ON THE INCLUSION OF ANALYTIC MAPS INTO ANALYTIC FLOWS

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§1. Introduction. Main result

Suppose we have a Hamiltonian system with periodic in time Hamiltonain. It is natural to consider the corresponding Poincaré map which is the shift for the period along trajectories of the system. The Poincaré map is obviously symplectic. The following question naturally arises. Given a symplectic map T. If there exists a Hamiltonian vector field with periodic in time Hamiltonain whose Poincaré map coincides with T?

This problem is called the problem of inclusion of a map into a flow in symplectic set up. We can formulate this problem for other classes of maps and the corresponding vector fields. For example, we can consider general maps and vector fields, reversible ones and so on. It is possible to consider also smooth and analytic set up.[1]

There are obvious obstacles for a map to be included into a flow. These obstacles are of purely topological nature. Consider for example, the diffeomorphism T of the torus $\mathbb{T}^2 = \{(\varphi_1, \varphi_2) \bmod 2\pi\}$ defined by the relation $(\varphi_1, \varphi_2) \to (-\varphi_2, \varphi_1)$. This map is symplectic with respect to the standard symplectic structure $d\varphi_1 \wedge d\varphi_2$, but it does not matter. The map T can not be included into any flow because (in particular) it sends the cycle $\gamma_1 = \{\varphi_1 = 0\}$ to $\gamma_2 = \{\varphi_2 = 0\}$. The cycles γ_1 and γ_2 are not homotopic. On the other hand, any Poincaré map corresponding to a periodic vector field, transforms any cycle into homotopic one. Generalizing this argument we see that any map which is not isotopic to the identity[2] cannot be included into a flow. Below we prove that, conversely, maps isotopic to the identity can be included into periodic flows.

First, several remarks about history of the problem. Douady [1] has proved that the problem has a positive solution in the smooth symplectic case provided the map can be defined by a generating function. Conley and Zehnder in their famous paper [2], containing proof of Arnold conjecture about the number of fixed points of a symplectic diffeomorphism of a torus, solved the problem for smooth

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$T \quad M \quad M \quad j \quad , \qquad M \quad M$

$T \quad M \quad M$

$s \quad , \qquad T \quad T \quad T \quad T$

diffeomorphisms of a torus which leave the center of mass fixed. The case of smooth general maps is almost trivial.

Analytic case turns out to be more complex. Douady [3], Kuksin [4], and Kuksin, Pöschel [5] solved the problem in analytic set up for maps which are close to integrable ones. Trifonov [6] solved the one for general maps.

Note that all the known proofs in analytic set up use essentially the Grauert theorem on the inclusion of an analytic manifold into Euclidean space (or modified versions of this theorem). Our proof is based on a special averaging method.

To formulate our main theorem we need some definitions and notation. Let M be an m-dimensional real-analytic manifold. Below we assume that M is compact (otherwise one should deal with a compact domain in M). Let χ be a subalgebra in the Lie algebra $(\mathcal{L}, [\,,])$ of all analytic vector fields on M, where $[\,,]$ is the ordinary commutator. The basic examples are as follows.

a) $\chi = \mathcal{L}$,

b) M is a symplectic manifold and χ is the algebra of Hamiltonian (or locally Hamiltonian) vector fields,

c) χ is the algebra of volume-preserving vector fields,

Reversible vector fields do not form a subalgebra in $\mathcal{L}$. Hence, we have to consider the reversible case separately. First, recall the definition. Let $I\colon M \to M$ be a real-analytic involution ($I \circ I = \mathrm{id}$). The vector field $u(z,t)$ on M is said to be reversible with respect to I (or I-reversible) if

$$u(z,t) = -dI\, u(Iz, -t). \tag{1.1}$$

We denote by X the subset of all analytic diffeomorphisms of M obtained as a result of the time 2π-shift along trajectories of the system

$$\dot z = u(z,t), \quad u \in \chi, \qquad z \in M, \quad t \in [0, 2\pi]. \tag{1.2}$$

We assume that the vector field u is C^2-smooth with respect to time. This smoothness condition is purely technical. For example, it can be considerably weakened in Hamiltonian case (see §2) and (still easier) in the general one.

Below it is convenient to continue the vector fields u by 2π-periodicity to all the real axis of time. Unfortunately, the obtained vector fields are in general discontinuous in t for $t = 2\pi n$, $n \in \mathbb{Z}$. However, we can achieve C^2-smoothness by changing time. Indeed, we can perform the change $t \to t'$, see Fig. 1. The function $t(t')$ is smooth, odd, strictly monotone, $t(0) = 0$, $t(2\pi) = 2\pi$, and the derivatives $d^l t/(dt')^l$, $l = 1, 2, 3$ vanish at the points $t' = 0$ and $t' = 2\pi$. Furthermore, we assume that the function $t(t') - t'$ is periodic. We put $\hat u(z,t') = u(z, t(t'))$. Obviously, $\hat u(z,\cdot)$ is C^2-smooth in time on all the real axis. Below we deal with the function $\hat u$ and skip prime in t' for brevity.

FIG. 1. THE FUNCTION $t(t')$.

We define the set X_I as a subset of X generated by I-reversible vector fields u.

Obviously, all diffeomorphisms from X (X_I) are isotopic to the identity inside X (X_I).

Theorem 1. [3] *Suppose that the map T belongs to X (respectively, X_I). Then there exists a vector field (respectively, an I-reversible vector field)*

$$U = U(z,t), \quad U \in \chi, \qquad z \in M, \quad t \in \mathbb{R}$$

which is analytic and 2π-periodic in t and such that time 2π-shift along its trajectories coincides with T.

As a corollary we obtain a possibility of inclusion of analytic maps into analytic flows in general, symplectic, volume-preserving and reversible cases.

Remark 1. Suppose that the map T is close to the one T_0 ($\mathrm{dist}(T, T_0) \sim \varepsilon$ in a complex neighborhood of M), where T_0 is already included into the flow generated by a periodic analytic vector field U_0. Then the vector field U can be chosen close to U_0 ($|U - U_0| \sim \varepsilon$ in a smaller complex neighborhood of M).

In particular, in the symplectic case, if T is close to an integrable map then a Hamiltonian system associated with T also can be chosen close to an integrable one and the orders of closeness are the same.

The structure of the rest part of the paper is as follows. In §2 we discuss Theorem 1 in the symplectic case (see Theorem 2). Note that assumptions about smoothness of initial isotopy of T with the identity are replaced in Theorem 2 just by assumptions of continuity. In §3 we present our main tool: the method of continuous averaging. We prove Theorem 2 in §§4-5. In §6 we reduce other cases of Theorem 1 to the symplectic one. In Appendices we gather various auxiliary assertions.

§2. Inclusion of a symplectic map into a Hamiltonian flow

Let $(\mathcal{M}, \omega)$ be a real-analytic symplectic m-dimensional manifold, where ω is a symplectic structure (a non-degenerate closed 2-form), and let the analytic map $T : \mathcal{M} \to \mathcal{M}$ be symplectic, i.e. $T^*\omega = \omega$.

We assume that T is continuously isotopic to the identity. More precisely, there exists a curve g^t, $t \in [0, 2\pi]$ in the set of analytic symplectic diffeomorphisms of $\mathcal{M}$ such that the following conditions are satisfied:

(1) g^0 is the identity map, $g^{2\pi} = T$,

(2) g^t is continuous with respect to t.

Lemma 2.1. *If T is continuously isotopic to the identity then the isotopy g^t can be chosen C^3-smooth with respect to time.*

Proof of this lemma is contained in Appendix 1. Below we assume that g^t is already chosen C^3-smooth.

Theorem 2. *Let $\mathcal{M}$ be compact. Then there exists an analytic in z and t, 2π-periodic in t function $H_*(z,t)$, $z \in \mathcal{M}$, $t \in \mathbb{R}$ such that the shift $z(0) \to z(2\pi)$ along trajectories of the Hamiltonian system $\dot{z} = v_{H_*}(z,t)$ coincides with T.*

Remark 1. If $\mathcal{M}$ is not compact, one should deal with a compact domain in $\mathcal{M}$.

Remark 2. In general the Hamiltonian function H_* is multivalued. In other words, the corresponding system is local Hamiltonain. However, H_* is single-valued provided the 1-form $\omega(\cdot, dg^t/dt)$ is exact for any $0 \quad t \quad 2\pi$.

Remark 3. If the map T is a perturbation (of order ε) of a map which is already included into a flow with the Hamiltonian H_0, then the function H_* can be regarded as a perturbation (of the same order) of H_0.

§3. Averaging method

Consider the Hamiltonian system

$$\dot{z} = v_{\hat{H}}(z), \qquad z \in \mathcal{M}, \tag{3.1}$$

where $\mathcal{M}$ is a symplectic manifold, $\hat{H}$ is a function on $\mathcal{M}$ and $v_{\hat{H}}$ is the Hamiltonian vector field corresponding to $\hat{H}$. We want to improve (in a certain sense) the Hamiltonian function $\hat{H}(z)$ by means of a canonical change of the variables. We construct the change as a shift along trajectories of another Hamiltonian system

$$dz/ds = v_F, \qquad F = F(z, s). \tag{3.2}$$

Suppose that the function F is known. The change of the variables depends on the rate s of the shift. We denote new variables by $z(s)$. In particular, $z(0) = z$.

Let $H(\cdot, s)$ be the Hamiltonian $\hat{H}$ written in the new variables. We have the relation $H(z(s), s) = \hat{H}(z)$. By differentiating this equality with respect to s and taking into account (3.1) we obtain the following well known equation:

$$H_s + \{F, H\} = 0.$$

Here the subscript s denotes partial derivative and $\{\, ,\}$ is the Poisson bracket.

Our basic idea is to choose the Hamiltonian F in the form of some operator acting on H, i.e. $F = \xi H$. If the operator ξ is chosen, we have the following Cauchy problem for the function H:

$$H_s = -\{\xi H, H\}, \qquad H(z, 0) = \hat{H}(z). \tag{3.3}$$

It turns out that in a number of problems it is possible to choose ξ and s such that the function $H(z, s)$ defines Hamiltonian equations which are easier to analyze in comparison with the initial ones (3.1).

We call the described above method of transformation $\hat{H} \to H$ by continuous averaging method.

The system (3.3) obviously has a non-autonomous analog. If the function $\hat{H}$ depends explicitly on t then H also depends on t and (3.3) takes the form

$$H_s = (\xi H)_t - \{\xi H, H\}, \qquad H(z, t, 0) = \hat{H}(z, t). \tag{3.4}$$

It is easy to write also non-Hamiltonian analogs of the system (3.3).

§4. Application of the averaging procedure

Now we present the proof of Theorem 2. First, we perform the change of the parameter $t \to t'$. The function $t(t')$ coincides with that defined in §1 (see Fig. 1). We put $\hat g^{t'} = g^{t(t')}$. Below we write t instead of t' for brevity. The following lemma is quite standard in Hamiltonian dynamics.

Lemma 4.1. *Each map $\hat g^t$, $t \in [0, 2\pi]$ is the shift $z(0) \to z(t)$ along trajectories of the locally Hamiltonian system $\dot z = v_{\hat H}(z, t)$, where the Hamiltonian $\hat H$ is analytic in z and C^2-smooth in t.*

Proof of the lemma is trivial. It is based on the fact that the vector field $d\hat g^t/dt$ is locally Hamiltonian, analytic in z and C^2-smooth in t.

We are going to deal with the function obtained as a result of 2π-periodic continuation of $\hat H$ to all the axis of t. We denote this function also by $\hat H$. Obviously, $\hat H$ is of class C^2 in t.

The system generated by the Hamiltonian $H_*(z_*, t)$ (see formulation of Theorem 2) is a result of application of the averaging procedure (see §3) to the system generated by $\hat H(\hat z, t)$. Thus the new Hamiltonian has the form

$$H_*(z, t) = H(z, t, s_0),$$

where the function H is a solution of the Cauchy problem (3.4) and $s_0 > 0$ is a small magnitude. We choose the operator ξ in the following way:

$$\xi H = i(H^+ - H^-),$$

$$H = \sum_{n \in} H^n(z, s) e^{int} = H^0 + H^+ + H^-, \quad H^\pm = \sum_{\pm n > 0} H^n(z, s) e^{int}. \qquad (4.1)$$

The equation (3.4) takes the form of the infinite system

$$\begin{aligned} H^0_s &= -2i \sum_{k=1}^{\infty} \{H^k, H^{-k}\}, \\ H^{\pm n}_s &= -nH^{\pm n} \pm i\{H^0, H^{\pm n}\} \mp 2i \sum_{k=1}^{\infty} \{H^{\pm n \pm k}, H^{\mp k}\}, \qquad n > 0, \end{aligned} \qquad (4.2)$$

$$H^k(z, 0) = \hat H^k(z), \qquad k \in \mathbb{Z}, \qquad (4.3)$$

where $\hat H^n(z)$ are Fourier coefficients of the function $\hat H(z, t)$.

It can be shown (see Lemma 5.1 below) that

$$|\hat H^n(z)| \quad \mathrm{const}/(|n| + 1)^2, \qquad z \in \mathcal{M}, \quad n \in \mathbb{Z}. \qquad (4.4)$$

Why the system (4.2) provides smoothing with respect to t? An informal answer is as follows. Let us remove nonlinear terms in (4.2). We obtain the system

$$H^n_s = -|n| H^n, \qquad n \in \mathbb{Z}.$$

This system (with initial conditions (4.3)) can be easily solved:

$$H^n(z, s) = e^{-|n|s} \hat H^n(z).$$

According to (4.4) these functions decrease exponentially in $|n|$ for any positive s.

It turns out that nonlinear terms in the system (4.2) do not hinder the averaging process for small s. Rigorous analysis of the system (4.2)–(4.3) is based on the majorant method.

§5. Majorant estimates

Since the manifold $\mathcal{M}$ is assumed to be compact, it is sufficient to use just a local analysis of the averaging procedure. We prove that the averaging system (4.2)–(4.3) has a solution on an interval $[0, s_0]$, $s_0 > 0$ in the vicinity of any point $z^0 \in \mathcal{M}$. The magnitude s_0 depends on the point z^0. Due to the compactness we can choose positive s_0 which is valid for any point.

Recall that for any two functions $f(z)$, $g(z)$, $z = (z_1, \dots, z_m)$ analytic at the point $z = 0$:

$$f(z) = \sum_\beta f_\beta z^\beta, \qquad g(z) = \sum_\beta g_\beta z^\beta,$$

$$\beta = (\beta_1, \dots, \beta_m), \quad \beta_j \quad 0, \quad z^\beta = z_1^{\beta_1} \cdots z_m^{\beta_m}$$

we say that g is a majorant for f ($g \gg f$) if for any multi-index β we have $g_\beta \quad |f_\beta|$.

Let the point z^0 be in the set $\mathcal{M}$. Obviously we can assume that the coordinates z^0 vanish: $z^0 = 0$.

Lemma 5.1. *The initial conditions* (4.3) *satisfy the estimates*

$$\hat{H}^n(z) \ll \frac{a}{(|n|+1)^2 w}, \qquad w = (b - z_1) \cdots (b - z_m) \tag{5.1}$$

with some constants a and b.

Proof of Lemma 5.1 is contained in Appendix 2.

Let us introduce new functions $G^n(z,s) = H^n(z,s)e^{|n|s}$. By using equations (4.2)–(4.3) we obtain the following system ($n \in \mathbb{N}$):

$$\begin{cases} G^0_s = -2i \sum_{k=1}^\infty \{G^k, G^{-k}\} e^{-2ks}, \\ G^{\pm n}_s = \pm i \{G^0, G^{\pm n}\} \pm 2i \sum_{k=1}^\infty \{G^{\mp k}, G^{\pm n \pm k}\} e^{-2ks}. \end{cases} \tag{5.2}$$

To get majorant estimates for the functions $G^n(z,s)$ we put

$$G^n(z,s) = \sum_{k=0}^\infty G^n_k(z,s). \tag{5.3}$$

We assume that

$$G^j_0(z,s) \equiv \hat{G}^j(z) \quad G^j_l(z,0) = 0, \qquad j \in \mathbb{Z},$$

$$(G^0_l)_s = -2i \sum_{u+v=l-1} \sum_{k=1}^\infty \{G^k_u, G^{-k}_v\} e^{-2ks},$$

$$(G^{\pm n}_l)_s = \pm i \sum_{u+v=l-1} \left(\{G^0_u, G^{\pm n}_v\} \pm 2 \sum_{k=1}^\infty \{G^{\mp k}_u, G^{\mp n \mp k}_v\} e^{-2ks} \right),$$

where $l, n \in \mathbb{N}$. Obviously, if the series (5.3) converge, they are solutions of the system (5.2). The functions G^k_l, $l > 0$ can be easily found recursively ($n \in \mathbb{N}$):

$$\begin{cases} G^0_l(z,s) = -2i \sum_{u+v=l-1} \int_0^s \sum_{k=1}^\infty \{G^k_u(z,\sigma), G^{-k}_v(z,\sigma)\} e^{-2k\sigma}\, d\sigma, \\ G^n_l(z,s) = i \sum_{u+v=l-1} \int_0^s \big(\{G^0_u(z,\sigma), G^n_v(z,\sigma)\} \\ \qquad\qquad + 2 \sum_{k=1}^\infty \{G^{-k}_u(z,\sigma), G^{n+k}_v(z,\sigma)\} e^{-2k\sigma} \big)\, d\sigma, \\ G^{-n}_l(z,s) = -i \sum_{u+v=l-1} \int_0^s \big(\{G^0_u(z,\sigma), G^{-n}_v(z,\sigma)\} \\ \qquad\qquad + 2 \sum_{k=1}^\infty \{G^k_u(z,\sigma), G^{-n-k}_v(z,\sigma)\} e^{-2k\sigma} \big)\, d\sigma. \end{cases}$$

Lemma 5.2. *For any integer n and $l \quad 0$ the following estimate holds*

$$G_l^n(z,s) \ll \frac{ac^l s^l}{(|n|+1)^2(l+1)^3 w^{2l+1}}, \tag{5.4}$$

where c is a positive constant and w is the same as in (5.1).

Lemma 5.2 implies Theorem 2.1 since the function $H_*(z,t) = H(z,t,s)$ is analytic in both z and t for any small $s > 0$. We prove this lemma by induction in l. Below we use several additional lemmas presented in Appendix 2.

The base of the induction follows from Lemma 5.1: $G_0^n(z,s) \ll a(|n|+1)^{-2}w^{-1}$.

Now suppose that the estimate (5.4) holds for any integer $k \in [0,l]$. Transforming the equation for G_{l+1}^0 to majorant formula and using Lemmas 8.2 and 8.1 (statement 3) we have:

$$\begin{aligned} G_{l+1}^0 &\ll 2\sum_{u+v=l}\int_0^s \sum_{k=1}^{\infty} \frac{T(2u+1)(2v+1)}{w^{1+(2u+1)+(2v+1)}} \frac{a^2c^l\sigma^l}{(k+1)^4(u+1)^3(v+1)^3}\,d\sigma \\ &= 2\frac{ac^l s^{l+1}}{w^{2l+3}} \frac{Ta}{l+1} \sum_{u+v=l} \frac{(2u+1)(2v+1)}{(u+1)^3(v+1)^3} \sum_{k=1}^{\infty} \frac{1}{(k+1)^4}. \end{aligned}$$

Applying Lemma 8.3 we see that:

$$G_{l+1}^0 \ll \frac{ac^{l+1}s^{l+1}}{(l+2)^3 w^{2l+5}} \frac{BTa(l+2)}{c(l+1)} \frac{\pi^4}{90} \ll \frac{ac^{l+1}s^{l+1}}{(l+2)^3 w^{2l+5}} \frac{\pi^4 BTa}{45c}.$$

Hence, the estimate (5.4) holds if we choose $c \quad \pi^4 BTa/45$.

Analogously we estimate the functions G_l^n:

$$\begin{aligned} G_{l+1}^n &\ll \sum_{u+v=l}\int_0^s \frac{a^2c^l\sigma^l T(2u+1)(2v+1)}{(u+1)^3(v+1)^3 w^{2l+5}} \\ &\qquad \times \Big(\frac{1}{(n+1)^2} + \sum_{k=1}^{\infty} \frac{2}{(k+1)^2(n+k+1)^2}\Big)d\sigma \\ &= \frac{ac^l s^{l+1}}{(l+2)^3(n+1)^2 w^{2l+5}} \frac{(l+2)BTa}{l+1} \Big(1 + \sum_{k=1}^{\infty} \frac{(n+1)^2}{(k+1)^2(n+k+1)^2}\Big) \\ &\ll \frac{ac^{l+1}s^{l+1}}{(l+2)^3(n+1)^2 w^{2l+5}} \frac{2BTa}{c} \frac{\pi^2}{6}. \end{aligned}$$

Consequently, if $c > BTa\max\{\pi^4/45, \pi^2/3\}$, the estimate (5.4) is valid.

§6. The case of arbitrary algebra χ.

We reduce Theorem 1 to Theorem 2 by using the following standard trick. As is well known, any system of differential equations

$$\dot z = v(z,t), \qquad z \in M$$

can be regarded in local coordinates as a part of the Hamiltonian system

$$\dot z = dH/dp, \quad \dot p = -dH/dz, \quad H(p,z,t) = \sum p_j v_j(z,t)$$

on the cotangent bundle $(T^*M,\ dp\wedge dz)$.

Let $T\in X$ (or $T\in X_I$) and $\hat u(z,t)$ be the corresponding vector field (see §1). The Hamiltonian function

$$\hat H(p,z,t)=\sum_j p_j\hat u_j(z,t) \tag{6.1}$$

is linear in momenta, analytic in z and C^2-smooth in t. Applying averaging procedure of sections 3-4 to this Hamiltonian we obtain the system

$$H_s=(\xi H)_t-\{\xi H,H\},\quad H(p,z,t,0)=\hat H(p,z,t), \tag{6.2}$$

$$\xi H(p,z,t,s)=i(H^+(p,z,t,s)-H^-(p,z,t,s)). \tag{6.3}$$

The system (6.1)–(6.3) is completely analogous to (3.4), (4.1). Existence of a solution $H(p,z,t,s)$ of the system (6.1)–(6.3) follows from majorant estimates of §5. Now Theorem 1 is a direct consequence of the following

Proposition 6.1. *The function $H(p,z,t,s)$ has the form*

$$H(p,z,t,s)=\sum_j p_j u_j(z,t,s),\qquad u(z,t,s)\in\chi. \tag{6.4}$$

Moreover, if $\hat u$ is I-reversible, u is also I-reversible.

Proposition 6.1 is a consequence of the following

Proposition 6.2. *For linear in p function H of form* (6.4) *the right-hand side of the equation* (6.2) *is also linear in p and has the form*

$$\sum_j p_j w_j(z,t,s),\qquad w(z,t,s)\in\chi. \tag{6.5}$$

Moreover, if $\hat u$ is I-reversible, the vector field w is also I-reversible.

First, we assume that $T\in X$. In this case Proposition 6.2 is proved with the help of the following two observations

(1) If the function H has the form (6.5), so do the functions ξH and $\partial\xi H/\partial t$,
(2) The following identity holds for any two vector fields v',v'' on M:

$$\{\sum_j p_j v'_j(z,t),\sum_j p_j v''_j(z,t)\}=\sum_j p_j V_j(z,t),\qquad V=[v',v''].$$

Let us consider the reversible case ($T\in X_I$) in more details. We call a vector field $v(z,t)$ on M symmetric with respect to I (or I-symmetric) if

$$v(z,t)=dI\,v(Iz,-t) \tag{6.6}$$

(cf. with (1.1)). Let S be the set of I-symmetric vector fields on M. (Recall that we denote by R the set of I-reversible vector fields.)

For any vector field $v(z,t)$ on M we put $\xi v=i(v^+-v^-)$, where the meaning of the superscripts $+$ and $-$ is the same as in (4.1). Then, obviously,

$$\xi\sum_j p_j v_j=\sum_j p_j(\xi v)_j.$$

Proposition 6.3. *For any two vector fields $v \in R$ and $w \in S$ the following relations hold*

$$\xi v \in S, \quad \partial w/\partial t \in R, \tag{6.7}$$
$$[v, w] \in R. \tag{6.8}$$

This proposition implies Proposition 6.2 in the reversible case. To prove Proposition 6.3 we note that for any I-reversible vector field

$$v(z,t) = \sum_{n\in} v^n(z)e^{int}$$

and any I-symmetric vector field

$$w(z,t) = \sum_{n\in} w^n(z)e^{int}$$

the Fourier coefficients v^n, w^l satisfy the equalities

$$v^n(z) = -dI\, v^{-n}(Iz), \quad w^l(z) = dI\, w^{-l}(Iz). \tag{6.9}$$

These equalities can be obtained from expansions of the equations (1.1) and (6.6) into Fourier series. Relations (6.7) easily follow from equalities (6.9).

Obviously, it is sufficient to prove relation (6.8) for the case

$$v = v^n(z)e^{int} + v^{-n}(z)e^{-int}, \quad w = w^l(z)e^{ilt} + w^{-l}(z)e^{-ilt}.$$

We have:

$$\begin{aligned}[v,w] &= [v^n, w^l]e^{i(n+l)t} + [v^{-n}, w^{-l}]e^{-i(n+l)t} \\ &\quad + [v^n, w^{-l}]e^{i(n-l)t} + [v^{-n}, w^l]e^{-i(n-l)t}.\end{aligned}$$

By using equalities (6.9) we get:

$$dI\,[v^n, w^l](z) = [dI\, v^n(z), dI\, w^l(z)] = [-v^{-n}(Iz), w^{-l}(Iz)] = -[v^{-n}, w^{-l}](Iz).$$

The equality $dI\,[v^n, w^{-l}](z) = -[v^{-n}, w^l](Iz)$ can be verified analogously. Proposition 6.3 is proved.

§7. Appendix 1. Proof of Lemma 2.1.

Lemma 2.1 is a consequence of the following

Proposition 7.1. *Lemma 2.1 holds for diffeomorphisms T close to the identity.*

Indeed, suppose that Proposition 7.1 is valid. Let us take the initial (continuous) isotopy g^t. We divide the interval $(0, 2\pi)$ by the points $0 = t_0 < t_1 < \ldots < t_K = 2\pi$ into a large amount of small intervals (t_{k-1}, t_k), $1 \quad k \quad K$. We have the equality

$$T = Q_K \circ \ldots \circ Q_1, \quad Q_k = g^{t_k} \circ g^{t_{k-1}}, \quad 1 \quad k \quad K.$$

All the maps Q_k are close to the identity. Hence, there exist smooth isotopies q_k^t which link Q_k with the identity. The isotopy $T_t = q_k^t \circ \ldots \circ q_1^t$ is the one we search for.

Proof of Proposition 7.1. We consider the graph $\mathcal{G}$ of T in the direct product $\mathcal{M} \times \mathcal{M}$. The set $\mathcal{G}$ consists of pairs $(z, T(z)) \in \mathcal{M} \times \mathcal{M}$. Since $\mathcal{M}$ is compact and T is close to the identity, $\mathcal{G}$ lies in a small neighborhood $\mathcal{N}$ of the diagonal $\mathcal{D} \subset \mathcal{M} \times \mathcal{M}$. (Recall that $\mathcal{D} = \{(z_1, z_2) \in \mathcal{M} \times \mathcal{M} \colon z_1 = z_2\}$).

The manifold $\mathcal{M} \times \mathcal{M}$ is endowed with the natural symplectic structure $\Omega = \omega_1 - \omega_2$, where ω_1 (respectively, ω_2) is the 2-form coinciding with ω on the first (respectively, the second) multiplier and vanishing on the second (respectively, the first) one. The manifolds $\mathcal{D}$ and $\mathcal{G}$ are Lagrangian with respect to Ω because the identity map and T are symplectic.

The neighborhood N is symplectomorphic to a neighborhood of the zero section $\mathcal{M}_0$ of the cotangent bundle $(T^*\mathcal{M}, dp \wedge dz)$. Moreover, we can assume that the symplectomorphism maps $\mathcal{D}$ onto $\mathcal{M}_0 \subset T^*\mathcal{M}$. Let $f \colon (N, \Omega) \to (T^*\mathcal{M}, dp \wedge dz)$ be this symplectomorphism and $\Lambda = f(\mathcal{G}) \subset T^*\mathcal{M}$. The manifold Λ projects one-to-one onto $\mathcal{M}_0$ under the natural projection $(p, z) \to (0, z)$. Hence, Λ can be determined by the relation $p = p(z)$. Since Λ is Lagrangian, the function $p(z)$ has the form $\partial S(z)/\partial z$, where S is (in general multivalued) real-valued function on M.

Consider the family $\Lambda_t, t \in [0,1]$ of Lagrangian manifolds in T^*M determining by the equation $p = (2\pi)^{-1} t\, \partial S/\partial z$. Obviously, $\Lambda_0 = M_0$, $\Lambda_{2\pi} = \Lambda$. The Lagrangian manifolds $\mathcal{G}_t = f^{-1}(\Lambda_t) \subset M \times M$ can be presented in the form $\mathcal{G}_t = \{(z_1, z_2) \colon z_2 = T_t(z_1)\}$, where T_t are some symplectic maps. Obviously, T_t are smooth with respect to t and

$$\mathcal{G}_0 = \mathcal{D}, \quad \mathcal{G}_{2\pi} = \mathcal{G}, \quad T_0 = \mathrm{id}, \quad T_{2\pi} = T.$$

Hence, T_t is the isotopy we search for.

§8. Appendix 2. Auxiliary statements.

Lemma 8.1. *The relation $\ll$ satisfies the following properties:*

(1) *If $f_1 \ll g_1$ and $f_2 \ll g_2$ then $f_1 + f_2 \ll g_1 + g_2$ and $f_1 f_2 \ll g_1 g_2$.*
(2) *If $f \ll g$ then for any j the estimate $df/dz_j \ll dg/dz_j$ holds.*
(3) *If $f(z, \lambda) \ll g(z, \lambda)$ for any $\lambda \in [a, b]$ then*

$$\int_a^b f(z, \lambda)\, d\lambda \ll \int_a^b g(z, \lambda)\, d\lambda.$$

(4) *For any positive b the estimate $1 \ll b/(b - z)$ holds.*
(5) *Suppose that $|f(z)| < c$ in the domain $\{z = (z_1, \ldots, z_m) : |z_j| < b,\ j = 1, \ldots, m\}$. Then $f(z) \ll cb^m/w$, where $w = (b - z_1) \cdots (b - z_m)$.*

Assertions (1)–(4) are obvious. Assertion (5) follows from the Cauchy formula.

Lemma 8.2. *Suppose that $f \ll w^{-k}$ and $g \ll w^{-l}$, where $w = \prod_{j=1}^m (b - z_j)$ ($m \in \mathbb{Z}$ is even). Then $\{f, g\} \ll Tklw^{-1-k-l}$, where $T = T(m, b)$ is some constant and $\{,\}$ is the Poisson bracket.*

Proof. We obtain majorant estimates for Poisson bracket using Lemma 8.1. Suppose that $z = (z_1, \ldots, z_m) = (p, q)$, where $p_j = z_j$, $q_j = z_{m/2+j}$ are canonical (symplectic) coordinates, $j \in [1, m/2]$. Then the following estimate holds:

$$\begin{aligned}
\{f,g\} &= \sum_j \Big(\frac{df}{dp_j}\frac{dg}{dq_j} - \frac{df}{dq_j}\frac{dg}{dp_j}\Big) \ll \sum_j \Big(\frac{dw^{-k}}{dp_j}\frac{dw^{-l}}{dq_j} + \frac{dw^{-k}}{dq_j}\frac{dw^{-l}}{dp_j}\Big) \\
&= \sum_j \Big(\frac{kw^{-k}}{b-p_j}\frac{lw^{-l}}{b-q_j} + \frac{kw^{-k}}{b-q_j}\frac{lw^{-l}}{b-p_j}\Big) = 2klw^{-k-l}\sum_{j=1}^{m}\frac{1}{(b-p_j)(b-q_j)} \\
&\ll 2klw^{-n-l-1}mb^{2(m-1)}.
\end{aligned}$$

Hence, we can put $T = mb^{2(m-1)}$. If the coordinates z are not canonical, Lemma 8.2 holds as well because it is always possible to pass to a symplectic coordinate system.

Lemma 8.3. *Let a, b, k be nonnegative integers. Then*

$$\sum_{a+b=k}\frac{(2a+1)(2b+1)}{(a+1)^3(b+1)^3} \quad \frac{B}{(k+2)^2}, \qquad B = 64\pi^2/3.$$

The proof follows from the estimates

$$\begin{aligned}
\sum_{a+b=k}\frac{(2a+1)(2b+1)}{(a+1)^3(b+1)^3} \quad & 4\sum_{a+b=k}\frac{1}{(a+1)^2(b+1)^2} \\
& 8\sum_{0 \quad a \quad k/2+1/2}\frac{1}{(a+1)^2(k/2+1/2)^2} \\
& \frac{32}{(k+1)^2}\sum_{a=0}^{\infty}\frac{1}{(a+1)^2} = \frac{32\pi^2}{6(k+1)^2} \quad \frac{64\pi^2}{3(k+2)^2}.
\end{aligned}$$

Proof of Lemma 5.1. Since, $\hat{H}$ is C^2-smooth in t and the function

$$\frac{\partial^2 H(z,t)}{\partial t^2} = \sum_{k\in}(-n^2)\hat{H}^n(z)e^{int}$$

is continuous, we have the estimate $|\hat{H}^n(z)| \quad c_1/(|n|+1)^2$ for $z \in \mathcal{M}, n \in \mathbb{Z}$, where c_1 is some constant. Moreover, we can assume that the same inequality holds if z belongs to some complex neighborhood of $\mathcal{M}$. Now Lemma 5.1 follows from statement (5) of Lemma 8.1. Since M is compact we can assume that the same inequality holds for each local coordinate system on M we deal with.

§8. Appendix 3. About the proof of Remark 3 to Theorem 2

Suppose that the map T is a perturbation of order ε of the map T_0. Then there exists a symplectic map Δ such that

(i) in some Riemannian metric $\mathrm{dist}\,(\Delta(z), z) \quad \mathrm{const}\,\varepsilon$, (ii) $T = \Delta \circ T_0$.

Note that according to Proposition 7.1 Δ is smoothly isotopic to the identity. Let δ^t, $t \in [0, 2\pi]$ be the corresponding isotopy linking the identity map and T_0. Then the smooth isotopy $g^t = \delta^t \circ g_0^t$ links the identity map with T. The (local)

Hamiltonian $\hat{H}$ of the flow g^t can be chosen smooth and $O(\varepsilon)$-close to the one $\hat{H}_0$ corresponding to g_0^t:

$$\hat{H}(z,t) = \hat{H}_0(z,t) + \varepsilon \hat{H}_*(z,t).$$

Let us put

$$H(z,t,s) = \hat{H}_0(z,t) + \varepsilon H_*(z,t,s), \qquad H|_{s=0} = \hat{H}, \quad H_*|_{s=0} = \hat{H}_*.$$

We assume that H satisfies the averaging system (3.4) with the operator ξ of the form

$$\xi H = \varepsilon i (H_*^+ - H_*^-),$$

where the meaning of the notations $H_*^{\pm}$ is the same as in (4.1). By using majorant estimates it is possible to verify that the averaging system has a solution for small values of $s \quad 0$. For any small positive s the function H is analytic in both z and t. Moreover, the relation $H = \hat{H}_0 + O(\varepsilon)$ preserves.

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