Remarks on the definition of hyperbolic tori of Hamiltonian systems

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Abstract

We show that under certain natural conditions the definition of a hyperbolic torus conventional for the general theory of dynamical systems is quite suitable for needs of the KAM-theory

1 Introduction

The goal of this paper is to discuss the definition of hyperbolic invariant tori of Hamiltonian systems. Hyperbolic tori were first studied by Poincaré. Arnold [1] named hyperbolic tori as whiskered and used them in his famous example of the phenomenon that is now called "Arnold diffusion". KAM-type results on the conservation of hyperbolic tori under small perturbations were obtained by Graff [16] and Zehnder [27]. In the recent years the notion of a hyperbolic torus was widely used in literature [25, 22, 7], but, as far as we know, there is no universally accepted definition. Usually a definition of hyperbolic torus is given by presenting some normal form of the Hamiltonian in its neighborhood. The goal of this note is to give an invariant (coordinate independent) definition of a hyperbolic turus and derive some consequences. This paper is mainly of the methodological nature and doesn't contain really new results. However, we hope that it will be useful for clarifying the subject.

The concept of hyperbolicity appeared in the general theory of dynamical systems, where a hyperbolic set is defined in terms of the so called stable and unstable foliations. Here we will show that under certain natural conditions the definition of a hyperbolic torus conventional for the general theory of dynamical systems is quite suitable for needs of the KAM-theory.

Recently there appeared the preprint [19], where an invariant definition of hyperbolic tori is also discussed.

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2 Definitions

Let a $2m$-dimensional symplectic manifold (M, ω) be the phase space of a Hamiltonian system with smooth Hamiltonian H. Let v_H be the Hamiltonian vector field on M and $g^t : M \to M$ the phase flow. Let $N \subset M$ be an invariant n-dimensional torus with frequency vector $\nu \in \mathbf{R}^n$. Thus N is the image of a smooth embedding $\Phi : \mathbf{T}^n \to M$ and the flow on N is linear: $\Phi(x + \nu t) = g^t(\Phi(x))$. Equivalently,

$$\partial_\nu \Phi = v_H \circ \Phi, \qquad \partial_\nu = \left\langle \nu, \frac{\partial}{\partial x} \right\rangle.$$

The linearized phase flow

$$Dg^t : T_N M \to T_N M, \qquad v \in T_x M \mapsto Dg^t(x)v \in T_{g^t(x)} M$$

is defined by a linear differential equation which is called the variational equation for the invariant torus N. The following definition was suggested in [6]. A slightly different invariant definition is used in [4].

Definition 2.1 *The torus N is said to be hyperbolic* [1] *if $n < m$ and there exist two smooth l-dimensional ($l = m - n$) subbundles $E^\pm$ of the bundle $T_N M$ such that*

1. *$E^\pm$ are invariant for the linearized phase flow:*

$$Dg^t(x)E^\pm_x = E^\pm_{g^t(x)}, \qquad x \in N, \quad t \in \mathbf{R}.$$

2. *The linearized flow is contracting on E^- and expanding on E^+ i.e., for some positive constants C and λ, we have*

$$\begin{aligned} \|Dg^t(x)\big|_{E^-_x}\| &\le Ce^{-\lambda t}, \qquad x \in N, \quad t \ge 0, \\ \|Dg^{-t}(x)\big|_{E^+_x}\| &\le Ce^{-\lambda t}, \qquad x \in N, \quad t \ge 0. \end{aligned}$$

Here $\|\cdot\|$ is the operator norm defined by any Riemannian metric on M. Since N is compact, the definition doesn't depend on the metric. Passing to a finite covering of a tubular neighborhood of N, without loss of generality we can assume that the bundles $E^\pm$ are oriented and hence trivial: $E^\pm = \mathbf{T}^n \times \mathbf{R}^l$.

Definition 2.2 *If the vector ν is nonresonant i.e., $\langle \nu, k \rangle \ne 0$ for all $k \in \mathbf{Z}^n \setminus \{0\}$, the invariant torus is said to be nonresonant. If ν is Diophantine i.e.,*

$$|\langle \nu, k \rangle| \ge \alpha |k|^{-\beta}, \qquad \alpha, \beta > 0,$$

the torus is said to be Diophantine.

[1] In the general theory of dynamical systems such torus would be called partially normally hyperbolic, because $E^+_x \oplus E^-_x \oplus T_x N \ne T_x M$. Here we follow the traditions of Hamiltonian dynamics and KAM-theory.

Definition 2.3 *A hyperbolic torus N is said to be nondegenerate if all bounded solutions of the variational equation are tangent to N. Thus $\|Dg^t(x)v\| \le C$ for all t implies that $v \in T_x N$.*

For $n = 1$ a hyperbolic torus is a hyperbolic periodic orbit. For Hamiltonian systems, periodic orbits appear in families parameterized by energy. Nondegeneracy means that the period is a nondegenerate function of energy.

Let $L(\mathbf{R}^l)$ be the set of linear operators (matrices) $\mathbf{R}^l \to \mathbf{R}^l$. For a smooth function $\Lambda : \mathbf{T}^n \to L(\mathbf{R}^l)$ consider a skew product system on $\mathbf{T}^n \times \mathbf{R}^l$:

$$\dot v = \Lambda(x)v, \quad \dot x = \nu, \qquad x \in \mathbf{T}^n, \quad v \in \mathbf{R}^l. \tag{2.1}$$

Its phase flow has the form

$$v(t) = \phi_t(x)v, \quad x(t) = x + \nu t, \tag{2.2}$$

where $\phi_t : \mathbf{T}^n \to L(\mathbf{R}^l)$, $\phi_0(x) = I$, is the fundamental matrix.

Definition 2.4 *The function $\Lambda : \mathbf{T}^n \to L(\mathbf{R}^l)$ is said to be ν-positive definite if there exist positive constants C, λ such that $\|\phi_{-t}(x)\| \le Ce^{-\lambda t}$ for all $x \in \mathbf{T}^n$ and $t \ge 0$. The function Λ is said to be ν-negative definite if $\|\phi_t(x)\| \le Ce^{-\lambda t}$, $t \ge 0$.*

In other words, the Lyapunov exponents of the skew product system (2.1) are positive or negative respectively.

Remark 2.1 *The duality of the systems*

$$\begin{cases} \dot v = \Lambda(x)v \\ \dot x = \nu \end{cases} \quad \text{and} \quad \begin{cases} \dot u = -\Lambda^T(x)u \\ \dot x = \nu \end{cases}$$

implies that Λ is ν-positive definite if and only if $-\Lambda^T$ is ν-negative definite.

The simplest example of a ν-positive definite function is $\Lambda =$ const, where eigenvalues of Λ have positive real parts. An equivalent condition is: $\Lambda + \Lambda^*$ is positive definite with respect to some Euclidean metric on $\mathbf{R}^l$, where the operator Λ^* is conjugate to Λ with respect to the same metric. Here the property of ν-positive definiteness does not depend on ν. For $l = 1$ equations (2.1) can be solved explicitly. Then, for ν nonresonant, ν-positive definiteness of Λ is equivalent to the condition $\int_{\mathbf{T}^n} \Lambda(x)\,dx > 0$.

As a model example, consider a system on $P = \mathbf{T}^n\{x\} \times \mathbf{R}^n\{y\} \times \mathbf{R}^l\{z_+\} \times \mathbf{R}^l\{z_-\}$ with Hamiltonian

$$H(x, y, z_+, z_-) = \langle \nu, y\rangle + \langle Ay, y\rangle/2 + \langle z_-, \Omega(x)z_+\rangle + O_3(y, z) \tag{2.3}$$

and the symplectic structure

$$\omega = dy \wedge dx + dz_- \wedge dz_+,$$

where the symmetric $(n \times n)$-matrix A is constant, and the matrix function Ω is ν-positive definite.

Proposition 2.1 *The invariant torus* $N = \{(x, y, z_+, z_-) : y = 0, z_\pm = 0\}$ *is hyperbolic.*

Indeed, the Hamilton equations have the form

$$\left\{\begin{array}{rll} \dot{x} = & \nu + Ay & +O_2(y,z), \\ \dot{y} = & & O_2(y,z), \\ \dot{z}_+ = & \Omega(x)z_+ & +O_2(y,z), \\ \dot{z}_- = & -\Omega^T(x)z_- & +O_2(y,z). \end{array}\right. \tag{2.4}$$

The variational equations on

$$T_N P = \mathbf{T}^n\{x\} \times \mathbf{R}^n\{\xi\} \times \mathbf{R}^n\{\eta\} \times \mathbf{R}^l\{\zeta_+\} \times \mathbf{R}^l\{\zeta_-\}$$

take the form

$$\left\{\begin{array}{rcl} \dot{x} & = & \nu, \\ \dot{\xi} & = & A\eta, \\ \dot{\eta} & = & 0, \\ \dot{\zeta}_+ & = & \Omega(x)\zeta_+, \\ \dot{\zeta}_- & = & -\Omega^T(x)\zeta_-. \end{array}\right. \tag{2.5}$$

The subbundles $E^+ = \mathbf{T}^n \times \mathbf{R}^l\{\zeta_+\}$ and $E^- = \mathbf{T}^n \times \mathbf{R}^l\{\zeta_-\}$ are obviously invariant. They satisfy Definition 2.1 of a hyperbolic torus due to the condition of ν-positive definiteness of Ω.

Definition 2.5 *Symplectic coordinates* (x, y, z_+, z_-) *in a finite covering of a neighborhood of a hyperbolic torus* N *are said to be canonical for* N *if* H *has the form (2.3), up to a constant. If there exist smooth canonical coordinates for a hyperbolic torus, the torus is said to be weakly reducible. A hyperbolic torus is said to be reducible if it is weakly reducible with a constant matrix* Ω.

Proposition 2.2 *A weakly reducible hyperbolic torus is nondegenerate if and only if* $\det A \neq 0$.

Indeed, for any bounded solution of variational equations (2.5), $\zeta_\pm = 0$ and $A\eta = 0$. All these solutions are tangent to N if and only if the matrix A is nondegenerate. ∎

In [16, 27, 25] a hyperbolic torus was defined in terms of canonical coordinates, where the matrix $\Lambda(x) + \Lambda^T(x)$ is positive definite for any $x \in \mathbf{T}^n$. Up to a change of variables, this is equivalent to a the weaker condition that for some positive definite matrix G, the matrix $G\Lambda(x) + \Lambda^T(x)G$ is positive definite for any $x \in \mathbf{T}^n$. The condition of ν-positive definiteness is weaker.

Proposition 2.3 *The function* $\Lambda : \mathbf{T}^n \to L(\mathbf{R}^l)$ *is* ν*-positive definite if and only if for some smooth family of positive definite matrices* $G : \mathbf{T}^n \to L(\mathbf{R}^l)$ *the matrix*

$$\partial_\nu G + G\Lambda + \Lambda^T G \tag{2.6}$$

is positive definite for all $x \in \mathbf{T}^n$.

The metric defined by G is called the Lyapunov metric.

Corollary 2.1 *A small perturbation of a ν-positive definite function is again ν-positive definite.*

Proof of Proposition 2.3. This proposition is a version of a result of Lyapunov (more general statement is presented in [23, 24]), but for the convenience of the reader we include the proof. Suppose that Λ is ν-positive definite. Take any family $K : \mathbf{T}^n \to L(\mathbf{R}^l)$ of symmetric positive definite matrices and put

$$G(x) = \int_{-\infty}^{0} \phi_s^T(x) K(x + \nu s) \phi_s(x)\, ds,$$

where ϕ_s is the fundamental matrix of the system (2.1). The integral exponentially converges due to the property of ν-positive definiteness. According to the definition

$$\frac{d}{dt}\Big|_{t=0} \langle G(x + \nu t)\phi_t(x)v, \phi_t(x)v\rangle = \langle K(x)v, v\rangle > 0 \quad \text{for any } v \neq 0.$$

This implies (2.6). On the other hand, if condition (2.6) holds, then

$$\frac{d}{dt}\Big|_{t=0} \langle G(x + \nu t)\phi_t(x)v, \phi_t(x)v\rangle \geq \lambda\langle v, v\rangle \quad \text{for any } v \neq 0,$$

where λ is a positive constant. This implies the required ν-positive definiteness. ■

A torus N is said to be isotropic if $\omega|_N = 0$.

Theorem 2.1 *Any Diophantine isotropic hyperbolic torus is weakly reducible. If $n = m - 1$ or $n = 1$, it is reducible. In the analytic case the canonical coordinates are analytic.*

Of course, this trivially holds also for $n = 0$ and $n = m$. Herman [17] proved that if the symplectic form ω is exact, then any nonresonant invariant torus is isotropic.

Theorem 2.1 gives a relation between the dynamical systems definition of a hyperbolic torus and the definition conventional for the KAM-theory. We prove Theorem 2.1 in §5. In §3–4 some simple applications are given.

3 Invariant manifolds and normal forms for a hyperbolic torus

The variational system (2.5) has two invariant isotropic subbundles

$$E^+ = \mathbf{T}^n\{x\} \times \mathbf{R}^l\{\zeta_+\} = \{\eta = 0, \zeta_- = 0\},$$
$$E^- = \mathbf{T}^n\{x\} \times \mathbf{R}^l\{\zeta_-\} = \{\eta = 0, \zeta_+ = 0\}.$$

Solutions belonging to the first (respectively, to the second) one, approach the hyperbolic torus exponentially when $t \to -\infty$ (respectively, $t \to +\infty$).

Manifolds with analogous properties exist in the system (2.4) as well. Graff proved [16] that in an analytic system with Hamiltonian (2.3), where $\Omega + \Omega^T$ is positive definite, there are stable and unstable invariant Lagrangian manifolds $W^\pm$ containing N such that $T_x W^\pm = T_x N \oplus E_x^\pm$ for any $x \in N$. They can be represented as $W^\pm = f_\pm(E^\pm)$, where $f_\pm : E^\pm \to \mathbf{T}^n\{x\} \times \mathbf{R}^n\{y\} \times \mathbf{R}^l\{z_-\} \times \mathbf{R}^l\{z_+\}$ are analytic maps of the form

$$f_+(\theta,\zeta) = (\theta + O_2(\zeta), O_2(\zeta), \zeta, O_2(\zeta)),$$
$$f_-(\theta,\zeta) = (\theta + O_2(\zeta), O_2(\zeta), O_2(\zeta), \zeta).$$

The Hamiltonian system restricted to $W^\pm$ has the form

$$\dot\vartheta = \nu, \quad \dot\zeta = \Lambda^+(\vartheta,\zeta)\zeta \qquad \text{on} \quad W^+,$$
$$\dot\vartheta = \nu, \quad \dot\zeta = -\Lambda^-(\vartheta,\zeta)\zeta \qquad \text{on} \quad W^-,$$

where the matrices $\Lambda^\pm + (\Lambda^\pm)^T$ for small $|\zeta|$ are positive definite.

Conjecture 3.1 *Manifolds $W^\pm$ exist and are unique for a weakly reducible Diophantine torus. For an analytic system, they are analytic.*

In this case, the matrices $\Lambda^\pm$ are ν-positive definite for small $|\zeta|$.

In the C^k category with large k, the existence (not uniqueness) of the stable and unstable manifolds is easy to prove by using Theorem 2.1 and standard hyperbolic technique. Let H be smooth. Fix large integer k.

Theorem 3.1 *Let N be a smooth isotropic Diophantine hyperbolic torus. Then it has m-dimensional Lagrangian stable and unstable manifolds $W^\pm$ of class C^k.*

For analytic H, this theorem doesn't imply that the manifolds $W^\pm$ are analytic.

Graff [16] (see also [27]) proved that reducible nondegenerate Diophantine hyperbolic tori of a real-analytic system survive a small perturbation. They just slightly deform and remain hyperbolic and real-analytic.

Conjecture 3.2 *The Graff theorem holds for weakly reducible Diophantine tori.*

We believe that the proof of both conjectures is straightforward and standard, however, the detailed argument is not short. Again, in the C^k category with large k, Conjecture 3.2 is an easy theorem.

Theorem 3.2 *Let k be sufficiently large. Then an isotropic nondegenerate Diophantine hyperbolic torus survives a small C^k perturbation of the Hamiltonian.*

This will be proved in the §4. In the analytic case Theorem 3.2 doesn't imply that the perturbed torus is analytic. To prove this, one needs KAM theory.

Eliasson [13] obtained the following normal form for an analytic system near a hyperbolic torus for $n = m - 1$.

Theorem 3.3 *Let N be a reducible nondegenerate $(m-1)$-dimensional Diophantine hyperbolic torus. Then there exist analytic canonical coordinates $(x, y, z_\pm)$ in a neighborhood of N such that*

$$H = \langle \nu, y \rangle + \lambda z_- z_+ + O_2(y, z_- z_+).$$

According to Theorem 2.1, the assumption of the reducibility for the torus N always holds if the form ω is exact.

The proof of Theorem 3.3 is of KAM nature: it is based on a Newton-type iterative procedure. The Eliasson coordinates are convenient for the studying the dynamics in a neighborhood of N.

Now, following [4], we give a normal form for a weakly reducible hyperbolic torus of a smooth system which can be obtained without KAM-theory. Eliasson's normal form gives more information, but it doesn't hold for tori of arbitrary codimension, even in the reducible case.

Theorem 3.4 *Let N be a smooth hyperbolic Diophantine isotropic invariant torus. Then for any k there exist canonical coordinates $x, y, z_\pm$ of class C^k in a neighborhood of N such that, up to a constant,*

$$\begin{aligned} H(x, y, z_-, z_+) = \langle \nu, y \rangle + \frac{1}{2}\langle Ay, y \rangle + \langle \Omega(x) z_-, z_+ \rangle \\ + O_3(y) + \langle O_1(y, z_-, z_+) z_-, z_+ \rangle. \end{aligned} \tag{3.1}$$

The proof is given in §4. Such a normal form is sufficient for proving the λ-lemma for invariant tori [5] that is needed in the problem of Arnold's diffusion [1].

4 Proofs of Theorems 3.1, 3.2, 3.4

We will use the method going back to Fenichel [14], see also [26]. Since the method is standard, we will be brief.

Sketch of the proof of Theorem 3.1. By Theorem 2.1, we can use canonical coordinates $x, y, z_\pm$ in a neighborhood of N. Perform the scaling

$$y = \mu \hat{y}, \quad z_\pm = \sqrt{\mu}\, \hat{z}_\pm, \quad H = \mu \hat{H},$$

where μ is a small positive formal parameter. This is a canonical transformation, and in the new variables, dropping the hats,

$$\hat{H} = \langle \nu, y \rangle + \langle z_-, \Omega(x) z_+ \rangle + O(\sqrt{\mu}). \tag{4.1}$$

For $\mu = 0$, the Hamiltonian system has the form

$$\dot{x} = \nu, \quad \dot{y} = 0, \quad \dot{z}_+ = \Omega(x) z_+, \quad \dot{z}_- = -\Omega^T(x) z_-. \tag{4.2}$$

For this system, $S = \{z_- = 0, |y|^2 + |z_+|^2 \le 1\}$ is an invariant manifold with boundary. It is not overflowing in the sense of Fenichel [14], because the Hamiltonian vector field $v_{\hat{H}}$ is tangent to the component $\{z_+ = 0, |y| = 1\}$ of the boundary ∂S. To deal with this difficulty, we use a standard trick [14]. Add to the vector field $v_{\hat H}$ a small vector field $\delta u = \delta f(|y|) y \frac{\partial}{\partial y}$, where $f(s) \ge 0$ is a smooth function equal to 1 for $s \ge 1$ and vanishing for $s \le 1/2$. Fix large k. For the modified vector field $v_{\hat H} + \delta u$, S is a C^k overflowing manifold, and the condition of Fenichel's theorem [14] holds provided $\delta > 0$ is small enough. It implies that there exists $\delta > 0$ such that for sufficiently small $\mu > 0$ the vector field $v_{\hat H} + \delta u$ has a C^{k-1} inflowing manifold close to S. But $u = 0$ in a neighborhood of N, and so the original system with Hamiltonian $\hat H$ has a C^{k-1} locally invariant manifold P of dimension $2n + l$ containing N and tending to S as $\mu \to 0$.

Similarly, there exists a C^{k-1} manifold $Q \subset P$ of dimension $n + l$ containing N and tending to the manifold $V = \{z_+ = z_- = 0, |y| \le 1\}$ as $\mu \to 0$.

Then consider the restriction of the system to P. The Lyapunov exponents on Q are small for small μ and δ, and the normal Lyapunov exponents are of order of the Lyapunov exponents for the variational equation on $E^\pm$. Hence for small μ and δ and every $q \in Q$ there exists a C^{k-2} local stable manifold $W_q^+ \subset P$, and these manifolds form a C^{k-2} family [14]. Hence $W_{\rm loc}^+ = \cup_{q \in N} W_q^+$ is a C^{k-2} manifold.

For $W_{\rm loc}^-$, the proof is similar. The manifolds $W_{\rm loc}^\pm$ are Lagrangian. This follows from the fact that N is isotropic and that the phase flow g^t makes any vector tangent to $W_{\rm loc}^+$ (respectively $W_{\rm loc}^-$) tending to the isotropic bundle TN when $t \to -\infty$ (respectively $t \to +\infty$). For the proof see e.g. [4]. ■

Sketch of the proof of Theorem 3.2. We use the same trick as in the proof of Theorem 3.1. Consider the Hamiltonian (4.1). If the original Hamiltonian is perturbed by εH_1, then

$$\hat H = \langle \nu, y \rangle + \langle z_-, \Omega z_+ \rangle + O(\sqrt{\mu}) + O(\varepsilon/\mu).$$

Consider the last two terms as a perturbation of the system (4.2). For this system, $V = \{z_+ = z_- = 0, |y| \le 1\}$ is an invariant manifold with boundary. We make it overflowing by adding a small vector field δu as in the proof of Theorem 3.1. Now for fixed k, small $\delta > 0$, and sufficiently small μ and ε/μ, we obtain a C^k symplectic invariant submanifold Q of dimension $2n$ that is a small perturbation of the manifold V. Restrict the Hamiltonian system to Q. Now we can apply the usual C^k KAM theory by using the nondegeneracy and Diophantine assumptions. This yields an invariant torus in Q provided that μ, ε/μ and δ are small enough.

■

Sketch of the proof of Theorem 3.4. By Theorem 2.1 we already know that H has the normal form (2.3). Represent the unstable manifold as a graph over E^-, and the stable manifold as a graph over E^+. Since they are Lagrangian manifolds of class C^k and tangent to the manifolds $\{y = 0, z_\mp = 0\}$ respectively, they are given by C^{k+1} generating functions $S^\pm(x, z_\pm) = O_3(z_\pm)$:

$$W^\pm = \{(x, y, z_-, z_+) \mid y = S_x^\pm, z_\mp = \pm S_{z_\pm}^\pm\}, \qquad S^\pm = S^\pm(x, z_\pm). \quad (4.3)$$

Let us perform a symplectic transformation

$$(x, y, z_-, z_+) \mapsto (x, Y, Z_-, Z_+)$$

with the generating function

$$S(x, Y, z_+, Z_-) = xY + z_+ Z_- + S^+(x, z_+).$$

The transformation is given by

$$\begin{array}{rcl} y &=& S_x = Y + S_x^+(x, z_+) \\ z_- &=& S_{z_+} = Z_- + S_{z_+}^+(x, z_+) \\ Z_+ &=& S_{Z_-} = z_+. \end{array}$$

In the new canonical variables, W^+ is given by the equations $Y = 0, Z_- = 0$. Let $\tilde{S}^-(x, Z_-)$ be the generating function for W_- in the new variables. Let us perform the second symplectic transformation

$$(x, Y, Z_-, Z_+) \mapsto (x, \eta, \zeta_-, \zeta_+)$$

with the generating function

$$\tilde{S}(x, \eta, Z_-, \zeta_+) = x\eta + Z_- \zeta_+ + \tilde{S}^-(x, Z_-).$$

Then

$$\begin{array}{rcl} Y &=& \tilde{S}_x = \eta + \tilde{S}_x^-(x, Z_-) \\ Z_+ &=& -\tilde{S}_{Z_-} = -\zeta_+ - \tilde{S}_{Z_-}^-(x, Z_-) \\ \zeta_- &=& \tilde{S}_{\zeta_+} = Z_-. \end{array}$$

In the new variables $W^\pm = \{\eta = 0, \zeta_\mp = 0\}$. Hence $H = O(\zeta_+)O(\zeta_-) + O(\eta)$ has the normal form (3.1).

5 Hyperbolic tori: reducibility

In this section we prove Theorem 2.1. The proof is based on several auxiliary propositions.

Proposition 5.1 *Suppose that the form ω is exact. Let N be a nonresonant hyperbolic torus. Then:*

1. *The torus N is isotropic i.e., the restriction of the symplectic form to the bundle TN vanishes.*
2. *For any $w \in N$ the subspaces $T_w N \oplus E_w^\pm$ are Lagrangian.*
3. *The form ω defines a nondegenerate bilinear form on $E_w^+ \times E_w^-$: if $v_+ \in E_w^+$ and $\omega(v_+, v_-) = 0$ for all $v_- \in E_w^-$, then $v_+ = 0$.*

Proof [4, 6]. The first statement follows from the Herman theorem [17].

To prove the second statement, take any two vectors $v, v' \in T_w N \oplus E_w^+$. Then

$$\omega(v, v') = \omega(Dg^t v, Dg^t v') \to 0, \qquad t \to -\infty.$$

Indeed, for any vector $v \in T_wN \oplus E^+_w$ we have

$$\|Dg^t v - Dg^t u\| \to 0, \quad \|Dg^t u\| \le C,$$

where $u \in T_wN$ is the component of v tangent to the torus.

The last statement follows from the first two and the nondegeneracy of ω. Indeed, suppose that for some $v_+ \in E^+_w$ we have $\omega(v_+, v_-) = 0$ for any $v_- \in E^-_w$. From statement *2* we have: $\omega(v_+, v) = 0$ for any $v \in T_wN$. Hence, the $(m+1)$-dimensional space spanned by $T_wN \oplus E^-_w$ and v_+ is isotropic. According to the nondegeneracy of ω, this implies that $v_+ = 0$. ∎

Proposition 5.2 *In a finite covering $\widetilde{U}$ of a neighborhood U of N in M there exist coordinates $(x, y, z_-, z_+) \in \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{R}^l_+ \times \mathbf{R}^l_-$ such that:*

1. *$\Phi(x) = (x, 0, 0, 0)$ for any $x \in \mathbf{T}^n$,*
2. *$\omega = dy \wedge dx + dz_+ \wedge dz_-$,*
3. *$E^+ = \mathbf{T}^n \times \mathbf{R}^l_+$, $E^- = \mathbf{T}^n \times \mathbf{R}^l_-$.*

Here to avoid misunderstandings we denote by $\mathbf{R}^l_\pm$ two copies of conjugate spaces $\mathbf{R}^l$. Thus $\mathbf{R}^l_+ = (\mathbf{R}^l_-)^*$.

Proof. By definition, $N = \Phi(\mathbf{T}^n)$. Let x be the standard coordinate in $\mathbf{T}^n$. After passing to a finite covering, the bundles $E^\pm$ become trivial. Since the bundle E^- is trivial, for any $w = \Phi(x)$ we have an isomorphism $E^-_w = \mathbf{R}^l\{z_-\}$ smoothly depending on $x \in \mathbf{T}^n$. We fix this isomorphism. By Proposition 5.1, $V = E^+_w \oplus E^-_w$ is a symplectic space. Hence $E^+_w = (E^-_w)^*$ with the isomorphism given by the symplectic form ω. Thus $V = \mathbf{R}^l_+ \oplus \mathbf{R}^l_-$ with the standard symplectic structure $dz_- \wedge dz_+$. Let $V^\perp = \{v \in T_wM \mid \omega(v, u) = 0 \ \forall u \in V\}$ be the symplectic complement of V. Then $V^\perp$ is a symplectic space of dimension $2n$, and it contains the Lagrangian subspace $T_wN = \mathbf{R}^n$. Hence there exists a symplectic isomorphism $V^\perp = T_wN \oplus T^*_wN = \mathbf{R}^n \oplus (\mathbf{R}^n)^*$ with the standard symplectic structure $dy \wedge dx$. Thus, for any $w = \Phi(x) \in N$, we have a symplectic isomorphism

$$T_wM = V^\perp \oplus V = \mathbf{R}^n\{x\} \oplus \mathbf{R}^{n*}\{y\} \oplus \mathbf{R}^l_+\{z_+\} \oplus \mathbf{R}^l_-\{z_+\}$$

smoothly depending on $x \in \mathbf{T}^n$. The bilinear form $\omega|_{T_wM}$ equals $dy \wedge dx + dz_+ \wedge dz_-$.

Using the exponential map $T_wM \to M$, we obtain a smooth map

$$f : \mathbf{T}^n\{x\} \times \mathbf{R}^n\{y\} \times \mathbf{R}^l_+\{z_+\} \times \mathbf{R}^l_-\{z_-\} \to M$$

defining the coordinates $x, y, z_\pm$ in a tubular neighborhood of the torus N such that

$$f^*\omega = dy \wedge dx + dz_+ \wedge dz_- + O(y, z).$$

Put

$$\Omega = dy \wedge dx + dz_+ \wedge dz_-.$$

The symplectic structures $f^*\omega$ and Ω coincide on the torus $\mathbf{T}^n$ and are homotopic in a neighborhood of $\mathbf{T}^n$ in the class of symplectic structures.

For example, the homotopy can be chosen as $af^*\omega + (1-a)\Omega$, $0 \le a \le 1$. According to the generalized Darboux theorem [3], there exists a diffeomorphism g in a neighborhood of the torus, such that g and Dg equal the identity on $\mathbf{T}^n$ and $g^*(f^*\omega) = \Omega$.

The map $f \circ g$ defines the coordinates we need. ∎

Proposition 5.3 *Suppose that for some vector $\nu \in \mathbf{R}^n$ the function $\Lambda : \mathbf{T}^n \to L(\mathbf{R}^l)$ is ν-positive definite or ν-negative definite. Then for any smooth function $v : \mathbf{T}^n \to \mathbf{R}^l$ there exists a unique smooth solution $u : \mathbf{T}^n \to \mathbf{R}^l$ of the equation*

$$\partial_\nu u(x) - \Lambda(x)u(x) = v(x). \tag{5.1}$$

For $\Lambda \equiv 0$ equation (5.1) has a unique solution provided the mean values of u and v over the torus $\mathbf{T}^n$ vanish and the vector ν is Diophantine.

If v and Λ are real-analytic, the function u is also real-analytic.

Proof. Consider the case of a ν-negative definite function Λ. Let $\phi_t(x)$ be the fundamental matrix of the system (2.1). Since Λ is ν-negative definite, the norm of $\phi_t(x)$ satisfies the estimate

$$\|\phi_t^{-1}(x)\| \le Ce^{-\lambda t}, \qquad t > 0 \tag{5.2}$$

for some positive constant λ. We will show that

$$u(x) = \int_{-\infty}^{0} \phi_s^{-1}(x)v(x+\nu s)\,ds. \tag{5.3}$$

Indeed, equation (5.1) is equivalent to the following one:

$$\dot{u}(x+\nu t) - \Lambda(x+\nu t)u(x+\nu t) = v(x+\nu t).$$

Searching for a solution in the form $u(x+\nu t) = \phi_t(x)\theta(x,t)$, we get

$$\theta(x,t) = \int_{-\infty}^{t} \phi_s^{-1}(x)v(x+\nu s)\,ds.$$

Putting $t = 0$, we obtain (5.3). The integral exponentially converges due to (5.2).

The solution (5.3) is unique since all nontrivial solutions of the homogeneous equation $\partial_\nu u(x) - \Lambda(x)u(x) = 0$ are unbounded on $\mathbf{T}^n$.

If v and Λ are real-analytic, $\phi_t(x)$ is also real-analytic in both x and t. The integral (5.3) exponentially converges in a complex neighborhood of $\mathbf{T}^n$. Hence, $u(x)$ is also real-analytic.

The case of a ν-positive definite function Λ is analogous. For $\Lambda \equiv 0$ and ν Diophantine equation (5.1) can be easily solved by using Fourier expansions. ∎

Proposition 5.4 *The coordinates (x, y, z_+, z_-) from Proposition 5.2 can be transformed to canonical ones for the torus N.*

Proof. Since $y = z_- = z_+ = 0$ is an invariant torus with frequency ω, in the coordinates (x, y, z_+, z_-) the Hamiltonian $H = \langle \nu, y\rangle + O_2(y, z)$ has the form

$$\begin{aligned} H \;=\; & \langle \nu, y\rangle + \langle A(x)y, y\rangle/2 + \langle z_+, B_+(x)y\rangle + \langle z_-, B_-(x)y\rangle \\ & + \langle z_-, C_-(x)z_-\rangle/2 + \langle z_-, C(x)z_+\rangle + \langle z_+, C_+(x)z_+\rangle/2 \\ & + O_3(y, z_\pm), \end{aligned}$$

where the $n \times n$-matrix A, the $l \times n$-matrices $B_\pm$, and the $l \times l$-matrices $C, C_\pm$ are functions of $x \in \mathbf{T}^n$. The matrices A and $C_\pm$ are symmetric.

The corresponding differential equations are as follows

$$\begin{cases} \dot x = & \nu + Ay + B_+^T z_+ + B_-^T z_- & +O_2(y, z), \\ \dot y = & & O_2(y, z), \\ \dot z_+ = & C_- z_- + C z_+ + B_- y & +O_2(y, z), \\ \dot z_- = & -C^T z_- - C_+ z_+ + B_+ y & +O_2(y, z). \end{cases}$$

By Proposition 5.2 the manifolds $\{y = 0, z_- = 0\}$ and $\{y = 0, z_+ = 0\}$ are invariant in linear approximation. This means that $C_- \equiv C_+ \equiv 0$. According to the definition of hyperbolic torus the function C is ν-positive definite.

To eliminate the terms containing $B_\pm$, we perform the canonical change $(x, y, z_+, z_-) \mapsto (\hat x, \hat y, \hat z_+, \hat z_-)$ with the generating function

$$S(x, z_+, \hat y, \hat z_-) = \langle x, \hat y\rangle + \langle z_+, \hat z_-\rangle + \langle z_+, b_+(x)\hat y\rangle + \langle z_-, b_-(x)\hat y\rangle,$$

where the matrices $b_\pm(x)$ are solutions of the equations

$$\partial_\nu b_\pm(x) \pm C(x) b_\pm(x) + B_\pm(x) = 0.$$

In the new variables

$$H = \langle \nu, \hat y\rangle + \langle \hat A(\hat x)\hat y, \hat y\rangle/2 + \langle \hat z_-, C(\hat x)\hat z_+\rangle + O_2(\hat y, \hat z_\pm)$$

with some matrix function $\hat A$.

The change $(\hat x, \hat y, \hat z_-, \hat z_+) \mapsto (x, y, z_-, z_+)$ with the generating function

$$S(\hat x, \hat z_+, y, z_-) = \langle \hat x, y\rangle + \langle \hat z_-, z_+\rangle + \langle a(\hat x)y, y\rangle,$$

where the matrix a is a solution of the equation

$$\partial_\nu a(x) + \hat A(x) = \overline{A}, \qquad \overline{A} = \frac{1}{(2\pi)^n}\int_{\mathbf{T}^n} \hat A(x)\, dx,$$

defines coordinates canonical for N. Indeed,

$$H = \langle \nu, y\rangle + \langle \overline{A} y, y\rangle/2 + \langle z_-, C(x)z_+\rangle + O_2(y, z).$$

■

Now suppose that $l = 1$. Then positive definiteness of C is equivalent to the condition $\int_{\mathbf{T}^n} C(x)\, dx = \overline{C} > 0$. Let us perform the change

$(x, y, z_+, z_-) \mapsto (X, Y, Z_+, Z_-)$ with generating function $S = \langle x, Y\rangle + u(x)z_+Z_-$, where

$$\partial_\nu u(x) + u(x)(C(x) - \overline{C}) = 0.$$

The last equation has a positive solution on the torus $\mathbf{T}^n$. In the new variables

$$H = \langle \nu, Y\rangle + \langle \overline{A}Y, Y\rangle/2 + \overline{C}Z_+Z_- + O_2(Y, Z).$$

This implies reducibility of N for $l = 1$. The reducibility in the case $n = 1$ follows from the Floquet theorem.

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