An averaging method for Hamiltonian systems, exponentially close to integrable ones

Dmitry V. Treschev

Abstract. Exponentially small separatrix splitting for a pendulum with rapidly oscillating suspension point and for the standard Chirikov map is studied by means of a new averaging method which is a continuous version of the Neishtadt averaging procedure. An asymptotic formula for the rate of the separatrix splitting is obtained.

1 Introduction

It is well known that many magnitudes, naturally associated with analytic dynamical systems, have an exponentially small order in a small parameter. Poincaré [1] was the first to observe exponentially small effects in Hamiltonian systems. Arnold [2] noticed that these effects are related to the possibility of diffusion in systems with three and more degrees of freedom. Zaslavsky et al showed that the diffusion may take place for isoenergetically degenerate systems with one and a half degrees of freedom [3-4]. In papers [5-6] (see also [3-4]) the width of stochastic layers is estimated by exponentially small magnitudes. Nekhoroshev [7] obtained explicit upper estimates for the speed of the Arnold diffusion in the case of steep unperturbed Hamiltonians. These estimates were essentially improved for quasiconvex systems by Lochak, Neishtadt [8–9] and Pöschel [10]. Neishtadt [11] showed that exponentially small effects can be studied by some special averaging method, applicable also in non-Hamiltonian case.

In order to find non-trivial lower estimates for Arnold diffusion, one has to study splitting of separatrices of hyperbolic (whiskered) tori in Hamiltonian systems, close to integrable. These hyperbolic tori are generated by resonance Liouville tori of integrable (unperturbed) systems. The existence of one-dimensional hyperbolic tori (periodic solutions) was established by Poincaré [1]. The corresponding theorem for dimensions greater than one was obtained in [12]. Arnold diffusion for systems of the class C^∞ was established by Douady [13]. The results of [13] imply in particular, Lyapunov instability of formally stable elliptic equilibria in C^∞ Hamiltonian systems with three and more degrees of freedom. Note that C^∞ case is simpler than analytic one.

Lazutkin [14] has obtained a non-trivial asymptotics of exponentially small splitting of separatrices in the standard Chirikov map. Methods of [14] were developed in subsequent papers [15–18].

Other results in the problem of exponentially small separatrix splitting (see Holmes, Marsden, and Scheurle [19]; Gelfreich [20]; Delshams and Seara [21]) concern Hamiltonian systems with one and a half degrees of freedom. These results are based on introducing of an additional auxiliary parameter. The presence of such a parameter simplifies the research, but makes the problem under consideration less natural. The main result of these papers is that usual Melnikov method gives the correct asymptotics of the splitting provided the additional parameter is really small.

Note that apart from Arnold diffusion and separatrix splitting some other exponentially small effects were studied in [22–23]. The main difference between these problems and ones, mentioned above is that here exponentially small effects are generated not by "very rapid", but on the contrary, by "very slow" perturbation of an integrable system.

In the present paper we point out an asymptotics of exponentially small splitting of separatrices for a pendulum with rapidly oscillating suspension point and for the standard Chirikov map. It seems interesting that both Hamiltonian systems and symplectic maps can be studied by means of a common universal method. It is necessary to note that our method is based on some complicated theorems which we formulate but do not prove here. A complete proof in the case of the pendulum can be found in [24] (see also [25]).

As a main tool in our investigation we use a continuous version of the Neishtadt averaging method [11]. We regard both problems in question as Hamiltonian systems with one and a half degrees of freedom. The averaging procedure reduces explicit dependence of the Hamiltonians on time to exponentially small terms. The obtained nearly autonomous systems can be studied by the usual Poincaré–Melnikov method.

2 Basic Results

2.1 Separatrix Splitting for a Pendulum

Consider the real Hamiltonian system

$$\dot{x} = \partial \widehat{H}/\partial y, \quad \dot{y} = -\partial \widehat{H}/\partial x, \tag{2.1}$$

$$\widehat{H} = \widehat{H}^0(x,y) + e^{it/\varepsilon}\widehat{H}^{+1}(x,y) + e^{-it/\varepsilon}\widehat{H}^{-1}(x,y). \tag{2.2}$$

Here 0 and ± 1 are indices. We assume that

$$\widehat{H}^0 = y^2/2 - \cos x - 1, \quad \widehat{H}^{\pm 1} = B_+^{\pm} e^{ix} + B_-^{\pm} e^{-ix}, \qquad B_\pm^\pm \in \mathbf{C}. \tag{2.3}$$

The system (2.1)–(2.3) describes motion of a pendulum with oscillating suspension point, ε^{-1} being the frequency of the oscillation. Since the Hamiltonian

$\widehat{H}(x,y,t,\varepsilon)$ is real-analytic, the functions $\widehat{H}(\bar{x},\bar{y},\bar{t},\varepsilon)$ and $\widehat{H}(x,y,t,\varepsilon)$ are complex conjugate for real ε. (Here the overbars denote the complex conjugation.) This implies that

$$\bar{B}_+^+ = B_-^-, \quad \bar{B}_-^+ = B_+^-.$$

One can study this problem by means of perturbation theory in two different cases

(1) $\varepsilon \sim 1$, $B_\pm^\pm$ are small;
(2) ε is small.

Separatrix splitting in case (1) can be studied by methods developed by Poincaré [1]. Moreover, Poincaré understood that in case (2) separatrix splitting is exponentially small.

In both cases (1) and (2) the system (2.1)–(2.3) is close to the following one:

$$\dot{x} = \partial \widehat{H}^0/\partial y, \quad \dot{y} = -\partial \widehat{H}^0/\partial x. \tag{2.4}$$

This is an ordinary mathematical pendulum. The phase portrait of the pendulum is well known (see Fig. 1).

The hyperbolic equilibrium $(x,y) = (\pi, 0)$ has the separatrix loop γ. For small ε or small $B_\pm^\pm$ the system (2.1)–(2.3) has a hyperbolic $2\pi\varepsilon$-periodic motion $p(t)$ which is close to the equilibrium $(\pi,0)$. There is a pair of asymptotic to $p(t)$ surfaces $\Gamma^\pm$ in the extended phase space

$$\widetilde{P} = (\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{R} \times (\mathbf{R}/(2\pi\varepsilon\mathbf{Z})) = \{x,y,t\}.$$

These surfaces are close to $\gamma \times (\mathbf{R}/(2\pi\varepsilon\mathbf{Z}))$, but generally speaking they do not coincide (see Fig. 2 which presents the section Π_c of the phase space $\widetilde{P}$ by a plane $t = c = \text{const}$). We measure the rate of their splitting by the area $\mathcal{A}_\mathrm{P}$ of a domain $D \subset \Pi_c$ bounded by the stable and unstable separatrices. Note that $\mathcal{A}_\mathrm{P}$ is a symplectic invariant, i.e. $\mathcal{A}_\mathrm{P}$ does not depend on coordinates on the symplectic manifold $(\Pi_c, dx \wedge dy)$. Moreover, it remains constant when c changes.

For case (1) we have the estimate

$$\mathcal{A}_\mathrm{P} = \frac{8\pi}{\varepsilon}\left|\frac{B_+^+ + B_-^+}{\sinh(\pi\varepsilon^{-1}/2)} - \frac{B_+^+ - B_-^+}{\cosh(\pi\varepsilon^{-1}/2)} + O(B_+^+, B_-^+)\right|. \tag{2.5}$$

It can be easily obtained with the help of the ordinary Poincaré–Melnikov integral.

For case (2)

$$\mathcal{A}_\mathrm{P} = \frac{4}{\varepsilon\exp(\pi\varepsilon^{-1}/2)}[|J_\mathrm{P}(B_-^+, B_-^-)| + O(\varepsilon\log\varepsilon^{-1})], \tag{2.6}$$

where the function J_{P} can be found from some nonlinear problem not containing any small parameter (see §VII). Moreover,

$$J_{\mathrm{P}}(B_-^+, B_-^-) = 4\pi B_-^+ j(4B_-^+ B_-^-), \quad j(0) = 2. \tag{2.7}$$

The function $j(z)$ is an entire real-analytic function of the complex variable z. Hence, it is natural to describe j by the Maclaurin series: $j(z) = \sum_{k=0}^{\infty} j_k z^k$. Together with C. Simo we have found numerically the following coefficients:

$$\begin{array}{lll} j_0 = 2, & j_3 = 3.21010 \cdot 10^{-3}, & j_6 = 2.1534 \cdot 10^{-8}, \\ j_1 = 0.65856738, & j_4 = 9.03367 \cdot 10^{-5}, & j_7 = 2.070 \cdot 10^{-10}, \\ j_2 = 6.651741 \cdot 10^{-2}, & j_5 = 1.6620 \cdot 10^{-6}, & j_8 = 1.53 \cdot 10^{-12}. \end{array}$$

Carles Simo noted an astonishing closeness of the numbers j_k to coefficients in the Maclaurin expansion of the function $2\sinh\sqrt{2z}/\sqrt{2z}$.

Note that the relations (2.5) and (2.6) coincide when ε and $B_\pm^\pm$ tend to zero simultaneously.

2.2 Separatrix Splitting in the Standard Chirikov Map

The standard Chirikov map (SCM) is a symplectic transformation of the cylinder $Z = \{x \bmod 2\pi, y\}$:

$$(x \bmod 2\pi, y) \to (\widehat{x} \bmod 2\pi, \widehat{y}), \qquad \widehat{y} = y + \varepsilon \sin x, \quad \widehat{x} = x + \widehat{y}.$$

For $\varepsilon = 0$ this map is integrable and the cylinder Z is foliated by invariant circles $S^1_{y_0} = \{y = y_0\}$. On each circle $S^1_{y_0}$ SCM is a rotation on the angle y_0. In particular, S^1_0 consists of fixed points. For nonzero ε circles, corresponding to rational y_0, disintegrate.

Now we discuss in more detail dynamics of SCM in the vicinity of S^1_0 for small $\varepsilon > 0$. The points $(x, y) = (0, 0)$ and $(x, y) = (\pi, 0)$ remain fixed. They become hyperbolic and elliptic respectively (see Fig. 3). Separatrices of the hyperbolic fixed point $(0, 0)$ split and intersect transversely. The rate of their splitting was investigated by Lazutkin. In particular, the area $\mathcal{A}_{\mathrm{SCM}}$ of a domain D bounded by stable and unstable separatrices can be estimated as follows:

$$\mathcal{A}_{\mathrm{SCM}} = \frac{2|\theta_1|}{\pi} e^{-\pi^2/\sqrt{\varepsilon}}(1 + O(\sqrt{\varepsilon})). \tag{2.8}$$

The constant $|\theta_1| = 1118.82770594090077842\ldots$ was evaluated from some nonlinear problem not containing the small parameter ε. In this paper we present another nonlinear problem not containing small parameters for evaluation of the constant θ_1.

Note that, the precision we can achieve when evaluating θ_1 numerically isn't as high as in the original methods [14–18]: estimates of $|\theta_1|$ we have obtained give us a relative error $\sim 10^{-5}$.

3 Transformation of Basic Equations

In this section we transform the problems under consideration to a form convenient for averaging.

3.1 Transformation of pendulum equations

The change $t \to \varepsilon t$ transforms the system (2.1)–(2.2) into the following one:

$$\dot{x} = \partial(\varepsilon \widehat{H})/\partial y, \quad \dot{y} = -\partial(\varepsilon \widehat{H})/\partial x, \tag{3.1}$$

$$\widehat{H} = \widehat{H}^0(x, y) + e^{it}\widehat{H}^{+1}(x, y) + e^{-it}\widehat{H}^{-1}(x, y). \tag{3.2}$$

For short the new time is denoted here as before by t. All our further constructions concerning the pendulum deal with the system (3.1),(3.2),(2.3) instead of (2.1)–(2.3). The system (3.1)–(3.2) can be regarded as a perturbation of a "slow pendulum"—the system with Hamiltonian $\varepsilon\widehat{H}^0$. Certainly the area $\mathcal{A}_{\mathrm{P}}$ may be calculated in the system (3.1),(3.2),(2.3) in the same way as in the initial one.

3.2 A Hamiltonian system corresponding to SCM

Symplectic maps are discrete analogs of Hamiltonian systems. This means that we can regard SCM as the first return map for some Hamiltonian system with one and a half degrees of freedom. The corresponding Hamiltonian function is not defined uniquely. The simplest form of this Hamiltonian is

$$H = \frac{1}{4\pi}y^2 + \varepsilon \cos x \sum_{k\in\mathbf{Z}} \delta(t + 2\pi k) \equiv \frac{1}{2\pi}\Big(\frac{y^2}{2} + \varepsilon \cos x \sum_{n\in\mathbf{Z}} e^{int}\Big), \tag{3.3}$$

where δ is the Dirak δ-function. Indeed, the equations

$$\dot{x} = \partial H/\partial y, \quad \dot{y} = -\partial H/\partial x \tag{3.4}$$

can be easily solved on any interval $[2\pi k - 0, 2\pi(k+1) - 0]$, $k \in \mathbf{Z}$ and the map $\big(x(2\pi k - 0), y(2\pi k - 0)\big) \to \big(x(2\pi k + 2\pi - 0), y(2\pi k + 2\pi - 0)\big)$ coincides with SCM.

We put

$$y_* = y/\sqrt{\varepsilon}, \quad x_* = x + \pi, \quad H_* = H/\sqrt{\varepsilon}, \quad \varepsilon_* = \sqrt{\varepsilon}/(2\pi).$$

The system (3.3)–(3.4) takes the form

$$\dot{x}_* = \partial(\varepsilon_* H_*)/\partial y_*, \quad \dot{y}_* = -\partial(\varepsilon_* H_*)/\partial x_*, \quad H_* = \frac{y_*^2}{2} - \cos x_* \sum_{n\in\mathbf{Z}} e^{int} - 1. \tag{3.5}$$

Here the term -1 is introduced into H_* just for convenience.

The system (3.5) can be also regarded as a perturbation of a "slow pendulum" (see subsection **a**). The area $\mathcal{A}_{\mathrm{SCM}} = \mathcal{A}_{\mathrm{SCM}}(\varepsilon)$ can be obtained from the area $\mathcal{A}^*_{\mathrm{SCM}} = \mathcal{A}^*_{\mathrm{SCM}}(\varepsilon_*)$ of a domain bounded by separatrices of the system (3.5) on a section $\Pi^*_c = \{(x_*, y_*, t) : t = c\}$ as follows:

$$\mathcal{A}_{\mathrm{SCM}}(\varepsilon) = \sqrt{\varepsilon}\mathcal{A}^*_{\mathrm{SCM}}(\varepsilon_*) = \sqrt{\varepsilon}\mathcal{A}^*_{\mathrm{SCM}}(\sqrt{\varepsilon}/(2\pi)). \tag{3.6}$$

4 Averaging Method

In [11] Neishtadt suggested an averaging procedure for studying exponentially small effects in Hamiltonian systems. The Neishtadt procedure is based on a large amount ($\sim \varepsilon^{-1}$) of successive symplectic changes which are close to identical one. In the present section we describe a continuous version of the Neishtadt method.

Consider the Hamiltonian system

$$\dot{z} = v_{\check{G}}(z), \quad z \in M, \tag{4.1}$$

where (M, ω) is a symplectic manifold, ω is a symplectic structure and $v_{\check{G}}$ is the Hamiltonian vector field on M, corresponding to the Hamiltonian $\check{G} : M \to \mathbf{R}$. We transform the Hamiltonian function by using the change of the variables

$$z \to z(s). \tag{4.2}$$

We assume that the change (4.2) is a shift along trajectories of the Hamiltonian system

$$dz/ds = v_F(z) \tag{4.3}$$

with Hamiltonian $F(z, s)$: $z(s) = Z(z, s)$, where the function $Z(z, s)$ is the solution of equations (4.3) with the initial conditions $Z(z, 0) = z$. In the new variables the Hamiltonian $G(z(s), s)$ of the system (4.1) satisfies the equality

$$G(z(s), s) = G(z, 0) \equiv \check{G}(z).$$

We differentiate this equality with respect to s, and obtain:

$$\partial G/\partial s + v_F G = 0.$$

Here v_F is regarded as the operator of differentiation along the vector field v_F. Hence we have:

$$G_s + \{F, G\} = 0, \tag{4.4}$$

where subscript s denotes the derivative with respect to s, and $\{\cdot, \cdot\}$ is the Poisson bracket: $\{f, g\} = v_f g = -v_g f$ for any two functions f and g on M.

Now let us turn to the analysis of equations (3.1)–(3.2) and (3.5). We try to weaken the rate of dependence of the functions $\widehat{H}$ and H_* on time t. The systems (3.1)–(3.2) and (3.5) can be obtained from the autonomous system

$$\frac{d}{d\tau}y = -\frac{\partial \check{G}}{\partial x}, \quad \frac{d}{d\tau}x = \frac{\partial \check{G}}{\partial y}, \quad \frac{d}{d\tau}E = -\frac{\partial \check{G}}{\partial t}, \quad \frac{d}{d\tau}t = \frac{\partial \check{G}}{\partial E},$$
$$\check{G}(x,y,t,E,\varepsilon) = \varepsilon\check{H}(x,y,t) + E, \quad \tau = t + \mathrm{const},$$

where $\check{H}$ equals either $\widehat{H}$ or H_*. (In the second case we drop the subindices $*$ for short.) We put

$$G(x,y,t,E,\varepsilon,s) = \varepsilon H(x,y,t,\varepsilon,s) + E,$$
$$G(x,y,t,E,\varepsilon,0) = \check{G}(x,y,t,E,\varepsilon), \quad H(x,y,t,\varepsilon,0) = \check{H}(x,y,t).$$

Furthermore, let $F = \varepsilon\xi H$, where ξ is some linear operator. Equation (4.4) takes the form $H_s + \{\xi H, \varepsilon H + E\} = 0$. After trivial transformations we obtain:

$$H_s = (\xi H)_t - \varepsilon\{\xi H, H\}, \tag{4.5}$$
$$H(x,y,t,\varepsilon,0) = \check{H}(x,y,t). \tag{4.6}$$

Now we define one of possible averaging operators ξ. Consider an expansion of H into the Fourier series:

$$H(x,y,t,\varepsilon,s) = H^0(x,y,\varepsilon,s) + H^+(x,y,t,\varepsilon,s) + H^-(x,y,t,\varepsilon,s),$$
$$H^\pm = \sum_{\pm n\in\mathbf{N}} H^n(x,y,\varepsilon,s)e^{int}. \tag{4.7}$$

We take

$$\xi H = i(H^+ - H^-).$$

Then equations (4.5) can be rewritten as follows:

$$\begin{aligned} H_s^0 &= -2i\varepsilon\{H^+, H^-\}^0, \\ H_s^\pm &= \pm iH_t^\pm \pm i\varepsilon\{H^0, H^\pm\} - 2i\varepsilon\{H^+, H^-\}^\pm, \end{aligned} \tag{4.8}$$

where the indices $0, +$ and $-$ denote “zero”, “positive” and “negative” parts of the Fourier series in the same way as in (4.7).

Now we give an informal analysis of this system. First, note that all non-linear terms are multiplied by ε. By neglecting these terms, we obtain the system

$$H_s^n = -|n|H^n, \qquad n \in \mathbf{Z},$$

where H^n are the Fourier coefficients (4.7). These equations can be easily solved:

$$H^n(x,y,\varepsilon,s) = e^{-|n|s}\check{H}^n(x,y). \tag{4.9}$$

We see that all the functions H^n, $n \neq 0$ rapidly tend to zero as s increases. Thus the system (4.8) actually realizes an averaging while the non-linear terms are small.

The role of nonlinear terms in (4.8) can be illustrated as follows. According to (4.9) we may suppose that the terms $-2i\varepsilon\{H^+,H^-\}^0$ and $-2i\varepsilon\{H^+,H^-\}^\pm$ play in (4.8) less significant role than $\pm i\varepsilon\{H^0,H^\pm\}$. So, consider instead of (4.8) the following system:

$$H^0_s = 0, \quad H^\pm_s = \pm i\partial H^\pm/\partial t \pm i\varepsilon\{H^0,H^\pm\}. \tag{4.10}$$

These equations can be solved explicitly. Indeed, introduce the one-parameter transformation group of the phase space $P = (\mathbf{R}/2\pi\mathbf{Z})\times\mathbf{R} = \{x,y\}$ — the phase flow g^t of the system

$$\dot{x} = \partial\check{H}^0/\partial y, \quad \dot{y} = -\partial\check{H}^0/\partial x. \tag{4.11}$$

Solutions of (4.10) have the form

$$H^n = e^{-|n|s}\check{H}^n \circ g^{i\varepsilon\lambda s}, \qquad \lambda = \mathrm{sgn}(n);$$

in particular, the function H^0 remains unchanged. The functions $\check{H}^n \circ g^{i\varepsilon\lambda s}$, generally speaking, have singularities for real x and y if s has the order $\sim \varepsilon^{-1}$. Hence we may suppose that our averaging procedure is defined in a neighborhood of a point (x,y) for $s = \Delta > 0$ provided all points of the interval $-\Delta \le s \le \Delta$ are not very close to singularities of the function $g^{i\varepsilon s}(x,y)$.

Although there exists an explicit solution of equations (4.10), we can not solve the complete system (4.8). We solve it approximately near the most essential singularities. The asymptotics of the separatrix splitting can be obtained from this approximate solution.

The system (4.8) contains infinitely many Fourier coefficients H^n as unknown functions. Note that in the case of the pendulum with rapidly oscillating suspension point there are only three nontrivial equations:

$$H^0_s = -2i\varepsilon\{H^{+1},H^{-1}\}, \quad H^{\pm 1}_s = -H^{\pm 1} \pm i\varepsilon\{H^0,H^{\pm 1}\}. \tag{4.12}$$

The corresponding initial conditions are

$$H^0(x,y,0) = \widehat{H}^0(x,y), \quad H^{\pm 1}(x,y,0) = \widehat{H}^{\pm 1}(x,y). \tag{4.13}$$

For SCM the system (4.8) can also be reduced to three equations. Indeed, we can look for its solution in the form

$$H^n(x,y,s) = \begin{cases} e^{-|n|s}H^{(+)}(x,y,s) & \text{for} \quad n > 0, \\ e^{-|n|s}H^{(-)}(x,y,s) & \text{for} \quad n < 0, \\ H^{(0)}(x,y,s) & \text{for} \quad n = 0, \end{cases} \tag{4.14}$$

where $H^{(\pm)}$ and $H^{(0)}$ are some new unknown functions. The system (4.8) is reduced to the following one:

$$\begin{aligned} H^{(0)}_s &= -\tfrac{2i\varepsilon e^{-2s}}{1-e^{-2s}}\{H^{(+)},H^{(-)}\}, \\ H^{(\pm)}_s &= \pm i\varepsilon\{H^{(0)},H^{(\pm)}\} - \tfrac{2i\varepsilon e^{-2s}}{1-e^{-2s}}\{H^{(+)},H^{(-)}\}. \end{aligned} \tag{4.15}$$

The initial conditions for this system have the form (see (3.5))

$$H^{(0)}(x,y,0) = y^2/2 - \cos x - 1 \equiv \widehat{H}^0(x,y), \quad H^{(\pm)}(x,y,0) = -\cos x. \tag{4.16}$$

5 Poincaré Method

In this section we consider the Poincaré method of the analysis of asymptotic surface splitting. More precisely, we discuss an application of this method to our problems.

The averaging procedure of the previous section enables us to reduce the explicit dependence of our Hamiltonians on time t to exponentially small terms. So we consider the Hamiltonian system

$$\dot{x} = \partial(\varepsilon H)/\partial y, \quad \dot{y} = -\partial(\varepsilon H)/\partial x \tag{5.1}$$

with the Hamiltonian

$$H = \mathcal{H}^0(x, y, \varepsilon) + \mathcal{H}^+(x, y, t, \varepsilon) + \mathcal{H}^-(x, y, t, \varepsilon), \tag{5.2}$$

$$\mathcal{H}^\pm(x, y, t, \varepsilon) = \textstyle\sum_{\pm n \in \mathbf{Z}} \sigma^{|n|} \mathcal{H}^n(x, y, \varepsilon) e^{itn}. \tag{5.3}$$

Here σ is an additional small parameter, $\sigma \ll \varepsilon$. It is convenient to assume henceforth that the Hamiltonian (5.2)–(5.3) is obtained from (3.2) or (3.5) as a result of our averaging procedure, and $\sigma \sim \exp(-\pi\varepsilon^{-1}/2)$.

Suppose that for small values of ε the system

$$\dot{x} = \partial(\varepsilon\mathcal{H}^0)/\partial y, \quad \dot{y} = -\partial(\varepsilon\mathcal{H}^0)/\partial x$$

has hyperbolic equilibrium positions $(x^0_\varepsilon, y^0_\varepsilon)$ with the solutions (separatrices) doubly asymptotic to them:

$$\gamma_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t)); \qquad \gamma_\varepsilon(t) \to (x^0_\varepsilon, y^0_\varepsilon) \quad \text{as} \quad t \to \pm\infty.$$

The corresponding separatrices of the system (5.1) split. We measure the rate of their splitting by the area $\mathcal{A}$ of a domain bounded by their segments (see §1 and Fig. 2).

We put

$$P^+(\varepsilon) = \int_{-\infty}^{\infty} \left[\mathcal{H}^{+1}(\gamma_\varepsilon(r), r, \varepsilon) - \mathcal{H}^{+1}(x^0_\varepsilon, y^0_\varepsilon, r, \varepsilon)\right] dr.$$

Using standard arguments of the Poincaré–Melnikov theory one can prove the following

Proposition 5.1 *The area $\mathcal{A}$ can be estimated as follows:*

$$\mathcal{A} = 4\sigma\varepsilon|P^+(\varepsilon)| + O(\sigma^2).$$

6 Time-energy coordinates

We have explained in section IV that singularities of solutions of the system (4.11) play an important role in analysis of the system (4.8). In both problems under consideration equations (4.11) coincide with (2.4). In this section we analyze complex singularities of solutions of the system (2.4) and introduce new coordinates in the complexified phase space.

Let $(x_\gamma(t), y_\gamma(t)) = g^t(0,2)$ be a separatrix solution of (2.4). This solution corresponds to the energy level $\widehat{H}^0 = 0$ and can be expressed explicitly with the help of the quadratures

$$t = \int_0^{x_\gamma(t)} f(0,u)\, du, \quad f(h,u) = \frac{1}{\sqrt{2(1+h+\cos u)}}.$$

Here the function f is supposed to be positive for real values of u close to zero.

If the function $x_\gamma(t)$ has a singularity at the point $t = \alpha$, then

$$\alpha = \int_0^{\pm i\infty} f(u)\, du.$$

The magnitude α in this relation depends on path of integration. The singularity with the minimal positive imaginary part corresponds to the ray $R = \{x \in \mathbf{C} : \operatorname{Re} x = 0,\ \operatorname{Im} x \geq 0\}$. It is easy to calculate:

$$\int_R f(u)\, du = i\pi/2.$$

Solutions of the system (2.4), whose initial conditions are close to $(0,2)$, have singularities close to the point $t = i\pi/2$. Let $(x(t), y(t))$ be such a solution and t_{sing} a corresponding singularity. We put $\nu = \exp(ix/2)$. For large values of $\operatorname{Im} x$ we have:

$$f(h,x) = \nu(1 + O(\nu)).$$

Thus for values of time t, close to t_{sing}

$$t_{\text{sing}} - t = \int_{x(t)}^{+i\infty} f(h,u)\, du = 2i\nu(t)\big(1 + O(\nu(t))\big),$$

where $\nu(t) = \exp(ix(t)/2)$. We see that

$$\nu(t) = \tfrac{t_{\text{sing}} - t}{2i}(1 + O(t_{\text{sing}} - t)),$$
$$y(t) = \dot{x}(t) = \tfrac{2}{i}\nu^{-1}\dot{\nu} = \tfrac{2i}{t_{\text{sing}} - t}(1 + O(t_{\text{sing}} - t)).$$

Therefore we have:

$$\lim_{t\to t_{\text{sing}}} \nu(t) = 0, \quad \lim_{t\to t_{\text{sing}}} y(t) = \infty, \quad \lim_{t\to t_{\text{sing}}} y(t)\nu(t) = 1. \tag{6.1}$$

It is convenient to introduce coordinates τ, h of the time–energy type for the system (2.4). The variable h is defined by the equality

$$h = \widehat{H}^0 = \frac{1}{2}(y^2 - \nu^{-2} - \nu^2) - 1. \tag{6.2}$$

The variable τ is canonically conjugate to h, and the function $\tau(x, y)$ vanishes for $\nu = 0$. According to relations (6.1) we claim that $\lim_{\tau\to 0} y(\tau, h)\nu(\tau, h) = 1$.

Proposition 6.1

$$\begin{aligned} \nu^{-1} &= -\sqrt{2h}i/\sin(\tau\sqrt{h/2}) + F_1(\tau, h\tau^2), && (6.3)\\ y &= -\sqrt{2h}i/\tan(\tau\sqrt{h/2}) + F_2(\tau, h\tau^2), && (6.4) \end{aligned}$$

where the functions $F_1(\tau, \sigma)$ *and* $F_2(\tau, \sigma)$ *are analytic at the point* $(\tau, \sigma) = (0, 0)$.

Corollary 1 *In the variables* τ, h *the functions* $\widehat{H}^0$, $\widehat{H}^{\pm}$, *and* H_*^n *have the form*

$$\begin{array}{ll} \widehat{H}^0 = h, & \widehat{H}^{\pm 1} = -\frac{4hB_-^{\pm}}{1-\cos(\tau\sqrt{2h})} + \tau^{-1}F^{\pm}(\tau, h\tau^2), \\ H_*^0 = h, & H_*^n = \frac{2h}{1-\cos(\tau\sqrt{2h})} + \tau^{-1}F_*(\tau, h\tau^2), \quad n \neq 0, \end{array} \tag{6.5}$$

where the functions $F^{\pm}(\tau, \sigma)$ *and* $F_*(\tau, \sigma)$ *are analytic at the point* $(0, 0)$.

The relations (6.5) can be obtained from (2.3) and (3.5) by a straightforward computation.

Remark 1 *The functions* $\tau\nu^{-1}$ *and* τy *are analytic in* τ *and* $\sigma = h\tau^2$ *at the point* $(\tau, \sigma) = 0$.

Proof of Proposition 6.1. Equality (6.2) can be solved with respect to y:

$$y = \pm\nu^{-1}\sqrt{1 + 2h\nu^2 + 2\nu^2 + \nu^4}. \tag{6.6}$$

We want relations (6.1) to hold for small $|\tau|$ and $|h\tau^2|$, so we must admit that the value of $|\nu|$ is small, and the value of $y\nu$ is close to 1. Thus taking in (6.6) the sign "+" we get:

$$y = \nu^{-1}\sqrt{1 + h\nu^2} + \varphi_1(\nu, h\nu^2), \tag{6.7}$$

where the function $\varphi_1(\nu, \sigma)$ is analytic at the point $(0, 0)$.

Let $S = S(x, h)$ be a generating function of the canonical transformation $x, y \to \tau, h$. Equality (6.7) is equivalent to the relation $y = \partial S/\partial x$. Consequently

$$S = \frac{2}{i}\int_0^{\nu} \nu^{-2}\sqrt{1 + 2h\nu^2}\, d\nu + \varphi_2(\nu, h\nu^2) + \varphi_3(h).$$

The function $\varphi_2(\nu,\sigma)$ is analytic at zero and φ_3 is arbitrary. We put $\varphi_3 = 0$ and obtain:

$$\tau = \frac{\partial S}{\partial h} = \frac{2}{i}\int_0^\nu \frac{d\nu}{\sqrt{1+2h\nu^2}} + \nu^2\varphi_4(\nu,h\nu^2) = \frac{2\mathrm{arcsinh}(\sqrt{2h}\nu)}{i\sqrt{2h}} + \nu^2\varphi_4(\nu,h\nu^2). \tag{6.8}$$

Here we denote by arcsinh the function inverse to the hyperbolic sinus. The function $\varphi_4(\nu,\sigma)$ is analytic at zero. According to the implicit function theorem equation (6.8) can be solved with respect to ν:

$$\nu = \tau\varphi_5(\tau,h\tau^2), \tag{6.9}$$

where the function φ_5 is analytic at zero.

To find $\varphi_5(0,h\tau^2)$ we put $\nu=\tilde\nu\beta$, $\tau=\tilde\tau\beta$, $h=\tilde h\beta^{-2}$, where β is some small parameter. We get from (6.8) and (6.9):

$$\begin{aligned} i\tilde\tau\sqrt{\tilde h/2} &= \mathrm{arcsinh}\Big(\sqrt{2\tilde h}\tilde\nu\Big) - i\beta\tilde\nu^2\sqrt{\tilde h/2}\,\varphi_4\big(\beta\tilde\nu,\tilde h\tilde\nu^2\big),\\ \tilde\nu &= \tilde\tau\varphi_5(\beta\tilde\tau,\tilde h\tilde\tau^2). \end{aligned}$$

We have:

$$\tilde\nu = \Big(i\sqrt{2\tilde h}\Big)^{-1}\sin\Big(\tau\sqrt{\tilde h/2}\Big) + O(\beta) = \tilde\tau\varphi_5(\beta\tilde\tau,\tilde h\tilde\tau^2).$$

Consequently

$$\tilde\nu = \Big(i\sqrt{2\tilde h}\Big)^{-1}\sin\Big(\tau\sqrt{\tilde h/2}\Big) + \beta\tilde\tau^2\varphi_6(\beta\tilde\tau,\tilde h\tilde\tau^2),$$

where the function

$$\varphi_6(\beta\tilde\tau,\tilde h\tilde\tau^2) = \big(\varphi_5(\beta\tilde\tau,\tilde h\tilde\tau^2) - \varphi_5(0,\tilde h\tilde\tau^2)\big)(\beta\tilde\tau)^{-1}$$

is analytic at zero. Returning to the original variables τ,h,ν we obtain (6.3). Then we obtain (6.4) from (6.7) and (6.3). Proposition 6.1 is proved.

7 Averaging in a Neighborhood of Singularities

We perform the change

$$\begin{gathered} \tau=\varepsilon\tilde\tau, \quad h=\varepsilon^{-2}\tilde h;\\ \tilde H^0=\varepsilon^2H^0, \quad \tilde H^{\pm1}=\varepsilon^2H^{\pm1};\\ \tilde H^{(0)}=\varepsilon^2H^{(0)}, \quad \tilde H^{(\pm)}=\varepsilon^2H^{(\pm)}, \end{gathered} \tag{7.1}$$

and put $\{f,g\}_\sim = f_{\tilde h} g_{\tilde\tau} - f_{\tilde\tau} g_{\tilde h}$ for any smooth f and g. The systems (4.12) and (4.15) take the form

$$\tilde H^0_s = -2i\{\tilde H^{+1}, \tilde H^{-1}\}_\sim, \quad \tilde H^{\pm 1}_s = -\tilde H^{\pm 1} \pm i\{\tilde H^0, \tilde H^{\pm 1}\}_\sim; \tag{7.2}$$

$$\begin{aligned} \tilde H^{(0)}_s &= -\tfrac{2ie^{-2s}}{1-e^{-2s}}\{\tilde H^{(+)}, \tilde H^{(-)}\}_\sim, \\ \tilde H^{(\pm)}_s &= \pm i\{\tilde H^{(0)}, \tilde H^{(\pm)}\}_\sim - \tfrac{2ie^{-2s}}{1-e^{-2s}}\{\tilde H^{(+)}, \tilde H^{(-)}\}_\sim. \end{aligned} \tag{7.3}$$

Initial conditions for these systems can be calculated from (6.5). Note that the terms $\tau^{-1}F^\pm$ and $\tau^{-1}F_*$ of (6.5) have the order ε after the scaling (7.1). We drop these terms and obtain the relations

$$\tilde H^0(\tilde\tau, \tilde h, 0) = \tilde h, \quad \tilde H^{\pm 1}(\tilde\tau, \tilde h, 0) = -4\tilde h B^\pm_- \Big(1 - \cos\Big(\tilde\tau\sqrt{2\tilde h}\Big)\Big)^{-1} \tag{7.4}$$

for the pendulum and

$$\tilde H^{(0)}(\tilde\tau, \tilde h, 0) = \tilde h, \quad \tilde H^{(\pm)}(\tilde\tau, \tilde h, 0) = 2\tilde h\Big(1 - \cos\Big(\tilde\tau\sqrt{2\tilde h}\Big)\Big)^{-1} \tag{7.5}$$

for SCM.

The nonlinear Cauchy problems (7.2),(7.4) and (7.3),(7.5) do not contain any small parameter. Now we give a procedure of obtaining the magnitudes $J_{\mathrm P}(B^+_-, B^-_-)$ and $|\theta_1|$ (see (2.6),(2.8)).

Proposition 7.1 *Let the positive constants $\tilde c, \tilde c_1$ be large. Then for any $s \ge 0$ each of the systems (7.2), (7.4) and (7.3), (7.5) has a solution in the domain*

$$D^s_0 = \{Im\,\tilde\tau \le -\tilde c - s,\ |(\tilde\tau + is)^2 \tilde h| \le \tilde c_1^{-1}\}.$$

Moreover, the following estimates hold in D^s_0:

$$|\tilde H^{\pm 1}(\tilde\tau, \tilde h, s)| \le \tilde A e^{-s}|\tilde\tau + is|^{-2} \tag{7.6}$$

for pendulum and

$$|\tilde H^{(\pm)}(\tilde\tau, \tilde h, s)| \le \tilde A|\tilde\tau + is|^{-2} \tag{7.7}$$

for SCM.

According to Proposition 7.1 the integrals

$$\begin{aligned} I_{\mathrm P}(s) &= \textstyle\int_{-\infty}^{\infty} e^{s+i(t-ic)}\tilde H^{+1}(t - ic - is, 0, s)\,dt, \\ I_{\mathrm{SCM}}(s) &= \textstyle\int_{-\infty}^{\infty} e^{i(t-ic)}\tilde H^{(+)}(t - ic - is, 0, s)\,dt \end{aligned} \tag{7.8}$$

exist for real $c \ge \tilde c$ and do not depend on c. We put

$$J_{\mathrm P} = \lim_{s\to+\infty} I_{\mathrm P}(s), \quad J_{\mathrm{SCM}} = \lim_{s\to+\infty} I_{\mathrm{SCM}}(s). \tag{7.9}$$

Proposition 7.2 *The limits* (7.9) *exist and the following estimates hold:*

$$J_{\rm P} - I_{\rm P}(s) = O(s^{-1}), \quad J_{\rm SCM} - I_{\rm SCM} = O(s^{-1}).$$

We shall show in the next section that for small $\varepsilon > 0$

$$\begin{aligned} \mathcal{A}_{\rm P} &= 4\varepsilon^{-1} e^{-\pi\varepsilon^{-1}/2}[|J_{\rm P}| + O(\varepsilon)], && (7.10) \\ \mathcal{A}^*_{\rm SCM}(\varepsilon_*) &= 4\varepsilon_*^{-1} e^{-\pi\varepsilon_*^{-1}/2}[|J_{\rm SCM}| + O(\varepsilon_*)]. && (7.11) \end{aligned}$$

According to (7.10) the function $J_{\rm P} = J_{\rm P}(B_-^+, B_-^-)$ is the same as in (2.6).

The relations (2.7) can be obtained as follows. Putting $\tilde{H}^{\pm 1} = 2B_-^{\pm} G^{\pm 1}$ and $a = 4B_-^+ B_-^-$, we transform the system (7.2)–(7.4) to the following one:

$$\begin{aligned} &\tilde{H}^0_s = -2ia\{G^{+1}, G^{-1}\}_\sim, \quad G^{\pm 1}_s = -G^{\pm 1} \pm i\{\tilde{H}^0, G^{\pm 1}\}_\sim; \\ &\tilde{H}^0(\tilde{\tau}, \tilde{h}, 0) = \tilde{h}, \quad G^{(\pm 1)}(\tilde{\tau}, \tilde{h}, 0) = -2\tilde{h}\Big(1 - \cos\Big(\tilde{\tau}\sqrt{2\tilde{h}}\,\Big)\Big)^{-1}, \end{aligned} \qquad (7.12)$$

This system contains only one parameter a and so the first relation (2.7) holds.

Now we put in (7.12) $a = 0$. Such a system can be easily solved. In particular, $\tilde{H}^0 \equiv \tilde{h}$ and $G^{+1}(\tilde{\tau}, \tilde{h}, s) = G^{+1}(\tilde{\tau} + is, \tilde{h}, 0)$. Consequently, for $a = 0$

$$\tilde{H}^{+1}(t - ic - is, 0, s) = 2B_-^+ G^{+1}(t - ic, \tilde{h}, 0) = -4B_-^+(t - ic)^{-2}.$$

We get

$$I_{\rm P}(s) = 8\pi B_-^+ = J_{\rm P} = 4\pi B_-^+ j(0).$$

Hence, the second relation (2.7) is proved.

In order to obtain the constant $|\theta_1|$ we use relations (3.6), (7.11), and the equality $\varepsilon_* = \sqrt{\varepsilon}/(2\pi)$. We have:

$$\mathcal{A}_{\rm SCM}(\varepsilon) = \sqrt{\varepsilon}\cdot 4\varepsilon_*^{-1} e^{-\pi\varepsilon_*^{-1}/2}[|J_{\rm SCM}| + O(\varepsilon_*)] = 8\pi e^{-\pi^2/\sqrt{\varepsilon}}[|J_{\rm SCM}| + O(\sqrt{\varepsilon})].$$

Therefore, according to (2.8)

$$|\theta_1| = 4\pi^2 J_{\rm SCM}.$$

8 Poincaré–Melnikov Integrals in the Averaged Systems

In this section we explain why the integrals (7.8) determine the separatrix splitting in the investigated systems. We consider the averaging systems (4.12)–(4.13) and (4.15)–(4.16) for $s = \widehat{s} = \widehat{c}/\varepsilon$, where $\pi/4 < \widehat{c} < \pi/2$.

Proposition 8.1 *The systems* (4.12)–(4.13) *and* (4.15)–(4.16) *have solutions for* $s = \widehat{s}$ *in a complex neighborhood of the separatrix* γ *of the pendulum* (2.4). *Moreover, the following estimates hold for* $(x, y) \in U$*:*

$$\begin{aligned} |H^0(x,y,\widehat{s}) - \widehat{H}^0(x,y)| &< \varepsilon^2 c', \quad |H^{\pm 1}(x,y,\widehat{s})| < c'' e^{-\widehat{s}}, \\ |H^{(0)}(x,y,\widehat{s}) - \widehat{H}^0(x,y)| &< \varepsilon^2 c', \quad |H^{(\pm)}(x,y,\widehat{s})| < c''. \end{aligned}$$

According to Proposition 8.1 we can put $\sigma = e^{-\widehat{s}}$ and use Proposition 5.1. We obtain:

$$\mathcal{A}_{\rm P} = 4e^{-\widehat{s}}\varepsilon |P^+_{\rm P}| + O(e^{-2\widehat{s}}), \quad \mathcal{A}_{\rm SCM} = 4e^{-\widehat{s}}\varepsilon |P^+_{\rm SCM}| + O(e^{-2\widehat{s}}),$$
$$P^+_{\rm P} = \int_{-\infty}^{\infty} e^{-\widehat{s}}\big[H^{+1}(\gamma_{\rm P}(t),\widehat{s}) - H^{+1}(\gamma_{\rm P}(\infty),\widehat{s})\big]\, dt,$$
$$P^+_{\rm SCM} = \int_{-\infty}^{\infty} e^{-\widehat{s}}\big[H^{(+)}(\gamma_{\rm SCM}(t),\widehat{s}) - H^{(+)}(\gamma_{\rm SCM}(\infty),\widehat{s})\big]\, dt,$$

where $\gamma_{\rm P}$ and $\gamma_{\rm SCM}$ are separatrix solutions of the systems with Hamiltonians $H^0(x, y, \widehat{s})$ and $H^{(0)}(x, y, \widehat{s})$ respectively.

Proposition 8.2 *The following estimates hold*:

$$\begin{aligned} P^+_{\rm P} &= e^{-\pi\varepsilon^{-1}/2+\widehat{s}}\varepsilon^{-2}(I_{\rm P}(\widehat{s}) + O(\varepsilon)), \\ P^+_{\rm SCM} &= e^{-\pi\varepsilon^{-1}/2+\widehat{s}}\varepsilon^{-2}(I_{\rm SCM}(\widehat{s}) + O(\varepsilon)). \end{aligned}$$

The estimates (7.10)–(7.11) obviously follow from Propositions 8.2 and 7.2. Proofs of Propositions 7.1–7.2 and 8.1–8.2 are based on an analysis of the systems (4.12)–(4.13) and (4.15)–(4.16) in a complex domain containing real separatrix of the pendulum (2.4). The analysis is carried out by the majorant method. We do not give these proofs in the present paper.

9 Evaluation of the Magnitudes $J_{\rm P}$ and $J_{\rm SCM}$

Apparently, the systems (7.2),(7.4) and (7.3),(7.5) cannot be solved explicitly. Straightforward computer integration of these systems seems to be complicated because they contain three independent variables.

In this section we present another approach to calculation of the magnitudes $J_{\rm P}$ and $J_{\rm SCM}$. For simplicity we deal here with the system (7.2),(7.4) in the form (7.12). Analysis of the system (7.3),(7.5) is similar.

We look for a solution of (7.12) in the form of power series in $\mu = -\tilde{\tau}^{-1}$ with coefficients depending on s and $\sigma = \tilde{\tau}^2\tilde{h}$. First we note that the change $\tilde{\tau}, \tilde{h} \to \mu, \sigma$ is symplectic. We denote for short the Poisson bracket in the variables μ, σ by $\{\ ,\ \}$, i.e. $\{f, g\} = f_\sigma g_\mu - f_\mu g_\sigma$. Putting

$$\tilde{H}^0 = (i\mu)^2 g_2^0 + (i\mu)^3 g_3^0 + \dots, \quad G^{\pm 1} = e^{-s}\big((i\mu)^2 g_2^\pm + (i\mu)^3 g_3^\pm + \dots\big), \quad (9.1)$$

we reduce the system (7.12) to the following recursive infinite system of ordinary differential equations:

$$\begin{cases} (g_2^0)_s = 0, \\ (g_2^{\pm})_s = 0, \end{cases} \quad \begin{cases} (g_k^0)_s = 2a \sum_{l+m=k+1} [m(g_l^+)_\sigma g_m^- - l g_l^+ (g_m^-)_\sigma], \\ (g_k^{\pm})_s = \mp \sum_{l+m=k+1} [m(g_l^0)_\sigma g_m^{\pm} - l g_l^0 (g_m^{\pm})_\sigma], \end{cases} \tag{9.2}$$

$$g_2^0(\sigma, 0) = -\sigma, \quad g_2^{\pm}(\sigma, 0) = 2\sigma(1 - \cos\sqrt{2\sigma})^{-1}, \tag{9.3}$$

$$g_k^0(\sigma, 0) = g_k^{\pm}(\sigma, 0) = 0, \qquad k \geq 3. \tag{9.4}$$

The following assertions can be easily proved by induction:
(i) $g_k^0(\sigma, s) \equiv 0$ for any odd $k \geq 3$
(ii) $g_k^+(\sigma, s) = (-1)^k g_k^-(\sigma, s)$.
The function $f(\sigma) = 2\sigma(1 - \cos\sqrt{2\sigma})^{-1}$ has the following Tailor expansion:

$$f(\sigma) = \sum_{m=0}^{\infty} f_m \sigma^m, \quad f_m = \frac{|(2n-1)B_{2n}|}{(2n)!} 2^{n+1},$$

where B_k are the Bernoulli numbers. In particular,

$$f_0 = 2, \quad f_1 = 1/3, \quad f_2 = 1/30, \quad f_3 = 1/378, \quad \ldots$$

To evaluate the magnitude $j(a)$ we can integrate termwise the series (9.1) for the function G^+. Since for $\mu = -(t - is - ic)$

$$\int_{-\infty}^{\infty} e^{s+i(t-ic)} (i\mu)^n \, dt = -\frac{2\pi}{(n-1)!},$$

we have

$$\int_{-\infty}^{\infty} e^{s+i(t-ic)} \tilde{H}^+(t - ic - is, 0, s) \, dt = -2\pi e^{-s} \sum_{k=2}^{\infty} \frac{g_k^+(0, s)}{(k-1)!}.$$

Therefore,

$$j(a) = -2\pi \lim_{s \to +\infty} e^{-s} \sum_{k=2}^{\infty} \frac{g_k^+(0, s)}{(k-1)!}. \tag{9.5}$$

Actually instead of the series (9.2) we evaluate its slightly modified version. We put $g_k^+ = \sum_{l=2}^{k} g_{l,k}^+$, where the functions $g_{l,k}^+$ satisfy the equations

$$\begin{aligned} (g_{l,k}^+)_s &= (k-1) g_{l,k-1}^+, \qquad l < k, \\ (g_{k,k}^+)_s &= -\sum_{j+m=k+1} [m(g_j^0)_\sigma g_m^+ - j g_j^0 (g_m^+)_\sigma] - (k-1) g_{k-1}^+. \end{aligned}$$

The relation (9.2) takes the form

$$j(a) = -2\pi \lim_{s \to +\infty} \sum_{l=2}^{\infty} \Sigma_l(s), \qquad \Sigma_l(s) = e^{-s} \sum_{k=l}^{\infty} \frac{g_{l,k}^+(0, s)}{(k-1)!}. \tag{9.6}$$

Differentiating the sums Σ_l we get:

$$\frac{d\Sigma_l}{ds} = e^{-s}\frac{1}{(l-1)!}\frac{d}{ds}g^+_{l,l}(0,s).$$

Since the magnitudes $\Sigma_l(0)$ vanish for $l > 2$, we have:

$$\Sigma_l(s) = \int_0^s e^{-\nu}\frac{1}{(l-1)!}\frac{d}{d\nu}g^+_{l,l}(0,\nu)\,d\nu, \qquad l > 2. \tag{9.7}$$

Obviously, $\Sigma_2 = 2$. The series (9.3)–(9.4) looks more convenient for numerical analysis in comparison with (9.2). In particular, for $a = 0$ all the sums Σ_l vanish but all the functions $g^+_k(0,s)$ are nonzero.

Unfortunately, the series (9.3) converges slowly, so we could not achieve good precision just by increasing the number of terms Σ^+_k kept in (9.3). Roughly speaking, the terms $\Sigma_k(\infty)$ have the form $c_2k^{-2} + c_3k^{-3} + \cdots$ with some constants c_j. Hence, we have only a power convergence and keeping K terms we get an error $\sim 1/K$. V. Gelfreich pointed out that the error can be reduced considerably by using extrapolation method. Indeed, let us put

$$p_N = -2\pi\sum_{l=2}^{N}\Sigma_l(\infty).$$

Then $j(a) - p_N = \mathrm{const}/N + O(N^2)$. Assume that we have two sums p_K and p_N with $K < N$. Then the following approximate equalities hold:

$$j(a) - p_K \approx c/K, \quad j(a) - p_N \approx c/N.$$

Calculating $j(a)$ from this linear system we obtain an error $\sim K^{-2}$. Analogously we can reduce the error to higher orders.

Using this method we have calculated the first 9 coefficients of Tailor series for the function $j(a)$ and the number J_{SCM}.

Acknowledgments. We are grateful to professors S. V. Bolotin, A. Delshams, V. G. Gelfreich, V. V. Kozlov, A. I. Neishtadt, and C. Simo for very helpful discussions.

Text of the paper was completed when the author was visiting at Centre de Recerca Matematica, Spain. We thank Profs. M. Castellet and C. Simo for hospitality.

The work was partially supported by Russian Foundation of Basic Research, International Science Foundation and INTAS.

References

[1] H. Poincaré, Les méthodes nouvelles de la mécanique celeste, V. 1–3, Gauthier-Villars, Paris, 1892, 1893, 1899.

[2] V. Arnol'd, Instability of dynamical systems with several degrees of freedom, Dokl. Acad. Nauk SSSR, V. 156, N. 1, P. 9–12, (1964) (in Russian), English transl. in Sov. Math. Dokl., V. 5, P. 581–589, (1964)

[3] V. V. Afanasiev, A. A. Chernikov, R. Z. Sagdeev and G. M. Zaslavsky, The Width of the Stochastic Web and Particle Diffusion Along the Web, Physics Letters A, V. 144, No. 4,5, P. 229–236, (1990).

[4] G. M. Zaslavsky, Stochastic Webs and Their Applications, Chaos V. 1, P. 1–12, (1991)

[5] N. N. Filonenko, R. Z. Sagdeev and G. M. Zaslavsky, Nuclear Fusion, V. 7, P. 253, (1967).

[6] G. M. Zaslavsky and N. N. Filonenko, Zh. Eksp. Teor. Fiz., V. 54, P. 1590, (1968).

[7] N. Nekhoroshev, An exponential estimate of stability time in Hamiltonian systems close to integrable, Usp. Mat. Nauk, V. 32, No. 5, (1977), (in Russian), English transl. in Russian Mat. Surveys, V. 32, No. 1, (1977)

[8] P. Lochak, Canonical perturbation theory via simultaneous approximation, Russian Math. Surveys, V. 47, No. 6 P. 57 (1991)

[9] P. Lochak and A. I. Neishtadt, Estimates of stability time for nearly integrable systems with a quasiconvex Hamiltonian, Chaos V. 2, No. 4 P. 495–499 (1992).

[10] J. Pöschel, On Nekhoroshev's estimate for quasiconvex Hamiltonians, Math. Z., (1993) V. 213, P. 187-216

[11] A. I. Neishtadt, The separation of motions in systems with rapidly rotating phase, Prikl. Mat. Mekh., V. 48, No. 2, (1984) P. 197–204 (in Russian) English transl. in J. Appl. Math. Mech. V. 48, No. 2, (1984) P. 133–139.

[12] D. V. Treshchev, A mechanism for the destruction of resonance tori in Hamiltonian systems Mat. Sb., V. 180, No. 10, (1989) P. 1325–1346, (in Russian), English transl. in Math. USSR-Sb., (1991) V. 68, No. 1, P. 181–203.

[13] R. Douady, Ann. Scient. Ec. Norm. Sup. Ser. IV, V. 21, P. 1–46, (1988).

[14] V. F. Lazutkin, Splitting of separatrices for standard Chirikov's mapping VINITI no 6372-84, 24 Sept. 1984, (in Russian).

[15] V. F. Lazutkin, I. G. Schachmannski and M. B. Tabanov, Splitting of separatrices for standard and semistandard mappings, Physica D, Vo. 40, P. 235–248, (1989).

[16] V. F. Lazutkin, On the width of the instability zone near the separatrices of a standard mapping, Soviet Math. Dokl., V. 42, No. 1, P. 5–9, (1991).

[17] V. G. Gelfreich, V. F. Lazutkin and N. V. Svanidze, Refined formula to separatrix splitting for the standard map, Physica D, V. 71, (1994) N. 2 P. 101

[18] V. G. Gelfreich, V. F. Lazutkin and M. B. Tabanov, Exponentially small splitting in Hamiltonian systems, Chaos, V. 1, (1991) N. 2, P. 137–142.

[19] P. Holmes, J. Marsden and J. Scheurle, Exponentially small splittings of separatrices with applications to KAM theory and degenerate bifurcations, Contemp. Math., V. 81, P. 213–244, (1988).

[20] V. G. Gelfreich, Splitting of separatrices of the rapidly forced pendulum, Preprint, (1990).

[21] A. Delshams and T. M. Seara, An asymptotic Expressions for the Splitting of Separatrices of the Rapidly Forced Pendulum, Comm. Math. Phys., V. 150, P. 433–463, (1992).

[22] A. M. Dykhne, Quantum transitions in the adiabatic approximation, Zh. Eksp. Teor. Fiz., V. 38, No. 2, P. 570–578, (1960) (in Russian), English transl. in Sov. Phys. JETP, V. 11, No. 2, P. 411–415, (1960).

[23] A. A. Slutskin, Motion of a one-dimensional non-linear oscillator under adiabatic conditions, Zh. Eksp. Teor. Fiz., V. 45, No. 4, P. 978–988, (1963), (in Russian), English transl. in Sov. Phys. JETP, V. 18, No. 3, P. 676–682, (1964).

[24] D. Treschev, Separatrix Splitting for a Pendulum with Rapidly Oscillating Suspension Point, Preprint no. 273, Centre de Recerca Matematica, Institute D'Estudis Catalans, Barcelona, Spain, (1994).

[25] D. Treschev, Separatrix Splitting for a Pendulum with Rapidly Oscillating Suspension Point, to appear in Russian J. of Math. Phys.