Continuous averaging in dynamical systems

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Abstract

The method of continuous averaging can be regarded as a combination of the Lie method, where a change of coordinates is constructed as a shift along solutions of a differential equation and the Neishtadt method, well-known in perturbation theory for ODE in the presence of exponentially small effects. This method turns out to be very effective in the analysis of one- and multi-frequency averaging, exponentially small separatrix splitting and in the problem of an inclusion of an analytic diffeomorphism into an analytic flow. We discuss general features of the method as well as the applications.

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1 The method

There are several problems in the perturbation theory, of real-analytic ordinary differential equations (ODE), where standard methods do not lead to satisfactory results. We mention as examples the problem of an inclusion of a diffeomorphism into a flow in the analytic set up, and the problem of quantitative description of exponentially small effects in dynamical systems. In this cases one of possible approaches is an application of the continuous averaging method. The method appeared as an extension of the Neishtadt averaging procedure [14]. We begin with the description of the method.

Let us transform the system

$$\dot{z} = \widehat{u}(z), \tag{1}$$

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by using the change of variables

$$z \mapsto Z(z,s). \tag{2}$$

Here z is a point of the manifold M, $\widehat{u}$ is a smooth vector field on M, s is a non-negative parameter, and the change (2) is defined as a shift along solutions of the equation[1]

$$dZ/ds = f(Z,s), \qquad Z(z,0) = z, \quad 0 \le s \le S. \tag{3}$$

Let the change $z \mapsto Z$ transform (1) to the following system:

$$\dot{Z} = u(Z,s). \tag{4}$$

Differentiating (4) with respect to s, we have:

$$\dot{f}(Z,s) = u_s(Z,s) + \partial_f u(Z,s) \quad \text{or} \quad u_s = [u,f].$$

Here ∂_f is the differential operator on M, corresponding to the vector field f, the subscript s denotes the partial derivative, and $[\cdot,\cdot]$ is the vector commutator: $[u_1,u_2] = \partial_{u_1} u_2 - \partial_{u_2} u_1$. Putting $f = \xi u$, where ξ is some fixed linear operator, we obtain the Cauchy problem

$$u_s = -[\xi u, u], \qquad u|_{s=0} = \widehat{u}. \tag{5}$$

We call the system (5) averaging. The equation $f = \xi u$ is crucial for our method. The vector field f is usually constructed as a series in the small parameter and not as a result of an application to u of an operator ξ, chosen in advance.

A nonautonomous analog of (5) can be easily constructed. If $\widehat{u}$ depends explicitly on t then $f = \xi u$ also depends on t and (5) should be replaced by the system

$$u_s = (\xi u)_t - [\xi u, u], \qquad u|_{s=0} = \widehat{u}(z,t). \tag{6}$$

Properties of the averaging system can be illustrated by the following example. Consider the non-autonomous real-analytic system

$$\dot{z} = \varepsilon \widehat{u}(z,t), \qquad z \in M. \tag{7}$$

Here ε is a small parameter, $\widehat{u}$ is 2π-periodic in t. Let us try to weaken the dependence of $\widehat{u}$ on time by the change (2)–(3) with $f = \xi u$. We put[2]

$$\xi u(z,t,s) = \sum_{k \in \mathbf{Z}} i\sigma_k u^k(z,s) e^{ikt}, \qquad \sigma_k = \operatorname{sign} k, \tag{8}$$

where u^k are Fourier coefficients in the expansion $u(z,t,s) = \sum_{k\in\mathbf{Z}} u^k(z,s)e^{ikt}$.

[1] Such method of constructing a change of variables is called the Lie method. The corresponding Hamiltonian version is called the Deprit-Hori method.

[2] Such an operator ξ is called the Hilbert transform.

Equation (6) takes the form

$$u^k_s = -|k|u^k + i\varepsilon\sigma_k[u^0, u^k] - 2i\varepsilon \textstyle\sum_{l+m=k, m<0<l}[u^l, u^m], \qquad (9)$$
$$u^k|_{s=0} = \widehat{u}^k, \qquad k \in \mathbf{Z}.$$

To have an idea of properties of this system, we skip in (9) the last term. The equations

$$u^k_s = -|k|u^k + i\varepsilon\,\sigma_k[u^0, u^k], \qquad u^k|_{s=0} = \widehat{u}^k, \qquad k \in \mathbf{Z}$$

can be solved explicitly:

$$u^k = e^{-|k|s}\widehat{u}^k \circ g^{i\varepsilon\sigma_k s}, \qquad (10)$$

where g^s is the time-s shift $z|_{t=0} \mapsto z|_{t=s}$ along solutions of the system $\dot{z} = \widehat{u}^0(z)$.

The complex singularities of the functions $\widehat{u}^k \circ g^s$ of the complex variable s prevent an unbounded continuation of the solutions (10) to all the set of positive s. Nevertheless, the functions (10) can be made exponentially small in ε since s can be chosen of order $\sim 1/\varepsilon$.

If ε is not small, the operator (8) can be used to smooth out the dependence of u on t. Indeed, for arbitrarily small $s > 0$ the Fourier coefficients u^k in (10) decrease exponentially fast in k even if $\widehat{u}$ is just continuous in t.

Certainly, these hewristic arguments cannot be regarded as a proof of the fact that systems of type (9) can be used for averaging or smoothing. Rigorous statements and estimates are based on the majorant method. In this paper we do not go into technical details, but present general ideas and applications.

If the vector field $\widehat{u}$ belongs to some subalgebra χ in the Lie algebra of vector fields on M, it is natural to look for the change (2) from the corresponding Lie group of diffeomorphisms. This means that f in (3) should be taken from χ. The same remains reasonable in the non-autonomous case. Note that the operator (8) is such that if $u(z,t,s) \in \chi$ for any t, the vector field ξu also belongs to χ for any t.

2 Applications

2.1 Fast phase averaging: one-frequency case

Mathematical models of various physical processes use systems of ODE which contain an angular variable changing much faster than other variables in the system. Taking the fast phase as a new time, we can rewrite the equations in the form

$$\dot{z} = \varepsilon\widehat{u}(z,t,\varepsilon), \qquad z \in M, \qquad (11)$$

where M is the m-dimensional phase space of the system, and ε is a small parameter. The vector field $\widehat{u}$ is assumed to be smooth and to depend on time 2π-periodically.

It is well known that by a change of the variables it is possible to weaken the dependence of the system (11) on time. In particular, by using the standard averaging method, it is easy to construct a 2π-periodic in t change of the variables $z \mapsto z_*$ such that the equations (11) take the form

$$\dot{z}_* = \varepsilon \widehat{u}^0(z_*) + \varepsilon^2 \widehat{u}_*(z_*, \varepsilon) + \varepsilon \tilde{u}(z_*, t, \varepsilon). \tag{12}$$

Here the only term in the right-hand side depending explicitly on time is $\varepsilon\tilde{u} = O(\varepsilon^K)$. The natural K is arbitrary and $\widehat{u}^0(z) = \frac{1}{2\pi}\int_0^{2\pi} \widehat{u}(z, t, 0)\, dt$.

Now suppose that $\widehat{u}$ is real-analytic in z. Poincaré noted in some example that in this case power series in ε presenting a change of variables eliminating time from the equations, exist but diverge: terms at ε^k in these series have the order $k!$. In a general situation this statement has been proved by Sauzin [21].

Neishtadt [14] noted that in this case it is possible to obtain in (12)

$$\tilde{u} = O(e^{-\alpha/\varepsilon}), \quad \alpha = \text{const} > 0, \tag{13}$$

where α is some constant (ε is assumed to be nonnegative). The method Neishtadt used to prove this assertion is based on a large (of order $1/\varepsilon$) number of successive changes of variables. These changes weaken gradually explicit dependence of the equations on time. Ramis and Schafke [19] obtained analogous results analyzing diverging series, produced by the standard averaging method.

It is known also that in general a constant $A > \alpha$ exists such that it is impossible to construct 2π-periodic in t change $z \mapsto z_*$, such that $\tilde{u} = O(e^{-A/\varepsilon})$. This statement follows, for example, from an estimate of the separatrix splitting rate in Hamiltonian systems of type (11) with one and a half degrees of freedom.

In this section we estimate a "maximal" α for which the estimate (13) is possible.

Suppose that the manifold M is real-analytic. We fix its complex neighborhood $M_{\mathbf{C}}$ and denote by g^t the phase flow of the averaged system[3]

$$\dot{z} = \widehat{u}^0(z). \tag{14}$$

Let Q be a compact in M and V_Q its neighborhood in $M_{\mathbf{C}}$. Suppose that for any real s such that $|s| < \alpha$ and for any point $z \in V_Q$ the map g^{is} is analytic at z and moreover, $g^{is}(z) \in M_{\mathbf{C}}$. We define the set

$$U_{Q,\alpha} = \bigcup_{-\alpha < s < \alpha} g^{is}(V_Q).$$

Theorem 1, [24]. *Let the positive constants $\alpha, \rho, \varepsilon_0$ be such that*

(1) $U_{Q,\alpha} \subset M_{\mathbf{C}}$.

(2) The vector field $\widehat{u}$ is analytic in z and C^2-smooth in t, ε on $U_{Q,\alpha} \times \mathbf{T} \times [0, \varepsilon_0]$.

Then for sufficiently small ε_0, there exists a 2π-periodic in t real-analytic in z map $F : V'_Q \times \mathbf{T} \times (0, \varepsilon_0) \to M_{\mathbf{C}}$, $\quad Q \subset V'_Q \subset V_Q$, such that

[3] It would be more correct to call (14) by the first approximation averaged system, written with respect to the fast time.

(a) The set V'_Q is open in $M_{\mathbf{C}}$,
(b) $F(z,t,\varepsilon) = z_* = z + O(\varepsilon)$,
(c) F transforms (11) into (12) and the following estimate holds:

$$|\tilde{u}(z,t,\varepsilon)| \le Ce^{-\alpha/\varepsilon}, \qquad z \in V'_Q, \quad t \in \Sigma_\rho, \quad \varepsilon \in [0,\varepsilon_0). \tag{15}$$

Theorem 1 means in particular, that in the case when components of the field $\widehat{u}$ are entire functions of z, the quantity α in (15) can be arbitrary positive number such that for all $s \in [-\alpha,\alpha]$ the maps $z \mapsto g^{is}(z)$ are holomorphic at any point $z \in Q$.

Proof of Theorem 1 is based on the continuous averaging. Namely, we solve the Cauchy problem (6), with ξ defined by (8). The required change of variables corresponds to the value $s = \alpha/\varepsilon$. The averaging can be performed inside a subalgebra χ in the Lie algebra of all real-analytic vector fields on M. In particular, if the initial vector field $\widehat{u}$ is Hamiltonian then $u_* = u(z,t,\varepsilon,\alpha/\varepsilon)$ is also Hamiltonian, and F is symplectic.

2.2 Averaging: multi-frequency case

Consider a real-analytic slow-fast system

$$\dot{x} = \omega + \varepsilon(\overline{u}(y) + \widehat{u}(x,y,\varepsilon)), \quad \dot{y} = \varepsilon(\overline{v}(y) + \widehat{v}(x,y,\varepsilon)), \qquad x \in \mathbf{T}^n,\ y \in \mathbf{R}^m. \tag{16}$$

Average in x of is assumed to be $O(\varepsilon)$. The frequency vector $\omega \in \mathbf{R}^n$ is constant and non-resonant.[4]

We try to weaken the dependence of the right-hand side of the system on the fast variables x by a near-identity change $(x \bmod 2\pi, y) \mapsto (x_\bullet \bmod 2\pi, y_\bullet)$. We put

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \overline{\omega} = \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \quad \overline{w} = \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}, \quad \widehat{w} = \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix}.$$

Note that the vector fields $\overline{\omega}$ and $\overline{w}$ commute. The system (16) takes the form

$$\dot{z} = \overline{\omega} + \varepsilon(\overline{w} + \widehat{w}). \tag{17}$$

A1. Diophantine condition. *We assume that $n \ge 2$ and the frequency vector ω is Diophantine*: there exist $\gamma_0, \gamma > 0$ such that

$$|\langle k,\omega\rangle| \ge \gamma_0 ||k||^{-\gamma}, \quad \text{for any } \ k \in \mathbf{Z}^n \setminus \{0\}. \tag{18}$$

To formulate the next assumption, we need some definition. Let g^t be the phase flow of the system

$$\dot{z} = \overline{w}(y). \tag{19}$$

[4] The case of non-constant frequencies ($\omega = \omega(y)$) can be reduced to this one in a small neighbourhood of an unperturbed (may be, resonant) torus $\{y = y^0\}$.

For any real-analytic function $f(y)$ with values in $\mathbf{C}^{n+m}$ and the vector $k \in \mathbf{Z}^n$ we put $f_k = f(y)e^{i\langle k,x\rangle}$. The function

$$\mathbf{g}_k^s f = e^{-i\langle k,x\rangle} g_*^{-is}(f_k \circ g^{is}), \qquad s \in \mathbf{C}$$

does not depend on x. Here g_*^t is the differential of the map g^t. Note that $\mathbf{g}_k^s$, $s \in \mathbf{R}$ include shifts along g^t in purely imaginary direction. We put

$$\begin{aligned}
\Sigma_q &= \{x \in \mathbf{C}^n/(2\pi\mathbf{Z})^n : |\operatorname{Im} x_j| \le q, \quad j = 1,\dots,n\}, \\
V_\nu &= \{y \in \mathbf{C}^m : \operatorname{Re} y \in (\overline{\mathcal{D}} + \nu) \subset \mathbf{R}^m, \quad |\operatorname{Im} y_l| \le \nu, \quad l = 1,\dots,m\},
\end{aligned}$$

where $\overline{\mathcal{D}}$ is a compact domain and $\overline{\mathcal{D}} + \nu$ is the ν-neighborhood of $\overline{\mathcal{D}}$.

Expand the function $\widehat{w}$ into the Fourier series:

$$\widehat{w}(z,\varepsilon) = \sum_{k\in\mathbf{Z}^n} \widehat{w}^k(y,\varepsilon)e^{i\langle k,x\rangle}.$$

A2. Analyticity. *Let the constant $\alpha > 0$ be such that for any real $s \in [-\alpha,\alpha]$, for any $k \in \mathbf{Z}$ and for any $(x,y) \in \overline{\Sigma}_q \times \overline{V}_\nu = \operatorname{closure}(\Sigma_q) \times \operatorname{closure}(V_\nu)$ the map $(x,y) \mapsto g^{is}(x,y)$ is analytic in x,y and the map $y \mapsto \mathbf{g}_k^s \widehat{w}^k(y)$ is analytic in y.*

Moreover, suppose that for some real ρ

$$|\mathbf{g}_k^s \widehat{w}^k|_{\overline{V}_\nu \times [0,\varepsilon_0)} \le \mu ||k||^{-\rho} e^{-q||k||}, \qquad k \neq 0.$$

The function $\widehat{w}^0$ is of order ε and we assume that $|\mathbf{g}_0^s \widehat{w}^0|_{\overline{V}_\nu \times [0,\varepsilon_0)} \le \varepsilon\mu_0$.

The constants $\mu, \mu_0, \nu, q, \rho, \gamma$ must satisfy some conditions [18]. Here we replace these conditions by more restrictive, but simple ones.

A3. $\mu, \mu_0, \nu, q, \rho, \gamma$ are positive, do not depend on ε, and $\rho > \gamma/(\gamma+1)$.

Theorem 2, [18]. *Suppose that assumptions* **A1**–**A3** *hold. Then there exists a change of variables*

$$z \mapsto z_\bullet = f(z,\varepsilon), \quad f : \Sigma_{2q/3} \times V_{2\nu/3} \times [0,\varepsilon_0) \to \mathbf{C}^n/(2\pi\mathbf{Z}^n) \times \mathbf{C}^m \tag{20}$$

such that f is analytic in z, smooth in ε, $f(z,\varepsilon) = z + O(\varepsilon)$, and the system (17) *takes the form*

$$\dot{z}_\bullet = \overline{\omega} + \varepsilon(\overline{w}(y_\bullet) + w_\bullet(z_\bullet,\varepsilon)). \tag{21}$$

Let $w_\bullet^0(y_\bullet,\varepsilon)$ be the average in $x_\bullet$ of $w_\bullet$. Then $w_\bullet^0(y_\bullet,0) = 0$. Moreover,

$$|w_\bullet - w_\bullet^0| \le C\mu\varepsilon^{\rho/(\gamma+1)} e^{-\overline{q}\varepsilon^{-1/(\gamma+1)}}, \qquad z_\bullet \in \Sigma_{q/2} \times V_{\nu/2}, \tag{22}$$

where C is a constant, not depending on ε and μ,

$$\overline{q} = (1+\gamma^{-1})(\gamma\gamma_0\alpha q^\gamma)^{1/(\gamma+1)}. \tag{23}$$

If the system (17) is Hamiltonian with respect to a certain symplectic structure Ω then (21) is also Ω-Hamiltonian and the change (20) is Ω-symplectic.

In [18] we also present another theorem which shows that the Fourier series $w_\bullet - w_\bullet^0$ can be divided into 2 parts: one is small and for another we have a sort of control.

Results analogous to Theorem 2 (without estimates for α) are contained in [22, 1,].

2.3 Exponentially small separatrix splitting

The phenomenon of exponentially small separatrix splitting was discovered by Poincaré [16]. Intensive quantitative studying of the problem was initiated by papers [7, 10] (see also [2]). The method proposed by Lazutkin with collaborators [12, 11, 5, 6] is based on an analysis of the separatrices in the complex domain. Another method was used in [4], where direct expansions of the Poincaré-Melnikov integral in an additional parameter are analyzed. The resurgent analysis is applyed to these problems in [20].

The main difficulty of the problem is that the traditional Poincaré-Melnikov method can not be applied directly. Indeed, its error has the order of square of the perturbation. Hence, the error considerably exceeds the expected result.

Exponentially small separatrix splitting can be studied with the help of the continuous averaging method. The main idea of to reduce the rate of the perturbation to an exponentially small quantity such that its square is much smaller than the result. Then the Poincaré-Melnikov method is applicable.

It is natural to measure the rate of the separatrix splitting by the area $\mathcal{A}_{\text{lobe}}$ of a lobe domain, bounded by segments $I^{s,u}$ of the stable (s) and unstable (u) separatrix such that I^s and I^u have the same boundary points, these points are homoclinic, and there are no other common points of I^s and I^u.

Consider the system with Hamiltonian

$$\widehat{H}(\widehat{x},\widehat{y},t/\varepsilon) = \widehat{y}^2/2 + (1+2B\cos(t/\varepsilon))\cos\widehat{x}, \tag{24}$$

where $\widehat{x},\widehat{y}$ are canonically conjugated variables and $\varepsilon > 0$ is small. Theorem 1 implies that there exists a symplectic change of coordinates $\hat{x},\hat{y} \mapsto x,y$ which is

i) close to the identity,

ii) $2\pi\varepsilon$-periodic in time,

iii) real-analytic in a complex neighborhood of the separatrices $\Gamma^{\pm}$ of the system with Hamiltonian $H_0 = \hat{y}^2/2 + \cos\hat{x}$

iv) such that the new Hamiltonian function takes the form:

$$H(x,y,t/\varepsilon) = H_0(x,y) + \varepsilon H_1(x,y,\varepsilon) + \exp(-c/\varepsilon)H_2(x,y,t,\varepsilon).$$

Here $H_0 = y^2/2 + \cos x$, the constant $c \in [0,\pi/2)$ is arbitrary, the functions H_1, H_2 are real-analytic in x,y in the vicinity of the unperturbed separatrices of the hyperbolic fixed point $(\widehat{x},\widehat{y}) = (0,0)$, smooth in $\varepsilon > 0$ and the function $\tilde{H}_2$ is analytic and 2π-periodic in t.

The ordinary Poincaré-Melnikov theory applied to this system for positive $\tilde{c}$ gives a correct asymptotics of the separatrix splitting, [25]. The following estimate holds:

$$\mathcal{A}_{\text{lobe}} = \frac{8\pi}{\varepsilon}\exp\Big(-\frac{\pi}{2\varepsilon}\Big)\big(Bf(B^2) + O(\frac{\varepsilon}{\log\varepsilon})\big),$$

where f is an entire real-analytic function, $f(0) = 2$. In [25, 23] numerical values of several Tailor coefficients of f are presented.

Analogous results for the Standard Chirikov Map and for some Hamiltonian systems with 2 degrees of freedom can be found in [23, 26, 15].

2.4 Inclusion of a map into a flow

In this section we consider the following problem: to present a given self-map of a manifold M as the time-one map (the Poincaré map) in some ODE system, generated by a periodic in time vector field.

The problem can be formulated for various classes of maps and the corresponding vector fields. For example, it is possible to consider generic maps and vector fields, reversible ones with respect to some involution, Hamiltonian, preserving a volume, etc. Here we discuss the analytic set up i.e., assuming that the map is real-analytic, we look for its inclusion into a flow generated by a real-analytic vector field. The problem in C^∞ category is much simpler.

The following construction is well-known. Given a diffeomorphism T of a manifold M onto itself consider the direct product $M \times [0,1]$ with the vector field $\partial/\partial t$, where t is the coordinate on $[0,1]$. The map T generates the identification

$$M \times \{0\} \sim M \times \{1\}, \quad (z,0) \sim (T(z),1).$$

This identification converts $M \times [0,1]$ into a manifold $\mathcal{M}$. Let $\pi : M \times [0,1] \to \mathcal{M}$ be the natural projection. The smooth vector field $\partial/\partial t$ generates on the surface $\pi(M \times \{0\}) \subset \mathcal{M}$ the Poincaré map coinciding with T.

This construction does not solve the problem we deal with because in general it is not clear if $\mathcal{M}$ is real-analytically diffeomorphic to $M \times \mathbf{T}^1$. Nevertheless, sometimes this can be proven.

The problem is solved in the symplectic set up for maps which are close to integrable, [3, 8, 9] and for generic maps [27].

Note that all the known proofs in the analytic set up use essentially the Grauert theorem on the inclusion of an analytic manifold into Euclidean space (or modified versions of this theorem). Our result is based on the method of continuous averaging.

First, note that any map which is not isotopic to the identity[5] obviously can not be included into a flow. Let M be an m-dimensional compact real-analytic manifold. Let χ be a closed subalgebra in the Lie algebra $(\mathcal{L}, [\,,])$ of all analytic vector fields on M.

We denote by X the subset of all analytic diffeomorphisms of M obtained as a result of the time-one shift along solutions of a system

$$\dot{z} = u(z,t), \qquad u(\cdot,t) \in \chi, \quad t \in [0,1], \quad z \in M \tag{25}$$

(it is not assumed that $u(z,0) = u(z,1)$). We assume that the vector field u is C^2-smooth with respect to time. This smoothness condition is technical. For example, it can be replaced by continuity in t in Hamiltonian case and in the general one.

Obviously, all diffeomorphisms from X are isotopic to the identity inside X.

[5] Two smooth maps $T_j : M' \to M''$, $j = 0,1$ (M' and M'' are manifolds) are called isotopic if there exists a family of maps $\widehat{T}_s : M' \to M''$ of the same smoothness class continuous in the parameter $s \in [0,1]$, such that $\widehat{T}_0 = T_0$ and $\widehat{T}_1 = T_1$. In other words, if T_0 can be continuously deformed into T_1.

Theorem 3 *Suppose that the map T belongs to X. Then there exists a vector field*

$$U = U(z,t), \qquad U(\cdot,t) \in \chi, \quad t \in \mathbf{R}, \quad z \in M$$

which is analytic in z and t, 2π-periodic in t, and such that the time-2π shift along its trajectories coincides with T.

As a corollary we obtain a possibility of the inclusion of analytic maps into analytic flows in general, symplectic, and volume-preserving cases.

The vector field U is obviously not unique.

It can be also proved that if T is reversible with respect to some involution $I : M \to M$, $I^2 = \mathrm{id}_M$ (i.e., $T \circ I = I \circ T^{-1}$), the corresponding vector field U, can be also regarded I-reversible: $U(z,t) = -dI\, U(Iz, -t)$.

Suppose that the map T is close to T_0 ($\mathrm{dist}(T, T_0) = \varepsilon$ in a complex neighborhood of M),[6] where T_0 is included into the flow generated by a periodic analytic vector field U_0. Then the vector field U can be chosen close to U_0 ($|U - U_0| = O(\varepsilon)$ in a smaller complex neighborhood of M). In particular, in the symplectic case if T is close to an integrable map, a Hamiltonian system associated with T also can be chosen close to an integrable one and the orders of closeness are the same.

Continuous averaging in the proof is used to smooth out the dependence of the original vector field (25) on time, [17].

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[6] The distance can be defined for example as follows:

$$\mathrm{dist}(T, T_0) = \sup_{z \in M'} \rho(T(z), T_0(z)),$$

where M' is a complex neighborhood of M, and ρ is some metric. The choice of the neighborhood and of the metric plays no role due to compactness of the manifold M.

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