^2_1$. So it has the form

$$\forall \mathcal{X}\, \forall Y^2\, \Psi$$

where $\Psi$ contains only unary set variables and no class variables. Again, we may assume that all unary set quantifiers in $\Psi$ are relativized to $\overline{\mathrm{Cof}}(X)$. Furthermore, since there is an analytical (i.e., definable in $\mathfrak{N}$ by a second-order formula) one-one function from the powerset of $\mathbb{N}$ onto $\mathbb{S}$, it can also be assumed that

$$\text{the class variable } \mathcal{X} \text{ ranges over the subsets of } \mathbb{S}.$$

Then we extend the definition of $\tau(\Psi)$ given in the proof of Theorem 5.2 by adding

$$\tau(\mathcal{X}(X)) \;:=\; \mathrm{Spec}(\mathfrak{a}).$$

It is easy to see that $\mathfrak{N} \vDash \Phi$ iff $\mathrm{Req}' \to \forall Y^2\, \tau(\Psi)$ is valid. Now apply the corresponding analogue of Corollary 4.4. □

Thus we have strengthened Theorems 3.1 and 3.2 in a crucial way.

## 6 The case of structures of type 1

The machinery developed in the previous section can be easily adapted to $\mathscr{L}_1$ by using the translation described in the proof of Theorem 6.2 in [1], which allows us to transfer complexity results from $\mathscr{L}_2$ to $\mathscr{L}_1$. To keep our presentation reasonably self-contained, we are going to sketch more direct arguments below.

Fix a special individual variable $\underline{u}$. Call a regular $\mathscr{L}_1(\varsigma)$-formula *flat* if each of its basic subformulas is of the form

$$\mu_{\underline{u}}(\phi) \;=\; \mu_{\underline{u}}(\psi) \quad \text{or} \quad \mu_{\underline{u}}(\phi) \;\leqslant\; \mathfrak{a}$$

where $\phi$ and $\psi$ belong to $\mathrm{Form}^{\circ}_{\varsigma}$, and $\mathfrak{a}$ is a field variable. Obviously, '$\mu_{\underline{u}}(\phi) \leqslant \mathfrak{a}$' must be omitted in the case of $\mathscr{L}_1^{\natural}$, i.e. if we exclude field variables.

**Theorem 6.1**

*Let $\varsigma$ be $\langle R^2 \rangle$ where $R$ is a binary predicate symbol. Then the validity problem for flat $\mathscr{L}_1^{\natural}(\varsigma)$-sentences is $\Pi^1_1$-hard, even if we confine ourselves to at most countable domains.*

*Proof.* Consider an arbitrary $\mathscr{L}_1(\varsigma)$-structure $\mathcal{M} = \langle D, \pi, \mathsf{p} \rangle$. Now, with each $d \in D$, associate the corresponding event

$$[\![d]\!] \;:=\; \{e \in D \mid \pi \models R(d, e)\}.$$

Denote by $\mathscr{D}$ the collection of all such events. If $x$ is a variable distinct from $\underline{u}$, let us write $[x]$ for $R(x, \underline{u})$. Here $R(x, \underline{u})$ may be read as '$x$ satisfies $P$ at $\underline{u}$'; so $\underline{u}$ is viewed as ranging over 'worlds'. Then we proceed as in the proof of Theorem 5.1, except that:

- $\mu$ is replaced by $\mu_{\underline{u}}$ throughout;
- the expressions $[\underline{u}]$, $\forall \underline{u}$ and $\exists \underline{u}$ are avoided. □

**Theorem 6.2**

*Let $\varsigma$ be as before. Then the validity problem for flat $\mathscr{L}_1(\varsigma)$-sentences is $\Pi^1_\infty$-hard, even if we confine ourselves to at most countable domains.*

*Proof.* Similar to the proof of Theorem 5.2 (but using the modified notation described in the proof of Theorem 6.1). □

**Theorem 6.3**

*Let $\varsigma$ be as before. Then the validity problem for flat $\mathscr{L}_1(\varsigma)$-sentences is $\Pi^2_1$-hard.*

*Proof.* Similar to the proof of Theorem 5.3. □

Thus we have strengthened Theorems 2.1 and 2.3 in a crucial way.

**Remark 6.4.** In effect, the necessity of having at least one binary predicate symbol in the case of $\mathscr{L}_1$ is justified by Theorem 5.1 in [1], which says that if $\varsigma$ consists of unary predicate symbols, then the validity problem for $\mathscr{L}_1(\varsigma)$-sentences is decidable. Moreover, we cannot obtain the complexity results in Section 5 from the those in this section by using the translation of $\mathscr{L}_1$ into $\mathscr{L}_2$ described in [1] (see Theorem 6.1), since the latter does not in any way make signatures smaller, so there is no hope of turning $\langle R^2 \rangle$ into $\langle U^1 \rangle$.

# 7 Concerning the upper bounds

It follows from [1] that $\Pi^1_2$ is an upper bound for the validity problems for $\mathscr{L}_1$ and $\mathscr{L}_2$, and if only at most countable structures are allowed, then $\Pi^1_2$ may be replaced by $\Pi^1_\infty$.[12] As for $\mathscr{L}_1^\natural$ and $\mathscr{L}_2^\natural$, if uncountable structures are not directly excluded, then a bit more work is needed.

Let $\mathcal{M}$ and $\mathcal{M}'$ be $\mathscr{L}_1(\varsigma)$-structures. We shall say that $\mathcal{M}'$ is a *substructure* of $\mathcal{M}$ if the following two conditions are met:

i. $\pi'$ is a substructure of $\pi$, as defined in first-order logic;

ii. $\mathsf{p}'$ is the restriction of $\mathsf{p}$ to $D'$.

Clearly, (ii) implies that $D \setminus D'$ has measure zero, i.e. $\{d \in D \mid \mathsf{p}(d) \neq 0\}$ must be a subset of $D'$. It turns out that the standard argument for the Löwenheim–Skolem theorem (in first-order logic) can be easily adapted to $\mathscr{L}_1^\natural$:

**Theorem 7.1**

*Let $\mathcal{M}$ be an $\mathscr{L}_1(\varsigma)$-structure, and $\kappa$ be a cardinal such that $\max\{|\varsigma|, \aleph_0\} \leqslant \kappa \leqslant |D|$. Then there exists a substructure $\mathcal{M}_\circ$ of $\mathcal{M}$ such that $D_\circ$ has cardinality $\kappa$, and for every $\mathscr{L}_1^\natural(\varsigma)$-formula $\phi$ and any function $\zeta$ from $\mathrm{Var}$ to $D_\circ$,*

$$\mathcal{M}_\circ \Vdash \phi[\zeta] \quad \Longleftrightarrow \quad \mathcal{M} \Vdash \phi[\zeta].$$

*Proof.* Without loss of generality, we may assume that $\varsigma$ contains the equality symbol. Let $S_0$ be a subset of $D$ that includes $\{d \in D \mid \mathsf{p}(d) \neq 0\}$ and has cardinality $\kappa$. As in first-order logic, we can build a sequence

$$S_0 \subseteq S_1 \subseteq S_2 \ \ldots$$

of subsets of $D$ of cardinality $\kappa$ such that for each $n \in \mathbb{N}$, every $\mathscr{L}_1^\natural(\varsigma)$-formula $\phi(x_1, \ldots, x_m, y)$ and any $d_1, \ldots, d_m \in S_n$,

$$\mathcal{M} \Vdash \exists y\, \phi(d_1, \ldots, d_m, y) \quad \Longrightarrow \quad \mathcal{M} \Vdash \phi(d_1, \ldots, d_m, e) \text{ for some } e \in S_{n+1}. \qquad (\star)$$

[12] Of course, when we speak of algorithmic problems, all signatures are assumed to be computable.

Take $D_\circ$ to be $\bigcup \{S_n \mid n \in \mathbb{N}\}$. Clearly, $D_\circ$ has cardinality $\kappa$. Further, $D_\circ$ contains all constants of $\pi$ and is closed under all functions of $\pi$.[13] Now consider the $\mathscr{L}_1(\varsigma)$-structure

$$\mathcal{M}_\circ \;=\; \langle D_\circ, \pi_\circ, \mathsf{p}_\circ \rangle$$

where $\pi_\circ$ is the substructure of $\pi$ with domain $D_\circ$, and $\mathsf{p}_\circ$ is the restriction of $\mathsf{p}$ to $D_\circ$. Evidently, $\mathcal{M}_\circ$ is a substructure of $\mathcal{M}$. We want to show that for every $\mathscr{L}_1^\natural(\sigma)$-formula $\phi$ and any function $\zeta$ from $\mathrm{Var}$ to $D_\circ$,

$$\mathcal{M}_\circ \Vdash \phi[\zeta] \quad \Longleftrightarrow \quad \mathcal{M} \Vdash \phi[\zeta]\,.$$

This can be done by induction on the depth of $\phi$. The main task here is to take care of the $\mathscr{L}^\natural(\varsigma)$-terms of the form $\mu_{(x_1,\ldots,x_k)}(\psi)$ occurring in $\phi$. Observe that the inductive hypothesis implies that for each $\mathscr{L}_1^\natural(\varsigma)$-formula $\psi$ with $\mathrm{dp}(\psi) < \mathrm{dp}(\phi)$ and any function $\eta$ from $\mathrm{Var}$ to $D_\circ$,

$$\begin{aligned}
\mathsf{P}^k\left(\left\{\vec{d} \in D^k \mid \mathcal{M} \Vdash \psi\left[\eta^{\vec{x}}_{\vec{d}}\right]\right\}\right) &= \mathsf{P}^k\left(\left\{\vec{d} \in D_\circ^k \mid \mathcal{M} \Vdash \psi\left[\eta^{\vec{x}}_{\vec{d}}\right]\right\}\right)\\
&= \mathsf{P}^k\left(\left\{\vec{d} \in D_\circ^k \mid \mathcal{M}_\circ \Vdash \psi\left[\eta^{\vec{x}}_{\vec{d}}\right]\right\}\right)\\
&= \mathsf{P}_\circ^k\left(\left\{\vec{d} \in D_\circ^k \mid \mathcal{M}_\circ \Vdash \psi\left[\eta^{\vec{x}}_{\vec{d}}\right]\right\}\right).
\end{aligned}$$

(where the first equality follows from $\mathsf{P}(D \setminus D_\circ) = 0$). The rest is easy. □

For our purposes, we only need:

**Corollary 7.2**

*An $\mathscr{L}_1^\natural(\varsigma)$-sentence is valid iff it is true in all $\mathscr{L}_1(\varsigma)$-structures with at most countable domains.*

*Proof.* $\boxed{\Longrightarrow}$ Trivial.

$\boxed{\Longleftarrow}$ Let $\phi$ be an $\mathscr{L}_1^\natural(\varsigma)$-sentence. Without loss of generality, we may assume that $\varsigma$ is finite. Suppose that $\phi$ is not valid, i.e. $\mathcal{M} \nVdash \phi$ for some $\mathscr{L}_1(\varsigma)$-structure $\mathcal{M}$. By Theorem 7.1, there exists a substructure $\mathcal{M}_\circ$ of $\mathcal{M}$ with $\mathcal{D}_\circ$ countable, such that $\mathcal{M}_\circ \nVdash \phi$. □

The case of $\mathscr{L}_2^\natural$ is analogous to that of $\mathscr{L}_1^\natural$. Let $\mathcal{M}$ and $\mathcal{M}'$ be $\mathscr{L}_2(\varsigma)$-structures. We shall say that $\mathcal{M}'$ is a *substructure* of $\mathcal{M}$ if the following conditions are met:

i. $\Omega'$ is a subset of $\Omega$, and $\pi'$ assigns to each $\omega \in \Omega'$ a substructure of $\pi(\omega)$;

ii. $\mathsf{p}'$ is the restriction of $\mathsf{p}$ to $\Omega'$.

Of course, (ii) implies that $\Omega \setminus \Omega'$ has measure zero.

**Theorem 7.3**

[13] To see this, assume that $\phi$ in $(\star)$ has the form $c = y$ or $f(x_1, \ldots, x_m) = y$ where $c$ is a constant symbol, and $f$ is an $m$-ary function symbol.

*Let $\mathcal{M}$ be an $\mathscr{L}_2(\varsigma)$-structure, $\kappa$ be a cardinal such that* $\max\{|\varsigma|, \aleph_0\} \leqslant \kappa \leqslant |D|$*, and $\Lambda$ be a subset of $\Omega$. Take*

$$\Omega_\circ \ := \ \Lambda \cup \{\omega \in \Omega \mid \mathsf{p}(\omega) \neq 0\}.$$

*Then there exists a substructure $\mathcal{M}_\circ$ of $\mathcal{M}$ such that $D_\circ$ has cardinality $\kappa$, and for every $\mathscr{L}_2^\natural(\varsigma)$-formula $\phi$, any $\omega \in \Omega_\circ$ and function $\zeta$ from* $\mathrm{Var}$ *to $D_\circ$,*

$$\mathcal{M}_\circ, \omega \Vdash \phi[\zeta] \quad \Longleftrightarrow \quad \mathcal{M}, \omega \Vdash \phi[\zeta].$$

*Proof.* For convenience, we may assume that $\varsigma$ contains the equality symbol. Clearly, we can build a sequence

$$S_0 \ \subseteq \ S_1 \ \subseteq \ S_2 \ \ldots$$

of subsets of $D$ of cardinality $\kappa$ such that for each $n \in \mathbb{N}$, every $\mathscr{L}_2^\natural(\varsigma)$-formula $\phi(x_1, \ldots, x_m, y)$, any $\omega \in \Omega_\circ$ and $d_1, \ldots, d_m \in S_n$,

$$\mathcal{M}, \omega \Vdash \exists y\, \phi(d_1, \ldots, d_m, y) \quad \Longrightarrow \quad \mathcal{M}, \omega \Vdash \phi(d_1, \ldots, d_m, e) \ \text{ for some } e \in S_{n+1}. \qquad (\star)$$

Take $D_\circ$ to be $\bigcup\{S_n \mid n \in \mathbb{N}\}$. Note that for each $\omega \in \Omega_\circ$, $D_\circ$ contains all constants of $\pi(\omega)$ and is closed under all functions of $\pi(\omega)$. Now consider the $\mathscr{L}_1(\varsigma)$-structure

$$\mathcal{M}_\circ \ = \ \langle D_\circ, \Omega_\circ, \pi_\circ, \mathsf{p}_\circ \rangle$$

where $\pi_\circ$ assigns to each $\omega \in \Omega_\circ$ the (unique) substructure of $\pi(\omega)$ with domain $D_\circ$, and $\mathsf{p}_\circ$ is the restriction of $\mathsf{p}$ to $\Omega_\circ$. Thus $\mathcal{M}_\circ$ is a substructure of $\mathcal{M}$. It remains to check that for every $\mathscr{L}_2^\natural(\sigma)$-formula $\phi$, any $\omega \in \Omega_\circ$ and function $\zeta$ from $\mathrm{Var}$ to $D_\circ$,

$$\mathcal{M}_\circ, \omega \Vdash \phi[\zeta] \quad \Longleftrightarrow \quad \mathcal{M}, \omega \Vdash \phi[\zeta].$$

This is done by induction on the depth of $\phi$. Observe that the inductive hypothesis implies that for each $\mathscr{L}_2^\natural(\varsigma)$-formula $\psi$ with $\mathrm{dp}(\psi) < \mathrm{dp}(\phi)$ and any function $\eta$ from $\mathrm{Var}$ to $D_\circ$,

$$\begin{aligned} \mathsf{P}(\{\omega \in \Omega \mid \mathcal{M}, \omega \Vdash \psi[\eta]\}) \ &= \ \mathsf{P}(\{\omega \in \Omega_\circ \mid \mathcal{M}, \omega \Vdash \psi[\eta]\}) \\ &= \ \mathsf{P}(\{\omega \in \Omega_\circ \mid \mathcal{M}_\circ, \omega \Vdash \psi[\eta]\}) \\ &= \ \mathsf{P}_\circ(\{\omega \in \Omega_\circ \mid \mathcal{M}_\circ, \omega \Vdash \psi[\eta]\}). \end{aligned}$$

(where the first equality follows from $\mathsf{P}(\Omega \setminus \Omega_\circ) = 0$). □

Again, we only need:

**Corollary 7.4**

*An $\mathscr{L}_2^\natural(\varsigma)$-sentence is valid iff it is true in all $\mathscr{L}_2(\varsigma)$-structures with at most countable universes.*

*Proof.* This is perfectly analogous to the proof of Corollary 7.2. □

As far as the validity problems for $\mathscr{L}_1^\natural$ and $\mathscr{L}_2^\natural$ are concerned, Corollaries 7.2 and 7.4 allow us to confine ourselves to at most countable structures, which can be treated as subsets of $\mathbb{N}$. Then it is straightforward to show that the corresponding problems belong to $\Pi_1^1$, by using some standard coding machinery described in [1, Section 5]; cf. [18, Section 9].

To sum up, all the lower bounds mentioned in Sections 5 and 6 turn out to be precise.

## 8 Further discussion

One may wonder what would change if we modify the semantics of $\mathscr{L}_1$ and $\mathscr{L}_2$ in natural ways. Here are two suggestions; see [9], for example.

i. We can try to allow more general (not necessarily discrete) probability measures.[14]

ii. We can generalize the semantics of $\mathscr{L}_2$ by allowing $\mathsf{P}$ to depend on the choice of a possible world, which is helpful for modeling more complex situations.

Briefly stated, the lower bound arguments provided in Sections 5 and 6 are somewhat hereditary: expanding the semantics in a reasonable way does not affect them, provided that all measures are real-valued. Roughly, this is because these arguments are not affected by adding more structures. However, the upper bounds are not as uniform as one might expect, and for natural reasons:

- arbitrary — or rather, non-discrete — probability spaces cannot be directly encoded as sets of natural numbers;
- a straightforward adaptation of the standard argument for the Löwenheim–Skolem theorem may lead from structures to non-structures, or 'weak' structures of a certain kind.

Still, in the case of (ii), if we continue to use discrete distributions, then the corresponding upper bound results can be obtained similarly. Furthermore, the completeness theorems in [9] imply the $\Pi^1_1$-boundedness of the corresponding validity problems; cf. [7, Section 3.3].

It may also be interesting to compare the present work with [18], which strengthens the earlier hardness results of [12] and [15] in a significant way.[15] In fact, the probabilistic languages studied in [18] are very different from $\mathscr{L}_1$ and $\mathscr{L}_2$, and unfortunately, there seems to be no nice translation of these into $\mathscr{L}_1$ or $\mathscr{L}_2$. Moreover, some of the relevant upper bound results require very different techniques; see [17]. Still, the lower bound arguments in [18] are based on similar ideas, and also utilize Corollaries 4.2 and 4.3. On the other hand, the 'flat' fragments described in [18] look richer; for instance, they allow inequalities between probabilities with coefficients in $\mathbb{N}$.

Finally, let us consider the $\Pi^1_1$-completeness result in [5]. Denote the corresponding language by $\mathscr{L}_H$. Briefly, $\mathscr{L}_H$ is rather similar to the quantifier-free fragment of $\mathscr{L}_1$ (where $\forall$ and $\exists$ are excluded); it has a more general semantics, but this plays no role in the lower bound argument. The $\Pi^1_1$-boundedness of the validity problem for $\mathscr{L}_H$ can be justified by the completeness of a suitable calculus. As for its $\Pi^1_1$-hardness, the proof given in [5] makes use of iterations of $\mu$ and employs a very rich signature which expands that of arithmetic; so the underlying technique is significantly

[14]In $\mathscr{L}_1$ and $\mathscr{L}_2$, the use of discrete distributions is motivated by the well-known measurability problem: existential individual quantifiers correspond to projections, but a projection of a measurable set is not necessarily measurable. In effect, there are less restrictive ways to avoid this problem.

[15]The technique of [19], which is useful for proving undecidability results, is based on coding finite simple graphs, whose first-order theory is only $\Pi^0_1$-complete; so it cannot give us high complexity lower bounds.

different from that of the proof of Theorem 6.1. Moreover, quantifiers over reals, which are crucial to Theorems 6.2 and 6.3, do not appear in [5], and for a good reason: an adequate infinitary calculus cannot handle more than $\Pi^1_1$.

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