Closures of asymptotic curves in two-dimensional symplectic map.

1 Introduction

Let T be a C^1-smooth area-preserving diffeomorphism of a two-dimensional manifold M and $\widehat{z} \in M$ a hyperbolic fixed point of the map T. The point $\widehat{z}$ generates four asymptotic curves (below they are called branches or separatrix branches). We denote the stable branches by $\Gamma^s_{1,2}$ and the unstable ones by $\Gamma^u_{1,2}$. We assume that the point $\widehat{z}$ does not belong to $\Gamma^{s,u}_{1,2}$.

Poincaré noticed [4] that in general the separatrices intersecting, form a very complicated network. Dynamics in the vicinity of this network is highly unstable and irregular. Because of this it is accepted to call this vicinity the stochastic layer and to characterize dynamics in the stochastic layer by the word "chaos".

The structure of such a chaos is weakly understood. It is known that in the stochastic layer there exists an invariant hyperbolic set on which T is isomorphic to the Smale horseshoe. The measure of this set however vanishes and the question what behavior is typical for trajectories in the stochastic layer remains open.

We mention here two questions well-known in mathematical folklore and related to the problems we deal with in this paper. Suppose that stable and unstable separatrices do not coincide: $\Gamma^s_1 \cup \Gamma^s_2 \neq \Gamma^u_1 \cup \Gamma^u_2$.

• Does the closure of the separatrices $\overline{\Gamma}^s_1 \cup \overline{\Gamma}^u_1 \cup \overline{\Gamma}^s_2 \cup \overline{\Gamma}^u_2$ have positive measure?

• If the answer to the first question is positive, is the map T restricted to this set ergodic?

In the present paper we prove two theorems.

Theorem 1 *Suppose that the following conditions hold.*

(1) *The set $\Gamma^s_1 \cap \Gamma^u_1$ is not empty.*

(2) *The curves Γ^s_1 and Γ^u_1 lie in an invariant domain $D \subset M$. The closure $\overline{D}$ is compact.*

Then the closure of the unstable branch $\overline{\Gamma}^u_1$ contains the stable one Γ^s_1.

Corollary 1.1 *Since conditions of the theorem are symmetric with respect to* Γ_1^s *and* Γ_1^u, *the inclusion* $\Gamma_1^u \subset \overline{\Gamma}_1^s$ *holds. Hence, the sets* $\overline{\Gamma}_1^s$ *and* $\overline{\Gamma}_1^u$ *coincide.*

Corollary 1.2 *If conditions of the theorem hold for three pairs*

$$\Gamma_i^s \text{ and } \Gamma_j^u, \qquad i,j \in \{1,2\}, \tag{1.1}$$

the sets $\overline{\Gamma}_1^s$, $\overline{\Gamma}_2^s$, $\overline{\Gamma}_1^u$, $\overline{\Gamma}_2^u$ *coincide.*

Let p and q be hyperbolic periodic points of T and let i and j respectively be their periods. Since p and q are hyperbolic fixed points for the maps T^i and T^j, we can define four branches $\Gamma_{1,2}^{s,u}(p)$ $\Gamma_{1,2}^{s,u}(q)$. We put

$$W_{1,2}^{s,u}(p) = \cup_{k=0}^{i-1} T^k(\Gamma_{1,2}^{s,u}(p)), \quad W_{1,2}^{s,u}(q) = \cup_{k=0}^{j-1} T^k(\Gamma_{1,2}^{s,u}(q)).$$

Theorem 2 *Suppose that the following conditions hold.*

(1′) *The set* $W_1^s(p) \cap W_1^u(q)$ *is not empty.*

(2′) *The set* $W_1^s(q) \cap W_1^u(p)$ *is not empty.*

(3′) *The asymptotic manifolds* $W_1^s(p)$, $W_1^u(p)$, $W_1^s(q)$ $W_1^u(q)$ *belong to an invariant domain* $D \subset M$. *The closure* $\overline{D}$ *is compact.*

Then the following inclusions take place:

$$W_1^s(p) \subset \overline{W}_1^u(q) \cup \overline{W}_1^s(q) \tag{1.2}$$
$$W_1^s(p) \subset \overline{W}_1^u(q) \cup \overline{W}_1^u(p) \tag{1.3}$$

Remark 1.1 *In the case* $p = q$ *conditions* **(1′)** *and* **(2′)** *coincide.*

Corollary 1.3 *Changing* q *by* p *and vice versa and (or) replacing* T *by* T^{-1} *we get instead of (1.2) the inclusions*

$$W_1^s(q) \subset \overline{W}_1^u(p) \cup \overline{W}_1^s(p)$$
$$W_1^u(p) \subset \overline{W}_1^u(q) \cup \overline{W}_1^s(q)$$
$$W_1^u(p) \subset \overline{W}_1^u(p) \cup \overline{W}_1^s(p).$$

In particular, **(1′)**–**(3′)** *imply* $\overline{W}_1^u(p) \cup \overline{W}_1^s(p) = \overline{W}_1^u(q) \cup \overline{W}_1^s(q)$.

Corollary 1.4 *If* $p = q$ *and conditions* **(1′)**–**(3′)** *hold, according to (1.3) and to the corresponding symmetric inclusion we have:* $\overline{W}_1^s(p) = \overline{W}_1^u(p)$.

Recall that points of the set

$$\big(W_1^s(p) \cup W_2^s(p)\big) \cap \big(W_1^u(q) \cup W_2^u(q)\big), \quad \big(W_1^s(q) \cup W_2^s(q)\big) \cap \big(W_1^u(p) \cup W_2^u(p)\big)$$

are called homoclinic if p and q lie on the same periodic trajectory, and heteroclinic otherwise.

Takens proved [5] that if M is compact, there exists a residual subset R in the set of area-preserving C^1-smooth diffeomorphismes of M such that for any $T \in R$ and any hyperbolic periodic point p of T the set of homoclinic points is dense on $W^{s,u}_{1,2}(p)$. Note that methods of [5] are essentially restricted to the C^1-topology.

The following conjecture was in fact formulated (in a weaker form) by Poincaré.

Conjecture 1 *If $p = q$ and conditions of Theorem 2 hold, the set of homoclinic points is dense on $W^s_1(p)$ and on $W^u_1(p)$.*

Mather [2] proved that if M is compact, for a C^r-generic ($r \geq 4$) area-preserving map any two branches of a hyperbolic periodic point have the same closure. The word "generic" is understood in the sense of Baire category.

Oliveira [3] obtained the following results related to the problems in question. Let T be a C^1 area-preserving diffeomorphism of a compact orientable surface. Assume L and K are branches of a hyperbolic fixed point with $L = K$ or $L \cap K = \emptyset$. If $K \cap \omega(L) \neq \emptyset$ then $K \subset \omega(L)$. (Here as ususal, $\omega(L)$ is ω-limit set of L.)

This result implies, [3] the following assertion. Let M be a compact orientable surface and $1 \leq r \leq \infty$. Then $L \subset \omega(L)$ for any branch L of a C^r-generic area-preserving map.

Note that the manifold M in Theorems 1–2 can be not compact, not orientable. It can even have a boundary. The further constructions depend on a certain integer r. The reader which will be satisfied by the case when M is a sphere or a plane can put $r = 0$ and do not pay any attention to geometric obgects (homology groups, intersection indices, etc.) appearing below. The basic geometric fact we use in the case $r = 0$ is very simple: two closed transversal to one another curves on a sphere or on a plane can not have exactly one common point.

In the general situation we consider the homology group $H_1(M, \partial M)$ of the manifold M with respect to the boundary ∂M with coefficients from $\mathbf{Z}_2$. Let G be the subgroup in $H_1(M, \partial M)$ generated by curves lying in D. We put $r = \operatorname{rank} G$. Since according to **(2)** or **(3′)** the closure $\overline{D}$ is compact, we have: $0 \leq r < \infty$.

Plan of the paper is as follows. In section 2 we prove Theorem 1 in the case $r = 0$. In section 3 we deal with the case $r > 0$. In section 4 we present a proof of Theorem 2.

Below we can assume that in the case of Theorem 1 the map T preserves each branch and in the case of Theorem 2 T preserves each of the eight manifolds $W^{s,u}_{1,2}(p)$, $W^{s,u}_{1,2}(q)$. Indeed, if T does not satisfy this condition, we just change T by T^2.

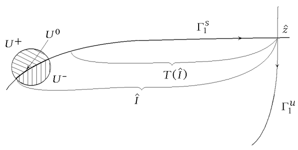

Figure 1:

2 The case $r = 0$.

Let $U \subset D$ be any open set such that $U \cap \Gamma_1^s \neq \emptyset$. Suppose that

$$U \cap \Gamma_1^u = \emptyset. \tag{2.1}$$

Then Theorem 1 is proved if we are able to obtain a contradiction with its conditions. Considering if necessary instead of U a smaller domain (which will be denoted also by U for brevity), we can assume that $U = U^+ \cup U^0 \cup U^-$, where $U^\pm$ are open, connected, and $U^0 \subset \Gamma_1^s$ is a connected interval (see Fig. 1). Let $\widehat{I}$ be the minimal connected piece of Γ_1^s such that $\widehat{z}$ is its endpoint and $U^0 \subset \widehat{I}$. We can assume that

$$U \cap T(\widehat{I}) \;=\; \emptyset, \tag{2.2}$$

$$U \cap T^{-r-1}(\widehat{I}) \;=\; U^0. \tag{2.3}$$

These equalities mean that U^0 and $U^\pm$ are sufficiently small.

Lemma 2.1 *There exists a natural $l > r+1$ such that*

$$U^+ \cap T^l(U^+) \neq \emptyset, \quad U^- \cap T^l(U^-) \neq \emptyset.$$

Proof of Lemma 2.1. Consider the map

$$T \times T : M \times M \to M \times M, \quad T \times T(z_1, z_2) = (T(z_1), T(z_2)).$$

This map preserves the measure $\sigma \times \sigma$, where σ is the area on M. The set $U^+ \times U^-$ lies inside the compact invariant set $\overline{D} \times \overline{D}$. Hence, according to the Poincaré Recurrence Theorem, for infinitely many naturals l

$$(T \times T)^l (U^+ \times U^-) \cap (U^+ \times U^-) \neq \emptyset.$$

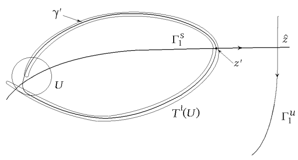

Figure 2:

The lemma is proved.

There exists a smooth closed curve γ' satisfying the following properties.
(a) $\gamma' \subset U'$, $\qquad U' = U \cup T^l(U)$.
(b) The set $\gamma' \cap T^l(U^0)$ consists of single point z' and the curves γ' and $T^l(U^0)$ intersect at z' transversely.

The curve γ' goes along the set $T^l(U^+)$ from the point z' to the set U^+. Then γ' passes through the interval U^0 to the set U^-, goes to the set $T^l(U^-)$ and returns to the point z' (see Fig. 2).

According to property **(a)** and equality (2.1) we have:

$$\gamma' \cap \Gamma_1^u = \emptyset. \tag{2.4}$$

Lemma 2.2 *There exists an interval $I \subset \Gamma_1^s$ satisfying the following two properties.*
(A) *Endpoints of I are $\widehat{z}$ and z_0, where z_0 is homoclinic.*
(B) $I \cap U' = T^{-r}(I) \cap U' = T^l(U^0)$.

Proof of Lemma 2.2. The interval $I' = T^{l-1}(\widehat{I}) \setminus T^l(\widehat{I})$ contains a homoclinic point. Indeed, otherwise we have a contradiction with assumption **(1)** because of the relation $\Gamma_1^s = \cup_{k \in \mathbf{Z}} T^k(I')$. We denote this homoclinic point by z_0. Hence, I is defined by **(A)**.

Now Lemma 2.2 follows from three inclusions:

$$T^l(U^0) \subset (I \cap U') \subset (T^{-r}(I) \cap U') \subset T^l(U^0).$$

The first two are obvious. Let us check the third one. Since $I \subset T^{l-1}(\widehat{I})$, we have: $T^{-r}(I) \subset T^{l-r-1}(\widehat{I})$. This relation together with the one $l > r+1$ imply

$$T^{-r}(I) \cap U \subset T^{l-r-1}(\widehat{I}) \cap U \subset T(\widehat{I}) \cap U = \emptyset$$

due to (2.2). Analogously,

$$T^{-r}(I) \cap T^l(U) \subset T^{l-r-1}(\widehat{I}) \cap T^l(U) \subset T^l\big(T^{-r-1}(\widehat{I}) \cap U\big) = T^l(U^0).$$

Here we have used assumption (2.3). Lemma 2.2 is proved.

According to property **(A)** the points $\widehat{z}$ and z_0 can be connected by an interval I^u of the unstable separatrix branch Γ_1^u. The curve $\sigma_0 = I \cup I^u$ is closed. According to the definition of γ' and property **(B)** the curves I and γ' have exactly one common point (the point z') and intersect at z' transversely. The curves I^u and γ' do not intersect because of (2.4). Hence, the intersection index (modulo 2)

$$\mathrm{ind}_2(\sigma_0, \gamma') = 1. \tag{2.5}$$

Proposition 2.1 *In the case $r = 0$, the intersection index of any two closed curves on D vanishes.*

Since the equality (2.5) contradicts to Proposition 2.1, Theorem 1 is proved for $r = 0$. Proposition 2.1 is a simple corollary of the following lemma.

For any closed curve $a \subset M$ let $\{a\}$ be the corresponding element of the homology group $H_1(M, \partial M)$.

Lemma 2.3 *For any two closed curves $a_1, a_2 \subset M$ their intersection index is determined by the elements $\{a_1\}, \{a_2\}$:*

$$\mathrm{ind}_2(a_1, a_2) = \alpha(\{a_1\}, \{a_2\}), \qquad \alpha : H_1(M, \partial M) \times H_1(M, \partial M) \to \mathbf{Z}_2.$$

The function α is bilinear.

Lemma 2.3 is a standard geometric fact. The proof can be found in [1]).

3 The case $r > 0$.

The map T is defined on the invariant domain $D \subset M$. Hence, the automotphism T_* of the homology group $H_1(M, \partial M)$ is also an automorphism of the invariant subgroup $G \subset H_1(M, \partial M)$. (Recall that the subgroup G is generated by the curves lying in D).

For any curves $u, v \subset M$ the following equality holds:

$$\alpha(\{u\}, \{v\}) = \alpha(T_*\{u\}, T_*\{v\}). \tag{3.1}$$

Lemma 3.1 *For any $0 \le k \le r$ the curve $\sigma_k = T^{-k}(\sigma_0)$, satisfies the equality $\alpha(\{\sigma_k\}, \{\gamma'\}) = 1$.*

Proof of Lemma 3.1. Any curve σ_k is a union $\sigma_k^s \cup \sigma_k^u$, where $\sigma_k^s \subset \Gamma_1^s$ and $\sigma_k^u \subset \Gamma_1^u$.

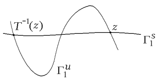

Figure 3:

Let $0 \leq k \leq r$. Then $I \subset \sigma_k^s \subset T^{-r}(I)$. Hence, by Lemma 2.2 $\sigma_k^s \cap \gamma' = z'$ and the intersection is transversal. Because of (2.4) we have: $\sigma_k^u \cap \gamma' = \emptyset$. Lemma 3.1 is proved.

For any homoclinic point $z \in \Gamma_1^s \cap \Gamma_1^u$ let us call by L-loop $\vartheta(z)$ the closed curve formed by segments of Γ_1^s and Γ_1^u, confined by z and its preimage $T^{-1}(z)$ (see Fig. 3). We put $\vartheta_0 = \vartheta(z_0)$ and $\vartheta_k = T^{-k}(\vartheta_0)$. For any natural q the following equality holds

$$\{\sigma_q\} = \{\sigma_0\} + \sum_{k=0}^{q-1} \{\vartheta_k\}. \tag{3.2}$$

Lemma 3.2 *For any L-loop ϑ_k, $k \in \mathbf{Z}$ the equality $\alpha(\{\vartheta_k\}, \{\gamma'\}) = 0$ holds.*

Proof of Lemma 3.2. According to Lemma 3.1 and the equalities (3.2) we have:

$$\alpha(\{\vartheta_k\}, \{\gamma'\}) = 0, \qquad 0 \leq k \leq r-1.$$

Consider the maximal m such that the vectors

$$\{\vartheta_0\}, \{\vartheta_1\}, \ldots, \{\vartheta_m\} \tag{3.3}$$

are linearly independent. Obviously, $m \leq r-1$.

Let $\mathcal{L}$ be linear hool of the vectors (3.3). Then $\{\vartheta_k\} \in \mathcal{L}$ for any $k \in \mathbf{Z}$. For any vector $\{v\}$ from the linear space $\mathcal{L}$ we have $\alpha(\{v\}, \{\gamma'\}) = 0$. Lemma 3.2 is proved.

According to identity (3.1) we have: $\alpha(T_*^l\{\sigma_0\}, T_*^l\{\gamma'\}) = \alpha(\{\sigma_0\}, \{\gamma'\}) = 1$. Since $T_*^l\{\sigma_0\} - \{\sigma_0\}$ equals a sum of L-loops $\{\vartheta_k\}$,

$$\alpha(T_*^l\{\sigma_0\}, \{\gamma'\}) = 1.$$

Hence,

$$\alpha(T_*^l\{\sigma_0\}, \{\gamma'\} + T_*^l\{\gamma'\}) = 0. \tag{3.4}$$

The curves γ' and $T^l(\gamma')$ go through the same connected open set $T^l(U)$. Hence, inside the set $U'' = U \cup T^l(U) \cup T^{2l}(U)$ there exists a closed curve γ'' of the homology class $\{\gamma'\} + T^l_*\{\gamma'\}$ which differs from $\gamma' \cup T^l(\gamma')$ only inside the set $T^l(U)$. Since $U'' \cap \Gamma^u_1 = \emptyset$, we have:

$$\gamma'' \cap \Gamma^u_1 = \emptyset. \tag{3.5}$$

The following relations hold:

$$T^l(\sigma_0) = T^l(I^u) \cup T^l(I), \quad T^l(I^u) \subset \Gamma^u_1, \quad T^l(I) \subset \Gamma^s_1.$$

According to Lemma 2.2 and to the obvious equality $U \cap T^l(I) = \emptyset$, we have: $T^l(I) \cap U'' = T^{2l}(U^0)$. Hence, $T^l(I) \cap \gamma'' = T^l(z')$ and the intersection is transversal. On the other hand, $T^l(I^u) \cap \gamma'' = \emptyset$ because of (3.5). Thus, $\mathrm{ind}_2(T^l(\sigma_0), \gamma'') = 1$, what contradicts to (3.4). Theorem 1 is proved.

4 Separatrices of periodic points

Let $U \subset D$ be an open set such that $U \cap W^s_1(p) \neq \emptyset$. Suppose that at least one of the equalities

$$U \cap (W^u_1(q) \cup W^s_1(q)) = \emptyset, \tag{4.1}$$
$$U \cap (W^u_1(q) \cup W^u_1(p)) = \emptyset. \tag{4.2}$$

holds. Then Theorem 2 will be proved when we obtain a contradiction with its conditions.

Considering the map T^i instead of T, we define the sets $U^{\pm}, U^0$. In particular, we again assume that U satisfies equalities (2.2)–(2.3) with T^i instead of T.

We can obviously assume that $U \cap \Gamma^s_1(p) \neq \emptyset$. (Otherwise we take $T^k(U)$ with a proper $k \in \{1, \ldots, i-1\}$ instead of U.) The closed curve $\gamma' \subset U'$ is defined in the same way as above. We can assume that γ' does not pass through points of the form $T^k(p)$ and $T^k(q)$, $k \in \mathbf{Z}$.

By using condition **(1′)** we get: $\Gamma^s_1(p) \cap W^u_1(q) \neq \emptyset$. Hence, Lemma 2.2 remains true if in its statement $z_0 \in \Gamma^s_1(p) \cap W^u_1(q)$.

For some $j^u, j^s \in \{0, \ldots, j-1\}$ we have:

$$z_0 \in T^{j^u}(\Gamma^u_1(q)) \cap \Gamma^s_1(p), \quad \Gamma^u_1(p) \cap T^{j^s}(\Gamma^s_1(q)) \neq \emptyset.$$

Let Λ' be the curve that goes from $T^{j^u}(q)$ to p and is formed by the segment of the branch $T^{j^u}(\Gamma^u_1(q))$ from the point $T^{j^u}(q)$ to the point z_0 and by the segment of the branch $\Gamma^s_1(p)$ from z_0 to p. Analogously, let Λ'' be the curve that goes from p to $T^{j^s}(q)$ and is formed by the segment of the branch $\Gamma^u_1(p)$ from p to some point $\tilde{z} \in \Gamma^u_1(p) \cap T^{j_s}(\Gamma^s_1(q))$ and by the segment of the branch $T^{j_s}(\Gamma^s_1(q))$ from $\tilde{z}$ to $T^{j_s}(q)$ (see Fig. 4).

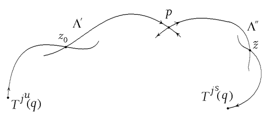

Figure 4:

Let ν be the least common multiple of the numbers j and $\Delta = j^s - j^u$. If $\Delta = 0$, we put $\nu = 0$. The curve

$$\Lambda' \cup \Lambda'' \cup T^{\Delta}(\Lambda') \cup T^{\Delta}(\Lambda'') \cup T^{2\Delta}(\Lambda') \cup T^{2\Delta}(\Lambda'') \cup \ldots \cup T^{\nu-\Delta}(\Lambda') \cup T^{\nu-\Delta}(\Lambda'')$$

is closed. The same is true for the curve

$$\begin{aligned} \sigma_0 = \sigma_0(k', k'') = \Lambda' \cup T^{ijk''}(\Lambda'') \cup T^{ijk'+\Delta}(\Lambda') \cup T^{ijk''+\Delta}(\Lambda'') \cup \ldots \\ \ldots \cup T^{ijk'+\nu-\Delta}(\Lambda') \cup T^{ijk'+\nu-\Delta}(\Lambda''), \end{aligned}$$

where k', k'' are arbitrary integers. Obviously,

$$\sigma_0 \in W_1^s(p) \cup W_1^u(p) \cup W_1^s(q) \cup W_1^u(q).$$

We choose $k' > 0$ so large that

$$\gamma' \cap \sigma_0 \cap W_1^s(p) = \gamma' \cap I = z'. \tag{4.3}$$

(We can do this because the set $(\sigma(k', k'') \cap W_1^s(p)) \setminus I$ for large $k' > 0$ is situated in a small neighborhood of the tragectory generated by p and the curve γ' does not contain points of this trajectory.)

Suppose that the relation (4.1) holds. Then choose $k'' < 0$ so large in absolute value that

$$\gamma' \cap \sigma_0 \cap W_1^u(p) = \emptyset. \tag{4.4}$$

(In the case (4.2) we take $k'' > 0$ such that $\gamma' \cap \sigma_0 \cap W_1^s(q) = \emptyset$ and use analogous arguments).

According to (4.1), (4.3), and (4.4) we have: $\sigma_0 \cap \gamma' = z'$, where the intersection is transversal. Hence, $\mathrm{ind}_2(\sigma_0, \gamma') = 1$. In the case $r = 0$ this index should vanish and we obtain the required contradiction. In the case $r > 0$ the arguments are the same as in section 3: a contradiction is obtained as a result of evaluation of the intersection index for the curves $T^{il}(\sigma_0)$ and γ'', where $\{\gamma''\} = \{\gamma'\} + T_*^{il}\{\gamma'\}$.

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References

[1] Dold A., Lectures on algebraic topology. Berlin: Springer, 1972.

[2] Mather J. N., Invariant subsets for area-preserving homeomorphisms of surfaces, Mathematical Analysis and Applications, Part B. Advances in Mathematics. Supplementary Studies, Vol. 7B, Leopoldo Nachbin, ed.

[3] Oliveira F. On the generic existence of homoclinic points, Ergodic Th. Dynam. Sys. (1987), V. 7, P. 567–595.

[4] Poincaré H., Les méthodes nouvelles de la mécanique celeste, V. 1–3: Gauthier-Villars, Paris, 1892, 1893, 1899.

[5] Takens F., Homoclinic points of conservative systems, Inv. Math. V. 18, 1972, P. 267–292.