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Quantum Observables: An Algebraic Aspect

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Abstract—Quantum observables are represented as series in noncommuting generators $\widehat{x}$ and $\widehat{p}$. The space of such series turns out to be an infinite-dimensional associative algebra and a Lie algebra. The concept of convergence is presented for such series. In this language, quantum objects turn out to be noncommutative analogues of classical objects. Quantum analogues are proved for several basic theorems of classical mechanics.

1. INTRODUCTION AND MOTIVATION

We start with some motivations. First, we present basic constructions of classical mechanics in terms of quantum mechanics (see, e.g., [5]).

Suppose, for simplicity, that the phase space of a classical mechanical system is $\mathbb{R}^{2n} = \{(x,p)\colon x \in \mathbb{R}^n,\, p \in \mathbb{R}^n\}$.

The space of C^∞-smooth functions on $\mathbb{R}^{2n}$ is an infinite-dimensional Lie algebra with respect to the Poisson bracket $\{\cdot,\cdot\}$,

$$\{f,g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} \right). \tag{1.1}$$

This algebra is called the space of classical observables.

To specify a system, one should choose an observable h (a Hamiltonian function). Then, on the space of classical observables, dynamics is defined by the partial differential equation

$$\dot f = \{h,f\} \qquad \text{for any classical observable } f. \tag{1.2}$$

In classical mechanics, one usually considers dynamics in the phase space:

$$\dot x = \frac{\partial h}{\partial p}, \qquad \dot p = -\frac{\partial h}{\partial x}. \tag{1.3}$$

Equation (1.2) is, in a sense, secondary with respect to the usual Hamiltonian equations (1.3). Indeed, let ϕ^t be the phase flow of system (1.3). Then the solution of (1.2) with the initial condition $f\big|_{t=0} = f_0$ has the form $f = f_0 \circ \phi^t$.

Quantum dynamics can be introduced similarly. Let

$$L_2 = L_2(\mathbb{R}^n), \qquad \mathbb{R}^n = \{x = (x_1,\dots,x_n)\},$$

denote the Hilbert space of square integrable functions on $\mathbb{R}^n$. The corresponding Hermitian product is given by

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}^n} \varphi(x)\overline{\psi}(x)\,dx.$$

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Quantum observables are Hermitian operators on L_2. For any two observables $\widehat{f}$ and $\widehat{g}$, their commutator is

$$[\widehat{f},\widehat{g}\,] = -\frac{1}{i\hbar}\big(\widehat{f}\circ\widehat{g} - \widehat{g}\circ\widehat{f}\,\big), \tag{1.4}$$

where $i=\sqrt{-1}$ and $\hbar$ is the Planck constant.

Specifying a quantum observable $\widehat{h}$, we define the dynamics by the Heisenberg equation:

$$\dot{\widehat{f}} = [\widehat{h},\widehat{f}\,] \qquad \text{for any quantum observable } \widehat{f}. \tag{1.5}$$

According to the standard quantization procedure, with the classical variables x_j and p_j, $j = 1,\dots,n$, we associate their quantum analogues $\widehat{x}_j$ and $\widehat{p}_j = -i\hbar\partial/\partial x_j$; here $\widehat{x}_j$ is the operator of multiplication by x_j. Following this rule, one can define quantum analogues of many classical dynamical quantities (such as momentum, angular momentum, potential energy, etc.). However, difficulties arise when a classical observable contains a product of two functions whose quantum analogues do not commute. In this case, the quantization is not uniquely defined. For example, with x_jp_j, one may associate either $\widehat{x}_j\circ\widehat{p}_j$, or $\widehat{p}_j\circ\widehat{x}_j$, or $\frac{1}{2}(\widehat{x}_j\circ\widehat{p}_j+\widehat{p}_j\circ\widehat{x}_j)$, $\dots$.

In this paper, we replace the standard language of quantum mechanics by a new one. This new language is more algebraic and has several advantages:

- classical mechanics becomes a natural projection of quantum mechanics,
- we do not use expansions in $\hbar$ and deal with convergent series,
- we obtain some spaces (associative and Lie algebras) of quantum observables that are closed with respect to the operations of composition $\circ$ and commutator $[\cdot,\cdot]$.

We do not claim that our approach is definitely better than the traditional one. The main difficulty is the interpretation of algebraic objects that we use as quantum observables in terms of operators on $L_2(\mathbb{R}^n)$.

2. ASSOCIATIVE ALGEBRA $\mathcal{QO}^{\mathrm{form}}$

We call the product

$$z = z_k\circ\ldots\circ z_1, \qquad z_j\in\{\widehat{x}_1,\dots,\widehat{x}_n,\widehat{p}_1,\dots,\widehat{p}_n\}, \quad j=1,\dots,k,$$

a monomial and $\deg z := k$ its degree. Below, due to the motivations from quantum mechanics, we refer to such monomials and their linear combinations as observables.

The observable

$$F_k = \sum_{\deg z = k} f_z z, \tag{2.1}$$

where f_z are complex constants, is called a homogeneous form of degree k: $\deg F_k = k$. We assume that the forms of degree zero are constants. The space of homogeneous forms of degree k is denoted by $\mathbf{F}_k$.

Below we consider observables that admit a formal expansion

$$F = \sum_{k=0}^{\infty} F_k(\widehat{x},\widehat{p}\,), \qquad F_k\in\mathbf{F}_k. \tag{2.2}$$

Let $\widetilde{\mathcal{QO}}^{\mathrm{form}}(0)$ (or simply $\widetilde{\mathcal{QO}}^{\mathrm{form}}$) denote the vector space of such observables. In $\widetilde{\mathcal{QO}}^{\mathrm{form}}$, different formal series (2.2) are regarded as distinct elements. The space $\widetilde{\mathcal{QO}}^{\mathrm{form}}$ is a free associative (noncommutative) algebra over $\mathbb{C}$ with respect to the composition $\circ$.

For $F \in \widetilde{\mathcal{QO}}^{\mathrm{form}}$, we say that $F = O_m(\widehat{x}, \widehat{p})$ if $F_0 = \ldots = F_{m-1} = 0$ in its expansion (2.2).

Setting $\mathbf{r}_j := \widehat{p}_j \circ \widehat{x}_j - \widehat{x}_j \circ \widehat{p}_j$, we consider the ideal $J \subset \widetilde{\mathcal{QO}}^{\mathrm{form}}$ generated by

$$\widehat{p}_j \circ \widehat{p}_k - \widehat{p}_k \circ \widehat{p}_j, \qquad 1 \le j, k \le n, \tag{2.3}$$

$$\widehat{x}_j \circ \widehat{x}_k - \widehat{x}_k \circ \widehat{x}_j, \qquad 1 \le j, k \le n, \tag{2.4}$$

$$\widehat{p}_j \circ \widehat{x}_k - \widehat{x}_k \circ \widehat{p}_j, \qquad j \ne k, \tag{2.5}$$

$$\mathbf{r}_j - \mathbf{r}_k, \qquad 1 \le j, k \le n, \tag{2.6}$$

$$\mathbf{r}_j \circ \widehat{p}_j - \widehat{p}_j \circ \mathbf{r}_j, \qquad 1 \le j \le n, \tag{2.7}$$

$$\mathbf{r}_j \circ \widehat{x}_j - \widehat{x}_j \circ \mathbf{r}_j, \qquad 1 \le j \le n. \tag{2.8}$$

Let $\mathcal{QO}^{\mathrm{form}}(0)$ (below, usually $\mathcal{QO}^{\mathrm{form}}$) be the quotient space $\widetilde{\mathcal{QO}}^{\mathrm{form}}/J$. We call $\mathcal{QO}^{\mathrm{form}}$ the algebra of formal quantum observables over $0 \in \mathbb{C}^{2n}$.

Note that the ideal J is generated only by certain homogeneous elements of $\widetilde{\mathcal{QO}}^{\mathrm{form}}$.

We denote the corresponding projection by

$$\pi\colon \widetilde{\mathcal{QO}}^{\mathrm{form}} \to \mathcal{QO}^{\mathrm{form}}.$$

Since $\pi(\mathbf{r}_1) = \ldots = \pi(\mathbf{r}_n)$, we denote $\mathbf{r} := \pi(\mathbf{r}_j) \in \mathcal{QO}^{\mathrm{form}}$. Traditionally, $\mathbf{r}$ is replaced by $-i\hbar$. However, we will not do this. The main reason is that expansions in $\hbar$ in quantum mechanics are usually divergent, while we would like to have a theory that deals with convergent series.

For any $F \in \mathcal{QO}^{\mathrm{form}}$, we say that $F = O_m(\widehat{x}, \widehat{p})$ if there exists $\widetilde{F} \in \widetilde{\mathcal{QO}}^{\mathrm{form}}$ such that $\widetilde{F} = O_m(\widehat{x}, \widehat{p})$ and $F = \pi(\widetilde{F})$.

For any $F, G \in \mathcal{QO}^{\mathrm{form}}$ such that $F = O_m(\widehat{x}, \widehat{p})$ and $G = O_l(\widehat{x}, \widehat{p})$, we have

$$F \circ G = O_{m+l}(\widehat{x}, \widehat{p}), \qquad F + G = O_k(\widehat{x}, \widehat{p}), \qquad k = \min\{m, l\}.$$

In particular, $\mathbf{r} = O_2(\widehat{x}, \widehat{p})$.

3. FORMAL CLASSICAL OBSERVABLES

We define the space $\mathcal{CO}^{\mathrm{form}}$ of classical formal observables as the space of formal series

$$F(x,p) = \sum_{\mu,\nu \in \mathbb{Z}_+^{2n}} f_{\mu,\nu} x^\mu p^\nu, \qquad x^\mu = x_1^{\mu_1} \ldots x_n^{\mu_n}, \quad p^\nu = p_1^{\nu_1} \ldots p_n^{\nu_n}.$$

Here x and p are regarded as ordinary coordinates in $\mathbb{R}^{2n}$. The space $\mathcal{CO}^{\mathrm{form}}$ is a formal infinite-dimensional associative (and commutative) algebra.

Let $\widehat{J} \subset \widetilde{\mathcal{QO}}^{\mathrm{form}}$ be the ideal generated by J and $\mathbf{r}_1, \ldots, \mathbf{r}_n$, and let $J_0 \subset \mathcal{QO}^{\mathrm{form}}$ be the ideal generated by $\mathbf{r}$.

Since $\mathbf{r}$ commutes with any $F \in \mathcal{QO}^{\mathrm{form}}$, we have

$$J_0 = \mathbf{r} \circ \mathcal{QO}^{\mathrm{form}}. \tag{3.1}$$

The following proposition is obvious.

Proposition 3.1. $\mathcal{CO}^{\mathrm{form}} \cong \widetilde{\mathcal{QO}}^{\mathrm{form}}/\widehat{J} \cong \mathcal{QO}^{\mathrm{form}}/J_0$.

The projections $\widetilde{\mathcal{QO}}^{\mathrm{form}} \to \mathcal{CO}^{\mathrm{form}}$ and $\mathcal{QO}^{\mathrm{form}} \to \mathcal{CO}^{\mathrm{form}}$ can be regarded as some kinds of averaging. Below we use the following notations for these projections:

$$\widetilde{\mathrm{aver}}\colon \widetilde{\mathcal{QO}}^{\mathrm{form}} \to \mathcal{CO}^{\mathrm{form}}, \qquad \mathrm{aver}\colon \mathcal{QO}^{\mathrm{form}} \to \mathcal{CO}^{\mathrm{form}}.$$

Corollary 3.1. *The maps* $\widetilde{\mathrm{aver}}$ *and* aver *are homomorphisms of associative algebras.*

Corollary 3.2. *The diagram*

$$\begin{tikzcd} \widetilde{\mathcal{QO}}^{\mathrm{form}} \arrow[rr, "\pi"] \arrow[dr, "\widetilde{\mathrm{aver}}"'] & & \mathcal{QO}^{\mathrm{form}} \arrow[dl, "\mathrm{aver}"] \\ & \mathcal{CO}^{\mathrm{form}} & \end{tikzcd}$$

is commutative.

Corollary 3.3. *For any* $\widetilde{F} \in J$ *and* $F \in J_0$, *we have* $\widetilde{\mathrm{aver}}\, \widetilde{F} = 0$ *and* $\mathrm{aver}\, F = 0$.

Equation (3.1) implies

Proposition 3.2. *The observable* $F \in \mathcal{QO}^{\mathrm{form}}$ *lies in* J_0 *if and only if*

$$F = \mathbf{r} \circ F_0, \qquad F_0 \in \mathcal{QO}^{\mathrm{form}}. \tag{3.2}$$

We say that a monomial $z \in \widetilde{\mathcal{QO}}^{\mathrm{form}}$ is of type $(\mu, \nu) \in \mathbb{Z}_+^{2n}$ if it contains exactly μ_j multipliers $\widehat{x}_j$ and ν_j multipliers $\widehat{p}_j$, $j = 1, \dots, n$. Obviously,

$$\deg z = |\mu| + |\nu| := \mu_1 + \ldots + \mu_n + \nu_1 + \ldots + \nu_n.$$

We say that

$$F^{\varkappa} = \sum_{\mathrm{type}\, z = \varkappa} f_z z, \qquad \varkappa = (\mu, \nu) \in \mathbb{Z}_+^{2n},$$

is a homogeneous form of type $\varkappa$. Then

$$\widetilde{\mathrm{aver}}\, F^{\varkappa} = \biggl(\sum_{\mathrm{type}\, z = \varkappa = (\mu,\nu)} f_z\biggr) x^{\mu} p^{\nu}.$$

Let $\mathbf{F}^{\varkappa} \subset \widetilde{\mathcal{QO}}^{\mathrm{form}}$ denote the space of homogeneous forms of type $\varkappa$. Note that for $n > 1$, the spaces $\pi(\mathbf{F}^{\varkappa})$ and $\pi(\mathbf{F}^{\rho})$, $\varkappa \neq \rho$, may have a nonzero intersection.

Any observable $F \in \widetilde{\mathcal{QO}}^{\mathrm{form}}$ can be expanded in homogeneous forms of type $\varkappa$:

$$F = \sum_{\varkappa \in \mathbb{Z}_+^{2n}} F^{\varkappa}, \qquad F^{\varkappa} \in \mathbf{F}^{\varkappa}. \tag{3.3}$$

Then

$$\widetilde{\mathrm{aver}}\, F = \sum_{\varkappa \in \mathbb{Z}_+^{2n}} \widetilde{\mathrm{aver}}\, F^{\varkappa}.$$

4. COMMUTATOR ON $\mathcal{QO}^{\mathrm{form}}$

The space $\mathcal{CO}^{\mathrm{form}}$ is a Lie algebra. The corresponding commutator $\{\cdot,\cdot\}$ is the Poisson bracket (1.1). In this section, we introduce a structure of a formal Lie algebra on $\mathcal{QO}^{\mathrm{form}}$.

Proposition 3.2 means that the following map is well-defined:

$$\Omega\colon J_0 \to \mathcal{QO}^{\mathrm{form}}, \qquad J_0 \ni F \mapsto \Omega(F) = F_0,$$

where $F_0 = F_0(F)$ is defined by (3.2). Informally speaking, Ω is the operator of division by $\mathbf{r}$.

For any $F, G \in \mathcal{QO}^{\mathrm{form}}$, we have $F \circ G - G \circ F \in J_0$. Define

$$[F, G] = \Omega(F \circ G - G \circ F). \tag{4.1}$$

The commutator (4.1) obviously satisfies the Jacobi identity (A.1) and the Leibnitz identity (A.4). Note that (4.1) is compatible with (1.4). Here, instead of dividing $F \circ G - G \circ F$ by $-i\hbar$, we divide it by $\mathbf{r}$; these operations are different from the algebraic point of view but are the same from the physical point of view.

For any two observables F and G, we define their symmetric (Jordan) product $(\![\cdot,\cdot]\!)$ as follows:

$$(\![F, G]\!) := \frac{1}{2}(F \circ G + G \circ F). \tag{4.2}$$

Proposition 4.1. *For any $F, G \in \mathcal{QO}^{\mathrm{form}}$,*

$$\operatorname{aver}(F \circ G) = \operatorname{aver}(\![F, G]\!) = \operatorname{aver} F \cdot \operatorname{aver} G, \tag{4.3}$$

$$\operatorname{aver}[F, G] = \{\operatorname{aver} F, \operatorname{aver} G\}. \tag{4.4}$$

Corollary 4.1. *The map*

$$\operatorname{aver}\colon \bigl(\mathcal{QO}^{\mathrm{form}}, \circ, [\cdot,\cdot]\bigr) \to \bigl(\mathcal{CO}^{\mathrm{form}}, \cdot, \{\cdot,\cdot\}\bigr)$$

is a homomorphism of associative algebras and Lie algebras.

Proof of Proposition 4.1. Equations (4.3) follow from Corollary 3.3. To prove (4.4), we note that both the left- and the right-hand sides of (4.4) are bilinear in F and G. Hence, it is sufficient to assume that F and G are monomials:

$$F = \pi(\widetilde{F}), \qquad G = \pi(\widetilde{G}), \qquad \widetilde{F} \in \mathbf{F}^{\varkappa}, \quad \widetilde{G} \in \mathbf{F}^{\nu}.$$

Now we use induction on $|\nu|$. The cases $|\nu| = 0$ and $|\nu| = 1$ are simple. Then assume that (4.4) is true for $|\nu| \le k$. Let $\deg \widetilde{G} = k+1$. We represent $\widetilde{G}$ as a product $\widetilde{G} = G' \circ G''$, $\deg G', \deg G'' \le k$. By the Leibnitz identity (A.4),

$$\operatorname{aver}[\widetilde{F}, \widetilde{G}] = \operatorname{aver}[\widetilde{F}, G' \circ G''] = \operatorname{aver}\bigl([\widetilde{F}, G'] \circ G''\bigr) + \operatorname{aver}\bigl(G' \circ [\widetilde{F}, G'']\bigr).$$

Using the induction hypothesis and (4.3), we get

$$\begin{aligned} \operatorname{aver}[\widetilde{F}, \widetilde{G}] &= \{\operatorname{aver} \widetilde{F}, \operatorname{aver} G'\} \cdot \operatorname{aver} G'' + \operatorname{aver} G' \cdot \{\operatorname{aver} \widetilde{F}, \operatorname{aver} G''\} \\ &= \{\operatorname{aver} \widetilde{F}, \operatorname{aver} \widetilde{G}\}. \quad \square \end{aligned}$$

It is easy to check that the brackets $[\cdot,\cdot]$ and $(\![\cdot,\cdot]\!)$ satisfy equations (A.2)–(A.4), (A.5), and (A.6).

5. BASES IN $\pi(\mathbf{F}^{\varkappa})$

5.1. xp-Basis. We obtain a basis in the vector space $\pi(\mathbf{F}^{\varkappa})$ by using an expansion in $\mathbf{r}$.

For any $k = (k_1, \dots, k_n) \in \mathbb{R}^n$, we say that $0 \preccurlyeq k$ if $0 \le k_j$ for any $j = 1, \dots, n$. We say that $k \preccurlyeq \alpha$ if $0 \preccurlyeq \alpha - k$.

Proposition 5.1. *Any $F \in \pi(\mathbf{F}^{\alpha,\beta})$ can be uniquely represented in the form*

$$F = \sum_{k \in \mathbb{Z}_+^n,\, k \preccurlyeq \alpha,\, k \preccurlyeq \beta} f_k\, \mathbf{r}^{|k|} \circ \pi\bigl(\widehat{x}^{\,\alpha-k} \circ \widehat{p}^{\,\beta-k}\bigr), \qquad |k| = k_1 + \ldots + k_n. \tag{5.1}$$

Corollary 5.1.

$$\dim \pi(\mathbf{F}^{\alpha,\beta}) = m_1 \cdot \ldots \cdot m_n, \qquad m_j = \min\{\alpha_j, \beta_j\} + 1. \tag{5.2}$$

Proof of Proposition 5.1. The existence of expansion (5.1) follows from the possibility of pushing $\pi(\widehat{p}_j)$ to the right and $\pi(\widehat{x}_j)$ to the left in the monomials of F with the help of the equation

$$\pi(\widehat{p}_j \circ \widehat{x}_j) - \pi(\widehat{x}_j \circ \widehat{p}_j) = \mathbf{r}.$$

The uniqueness of expansion (5.1) means that equation (5.1) with $F = 0$ holds if and only if all the coefficients f_k vanish. This fact can be easily proved by induction on $|k|$. $\square$

The observables

$$\mathbf{r}^{|k|} \circ \widehat{x}^{\alpha-k} \circ \widehat{p}^{\beta-k}, \qquad k \in \mathbb{Z}_+^n, \quad k \preccurlyeq \alpha, \quad k \preccurlyeq \beta,$$

form an xp-basis in the vector space $\pi(\mathbf{F}^{\alpha,\beta})$. In the traditional language, an expansion in the xp-basis is associated with the xp-symbol of an observable (see, for example, [2]).

5.2. Primitive basis. We call a monomial z, $\operatorname{type} z = \varkappa \in \mathbb{Z}_+^{2n}$, *composite* if $\pi(z)$ can be represented as a nontrivial convex combination

$$\pi(z) = \vartheta'\pi(z') + \vartheta''\pi(z'') + \ldots + \vartheta^{(s)}\pi\big(z^{(s)}\big),$$
$$\operatorname{type} z' = \operatorname{type} z'' = \ldots = \operatorname{type} z^{(s)} = \varkappa,$$
$$\vartheta', \vartheta'', \ldots, \vartheta^{(s)} > 0, \qquad \vartheta' + \vartheta'' + \ldots + \vartheta^{(s)} = 1, \qquad s > 1.$$

Other z are called *primitive*. We define

$$\mathbf{F}^{\varkappa}_{\mathrm{prim}} = \big\{\pi(z)\colon\ \operatorname{type} z = \varkappa,\ z \text{ is primitive}\big\}.$$

Proposition 5.2. *Consider the case $n = 1$. For any $(\mu,\nu) \in \mathbb{Z}_+^2$ such that $\mu \le \nu$, the primitive monomials in $\pi(\mathbf{F}^{\varkappa})$ are*

$$\pi(\widehat{x}^{\mu} \circ \widehat{p}^{\nu}),\ \pi(\widehat{x}^{\mu-1} \circ \widehat{p}^{\nu} \circ \widehat{x}),\ \pi(\widehat{x}^{\mu-2} \circ \widehat{p}^{\nu} \circ \widehat{x}^2),\ \ldots,\ \pi(\widehat{p}^{\nu} \circ \widehat{x}^{\mu}). \tag{5.3}$$

In the case $\mu \ge \nu$, the primitive monomials are as follows:

$$\pi(\widehat{p}^{\nu} \circ \widehat{x}^{\mu}),\ \pi(\widehat{p}^{\nu-1} \circ \widehat{x}^{\mu} \circ \widehat{p}),\ \pi(\widehat{p}^{\nu-2} \circ \widehat{x}^{\mu} \circ \widehat{p}^2),\ \ldots,\ \pi(\widehat{x}^{\mu} \circ \widehat{p}^{\nu}). \tag{5.4}$$

We prove Proposition 5.2 in Appendix B.

Let $\mathbf{F}_j^{\mu_j,\nu_j}$, $\mu_j, \nu_j \in \mathbb{Z}_+$, be the space of homogeneous forms of type (μ_j, ν_j) with respect to $\widehat{x}_j$ and $\widehat{p}_j$, and let $\mathbf{F}_{j(\mathrm{prim})}^{\mu_j,\nu_j}$ be the set of the corresponding primitive monomials.

Obviously, for $n > 1$, the set $\mathbf{F}^{\mu,\nu}_{\mathrm{prim}}$ may contain only the products

$$z_1 \circ \ldots \circ z_n, \qquad z_j \in \mathbf{F}_{j(\mathrm{prim})}^{\mu_j,\nu_j}, \quad j = 1, \ldots, n. \tag{5.5}$$

Proposition 5.3. *For any $\mu, \nu \in \mathbb{Z}_+^n$, the set $\mathbf{F}^{\mu,\nu}_{\mathrm{prim}}$ contains all products* (5.5), *and monomials $\pi(z) \in \mathbf{F}^{\mu,\nu}_{\mathrm{prim}}$ form a basis in the vector space $\pi(\mathbf{F}^{\mu,\nu})$.*

5.3. Norm on $\pi(\mathbf{F}_m)$. For any $G\in\mathbf{F}_m$, $G=\sum_{\deg z=m} g_z z$, we set

$$\|G\| = \sum_{\deg z=m} |g_z|.$$

For any $F\in\pi(\mathbf{F}_m)$, we set

$$\|F\| = \inf_{G\in\mathbf{F}_m,\,\pi(G)=F} \|G\|.$$

The definition of the primitive basis implies the following proposition.

Proposition 5.4. *For any $F\in\mathbf{F}_m$, we have $\|F\|=\sum|f_z|$, where f_z are the coefficients in the expansion of F in the primitive basis.*

Note that according to Proposition A.3,

$$\|l!\,\mathbf{r}^l\| = 2^l.$$

This equation explains why $l!\,\mathbf{r}^l$ naturally appears instead of $\mathbf{r}^l$ in the expansions in powers of $\mathbf{r}$ below. The reason is that $\mathbf{r}^l$ are "too small" for large l. This fact is also responsible for the divergence of expansions in powers of the Planck constant in the traditional language of quantum mechanics.

6. MAJORANTS

For any $f=\sum f_{\mu,\nu}x^\mu p^\nu\in\mathcal{CO}^{\mathrm{form}}$, we say that $\varphi=\sum\varphi_{\mu,\nu}x^\mu p^\nu\in\mathcal{CO}^{\mathrm{form}}$ is a *majorant* for f ($f\ll\varphi$) if

$$|f_{\mu,\nu}|\le\varphi_{\mu,\nu} \qquad \text{for all} \quad (\mu,\nu)\in\mathbb{Z}_+^{2n}.$$

Let $\{\{\cdot,\cdot\}\}$ be the "majorant bracket,"

$$\{\{f(x,p),g(x,p)\}\} = \sum_{j=1}^n\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}+\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}\right).$$

The following proposition is obvious.

Proposition 6.1. *Suppose that $f\ll\varphi$, $g\ll\psi$, and $a,b\in\mathbb{C}$. Then*

$$af+bg\ll|a|\varphi+|b|\psi, \qquad fg\ll\varphi\psi,$$
$$\frac{\partial f}{\partial p_j}\ll\frac{\partial\varphi}{\partial p_j}, \qquad \frac{\partial f}{\partial x_j}\ll\frac{\partial\varphi}{\partial x_j}, \qquad j=1,\dots,n,$$
$$\{f,g\}\ll\{\{\varphi,\psi\}\}.$$

Consider an observable $\widetilde f=\sum\widetilde f_z z\in\widetilde{\mathcal{QO}}^{\mathrm{form}}$. We say that $f=\sum f_{\mu,\nu}x^\mu p^\nu\in\mathcal{CO}^{\mathrm{form}}$ is the *absolute average* of $\widetilde f$ ($f=\operatorname{Aver}\widetilde f$) if

$$f_{\mu,\nu}=\sum_{\mathrm{type}\,z=(\mu,\nu)}|\widetilde f_z| \qquad \text{for any} \quad (\mu,\nu)\in\mathbb{Z}_+^{2n}.$$

We say that $\varphi\in\mathcal{CO}^{\mathrm{form}}$ is a *majorant* for $\widetilde f\in\widetilde{\mathcal{QO}}^{\mathrm{form}}$ ($\widetilde f\ll\varphi$) if $\operatorname{Aver}\widetilde f\ll\varphi$.

If $f\in\mathcal{QO}^{\mathrm{form}}$, the relation $f\ll\varphi$ means by definition that φ is a majorant for some $\widetilde f\in\widetilde{\mathcal{QO}}^{\mathrm{form}}$, $\pi(\widetilde f)=f$.

Proposition 6.2. *Suppose that* $f, g \in \mathcal{QO}^{\text{form}}$ *and* $\varphi, \psi \in \mathcal{CO}^{\text{form}}$ *are such that* $f \ll \varphi$ *and* $g \ll \psi$. *Then, for any* $a, b \in \mathbb{C}$,

$$af + bg \ll |a|\varphi + |b|\psi, \qquad f \circ g \ll \varphi\psi, \qquad [f, g] \ll \{\{\varphi, \psi\}\}.$$

Proof. The first relation is obvious, and the other two follow from the fact that

$$\operatorname{Aver}(\widetilde{f} \circ \widetilde{g}) \ll \operatorname{Aver} \widetilde{f} \cdot \operatorname{Aver} \widetilde{g}, \qquad \operatorname{Aver}[\widetilde{f}, \widetilde{g}\,] \ll \{\{\operatorname{Aver} \widetilde{f}, \operatorname{Aver} \widetilde{g}\}\}$$

for any $\widetilde{f}, \widetilde{g} \in \widetilde{\mathcal{QO}}^{\text{form}}$. $\square$

Let $F, F_0 \in \mathbf{F}^{(\mu,\nu)}$ be homogeneous forms of the type $(\mu, \nu) \in \mathbb{Z}_+^{2n}$.

Proposition 6.3. *Suppose that* $\pi(F) = \pi(F_0)$ *and the* π*-projection of any monomial of* F_0 *is a primitive monomial. Let* $a, a_0 \in \mathbb{R}$ *be such that*

$$\operatorname{Aver} F = a x^\mu p^\nu, \qquad \operatorname{Aver} F_0 = a_0 x^\mu p^\nu.$$

Then $0 \le a_0 \le a$.

This proposition is analogous to Proposition 5.4 and means that the expansion in the primitive basis provides an optimal majorant for $F \in \mathcal{QO}^{\text{form}}$.

7. ANALYTIC OBSERVABLES

We say that $F = \sum_{\mu,\nu} f_{\mu,\nu} x^\mu p^\nu \in \mathcal{CO}^{\text{form}}$ lies in $\mathcal{CO}(c, a)$ if the coefficients $f_{\mu,\nu}$ are estimated as follows. For some $c, a > 0$,

$$|f_{\mu,\nu}| \le c a^{|\mu|+|\nu|}.$$

We set $\mathcal{CO} = \bigcup_{c,a>0} \mathcal{CO}(c, a)$. Hence, $\mathcal{CO}$ is the space of functions that are analytic at the origin of $\mathbb{C}^{2n}$. The structures of an associative algebra and a Lie algebra on $\mathcal{CO}$ are the same as those on $\mathcal{CO}^{\text{form}}$.

We say that $\widetilde{F} \in \widetilde{\mathcal{QO}}^{\text{form}}$ lies in the space $\widetilde{\mathcal{QO}}$ if

$$\operatorname{Aver} \widetilde{F} \in \mathcal{CO}.$$

We say that $F \in \mathcal{QO}^{\text{form}}$ belongs to the space $\mathcal{QO}$ if there exists $\widetilde{F} \in \widetilde{\mathcal{QO}}$ such that $\pi(\widetilde{F}) = F$.

Proposition 7.1. *The space* $\widetilde{\mathcal{QO}}$ *is invariant with respect to the operation* $\circ$. *The space* $\mathcal{QO}$ *is invariant with respect to the operations* $\circ$ *and* $[\cdot, \cdot]$.

Proof. For any $\widetilde{F}, \widetilde{G} \in \widetilde{\mathcal{QO}}$, we have $\operatorname{Aver} \widetilde{F}, \operatorname{Aver} \widetilde{G} \in \mathcal{CO}$. Therefore, by Proposition 6.2,

$$\operatorname{Aver}(\widetilde{F} \circ \widetilde{G}) \ll \operatorname{Aver} \widetilde{F} \cdot \operatorname{Aver} \widetilde{G} \in \mathcal{CO}.$$

For $F, G \in \mathcal{QO}$, let $\widetilde{F}, \widetilde{G} \in \widetilde{\mathcal{QO}}$ be such that $\pi(\widetilde{F}) = F$ and $\pi(\widetilde{G}) = G$. Then, by Proposition 6.2,

$$F \circ G \ll \operatorname{Aver}(\widetilde{F} \circ \widetilde{G}) \in \mathcal{CO}, \qquad [F, G] \ll \{\{\operatorname{Aver} \widetilde{F}, \operatorname{Aver} \widetilde{G}\}\} \in \mathcal{CO}. \quad \square$$

Obviously, $\widetilde{\operatorname{aver}}(\widetilde{\mathcal{QO}}) = \operatorname{aver}(\mathcal{QO}) = \mathcal{CO}$. By (4.3) and (4.4), aver is a homomorphism of the associative and Lie algebras $\mathcal{QO}$ and $\mathcal{CO}$.

8. ANALYTICITY AND THE xp-EXPANSION

In this section, we present the analyticity condition in terms of expansions in xp-bases.

Proposition 8.1. *Suppose that $F\in\mathcal{QO}^{\mathrm{form}}$ has the form*

$$F=\sum_{\alpha,\beta,\gamma} f_{\alpha,\beta,\gamma}\alpha!\,\mathbf{r}^\alpha\circ\pi\bigl(\widehat{x}^\beta\circ\widehat{p}^\gamma\bigr),\qquad \alpha\in\mathbb{Z}_+,\quad \beta,\gamma\in\mathbb{Z}_+^n, \tag{8.1}$$

where the coefficients $f_{\alpha,\beta,\gamma}$ satisfy the following exponential estimate:

$$|f_{\alpha,\beta,\gamma}|<Ca^{\alpha+|\beta|+|\gamma|},\qquad C,a>0. \tag{8.2}$$

Then $F\in\mathcal{QO}$.

Proof. By Corollary A.2,

$$F\ll\sum_{\alpha,\beta,\gamma}Ca^{\alpha+|\beta|+|\gamma|}(2x_1p_1)^\alpha x^\beta p^\gamma=\frac{C}{1-2ax_1p_1}\prod_{j=1}^n\frac{1}{(1-ax_j)(1-ap_j)}.\qquad\square$$

Proposition 8.2. *Any $F\in\mathcal{QO}$ admits expansion* (8.1) *in which the coefficients $f_{\alpha,\beta,\gamma}$ are estimated by* (8.2).

Proof. Consider the case $n=1$. For any monomial z, type $z=(\beta,\gamma)$, $\beta<\gamma$,

$$z=\sum_{j=0}^{\alpha}K_j\,j!\,\mathbf{r}^j\circ\widehat{x}^{\beta-j}\circ\widehat{p}^{\gamma-j},\qquad |K_j|\le C_\beta^jC_\gamma^j,$$

where C_u^v are binomial coefficients.

Since F is analytic, its homogeneous forms of type β,γ admit the estimate $F_{\beta,\gamma}\ll C(a')^{\beta+\gamma}x^\beta p^\gamma$, where we can assume that $2a'>1$. Hence,

$$\begin{aligned}F&\ll\sum_{\alpha,\beta',\gamma'}C(a')^{\beta'+\gamma'}C_{\beta'}^\alpha C_{\gamma'}^\alpha\alpha!\,\mathbf{r}^\alpha x^{\beta'-\alpha}p^{\gamma'-\alpha}=\sum_{\alpha,\beta,\gamma}C(a')^{\beta+\gamma+2\alpha}C_{\beta+\alpha}^\alpha C_{\gamma+\alpha}^\alpha\alpha!\,\mathbf{r}^\alpha x^\beta p^\gamma\\&\ll\sum_{\alpha,\beta,\gamma}C(2a')^{\beta+\gamma+2\alpha}\alpha!\,\mathbf{r}^\alpha x^\beta p^\gamma\ll\sum_{\alpha,\beta,\gamma}Ca^{\beta+\gamma+\alpha}\alpha!\,\mathbf{r}^\alpha x^\beta p^\gamma,\end{aligned}$$

where $a=(2a')^2$. In the case $n>1$, the arguments are analogous. $\square$

9. HERMITIAN OBSERVABLES

For any monomial $z=z_m\circ\ldots\circ z_1\in\mathbf{F}_m$ and $f\in\mathbb{C}$, we set $\widetilde{\mathcal{I}}(fz)=\overline{f}\,z_1\circ\ldots\circ z_m$, where $\overline{f}$ is the complex conjugate of f. By linearity, $\widetilde{\mathcal{I}}$ is continued to the involution

$$\widetilde{\mathcal{I}}\colon\widetilde{\mathcal{QO}}^{\mathrm{form}}\to\widetilde{\mathcal{QO}}^{\mathrm{form}},\qquad \widetilde{\mathcal{I}}\circ\widetilde{\mathcal{I}}=\mathrm{id}.$$

We say that an observable $F\in\widetilde{\mathcal{QO}}^{\mathrm{form}}$ is Hermitian at zero if $F=\widetilde{\mathcal{I}}F$ and skew-Hermitian at zero if $F=-\widetilde{\mathcal{I}}F$.

Proposition 9.1. $\widetilde{\mathcal{I}}(J)=J$.

Proof. Indeed, this simple statement follows from the fact that all generators (2.3)–(2.8) of J are either Hermitian or skew-Hermitian at zero. $\square$

For any $\widetilde{F}\in\widetilde{\mathcal{QO}}^{\mathrm{form}}$ and $F=\pi(\widetilde{F})$, we set $\mathcal{I}(F)=\pi(\widetilde{\mathcal{I}}\widetilde{F})$. Due to Proposition 9.1, the observable $\mathcal{I}(F)$ is well-defined.

We obtain the involution $\mathcal{I}\colon \mathcal{QO}^{\mathrm{form}} \to \mathcal{QO}^{\mathrm{form}}$ such that the diagram

$$\begin{tikzcd} \widetilde{\mathcal{QO}}^{\mathrm{form}} \arrow[r, "\widetilde{\mathcal{I}}"] \arrow[d, "\pi"'] & \widetilde{\mathcal{QO}}^{\mathrm{form}} \arrow[d, "\pi"] \\ \mathcal{QO}^{\mathrm{form}} \arrow[r, "\mathcal{I}"] & \mathcal{QO}^{\mathrm{form}} \end{tikzcd}$$

is commutative.

Remark 9.1. Let $\mathrm{conj}\colon \mathcal{CO}^{\mathrm{form}} \to \mathcal{CO}^{\mathrm{form}}$ be the complex conjugation. Then the diagrams

$$\begin{tikzcd} \widetilde{\mathcal{QO}}^{\mathrm{form}} \arrow[r, "\widetilde{\mathcal{I}}"] \arrow[d, "\widetilde{\mathrm{aver}}"'] & \widetilde{\mathcal{QO}}^{\mathrm{form}} \arrow[d, "\widetilde{\mathrm{aver}}"] \\ \mathcal{CO}^{\mathrm{form}} \arrow[r, "\mathrm{conj}"] & \mathcal{CO}^{\mathrm{form}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{QO}^{\mathrm{form}} \arrow[r, "\mathcal{I}"] \arrow[d, "\mathrm{aver}"'] & \mathcal{QO}^{\mathrm{form}} \arrow[d, "\mathrm{aver}"] \\ \mathcal{CO}^{\mathrm{form}} \arrow[r, "\mathrm{conj}"] & \mathcal{CO}^{\mathrm{form}} \end{tikzcd}$$

are obviously commutative.

We say that an observable $F \in \mathcal{QO}^{\mathrm{form}}$ is Hermitian at zero if $F = \mathcal{I}F$.

Proposition 9.2. *The following statements are equivalent*:

(1) $F \in \mathcal{QO}^{\mathrm{form}}$ *is Hermitian at zero.*

(2) *The expansion of* F *in the primitive basis,* $F = \sum_{\kappa\in\mathbb{Z}_+^{2n}} \sum_{z\in\mathbf{F}^{\kappa}_{\mathrm{prim}}} f_z z$, *is symmetric*: $f_{\mathcal{I}z} = \overline{f}_z$ *for any* z.

The space of observables that are Hermitian at zero forms a Lie subalgebra in $\mathcal{QO}$. We denote it by $\mathcal{QO}^{\mathrm{H}}$. Below (Section 14), we show that if an observable is Hermitian at zero, it is Hermitian everywhere in its domain of definition.

The product of two Hermitian observables is not Hermitian in general. However, for any $F, G \in \mathcal{QO}^{\mathrm{H}}$, we have $(\![F, G]\!) \in \mathcal{QO}^{\mathrm{H}}$.

10. RIGHT INVERSE FOR aver

The map aver has no right inverse homomorphism

$$\mathrm{Op}\colon \mathcal{CO}^{\mathrm{form}} \to \mathcal{QO}^{\mathrm{form}}, \qquad \mathrm{aver} \circ \mathrm{Op} = \mathrm{id}_{\mathcal{CO}^{\mathrm{form}}}.$$

We do not prove this simple fact here but just mention one similar result, the Groenvald–Van Hove theorem (see, for example, [1, 6]).

Theorem. *There is no linear map* $\mathrm{Op}\colon \mathcal{CO}^{\mathrm{form}} \to \mathcal{QO}^{\mathrm{form}}$ *satisfying the following properties*: *for any* $F, G \in \mathcal{CO}^{\mathrm{form}}$,

(1) $[\mathrm{Op}\, F, \mathrm{Op}\, G] = \mathrm{Op}\{F, G\}$,

(2) $\mathrm{Op}\, x_j = \widehat{x}_j$ *and* $\mathrm{Op}\, p_j = \widehat{p}_j$,

(3) $\mathrm{Op}\, \overline{F} = \mathcal{I}(\mathrm{Op}\, F)$.

In a more algebraic language, this theorem means that there is no homomorphism of Lie algebras $\mathrm{Op}\colon \mathcal{CO} \to \mathcal{QO}^{\mathrm{H}}$ satisfying conditions (2) and (3).

It is well known that such a homomorphism exists for the subalgebras formed by observables of degree at most 2. Moreover, this homomorphism is unique. It is defined by the equations

$$\mathrm{Op}(p_j p_k) = \widehat{p}_j \circ \widehat{p}_k, \qquad \mathrm{Op}(x_j x_k) = \widehat{x}_j \circ \widehat{x}_k, \qquad \mathrm{Op}(p_j x_k) = \frac{1}{2}(\widehat{p}_j \circ \widehat{x}_k + \widehat{x}_j \circ \widehat{p}_k)$$

for any $1 \le j, k \le n$.

Various maps $\mu\colon \mathcal{CO}^{\mathrm{form}} \to \mathcal{QO}^{\mathrm{form}}$ that are right inverses for aver have been considered in the literature. None of these maps is a homomorphism. It is clear that to specify μ, it is sufficient to define it on the monomials $x^\alpha p^\beta$, $\alpha,\beta \in \mathbb{Z}_+^n$.

If $\mu(F) = \widehat{F}$, then F is called a symbol of the quantum observable $\widehat{F}$. Various maps μ define xp-symbols, px-symbols, Weyl, Wick, and anti-Wick symbols, etc.

For example,

- xp-symbols correspond to a map μ such that $\mu(x^\alpha p^\beta) = \widehat{x}^\alpha \circ \widehat{p}^\beta$;
- px-symbols correspond to $\mu(x^\alpha p^\beta) = \widehat{p}^\beta \circ \widehat{x}^\alpha$;
- for the Weyl symbols, we have

$$\mu(x^\alpha p^\beta) = \prod_{j=1}^{n} \frac{1}{C_{\alpha_j+\beta_j}^{\beta_j}} \sum_{\mathrm{type}\, z_j=(\alpha_j,\beta_j)} \pi(z_j),$$

  where z_j is a monomial in $\widetilde{\mathcal{QO}}_j^{\mathrm{form}}$, which is the space of quantum observables with $n=1$ and the generators $\widehat{x}_j$ and $\widehat{p}_j$.

11. IMPLICIT FUNCTION THEOREM

We say that observables $F_1(\widehat{x},\widehat{p}),\dots,F_m(\widehat{x},\widehat{p}) \in \widetilde{\mathcal{QO}}$ are independent at zero if the functions $\widetilde{\mathrm{aver}}\, F_1(x,p),\dots,\widetilde{\mathrm{aver}}\, F_m(x,p) \in \mathcal{CO}$ are independent at zero. An analogous definition is used for $F_1,\dots,F_m \in \mathcal{QO}$.

Theorem 1 (implicit function theorem in $\widetilde{\mathcal{QO}}$)**.** *Let*

$$X_1(\widehat{x},\widehat{p}),\dots,X_n(\widehat{x},\widehat{p}),\ P_1(\widehat{x},\widehat{p}),\dots,P_n(\widehat{x},\widehat{p}) \in \widetilde{\mathcal{QO}} \tag{11.1}$$

be observables that are independent at zero and are such that $X(0,0) = P(0,0) = 0$. Then there exist $u_1,\dots,u_n,v_1,\dots,v_n \in \widetilde{\mathcal{QO}}$ such that $\widehat{x}_j = u_j(X,P)$ and $\widehat{p}_j = v_j(X,P)$.

Corollary 11.1 (implicit function theorem in $\mathcal{QO}$)**.** *Let*

$$X_1(\widehat{x},\widehat{p}),\dots,X_n(\widehat{x},\widehat{p}),\ P_1(\widehat{x},\widehat{p}),\dots,P_n(\widehat{x},\widehat{p}) \in \mathcal{QO}$$

be observables that are independent at zero and are such that $X(0,0) = P(0,0) = 0$ and

$$[X_j,X_k] = [P_j,P_k] = 0, \qquad [P_j,X_k] = \delta_{jk} \qquad \textit{for any} \quad 1 \le j,k \le n. \tag{11.2}$$

Then there exist $u_1,\dots,u_n,v_1,\dots,v_n \in \mathcal{QO}$ such that

$$\pi(\widehat{x}_j) = \widehat{u}_j(X,P), \qquad \pi(\widehat{p}_j) = \widehat{v}_j(X,P).$$

Indeed, equations (11.2) imply that for any $1 \le j,k \le n$,

$$X_j \circ X_k - X_k \circ X_j = P_j \circ P_k - P_k \circ P_j = 0, \qquad P_j \circ X_k - X_k \circ P_j = \delta_{jk}\mathbf{r}.$$

Therefore, $F(X,P) = 0$ for any observable $F \in J$. Hence, we can set $\widehat{u} = \pi(u)$ and $\widehat{v} = \pi(v)$, where $u,v \in \widetilde{\mathcal{QO}}$ are constructed in Theorem 1.

Proof of Theorem 1. Let

$$\begin{pmatrix} X \\ P \end{pmatrix} = L \begin{pmatrix} \widehat{x} \\ \widehat{p} \end{pmatrix} + F(\widehat{x},\widehat{p}), \qquad F(\widehat{x},\widehat{p}) = \sum_{k=2}^{\infty} F_k(\widehat{x},\widehat{p}), \tag{11.3}$$

where $L\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear operator and F_k are vector-valued homogeneous forms of degree k. We set

$$\begin{pmatrix} \widehat{X} \\ \widehat{P} \end{pmatrix} = L^{-1} \begin{pmatrix} X \\ P \end{pmatrix}, \qquad \widehat{F}(\widehat{x}, \widehat{p}\,) = L^{-1} F(\widehat{x}, \widehat{p}\,), \qquad \widehat{F}_k(\widehat{x}, \widehat{p}\,) = L^{-1} F_k(\widehat{x}, \widehat{p}\,).$$

Then equation (11.3) takes the form

$$\begin{pmatrix} \widehat{X} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{x} \\ \widehat{p} \end{pmatrix} + \widehat{F}(\widehat{x}, \widehat{p}\,). \tag{11.4}$$

We can assume that for some $a > 0$,

$$F_k(\widehat{x}, \widehat{p}\,) \ll q a^k \xi^k, \qquad k \ge 2, \quad \xi = x_1 + \ldots + x_n + p_1 + \ldots + p_n, \tag{11.5}$$

where the majorant inequalities (11.5) should be understood in the sense that $qa^k\xi^k$ is a majorant for each component of the vector F_k. Then

$$\widehat{F}_k(\widehat{x}, \widehat{p}\,) \ll q_* a^k \xi^k, \qquad k \ge 2, \quad q_* = \mathbf{L} q, \tag{11.6}$$

where $\mathbf{L} = \|L^{-1}\|_1$ is the l_1-norm of L^{-1}.

We want to solve equations (11.4) with respect to $\widehat{x}$ and $\widehat{p}$; i.e., we want to construct observables $\widehat{x} = u(\widehat{X}, \widehat{P})$ and $\widehat{p} = v(\widehat{X}, \widehat{P})$ that satisfy (11.4).

Equation (11.4) can be written as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{M} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{11.7}$$

where

$$\mathcal{M}\colon \begin{pmatrix} u(\widehat{X}, \widehat{P}) \\ v(\widehat{X}, \widehat{P}) \end{pmatrix} \mapsto \begin{pmatrix} \widehat{X} \\ \widehat{P} \end{pmatrix} - \widehat{F}(u, v).$$

Let $\vartheta(\widehat{X}, \widehat{P})$ denote the $2n$-dimensional vector $\left(\begin{smallmatrix} u(\widehat{X}, \widehat{P}) \\ v(\widehat{X}, \widehat{P}) \end{smallmatrix}\right)$. Consider the sequence $\vartheta^0, \vartheta^1, \ldots,$ where

$$\vartheta^0 = \begin{pmatrix} \widehat{X} \\ \widehat{P} \end{pmatrix}, \qquad \vartheta^{s+1} = \mathcal{M}(\vartheta^s).$$

The following proposition is obvious.

Proposition 11.1. *The observables ϑ^{s+1} and ϑ^s coincide in all orders less than $s+1$.*

Corollary 11.2. *The functions $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \vartheta^s(\widehat{X}, \widehat{P})$ satisfy* (11.7) *in all orders less than $s+1$.*

Corollary 11.3. *The sequence $\vartheta^0, \vartheta^1, \ldots$ converges order by order to $\vartheta \in \widetilde{\mathcal{QO}}^{\mathrm{form}}$, a formal solution of* (11.7).

Proof of Proposition 11.1. The proof is performed by induction. □

Proposition 11.2. *The limit ϑ lies in $\widetilde{\mathcal{QO}}$.*

Proof. We set $\vartheta(\widehat{X}, \widehat{P}) = \sum_{k=1}^{\infty} \vartheta_k(\widehat{X}, \widehat{P})$, where ϑ_k is a homogeneous form of degree k. Below we replace the arguments $\widehat{X}$ and $\widehat{P}$ of the functions ϑ and ϑ_k by $\widehat{x}$ and $\widehat{p}$.

Given an analytic function $g(\xi)$ with ξ satisfying (11.5), we say that $\vartheta(\widehat{x}, \widehat{p}\,) \ll g(\xi)$ if $g(\xi)$ is a majorant for each component of the vector ϑ. Assuming that $\vartheta_k(\widehat{x}, \widehat{p}\,) \ll a^k q_k \xi^k$, we will estimate the coefficients q_k.

We define $f(\xi)=\sum_{k=0}^{\infty}a^{k+2}q_{k+2}\xi^k$. Then

$$\vartheta(\widehat{x},\widehat{p}\,)\ll\xi+\xi^2f(\xi).$$

By (11.7),

$$\vartheta_1(\widehat{x},\widehat{p}\,)=\begin{pmatrix}\widehat{x}\\ \widehat{p}\end{pmatrix},\qquad \vartheta_s(\widehat{x},\widehat{p}\,)=-\sum_{k=2}^{s}\bigl(\widehat{F}_k(\vartheta(\widehat{x},\widehat{p}\,))\bigr)_s,\qquad s\ge 2, \tag{11.8}$$

where $(\cdot)_s$ denotes a homogeneous form of degree s in $\widehat{x}$ and $\widehat{p}$. It is important to note that the right-hand side of the second equation in (11.8) depends only on the forms $\vartheta_1,\dots,\vartheta_{s-1}$.

Equations (11.8) and (11.6) imply that

$$\vartheta_s(\widehat{x},\widehat{p}\,)\ll\left(\sum_{k=2}^{s}q_*a^k(2n)^k\bigl(\xi+\xi^2f(\xi)\bigr)^k\right)_s,\qquad s\ge 2,$$

where the subscript s denotes the coefficient at ξ^s. This means that

$$\vartheta(\widehat{x},\widehat{p}\,)\ll\xi+\sum_{k=2}^{\infty}q_*a^k(2n)^k\bigl(\xi+\xi^2f(\xi)\bigr)^k.$$

Hence, we can take an f that satisfies the equation

$$\xi^2f(\xi)=\sum_{k=2}^{\infty}q_*a^k(2n)^k\bigl(\xi+\xi^2f(\xi)\bigr)^k,$$

which is equivalent to

$$\bigl(1-2an(\xi+\xi^2f)\bigr)f=4n^2q_*a^2(1+\xi f)^2.$$

This equation, quadratic in f, has an analytic solution

$$f(\xi)=\frac{1-A\xi-\sqrt{(1-A\xi)^2-B^2\xi^2}}{2C\xi^2},$$

$$A=2na(1+4naq_*),\qquad B^2=32n^3a^3q_*(1+2naq_*),\qquad C=2na(1+2naq_*).$$

Obviously, $0<B<A$ and $0<C<A$. By Proposition C.2, we have

$$f(\xi)\ll\frac{B(A+B)}{8C(1-(A+B)\xi)}.\qquad\square$$

Theorem 1 is proved. $\square$

Remark 11.1. If observables (11.1) are Hermitian, then the observables u and v constructed in Theorem 1 are also Hermitian.

12. AUTOMORPHISMS OF $\mathcal{QO}$

Let $\widehat{X}_1(\widehat{x},\widehat{p}\,),\dots,\widehat{X}_n(\widehat{x},\widehat{p}\,),\widehat{P}_1(\widehat{x},\widehat{p}\,),\dots,\widehat{P}_n(\widehat{x},\widehat{p}\,)\in\mathcal{QO}$ be observables that satisfy (11.2). We call such set of observables canonical.

We define a map

$$\mathcal{A}\colon\ \mathcal{QO}\to\mathcal{QO},\qquad \mathcal{QO}\ni F(\widehat{x},\widehat{p}\,)\mapsto\mathcal{A}(F)(\widehat{x},\widehat{p}\,)=F\bigl(\widehat{X}(\widehat{x},\widehat{p}\,),\widehat{P}(\widehat{x},\widehat{p}\,)\bigr).$$

Proposition 12.1. *The map $\mathcal{A}$ is an automorphism of* $(\mathcal{QO}, \circ, [\,\cdot\,,\cdot\,])$; *i.e.,*

(a) *$\mathcal{A}$ is linear,*

(b) $\mathcal{A}(F \circ G) = \mathcal{A}(F) \circ \mathcal{A}(G)$,

(c) $\mathcal{A}(\mathbf{r}) = \mathbf{r}$,

(d) $\mathcal{A}([F, G]) = [\mathcal{A}(F), \mathcal{A}(G)]$.

Proof. Assertions (a), (b), and (c) are obvious, and (c) implies (d). $\square$

We call the automorphism $\mathcal{A}$ canonical. The implicit function theorem implies that for any canonical automorphism $\mathcal{A}$, there exists an inverse automorphism $\mathcal{A}^{-1}$. Canonical automorphisms form a group $\mathrm{Aut}(\mathcal{QO})$.

Consider a map $a\colon \mathcal{CO} \to \mathcal{CO}$ generated by a symplectic change of variables

$$(x, p) \mapsto (X(x, p), P(x, p)), \qquad X = \operatorname{aver} \widehat{X}, \quad P = \operatorname{aver} \widehat{P}.$$

More precisely,

$$\mathcal{CO} \ni f(x, p) \mapsto a(f)(x, p) = f(X(x, p), P(x, p)).$$

Then the diagram

$$\begin{tikzcd} \mathcal{QO} \arrow[r, "\mathcal{A}"] \arrow[d, "\mathrm{aver}"'] & \mathcal{QO} \arrow[d, "\mathrm{aver}"] \\ \mathcal{CO} \arrow[r, "a"] & \mathcal{CO} \end{tikzcd}$$

is commutative.

Obviously, a is an automorphism of $\bigl(\mathcal{CO}, \cdot, \{\,\cdot\,,\cdot\,\}\bigr)$. We say that a is compatible with $\mathcal{A}$. For an automorphism $a\colon \mathcal{CO} \to \mathcal{CO}$, the corresponding $\mathcal{A}\colon \mathcal{QO} \to \mathcal{QO}$ that is compatible with a is not unique.

An automorphism $\mathcal{A}$ is called Hermitian if it preserves $\mathcal{QO}^{\mathrm{H}}$, i.e., $\mathcal{A}(\mathcal{QO}^{\mathrm{H}}) \subset \mathcal{QO}^{\mathrm{H}}$. Any Hermitian automorphism can be regarded as an automorphism of the Lie–Jordan algebra $(\mathcal{QO}^{\mathrm{H}}, [\![\,\cdot\,,\cdot\,]\!], [\,\cdot\,,\cdot\,])$. Below $\mathrm{Aut}(\mathcal{QO}^{\mathrm{H}})$ denotes the group of such automorphisms. Obviously, $\mathcal{A} \in \mathrm{Aut}(\mathcal{QO}^{\mathrm{H}})$ if and only if the observables

$$\mathcal{A}(\widehat{x}_1), \dots, \mathcal{A}(\widehat{x}_n), \mathcal{A}(\widehat{p}_1), \dots, \mathcal{A}(\widehat{p}_n)$$

are Hermitian.

13. THE OPERATORS $e^{[H,\cdot]}$

For any $H, F \in \mathcal{QO}$, we set

$$e^{[H,\cdot]}F = F + \frac{1}{1!}[H, F] + \frac{1}{2!}[H, [H, F]] + \dots. \tag{13.1}$$

Obviously, $F(\widehat{x}, \widehat{p}, t) := e^{t[H,\cdot]}F_0(\widehat{x}, \widehat{p})$ is a solution of the Heisenberg equation

$$\dot{F} = [H, F], \qquad F\big|_{t=0} = F_0, \qquad F_0, H \in \mathcal{QO}.$$

If H is quadratic (i.e., $H \in \pi(\mathbf{F}_2)$), then the corresponding change $(\widehat{x}, \widehat{p}) \mapsto e^{[H,\cdot]}(\widehat{x}, \widehat{p})$ is linear. In particular, the corresponding automorphism $\mathcal{A}$ preserves the spaces $\mathbf{F}_k$, $k \in \mathbb{Z}_+$. The group of such automorphisms is isomorphic to the group of linear symplectic self-maps of $\mathbb{R}^{2n}$.

Proposition 13.1. *For any $H, F \in \mathcal{QO}$ such that*

$$F \ll \frac{A}{a-\xi}, \qquad H \ll \frac{B\xi^s}{b-\xi}, \qquad 0 < a < b, \quad A, B > 0, \quad \xi = x_1 + \ldots + x_n + p_1 + \ldots + p_n,$$

the estimate $e^{t[H,\cdot]F} \ll \Phi(\xi, t)$ holds, where

$$\Phi = \begin{cases} \dfrac{A}{a - \mu t - \xi} & \text{if } s = 1, \\ \dfrac{A}{a - e^{\mu t}\xi} & \text{if } s = 2, \\ \dfrac{A}{a - \xi - (s-2)a\mu t\xi^{s-2}} & \text{if } s \geq 3, \end{cases} \qquad \mu = \frac{2nsbB}{(b-a)^2}.$$

Remark 13.1. The case $s = 0$ can always be reduced to the case $s = 1$ if we replace $H(\widehat{x}, \widehat{p})$ by $H(\widehat{x}, \widehat{p}) - H(0, 0)$.

Corollary 13.1. *For any $H \ll \frac{B\xi^s}{b-\xi}$ with $s \geq 2$, the operator $e^{t[H,\cdot]}\colon \mathcal{QO} \to \mathcal{QO}$ is well-defined for any $t \in \mathbb{C}$.*

Proof of Proposition 13.1. We define

$$W_0(\widehat{x}, \widehat{p}) = F(\widehat{x}, \widehat{p}), \qquad W_{m+1} = [H, W_m] \quad \text{for any} \quad m \in \mathbb{Z}_+.$$

The basic fact used in the proof is that

$$W_m \ll \left(\mu\xi^{s-1}\frac{\partial}{\partial\xi}\right)^m \frac{A}{a-\xi}. \tag{13.2}$$

This estimate is easily checked by induction. For $m = 0$, it is obvious. Suppose that it is true for $m = k$. Then

$$W_{k+1} \ll \left\{\!\!\left\{\frac{B\xi^s}{b-\xi}, \left(\mu\xi^{s-1}\frac{d}{d\xi}\right)^k \frac{A}{a-\xi}\right\}\!\!\right\}.$$

Since $\frac{d}{d\xi}\frac{\xi^s}{b-\xi} \ll \frac{sb\xi^{s-1}}{(b-\xi)^2}$ for $s \geq 1$, we have

$$W_{k+1} \ll \frac{2nsbB\xi^{s-1}}{(b-\xi)^2}\frac{d}{d\xi}\left(\left(\mu\xi^{s-1}\frac{d}{d\xi}\right)^k \frac{A}{a-\xi}\right).$$

Now we note that

$$\frac{d}{d\xi}\left(\left(\mu\xi^{s-1}\frac{d}{d\xi}\right)^k \frac{A}{a-\xi}\right) = \frac{1}{a-\xi}P\left(\frac{1}{a-\xi}, \xi\right)$$

for some polynomial $P(u, v)$ and

$$\frac{1}{(b-\xi)(a-\xi)} \ll \frac{1}{(b-a)(a-\xi)}$$

for any $0 < a < b$. Hence,

$$W_{k+1} \ll \frac{2nsbB\xi^{s-1}}{(b-a)^2}\frac{d}{d\xi}\left(\left(\mu\xi^{s-1}\frac{d}{d\xi}\right)^k \frac{A}{a-\xi}\right) = \left(\mu\xi^{s-1}\frac{d}{d\xi}\right)^{k+1}\frac{A}{a-\xi}.$$

Estimate (13.2) implies that

$$e^{t[H,\cdot]}F \ll \frac{A}{a - g_s^t(\xi)},$$

where g_s^t is the phase flow of the ordinary differential equation $\dot{\xi} = \mu\xi^{s-1}$. It remains to apply the equations

$$g_s^t(\xi) = \begin{cases} \xi + \mu t & \text{if } s = 1, \\ e^{\mu t}\xi & \text{if } s = 2, \\ \dfrac{\xi}{(1-(s-2)\mu t\xi^{s-2})^{1/(s-2)}} \ll \dfrac{\xi}{1-(s-2)\mu t\xi^{s-2}} & \text{if } s \geq 3. \end{cases}$$

The proposition is proved. □

Proposition 13.2. *For any $H \in \mathcal{QO}$, $H = O_2(\widehat{x}, \widehat{p})$, the map $e^{[H,\cdot]}\colon \mathcal{QO} \to \mathcal{QO}$ is a canonical automorphism.*

Proof. Take any $F, G \in \mathcal{QO}$. By Corollary 13.1, the observables $e^{[H,\cdot]}F$ and $e^{[H,\cdot]}G$ lie in $\mathcal{QO}$. Hence,

$$\begin{aligned} e^{[H,\cdot]}F \circ e^{[H,\cdot]}G &= F \circ G + \frac{1}{1!}\big([H,F] \circ G + F \circ [H,G]\big) + \dots \\ &= F \circ G + \frac{1}{1!}[H, F \circ G] + \dots = e^{[H,\cdot]}F \circ G, \\ \big[e^{[H,\cdot]}F, e^{[H,\cdot]}G\big] &= [F,G] + \frac{1}{1!}\big([[H,F],G] + [F,[H,G]]\big) + \dots \\ &= [F,G] + \frac{1}{1!}[H,[F,G]] + \dots = e^{[H,\cdot]}[F,G]. \quad \square \end{aligned}$$

Remark 13.2. For any $H \in \mathcal{QO}^{\mathrm{H}}$, $H = O_2(\widehat{x}, \widehat{p})$, the corresponding automorphism $e^{[H,\cdot]}$ is Hermitian.

14. ANALYTIC CONTINUATION

The simplest canonical automorphism of $\mathcal{QO}$ is generated by the observables

$$X = \widehat{x} + x^0, \qquad P = \widehat{p} + p^0, \qquad (x^0, p^0) = \mathrm{const} \in \mathbb{C}^{2n}.$$

We say that an observable $F(\widehat{x}, \widehat{p}) \in \mathcal{QO}$ can be continued analytically to the point $(x^0, p^0) \in \mathbb{C}^{2n}$ if $F(\widehat{x} + x^0, \widehat{p} + p^0) \in \mathcal{QO}$. Here we mean that if we substitute $\widehat{x} + x^0$ and $\widehat{p} + p^0$ for $\widehat{x}$ and $\widehat{p}$ in the expansion of F in homogeneous forms (more precisely, in the expansion of some $\widetilde{F} \in \widetilde{\mathcal{QO}}$, $\pi(\widetilde{F}) = F$) and re-expand the result in $\widehat{x}$ and $\widehat{p}$, we obtain an element of $\mathcal{QO}$.

Analytic continuation can be performed several times. Hence, like holomorphic functions, the observables from $\mathcal{QO}$ may be analytically continued (if this is possible) along curves on $\mathbb{C}^{2n}$.

We call an observable $F \in \mathcal{QO}$ analytic on a set $D \subset \mathbb{C}^{2n}$ if F can be analytically continued to D. In this case, we write $F \in \mathcal{QO}(D)$.

Any observable $F \in \mathcal{QO}(D)$, $D \subset \mathbb{C}^{2n}$, defines a map

$$\mathcal{F}\colon D \to \mathcal{QO}, \qquad D \ni (x,p) \mapsto \mathcal{F}(x,p;\widehat{x},\widehat{p}) = F(x + \widehat{x}, p + \widehat{p}).$$

We have the following obvious equation: for any point $(x^0, p^0) \in D$,

$$\mathcal{F}(x^0 + x, p^0 + p; \widehat{x}, \widehat{p}) = \mathcal{F}(x^0, p^0; \widehat{x} + x, \widehat{p} + p), \tag{14.1}$$

where $(x,p) \in \mathbb{C}^{2n}$ is an arbitrary small vector.

Hence, we obtain an equivalent definition of $\mathcal{QO}(D)$. We say that F is analytic in D if it is defined by the equation

$$F(x+\widehat{x}, p+\widehat{p}) = \mathcal{F}(x,p;\widehat{x},\widehat{p}), \qquad (x,p)\in D,$$

for some $\mathcal{F}\colon D\to\mathcal{QO}$ satisfying (14.1).

If $D\subset\mathbb{R}^{2n}$ and $F\big(\overline{\widehat{x}},\overline{\widehat{p}}\big)=\overline{F}(\widehat{x},\widehat{p})$, then we say that F is real-analytic on D.

An observable F is said to be Hermitian at the point (x^0,p^0) if $F(\widehat{x}+x^0,\widehat{p}+p^0)$ is Hermitian at zero.

Proposition 14.1. *Suppose that $F\in\mathcal{QO}(D)$ is Hermitian at zero. Then it is Hermitian at any point $(x^0,p^0)\in D$.*

Proof. It is sufficient to verify that this statement holds for $F=\pi(fz+\overline{f}\mathcal{I}z)$ with any monomial z. In this case, the proposition is obvious. $\square$

Proposition 14.2. *Suppose that $F\in\mathcal{CO}$ is a majorant for $\widehat{F}\in\mathcal{QO}$ $(\widehat{F}\ll F)$ and F is analytic in the domain*

$$D(x^0,p^0)=\big\{(x,p)\in\mathbb{C}^{2n}\colon\ |x_j|\le x_j^0,\ |p_j|\le p_j^0,\ j=1,\dots,n\big\}.$$

Then $\widehat{F}\in\mathcal{QO}(D(x^0,p^0))$.

Proof. Indeed, let $\widehat{G}\in\widetilde{\mathcal{QO}}$ be such that

$$\pi(\widehat{G})=\widehat{F}, \qquad \widehat{G}\ll F.$$

Since the function

$$f(x,p)=F\big(x_1+|\widetilde{x}_1|,\dots,x_n+|\widetilde{x}_n|,p_1+|\widetilde{p}_1|,\dots,p_n+|\widetilde{p}_n|\big)$$

is analytic at zero for any $(\widetilde{x},\widetilde{p})$, the observable $\widetilde{G}(\widehat{x},\widehat{p})=\widehat{G}(\widehat{x}+\widetilde{x},\widehat{p}+\widetilde{p})$ satisfies

$$\widetilde{G}(\widehat{x},\widehat{p})\ll f(x,p), \qquad \pi\big(\widetilde{G}(\widehat{x},\widehat{p})\big)=\widehat{F}(\widehat{x}+\widetilde{x},\widehat{p}+\widetilde{p}). \quad \square$$

If $F\in\mathcal{QO}(D)$, we have $F\in\mathcal{QO}(D_1)$ for any $D_1\subset D$. Hence, we have a natural restriction map $\mathcal{QO}(D)\to\mathcal{QO}(D_1)$.

15. FUNCTION OF AN OBSERVABLE

Let $F_1,\dots,F_k\in\mathcal{QO}(D)$ be commuting observables:

$$[F_j,F_s]=0, \qquad 1\le j,s\le k.$$

Consider the vector-valued function $\operatorname{aver}F$, where $F=(F_1,\dots,F_k)$. Denote $D_0=F(D)\subset\mathbb{C}^k$.

Proposition 15.1. *Let $f\colon D_0\to\mathbb{C}$ be an analytic function. Then $f(F)\in\mathcal{QO}(D)$ is well-defined and* $\operatorname{aver}f(F)=f(\operatorname{aver}F)$.

Moreover, if $F_1,\dots,F_k\in\mathcal{QO}^{\mathrm{H}}$ and f is real-analytic, we have $f(F)\in\mathcal{QO}^{\mathrm{H}}$.

Proof. For any $(x^0,p^0)\in D$, we have $F_s(\widehat{x}+x^0,\widehat{p}+p^0)\in\mathcal{QO}$, $s=1,\dots,k$. Set $z^0=\operatorname{aver}F(x^0,p^0)\in D_0$. Then

$$F\big(\widehat{x}+x^0,\widehat{p}+p^0\big)-z^0=\Phi(\widehat{x},\widehat{p})=O_1(\widehat{x},\widehat{p}). \tag{15.1}$$

Since $f(z+z^0)$ is analytic in z at the point $z=0\in\mathbb{C}^k$,

$$f(z+z^0)=\sum_{l\in\mathbb{Z}_+^k}f_lz^l, \qquad f_l=f_l(z^0)\in\mathbb{C},$$

we can define

$$f\big(F(x^0+\widehat{x},p^0+\widehat{p})\big)=\sum_{l\in\mathbb{Z}_+^k} f_l\Phi^l\in\mathcal{QO}^{\mathrm{form}}. \tag{15.2}$$

Since the observables $\Phi_1,\dots,\Phi_k$ commute, any monomial Φ^l, $l\in\mathbb{Z}_+^k$, is well-defined.

Using majorants, one can easily verify that $f\big(F(x^0+\widehat{x},p^0+\widehat{p})\big)\in\mathcal{QO}$.

If the observables $F_1,\dots,F_k$ are Hermitian and f is real-analytic, we take $(x^0,p^0)\in D\cap\mathbb{R}^{2n}$ in (15.1) and (15.2):

$$f(F(\widehat{x},\widehat{p}))=\sum_{l\in\mathbb{Z}_+^k}\mathbf{f}_lF^l,$$

where the coefficients $\mathbf{f}_l$ are real. We have $\mathcal{I}(\mathbf{f}_lF^l)=\mathbf{f}_lF^l$ because $F_j=\mathcal{I}(F_j)$ commute. Hence, $f(F)$ is Hermitian at the point (x^0,p^0), and therefore, it is Hermitian everywhere in D. □

16. DARBOUX THEOREM

Theorem 2. *Let $P_1,\dots,P_n\in\mathcal{QO}$ be commuting observables ($[P_j,P_s]=0$, $1\le j,s\le n$) that are independent at zero. Then there exists an automorphism $\mathcal{A}\in\mathrm{Aut}(\mathcal{QO})$ such that*

$$\mathcal{A}(P_j)=\widehat{p}_j,\qquad j=1,\dots,n.$$

Corollary 16.1 (noncommutative Darboux theorem)**.** *Let $P_1,\dots,P_n\in\mathcal{QO}$ be commuting observables that are independent at zero. Then there exist $X_1,\dots,X_n\in\mathcal{QO}$ such that the set of observables $X_1,\dots,X_n,P_1,\dots,P_n$ is canonical.*

Moreover, suppose that $\widetilde{X}_1,\dots,\widetilde{X}_n,P_1,\dots,P_n$ is another canonical set with $\widetilde{X}_1,\dots,\widetilde{X}_n\in\mathcal{QO}$. Then

$$\widetilde{X}_j=X_j+\big[\widehat{\Phi}(P_1,\dots,P_n,\mathbf{r}),X_j\big],\qquad j=1,\dots,n,$$

where $\widehat{\Phi}\in\mathcal{QO}$ is an analytic observable that depends only on P and $\mathbf{r}$.

To show that the noncommutative Darboux theorem (NDT) follows from Theorem 2, we define $X_j(\widehat{x},\widehat{p})=\mathcal{A}^{-1}(\widehat{x}_j)$. Then, obviously, the observables X and P form a canonical set.

To prove the second part of NDT, we note that for any $s=1,\dots,n$, the observable $\widetilde{X}_s-X_s$ commutes with all P_j, $j=1,\dots,n$. Hence, $\Phi_s:=\widetilde{X}_s-X_s$ depend only on P and $\mathbf{r}$. It remains to use the fact that the set of analytic observables

$$X_1+\Phi_1(P,\mathbf{r}),\ \dots,\ X_n+\Phi_n(P,\mathbf{r}),\ P_1,\ \dots,\ P_n$$

is canonical if and only if $\Phi_j(P,\mathbf{r})=\big[\widehat{\Phi}(P,\mathbf{r}),X_j\big]$ for some $\widehat{\Phi}\in\mathcal{QO}$. □

Remark 16.1. If $P_1,\dots,P_n\in\mathcal{QO}^{\mathrm{H}}$, it is possible to choose an automorphism $\mathcal{A}\in\mathrm{Aut}(\mathcal{QO}^{\mathrm{H}})$ in Theorem 2.

17. PROOF OF THEOREM 2

The proof is based on a rapidly converging procedure similar to the Newton method. We construct the automorphism $\mathcal{A}$ as the limit

$$\mathcal{A}=\lim_{m\to\infty}\mathcal{A}_m,\qquad \mathcal{A}_{m+1}=e^{[\chi_m,\cdot]}\circ\mathcal{A}_m,\qquad m=0,1,\dots.$$

The automorphism $\mathcal{A}_0$ is linear. It reduces $P_1,\dots,P_n$ to the form

$$\mathcal{A}_0P_j=\mathrm{const}_j+\widehat{p}_j+P_j^{(0)},\qquad P_j^{(0)}=O_2(\widehat{x},\widehat{p}),\qquad P_j^{(0)}\ll\frac{\mu_0\xi^2}{1-\lambda_0\xi},\qquad j=1,\dots,n.$$

In what follows, we assume without loss of generality that $\mathrm{const}_j=0$.

We denote

$$\mathcal{A}_m P_j = \widehat{p}_j + P_j^{(m)}, \qquad j = 1, \dots, n.$$

Below we will see that $P_j^{(m)} = O_{2^m+1}(\widehat{x}, \widehat{p})$.

Let $\check{P}_j^{(m)}$ be the polynomial part of $P_j^{(m)}$ of degree less than $2^{m+1}+1$:

$$P_j^{(m)} = \check{P}_j^{(m)} + Q_j^{(m)}, \qquad \deg \check{P}_j^{(m)} \le 2^{m+1}, \quad Q_j^{(m)} = O_{2^{m+1}+1}(\widehat{x}, \widehat{p}). \tag{17.1}$$

We define the Hamiltonian χ_m as a solution of the system

$$[\widehat{p}_j, \chi_m] = \check{P}_j^{(m)}. \tag{17.2}$$

Lemma 17.1. *Suppose that* $P_j^{(m)} = O_{2^m+1}(\widehat{x}, \widehat{p})$, $j = 1, \dots, n$, *and, moreover,*

$$P_j^{(m)}(\widehat{x}, \widehat{p}) \ll \frac{\mu_m \xi^{2^m+1}}{1 - \lambda_m \xi}, \qquad j = 1, \dots, n, \quad \xi = x_1 + \dots + x_n + p_1 + \dots + p_n,$$

for some constants μ_m *and* λ_m. *Then*

$$Q_j^{(m)}(\widehat{x}, \widehat{p}) \ll \frac{\mu_m \lambda_m^{2^m} \xi^{2^{m+1}+1}}{1 - \lambda_m \xi}, \tag{17.3}$$

and system (17.2) *has a solution* $\chi_m = O_{2^m+2}(\widehat{x}, \widehat{p})$,

$$\chi_m(\widehat{x}, \widehat{p}) \ll \frac{n \mu_m \xi^{2^m+2}}{1 - \lambda_m \xi}. \tag{17.4}$$

We prove Lemma 17.1 in Appendix E.

Proposition 17.1. $P_j^{(m+1)} = U_j^{(m+1)} + V_j^{(m+1)}$, *where*

$$U_j^{(m+1)} = -\frac{1}{0!} Q_j^{(m)} - \frac{1}{1!} \big[\chi_m, Q_j^{(m)}\big] - \frac{1}{2!} \big[\chi_m, \big[\chi_m, Q_j^{(m)}\big]\big] - \dots,$$

$$V_j^{(m+1)} = \left(\frac{1}{1!} - \frac{1}{2!}\right) \big[\chi_m, \check{P}_j^{(m)}\big] + \left(\frac{1}{2!} - \frac{1}{3!}\right) \big[\chi_m, \big[\chi_m, \check{P}_j^{(m)}\big]\big] + \dots.$$

Proof. We have

$$\widehat{p}_j + P_j^{(m+1)} = \widehat{p}_j + P_j^{(m)} + \frac{1}{1!} \big[\chi_m, \widehat{p}_j + P_j^{(m)}\big] + \frac{1}{2!} \big[\chi_m, \big[\chi_m, \widehat{p}_j + P_j^{(m)}\big]\big] + \dots.$$

Now it remains to apply (17.1) and (17.2). ☐

Corollary 17.1. *If* $P_j^{(m)} = O_{2^m+1}(\widehat{x}, \widehat{p})$, *we have* $P_j^{(m+1)} = O_{2^{m+1}+1}(\widehat{x}, \widehat{p})$.

Lemma 17.2. *For any* $m \in \mathbb{Z}_+$ *and* $\sigma_m > 0$,

$$U_j^{(m+1)} \ll \mu_m \lambda_m^{2^m} \frac{\xi^{2^{m+1}+1}}{1 - A_m - (1+\sigma_m)\lambda_m \xi},$$

$$V_j^{(m+1)} \ll \frac{2n^2 \mu_m^2 (2^m+3)^2 (1+\sigma_m)^3}{\sigma_m^3} \, \frac{\xi^{2^{m+1}+1}}{1 - A_m - (1+\sigma_m)\lambda_m \xi},$$

$$A_m = 6n^2 \mu_m \lambda_m^{-2^m} (2^{m+1}+3) \frac{(1+\sigma_m)^2}{\sigma_m^2}. \tag{17.5}$$

We prove Lemma 17.2 in Appendix F.

Thus, we have obtained the sequence A_m inductively. Below we will show that $A_m \le 1/2$ and $A_m \to 0$ as $m \to \infty$. Three sequences take part in the construction of $P_j^{(m)}$: $\{\mu_m\}$, $\{\lambda_m\}$, and $\{\sigma_m\}$, $m \in \mathbb{Z}_+$, where

$$\begin{aligned}\mu_{m+1} &\ge \frac{\mu_m \lambda_m^{2^m}}{1 - A_m} + \frac{2n^2 \mu_m^2 (2^m + 3)^2 (1 + \sigma_m)^3}{\sigma_m^3 (1 - A_m)}, \qquad (17.6)\\ \lambda_{m+1} &= \frac{1 + \sigma_m}{1 - A_m} \lambda_m.\end{aligned}$$

We set

$$\lambda_{-1} = 1, \qquad \mu_m = \mu_0 \lambda_{m-1}^{2^m}, \qquad \sigma_m = \sigma_0 (m+1) \cdot 2^{-m}.$$

Proposition 17.2. *There exist constants c_λ and c_σ such that for any λ_0 and σ_0 satisfying*

$$\lambda_0 \ge c_\lambda n^2 (\mu_0 + 1), \qquad \sigma_0 \ge c_\sigma n \sqrt{\mu_0 + 1}, \tag{17.7}$$

inequalities (17.6) *hold and*

$$A_m \le 2^{-m-1} \qquad \textit{for any} \quad m \in \mathbb{Z}_+. \tag{17.8}$$

Proof. Since the sequences λ_m, μ_m, σ_m, and A_m are fixed, we see that (17.6) and (17.8) hold provided that

$$\begin{aligned}1 &\ge 2 \left(\frac{\lambda_{m-1}}{\lambda_m} \right)^{2^m} + 4n^2 \mu_0 (2^m + 3)^2 \left(\frac{1 + \sigma_m}{\sigma_m} \right)^3 \left(\frac{\lambda_{m-1}}{\lambda_m} \right)^{2^{m+1}},\\ 1 &\ge 6n^2 \mu_0 \cdot 2^{m+2} (2^{m+3} + 3) \left(\frac{1 + \sigma_m}{\sigma_m} \right)^2 \left(\frac{\lambda_{m-1}}{\lambda_m} \right)^{2^m}.\end{aligned}$$

It is easy to see that for $m = 0$, these inequalities hold if c_λ is sufficiently large, and for $m > 0$, they hold if c_σ is sufficiently large. $\square$

Remark 17.1. If the observables $P_1, \dots, P_n$ in Theorem 2 are Hermitian, the automorphism $\mathcal{A}$ can also be chosen Hermitian.

18. QUANTUM OBSERVABLES OVER A SYMPLECTIC MANIFOLD

Let $(M, \{\cdot, \cdot\})$ be a real-analytic Poisson manifold with the nondegenerate Poisson bracket $\{\cdot, \cdot\}$. Let $\mathcal{CO}(M)$ be the Lie algebra of analytic functions on M.

Consider an open covering of M by coordinate neighborhoods V_j with canonical coordinates $(x, p)_j$. Let $\mathcal{CO}(V_j)$ be the Lie algebra of analytic functions on V_j, and let $\mathcal{QO}(V_j)$ denote the algebra of quantum observables over V_j. We have the homomorphisms

$$\mathrm{aver}_j \colon \mathcal{QO}(V_j) \mapsto \mathcal{CO}(V_j).$$

We set $(x, p) = (x, p)_j$ and $(\xi, \eta) = (x, p)_k$ for short. The transfer functions $\tau_{j,k}$,

$$\tau_{j,k}(x, p) = (\xi, \eta),$$

which are defined on $V_j \cap V_k$, induce isomorphisms $T_{j,k} \colon \mathcal{CO}(V_k)\big|_{V_j \cap V_k} \to \mathcal{CO}(V_j)\big|_{V_j \cap V_k}$: for any $f_k = f_k(\xi, \eta) \in \mathcal{CO}(V_k)\big|_{V_j \cap V_k}$,

$$f_k \to f_j = T_{j,k} f_j \in \mathcal{CO}(V_j)\big|_{V_j \cap V_k}, \qquad f_j(x, p) = f_k \circ \tau_{j,k}(x, p).$$

Suppose that for any j and k, there is an isomorphism $\widehat{T}_{j,k}\colon \mathcal{QO}(V_k)\big|_{V_j\cap V_k} \to \mathcal{QO}(V_j)\big|_{V_j\cap V_k}$ that is compatible with $T_{j,k}$, i.e., the diagrams

$$\begin{array}{ccc} \mathcal{QO}(V_k)\big|_{V_j\cap V_k} & \xrightarrow{\widehat{T}_{j,k}} & \mathcal{QO}(V_j)\big|_{V_j\cap V_k} \\ \Big\downarrow \scriptstyle{\mathrm{aver}_k} & & \Big\downarrow \scriptstyle{\mathrm{aver}_j} \\ \mathcal{CO}(V_k)\big|_{V_j\cap V_k} & \xrightarrow{T_{j,k}} & \mathcal{CO}(V_j)\big|_{V_j\cap V_k} \end{array}$$

are commutative. Moreover, suppose that for any j, k, and l such that $V_j\cap V_k\cap V_l\neq\varnothing$,

$$\widehat{T}_{j,k}\circ\widehat{T}_{k,l}\circ\widehat{T}_{l,j}=\mathrm{id}.$$

Then these isomorphisms define the algebra $\mathcal{QO}(M)$, which is called the algebra of quantum observables over M.

Apparently, the existence of such isomorphisms $\widehat{T}_{j,k}$ is not trivial in the general situation. However, there are some simple examples.

Example 1. Let M be a cotangent bundle T^*N, $\dim N = n$, with the standard Poisson structure. Consider a covering U_j of the manifold N. Then $V_j = T^*U_j$ is a covering of M. If x are coordinates on U_j, we have canonical coordinates (x,p) on V_j. In this case, $\tau_{j,k}$ have the form

$$(\xi,\eta)=\tau_{j,k}(x,p)=\Big(\gamma_{j,k}(x),\big(B^T_{j,k}(x)\big)^{-1}p\Big), \qquad B_{j,k}(x)=\left(\frac{\partial\gamma_{j,k}(x)}{\partial x}\right),$$

where $B^T_{j,k}$ is the transpose of the matrix $B_{j,k}$. We set

$$\widehat{T}_{j,k}\big(\widehat{\xi}\big)=\gamma_{j,k}(\widehat{x}), \qquad \widehat{T}_{j,k}(\widehat{\eta})=\frac{1}{2}\Big(\big(B^T_{j,k}(\widehat{x})\big)^{-1}\circ\widehat{p}+\big(\widehat{p}^{\,T}\circ B^{-1}_{j,k}(\widehat{x})\big)^T\Big). \tag{18.1}$$

There exists a unique isomorphism $\widehat{T}_{j,k}\colon \mathcal{QO}(V_k)\big|_{V_j\cap V_k}\to\mathcal{QO}(V_j)\big|_{V_j\cap V_k}$ satisfying (18.1).

Example 2. The group $(\mathbb{Z},+)$ acts on the algebra $\mathcal{QO}(\mathbb{R}^n\times D)$, $D\subset\mathbb{R}^n$:

$$l\in\mathbb{Z},\ F\in\mathcal{QO}(\mathbb{R}^n\times D)\ \longrightarrow\ l(F)(\widehat{x},\widehat{p})=F(\widehat{x}+e_jl,\widehat{p}),$$

where e_j is the jth basis vector in $\mathbb{R}^n$. The fixed points of this action are called observables that are 2π-periodic in x_j. They form a subalgebra in $\big(\mathcal{QO}(\mathbb{R}^n\times D),\circ,[\cdot,\cdot]\big)$.

It is natural to identify the subalgebra of observables that are 2π-periodic in all x_j, $j=1,\dots,n$, with $\mathcal{QO}(\mathbb{T}^n\times D)$.

19. LIOUVILLE THEOREM

Let M be a real-analytic symplectic manifold, $\dim M = 2n$, such that the algebra $\mathcal{QO}(M)$ is defined. Suppose that n analytic Hermitian observables $\widehat{F}_1,\dots,\widehat{F}_n\in\mathcal{QO}^{\mathrm{H}}(M)$ are such that

$$[\widehat{F}_j,\widehat{F}_k]=0, \qquad 1\le j,k\le n.$$

Then the real-analytic functions $F_j=\mathrm{aver}\,\widehat{F}_j$, $j=1,\dots,n$, form a commutative set:

$$\{F_j,F_k\}=0, \qquad 1\le j,k\le n.$$

Let $D\subset M$ be a domain on which the classical angle–action variables

$$(\varphi,I)\in\mathbb{T}^n\times D_0, \qquad D_0\subset\mathbb{R}^n,$$

are defined. The change of the variables

$$\tau\colon\ \mathbb{T}^n \times D_0 \to D, \qquad (\varphi, I) \mapsto (x,p) = \tau(\varphi, I)$$

is symplectic. Therefore, it generates an isomorphism a of the Lie algebras $\mathcal{CO}(D)$ and $\mathcal{CO}(\mathbb{T}^n \times D_0)$:

$$\mathcal{CO}(D) \ni f \mapsto a(f) = f \circ \tau \in \mathcal{CO}(\mathbb{T}^n \times D_0).$$

Consider the algebra $\mathcal{QO}(\mathbb{T}^n \times D_0)$ with generators $\widehat{\varphi}$ and $\widehat{I}$.

Theorem 3. *For any point $z^0 = \tau(\varphi^0, I^0) \in D$, there exists a neighborhood $U \subset D_0$ of the point I^0 and a Hermitian isomorphism*

$$\mathcal{A}\colon \mathcal{QO}(D_\bullet) \to \mathcal{QO}(\mathbb{T}^n \times U), \qquad D_\bullet = \tau(\mathbb{T}^n \times U),$$

such that the diagram

$$\begin{tikzcd} \mathcal{QO}(D_\bullet) \arrow[r, "\mathcal{A}"] \arrow[d, "\text{aver}"'] & \mathcal{QO}(\mathbb{T}^n \times D_0) \arrow[d, "\text{aver}"] \\ \mathcal{CO}(D_\bullet) \arrow[r, "a"] & \mathcal{CO}(\mathbb{T}^n \times D_0) \end{tikzcd}$$

is commutative and

$$\mathcal{A}(F_j) = \mathcal{F}_j(\widehat{I}, \mathbf{r}), \qquad j = 1, \dots, n. \tag{19.1}$$

Remark 19.1. It is natural to call the observables $\widehat{I}_j = \mathcal{A}(\widehat{p}_j)$ and $\widehat{\varphi}_j = \mathcal{A}(\widehat{x}_j)$ quantum action–angle variables.

Remark 19.2. We believe that $\mathcal{A}$ can always be continued to a Hermitian isomorphism of $\mathcal{QO}(D)$ and $\mathcal{QO}(\mathbb{T}^n \times D_0)$.

Proof of Theorem 3. Let $I_j = \Phi_j(F)$, $j = 1, \dots, n$, be classical actions. We define $\widehat{J}_j = \Phi_j(\widehat{F}) \in \mathcal{QO}(D)$. Since the observables $\widehat{F}$ commute, these equations make sense. Obviously, $J_j :=$ aver $\widehat{J}_j = I_j$ and $\widehat{F}_j$ are functions of $\widehat{J}$:

$$\widehat{F}_j = \mathcal{F}_j^0(\widehat{J}(\widehat{x}, \widehat{p})). \tag{19.2}$$

Let $B_0 \subset D$ be a ball on which, by virtue of the noncommutative Darboux theorem, the set of observables can be extended to a canonical set by adding a certain $\widehat{\psi} = (\widehat{\psi}_1, \dots, \widehat{\psi}_n) \in \mathcal{QO}^{\mathrm{H}}(B_0)$.

Let $\mathcal{A}_0\colon \mathcal{QO}(V_0) \to \mathcal{QO}(B_0)$, $V_0 \subset \mathbb{R}^{2n}_{\psi,J}$, be the corresponding Hermitian isomorphism:

$$\mathcal{A}_0(\widehat{\psi}_j) = \widehat{x}_j, \qquad \mathcal{A}_0(\widehat{J}_j) = \widehat{p}_j, \qquad j = 1, \dots, n,$$

and $a_0\colon \mathcal{CO}(V_0) \to \mathcal{CO}(B_0)$ be its average: aver $\circ\, \mathcal{A}_0 = a_0 \circ$ aver.

Lemma 19.1. *For sufficiently small $t \in \mathbb{R}$, the observables $\widehat{\psi}_l(\widehat{x}, \widehat{p})$ and $e^{t[\widehat{J}_j, \cdot]}\widehat{\psi}_l(\widehat{x}, \widehat{p})$ coincide on $B_0 \cap g^t_{J_j} B_0$, where $g^t_{J_j}\colon D \to D$ is the flow of the Hamiltonian system with Hamiltonian J_j.*

Proof. This statement becomes obvious in the canonical coordinates $\widehat{\psi}, \widehat{J}$. $\square$

Lemma 19.2. *For any $j, l = 1, \dots, n$,*

$$a_0\Big(e^{2\pi\{J_j, \cdot\}} J_l\Big) = a_0(J_l), \qquad a_0\Big(e^{2\pi\{J_j, \cdot\}} \psi_l - 2\pi\delta_{lj}\Big) = a_0(\psi_l).$$

Proof. These equations follow from the definition of the classical action–angle variables. $\square$

Lemma 19.3. *For any $j,l=1,\dots,n$,*

$$\mathcal{A}_0\Big(e^{2\pi\{\widehat{J}_j,\cdot]}\widehat{J}_l\Big)=\mathcal{A}_0(\widehat{J}_l),\qquad \mathcal{A}_0\Big(e^{2\pi\{\widehat{J}_j,\cdot\}}\widehat{\psi}_l-2\pi\delta_{lj}\Big)=\mathcal{A}_0(\widehat{\psi}_l+\widehat{\Psi}_{lj}),\tag{19.3}$$

where

$$\widehat{\Psi}_{lj}=\mathbf{r}\circ\big[\widehat{\Lambda}_j(\widehat{J},\mathbf{r}),\widehat{\psi}_l\big]\tag{19.4}$$

for some observables $\widehat{\Lambda}_j\in\mathcal{QO}(V_0)$, $\widehat{\Lambda}_j=\widehat{\Lambda}_j(\widehat{J},\mathbf{r})$.

Proof. The first equation in (19.3) is obvious. To prove (19.4), we apply aver to the second equation in (19.3). By Lemma 19.2, we see that aver $\widehat{\Psi}_{lj}=0$. Now equation (19.4) follows from the fact that the set of observables

$$\widehat{\psi}_l+\widehat{\Psi}_{lj},\ \widehat{J}_l,\qquad l=1,\dots,n,$$

is canonical for any $j=1,\dots,n$. $\square$

We define

$$\widehat{I}_j:=\widehat{J}_j-\frac{1}{2\pi}\mathbf{r}\circ\widehat{\Lambda}_j,\qquad \mathcal{A}_{0j}\Psi\big(\widehat{\psi},\widehat{J}\,\big):=\mathcal{A}_0\Psi\big(\widehat{\psi}-2\pi e_j,\widehat{J}\,\big),\tag{19.5}$$

where $\Psi\in\mathcal{QO}(V_0)$ is an arbitrary observable and $e_j\in\mathbb{R}^n$ is the jth unit vector.

Lemma 19.4. *The maps*

$$\mathcal{S}_j:=\mathcal{A}_{0j}\circ e^{2\pi[\widehat{I}_j,\cdot]}\circ\mathcal{A}_0^{-1}\colon\ \mathcal{QO}(B_0)\to\mathcal{QO}(B_0)$$

are identity automorphisms.

Proof. It is sufficient to verify that $\mathcal{S}_j$ acts as the identity on some canonical set of observables in $\mathcal{QO}(B_0)$, for example, on $\widehat{\psi}(\widehat{x},\widehat{p}\,)$, $\widehat{J}(\widehat{x},\widehat{p}\,)$. We have

$$\begin{aligned}\mathcal{S}_j(\psi_l(\widehat{x},\widehat{p}\,))&=\mathcal{A}_{0j}\circ e^{2\pi[\widehat{I}_j,\cdot]}\psi_l=\mathcal{A}_{0j}\circ e^{-\mathbf{r}\circ[\widehat{\Lambda}_j,\cdot]}\big(\widehat{\psi}_l+2\pi\delta_{lj}+\widehat{\Psi}_{lj}\big)\\&=\mathcal{A}_{0j}\big(\widehat{\psi}_l-\mathbf{r}\circ\big[\Lambda_j,\widehat{\psi}_l\big]+2\pi\delta_{lj}+\widehat{\Psi}_{lj}\big)=\mathcal{A}_{0j}\big(\widehat{\psi}_l+2\pi\delta_{lj}\big)=\widehat{\psi}_l(\widehat{x},\widehat{p}\,).\end{aligned}$$

Here the second equality follows from (19.3), and the fourth equality from (19.4).

The equation $\mathcal{S}_j(\widehat{J})=\widehat{J}$ is obvious.

Now we define $\widehat{\varphi}$ that are canonically conjugate to $\widehat{I}$; we first do this locally and then use continuation by $e^{[\widehat{I},\cdot]}$. Lemma 19.4 implies the periodicity of the quantum angles $\widehat{\varphi}$. Equation (19.1) follows from (19.2) and the first equation in (19.5). Theorem 3 is proved. $\square$

Example 1 (harmonic oscillator)**.** For $H=\frac{1}{2}(\widehat{p}^2+\widehat{x}^2)$, the action is $\widehat{I}=H$ and the angle is $\widehat{\varphi}=\frac{1}{i}\ln(\widehat{p}-i\widehat{x})$ or $\widehat{\varphi}=-\frac{1}{i}\ln(\widehat{p}+i\widehat{x})$. If we need a Hermitian observable, we can take $\widehat{\varphi}=\frac{1}{2i}\big(\ln(\widehat{p}-i\widehat{x})-\ln(\widehat{p}+i\widehat{x})\big)$.

Example 2 (separation of variables)**.** Consider a Hamiltonian

$$H=\Phi\big(\widehat{H}_1(\widehat{x}_1,\widehat{p}_1),\dots,\widehat{H}_n(\widehat{x}_n,\widehat{p}_n)\big),$$

where Φ is an analytic function. Such a system is obviously Liouville integrable with $\widehat{F}_j=\widehat{H}_j(\widehat{x}_j,\widehat{p}_j)$. In this case, we say that the variables separate.

Other examples of quantum Liouville integrable systems can be found in [3, 7, 8]. Some analogies between quantum and classical problems of integrability are discussed in [9, 10].

In the action–angle variables, quantum dynamics becomes very simple. Indeed, consider the Heisenberg equation

$$\dot{\mathcal{F}} = [\mathcal{H}, \mathcal{F}], \qquad \mathcal{F}\big|_{t=0} = \mathcal{F}_0, \qquad \mathcal{F}_0, \mathcal{H} \in \mathcal{QO}(\mathbb{T}^n \times D), \quad D \subset \mathbb{R}^n.$$

Suppose that the Hamiltonian does not depend on $\widehat{\varphi}$, i.e., $\mathcal{H} = \mathcal{H}(\widehat{I}\,)$.

We define a quantum frequency vector

$$\omega = (\omega_1, \dots, \omega_n), \qquad \omega_j = D_j \mathcal{H}(\widehat{I}\,), \quad j = 1, \dots, n,$$

where D_j is the jth partial derivative.

If $\mathcal{F}_0 = \widehat{\varphi}_j$, the equation can be solved easily: $\mathcal{F}(\widehat{\varphi}, \widehat{I}, t) = \widehat{\varphi}_j + \omega_j(\widehat{I}\,)t$. Hence, for any $\mathcal{F}_0(\widehat{\varphi}, \widehat{I}\,) \in \mathcal{QO}(\mathbb{T}^n \times D)$, we have

$$\mathcal{F}(\widehat{\varphi}, \widehat{I}, t) = \Phi^t(\mathcal{F}_0)(\widehat{\varphi}, \widehat{I}\,),$$

where $\Phi^t = e^{[t\mathcal{H},\cdot]} \colon \mathcal{QO}(\mathbb{T}^n \times D) \to \mathcal{QO}(\mathbb{T}^n \times D)$ is the canonical isomorphism defined by the equations $\Phi^t(\widehat{\varphi}, \widehat{I}\,) = (\widehat{\varphi} + \omega(\widehat{I}\,)t, \widehat{I}\,)$. In fact, for any $\mathcal{G} \in \mathcal{QO}(\mathbb{T}^n \times D)$, we have

$$\Phi^t(\mathcal{G})(\widehat{\varphi}, \widehat{I}\,) = \mathcal{G}(\widehat{\varphi} + \omega(\widehat{I}\,)t, \widehat{I}\,).$$

In the original coordinates, the solution of the Heisenberg equation

$$\dot{F} = [H, F], \qquad F|_{t=0} = F_0,$$

has the form

$$F(\widehat{x}, \widehat{p}\,) = \mathcal{A}^{-1} \circ \Phi^t \circ \mathcal{A}(F_0)(\widehat{x}, \widehat{p}\,).$$

20. EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR

In addition to simple dynamics (in terms of the solutions of the Heisenberg equation), quantum Liouville integrability implies some important properties of the spectrum of the corresponding Schrödinger operator. In this section, we present several conditional results that partially confirm this fact. In the traditional language, some analogous statements are presented in [4].

Below we associate with $\mathbf{r}$ the skew-Hermitian operator $-i\hbar \cdot \mathrm{id}$ on $L_2(\mathbb{R}^n)$.

Proposition 20.1. *Suppose that observables $F_1, \dots, F_k, \Phi \in \mathcal{QO}(\mathbb{R}^{2n})$ satisfy the equations*

$$[F_j, F_l] = 0, \qquad [F_j, \Phi] = i\Phi \circ \mu_j(F, i\mathbf{r}). \tag{20.1}$$

Here $\mu_j \colon \mathbb{R}^{k+1} \to \mathbb{R}$ are some functions that are real-analytic in the first k arguments and smooth in the last argument. Suppose also that $F_1, \dots, F_k$ and Φ can be associated with operators on $L_2(\mathbb{R}^n)$. Let $\psi \in L_2(\mathbb{R}^n)$ be an eigenfunction for F_j,

$$F_j \cdot \psi = \lambda_j \psi, \qquad j = 1, \dots, k, \tag{20.2}$$

that belongs to the domain of the operator Φ. Then the function $\varphi := \Phi \cdot \psi$ lies in the domain of the operators $F_1, \dots, F_k$ and

$$F_j \cdot \varphi = (\lambda_j + \hbar\mu_j(\lambda, \hbar))\, \varphi.$$

Proof. Direct calculations yield

$$\begin{aligned} F_j \cdot \varphi &= F_j \circ \Phi \cdot \psi = \Phi \circ F_j \cdot \psi + \mathbf{r} \circ [F_j, \Phi] \cdot \psi = \lambda_j \Phi \cdot \psi + i\mathbf{r} \circ \Phi \circ \mu_j(F, i\mathbf{r}) \cdot \psi \\ &= (\lambda_j + \hbar\mu_j(\lambda, \hbar))\varphi. \quad \square \end{aligned}$$

Proposition 20.2. *For any $F = F(\widehat{p}, i\mathbf{r}) \in \mathcal{QO}^{\text{form}}$ and $k \in \mathbb{C}^n$, the following formal identity holds*:

$$\big[F(\widehat{p}, i\mathbf{r}), e^{i\langle k,\widehat{x}\rangle}\big] = e^{i\langle k,\widehat{x}\rangle} \circ \frac{F(\widehat{p} + ik\mathbf{r}, i\mathbf{r}) - F(\widehat{p}, i\mathbf{r})}{\mathbf{r}}.$$

Proof. 1. The case $F = \widehat{p}_j$ is obvious.

2. In the case $F = \widehat{p}^\alpha$, $\alpha \in \mathbb{Z}^n_+$, we apply induction on $|\alpha|$ and use the identity

$$\begin{aligned}\big[F_1 \circ F_2, e^{i\langle k,\widehat{x}\rangle}\big] &= \big[F_1, e^{i\langle k,\widehat{x}\rangle}\big] \circ F_2 + F_1 \circ \big[F_2, e^{i\langle k,\widehat{x}\rangle}\big] \\ &= \big[F_1, e^{i\langle k,\widehat{x}\rangle}\big] \circ F_2 + \big[F_2, e^{i\langle k,\widehat{x}\rangle}\big] \circ F_1 + \mathbf{r} \circ \big[F_1, \big[F_2, e^{i\langle k,\widehat{x}\rangle}\big]\big],\end{aligned}$$

which holds for any $F_1, F_2 \in \mathcal{QO}^{\text{form}}$.

3. In the case of arbitrary F, we use the linearity. $\square$

If we use $\Phi = e^{i\langle k,\widehat{x}\rangle}$ in Proposition 20.1, we obtain the following result.

Corollary 20.1. *Suppose that a system of commuting observables $F_1, \dots, F_n$, $[F_j, F_l] = 0$, admits global angle–action variables $\widehat{\varphi}$ and $\widehat{I}$*:

$$F = \mathcal{F}(\widehat{I}, i\mathbf{r}), \qquad \widehat{I} = \mathcal{M}(F, i\mathbf{r}).$$

Suppose that the observables $F_1, \dots, F_n$ and $e^{i\langle k,\widehat{\varphi}\rangle}$ can be associated with operators on $L_2(\mathbb{R}^n)$. Let $\psi \in L_2(\mathbb{R}^n)$ be a common eigenfunction for F_j,

$$F_j \cdot \psi = \lambda_j \psi, \qquad j = 1, \dots, n.$$

Suppose also that $\psi_k = e^{i\langle k,\widehat{\varphi}\rangle} \cdot \psi$ is defined and belongs to $L_2(\mathbb{R}^n)$.

Then

$$F_j \cdot \psi_k = \mathcal{F}_j\big(\mathcal{M}(\lambda, \hbar) + \hbar k, \hbar\big)\, \psi_k.$$

Proof. Indeed, by Proposition 20.1,

$$F_j \cdot \psi_k = (\lambda_j + \hbar\mu_j(\lambda, \hbar))\, \psi_k,$$

where the functions $\mu_j(F, i\mathbf{r})$ satisfy the equation

$$\big[F_j, e^{i\langle k,\widehat{\varphi}\rangle}\big] = ie^{i\langle k,\widehat{\varphi}\rangle} \circ \mu_j(F, i\mathbf{r}).$$

By Proposition 20.2,

$$\mu_j(F, i\mathbf{r}) = \frac{\mathcal{F}_j(\widehat{I} + ik\mathbf{r}, i\mathbf{r}) - \mathcal{F}_j(\widehat{I}, i\mathbf{r})}{i\mathbf{r}} = \frac{\mathcal{F}_j(\mathcal{M}(F, i\mathbf{r}) + i\mathbf{r}k, i\mathbf{r}) - F_j}{i\mathbf{r}}.$$

Therefore, $\hbar\mu_j(\lambda, \hbar) = \mathcal{F}_j(\mathcal{M}(\lambda, \hbar) + \hbar k, \hbar) - \lambda_j$. $\square$

APPENDICES

A. FORMULAS

A.1. Identities for the commutator. (a) Jacobi identities:

$$[[F,G],H]+[[G,H],F]+[[H,F],G]=0, \tag{A.1}$$

$$[(\!(F,G)\!),H]+[(\!(G,H)\!),F]+[(\!(H,F)\!),G]=0, \tag{A.2}$$

$$(\!([F,G],H)\!)+(\!([G,H],F)\!)+(\!([H,F],G)\!)=0. \tag{A.3}$$

(b) Leibnitz identities:

$$[F\circ G,H]=F\circ[G,H]+[F,H]\circ G, \tag{A.4}$$

$$[(\!(F,G)\!),H]=(\!(F,[G,H])\!)+(\!(G,[F,H])\!). \tag{A.5}$$

(c) The "bac–cab" rule:

$$\mathbf{r}^2[F,[G,H]]=(\!((\!(G,F)\!),H)\!)-(\!((\!(H,F)\!),G)\!). \tag{A.6}$$

A.2. Relations $\sim$ and $\approx$.

Notation. Let $n=1$. For any monomial $z=\pi(\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta}\circ\widehat{p}^{\,\gamma})$, we write $z\overset{\beta}{\sim}u^\gamma v^\alpha$.

Similarly, for any monomial $\zeta=\pi(\widehat{x}^{\alpha}\circ\widehat{p}^{\,\beta}\circ\widehat{x}^{\gamma})$, we write $\zeta\overset{\beta}{\approx}\widetilde{u}^\gamma\widetilde{v}^\alpha$.

If $\alpha+\gamma\le\beta$, the monomials $\pi(\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta}\circ\widehat{p}^{\,\gamma})$ and $\pi(\widehat{x}^{\alpha}\circ\widehat{p}^{\,\beta}\circ\widehat{x}^{\gamma})$ can be regarded as elements of the primitive basis; however, we usually do not assume that this inequality holds.

Proposition A.1. *Let $n=1$. For any $\alpha,\beta,\gamma\ge 0$, let $z=\pi(\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta}\circ\widehat{p}^{\,\gamma})$ and $\zeta=\pi(\widehat{x}^{\alpha}\circ\widehat{p}^{\,\beta}\circ\widehat{x}^{\gamma})$. Then*

$$\pi(\widehat{p}\circ z)\overset{\beta+1}{\sim}u^\gamma v^{\alpha+1},\qquad \pi(\widehat{x}\circ z)\overset{\beta+1}{\sim}\left(1+\frac{u-v}{\beta+1}\frac{\partial}{\partial v}\right)u^\gamma v^\alpha, \tag{A.7}$$

$$\pi(\widehat{p}\circ \zeta)\overset{\beta+1}{\approx}\left(1+\frac{u-v}{\beta+1}\frac{\partial}{\partial \widetilde{v}}\right)\widetilde{u}^\gamma \widetilde{v}^\alpha,\qquad \pi(\widehat{x}\circ \zeta)\overset{\beta+1}{\approx}\widetilde{u}^\gamma\widetilde{v}^{\alpha+1}, \tag{A.8}$$

$$\pi(\mathbf{r}\circ z)\overset{\beta+1}{\sim}\frac{v-u}{\beta+1}u^\gamma v^\alpha,\qquad \pi(\mathbf{r}\circ \zeta)\overset{\beta+1}{\approx}\frac{\widetilde{u}-\widetilde{v}}{\beta+1}\widetilde{u}^\gamma\widetilde{v}^\alpha. \tag{A.9}$$

Proof. The first relation (A.7) is obvious, while the second one follows from the identity

$$\pi(\widehat{x}\circ z)=\pi\big(\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta+1}\circ\widehat{p}^{\,\gamma}\big)+\frac{\alpha}{\beta+1}\pi\Big(\widehat{p}^{\,\alpha-1}\circ\widehat{x}^{\beta+1}\circ\widehat{p}^{\,\gamma+1}-\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta+1}\circ\widehat{p}^{\,\gamma}\Big).$$

This identity follows from a simpler one,

$$(\beta+1)\pi\big(\widehat{x}\circ\widehat{p}^{\,\alpha}\circ x^{\beta}-\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta+1}\big)=\alpha\pi\big(\widehat{p}^{\,\alpha-1}\circ\widehat{x}^{\beta+1}\circ\widehat{p}-\widehat{p}^{\,\alpha}\circ\widehat{x}^{\beta+1}\big).$$

To verify the latter identity, it is sufficient to use the following trivial equations:

$$\pi(\widehat{x}\circ\widehat{p}^{\,\alpha}-\widehat{p}^{\,\alpha}\circ\widehat{x})=-\alpha\pi(\widehat{p}^{\,\alpha-1}),\qquad \pi\big(\widehat{x}^{\beta+1}\circ\widehat{p}-\widehat{p}\circ\widehat{x}^{\beta+1}\big)=-(\beta+1)\pi(\widehat{x}^{\beta}).$$

Relations (A.8) can be checked similarly, while (A.9) follow from (A.7) and (A.8). □

A.3. Forms $\Sigma_{\alpha,\beta}$. We set

$$\Sigma_{\alpha,\beta} := \sum_{\mathrm{type}\, z=(\alpha,\beta)} \pi(z).$$

Obviously,

$$\operatorname{Aver} \Sigma_{\alpha,\beta} = C^{\alpha}_{\alpha+\beta} x^{\alpha} p^{\beta}. \tag{A.10}$$

Proposition A.2. *Suppose that $n = 1$. Then*

$$\Sigma_{\alpha,\beta} = C^{\beta}_{\alpha+\beta} \cdot 2^{-\beta} \sum_{j=0}^{\beta} C^{j}_{\beta}\, \pi\big(\widehat{p}^{\,j} \circ \widehat{x}^{\alpha} \circ \widehat{p}^{\,\beta-j}\big). \tag{A.11}$$

Corollary A.1.

$$\Sigma_{\alpha,\beta} \overset{\alpha}{\sim} C^{\beta}_{\alpha+\beta} \cdot 2^{-\beta} (u+v)^{\beta}, \qquad \Sigma_{\alpha,\beta} \overset{\beta}{\approx} C^{\alpha}_{\alpha+\beta} \cdot 2^{-\alpha} (\widetilde{u}+\widetilde{v})^{\alpha}. \tag{A.12}$$

Proof of Proposition A.2. Equation (A.11) obviously holds if either $\alpha = 0$ or $\beta = 0$. Then

$$\Sigma_{\alpha,\beta} = \pi(\widehat{x}) \circ \Sigma_{\alpha-1,\beta} + \pi(\widehat{p}\,) \circ \Sigma_{\alpha,\beta-1}. \tag{A.13}$$

Applying equation (A.13) several (finitely many) times, we finally express $\Sigma_{\alpha,\beta}$ in terms of $\Sigma_{\alpha_k,\beta_k}$, $k = 1, \dots, K$, where $\alpha_k \beta_k = 0$ for any $k = 1, \dots, K$. Hence, it is sufficient to prove (A.11) assuming that $\Sigma_{\alpha-1,\beta}$ and $\Sigma_{\alpha,\beta-1}$ satisfy (A.11).

Following this plan, we evaluate two terms on the right-hand side of (A.13). By Proposition A.1,

$$\pi(\widehat{x}) \circ \Sigma_{\alpha-1,\beta} \overset{\alpha}{\sim} \left(1 + \frac{u-v}{\alpha} \frac{\partial}{\partial v}\right) C^{\beta}_{\alpha+\beta-1} \cdot 2^{-\beta} (u+v)^{\beta},$$

$$\pi(\widehat{p}\,) \circ \Sigma_{\alpha,\beta-1} \overset{\alpha}{\sim} C^{\beta-1}_{\alpha+\beta-1} \cdot 2^{1-\beta} v (u+v)^{\beta-1}.$$

Summing up these relations, we get

$$\pi(\widehat{x}) \circ \Sigma_{\alpha-1,\beta} + \pi(\widehat{p}\,) \circ \Sigma_{\alpha,\beta-1} \overset{\alpha}{\sim} C^{\beta}_{\alpha+\beta} \cdot 2^{-\beta} (u+v)^{\beta}. \qquad \square$$

A.4. The forms $\mathbf{r}^l$.

Proposition A.3. *Suppose that $n = 1$. Then*

$$l!\,\mathbf{r}^l = \sum_{j=0}^{l} (-1)^j C^j_l \pi\big(\widehat{x}^j \circ \widehat{p}^{\,l} \circ \widehat{x}^{l-j}\big) = \sum_{j=0}^{l} (-1)^j C^j_l \pi\big(\widehat{p}^{\,l-j} \circ \widehat{x}^l \circ \widehat{p}^{\,j}\big),$$

$$C^{\beta}_{l+\beta}\, l!\, \pi(\mathbf{r}^l \circ \widehat{p}^{\,\beta}) = \sum_{j=0}^{l} (-1)^j C^j_l \pi\big(\widehat{x}^j \circ \widehat{p}^{\,\beta+l} \circ \widehat{x}^{l-j}\big),$$

$$C^{\alpha}_{l+\alpha}\, l!\, \pi(\mathbf{r}^l \circ \widehat{x}^{\alpha}) = \sum_{j=0}^{l} (-1)^j C^j_l \pi\big(\widehat{p}^{\,l-j} \circ \widehat{x}^{\alpha+l} \circ \widehat{p}^{\,j}\big).$$

Corollary A.2. $\operatorname{Aver}(l!\,\mathbf{r}^l) = (2xp)^l$.

Proof of Proposition A.3. Induction on l with the help of (A.9). $\square$

Proposition A.4. *For $n = 1$, let $F \in \pi(\mathbf{F}^{\alpha,\beta})$. Then*

$$F = \sum_{j=0}^{\min\{\alpha,\beta\}} f_j\, j!\, \mathbf{r}^j \circ \sigma_{\alpha-j,\beta-j}, \qquad \sigma_{k,l} = (C^k_{k+l})^{-1}\Sigma_{k,l},$$

where the coefficients f_j satisfy the estimate

$$|f_j| \le 2^{\alpha+\beta-j}\|F\|.$$

Proof. Suppose, for definiteness, that $\alpha \le \beta$. Expanding F in the primitive basis, we get

$$F \overset{\beta}{\approx} \sum_{k=0}^{\alpha} q_k \widetilde{u}^k \widetilde{v}^{\alpha-k}, \qquad \|F\| = \sum_{k=0}^{\alpha} |q_k|. \tag{A.14}$$

By Propositions A.1 and A.2,

$$F \overset{\beta}{\approx} \sum_{j=0}^{\alpha} \frac{f_j \cdot 2^{j-\alpha}}{C^j_\beta} (\widetilde{u} - \widetilde{v})^j (\widetilde{v} + \widetilde{u})^{\alpha-j}. \tag{A.15}$$

Equating the polynomials on the right-hand sides of (A.14) and (A.15), we obtain the estimate

$$|f_j| \cdot 2^{j-\alpha}/C^j_\beta \le 2^{-\alpha} C^j_\alpha \|F\|.$$

It remains to apply the inequalities $C^j_\beta \le 2^\beta$ and $C^j_\alpha \le 2^\alpha$. □

B. PRIMITIVE MONOMIALS

B.1. Some identities.

Proposition B.1. *In the case $n = 1$, the following identity holds for any $\alpha, \beta \in \mathbb{Z}_+$:*

$$\pi\big(\widehat{x}^\alpha \circ \widehat{p}^{\alpha+\beta} \circ \widehat{x}^\beta\big) = \pi\big(\widehat{p}^\beta \circ \widehat{x}^{\alpha+\beta} \circ \widehat{p}^\alpha\big). \tag{B.1}$$

Corollary B.1. *The sets of monomials* (5.3) *and* (5.4) *coincide for $\mu = \nu$.*

Proof of Proposition B.1. First, note that identity (B.1) is equivalent to

$$\big[\pi(\widehat{x}^\alpha \circ \widehat{p}^\alpha), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] = 0. \tag{B.2}$$

This equation is obvious if $\alpha = 0$ or $\beta = 0$. To prove (B.2) in the general case, we use induction on $\alpha + \beta$ and the identity

$$\begin{aligned}\big[\pi(\widehat{x}^\alpha \circ \widehat{p}^\alpha), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] &= \pi(\widehat{x}) \circ \big[\pi(\widehat{x}^{\alpha-1} \circ \widehat{p}^{\alpha-1}), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \circ \pi(\widehat{p}) \\ &\quad + \beta\mathbf{r} \circ \big[\pi(\widehat{x}^\alpha \circ \widehat{p}^\alpha), \pi(\widehat{p}^{\beta-1} \circ \widehat{x}^{\beta-1})\big].\end{aligned} \tag{B.3}$$

It remains to prove (B.3). By the Leibniz identity,

$$\begin{aligned}\big[\pi(\widehat{x}^\alpha \circ \widehat{p}^\alpha), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] &= \big[\pi(\widehat{x}), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \circ \pi(\widehat{x}^{\alpha-1} \circ \widehat{p}^\alpha) + \pi(\widehat{x}) \circ \big[\pi(\widehat{x}^{\alpha-1} \circ \widehat{p}^\alpha), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \\ &= -\beta\pi\big(\widehat{p}^{\beta-1} \circ \widehat{x}^{\beta+\alpha-1} \circ \widehat{p}^\alpha\big) + \pi(\widehat{x}) \circ \big[\pi(\widehat{x}^{\alpha-1} \circ \widehat{p}^{\alpha-1}), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \circ \pi(\widehat{p}) \\ &\quad + \pi(\widehat{x}^\alpha \circ \widehat{p}^{\alpha-1}) \circ \big[\pi(\widehat{p}), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \\ &= \pi(\widehat{x}) \circ \big[\pi(\widehat{x}^{\alpha-1} \circ \widehat{p}^{\alpha-1}), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \circ \pi(\widehat{p}) \\ &\quad - \beta\pi\big(\widehat{p}^{\beta-1} \circ \widehat{x}^{\beta+\alpha-1} \circ \widehat{p}^\alpha\big) + \beta\pi\big(\widehat{x}^\alpha \circ \widehat{p}^{\beta+\alpha-1} \circ \widehat{x}^{\beta-1}\big) \\ &= \pi(\widehat{x}) \circ \big[\pi(\widehat{x}^{\alpha-1} \circ \widehat{p}^{\alpha-1}), \pi(\widehat{p}^\beta \circ \widehat{x}^\beta)\big] \circ \pi(\widehat{p}) + \beta\mathbf{r} \circ \big[\pi(\widehat{x}^\alpha \circ \widehat{p}^\alpha), \pi(\widehat{p}^{\beta-1} \circ \widehat{x}^{\beta-1})\big]. \quad \square\end{aligned}$$

B.2. Proof of Proposition 5.2. Let $z \in \pi(\mathbf{F}^{\alpha,\beta})$, $0 \le \alpha \le \beta$, be a monomial that does not belong to the list (5.3). To show that it is composite, we use induction on α. First, we consider the case when

$$z = \pi(\widehat{p}^{\mu} \circ \widehat{x}^{\nu} \circ \widehat{p}^{\lambda}), \qquad \mu, \nu, \lambda > 0, \quad \nu < \mu + \lambda.$$

If $\nu = 1$, then z is composite due to the obvious identity

$$\pi(\widehat{p} \circ \widehat{x} \circ \widehat{p}) = \frac{1}{2}\pi\big(\widehat{p}^2 \circ \widehat{x} + \widehat{x} \circ \widehat{p}^2\big). \tag{B.4}$$

Suppose that the statement is true for all positive $\nu < \nu_0 < \mu + \lambda$. Consider the case $\nu = \nu_0$. We can present z in the form $\pi(\widehat{p}^a \circ \widetilde{z} \circ \widehat{p}^b)$, $a, b \ge 0$, so that $\widetilde{z} = \pi(\widehat{p}^{\mu_0} \circ \widehat{x}^{\nu_0} \circ \widehat{p}^{\lambda_0})$ with

$$\mu_0, \lambda_0 > 0, \qquad \nu_0 < \mu_0 + \lambda_0, \qquad \text{and} \qquad \nu_0 > \min\{\mu_0, \lambda_0\}.$$

Let, for definiteness, $\nu_0 > \mu_0$. Then, by (B.1),

$$\widetilde{z} = \pi\big(\widehat{x}^{\nu_0 - \mu_0} \circ \widehat{p}^{\nu_0} \circ \widehat{x}^{\mu_0} \circ \widehat{p}^{\lambda_0 + \mu_0 - \nu_0}\big).$$

By the induction hypothesis, $\pi(\widehat{p}^{\nu_0} \circ \widehat{x}^{\mu_0} \circ \widehat{p}^{\lambda_0 + \mu_0 - \nu_0})$ is composite. Hence, $\widetilde{z}$ and z are composite too.

Now we turn to the general case. We have

$$z = \pi\big(\widehat{x}^{\alpha_0} \circ \widehat{p}^{\beta_1} \circ \widehat{x}^{\alpha_1} \circ \widehat{p}^{\beta_2} \circ \widehat{x}^{\alpha_2} \circ \ldots \circ \widehat{p}^{\beta_k} \circ \widehat{x}^{\alpha_k} \circ \widehat{p}^{\beta_{k+1}} \circ \widehat{x}^{\alpha_{k+1}}\big)$$

for some positive $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_{k+1}$, and $\alpha_0 \ge 0$, $\alpha_{k+1} \ge 0$,

$$0 \le \sum_{j=1}^{k} \alpha_j \le \sum_{j=1}^{k+1} \beta_j. \tag{B.5}$$

It is important that $k \ge 1$ (otherwise, z belongs to the list (5.3)).

Now note that at least one of the following inequalities holds:

$$\alpha_j < \beta_j + \beta_{j+1}, \qquad j = 1, \ldots, k. \tag{B.6}$$

Indeed, if $k > 1$ and all inequalities (B.6) fail to hold, we have a contradiction with (B.5). In the case of $k = 1$, it may happen that $\alpha_1 = \beta_1 + \beta_2$. However, in this case, by (B.1), $z = \pi(\widehat{x}^{\beta_2} \circ \widehat{p}^{\alpha_1} \circ \widehat{x}^{\beta_1})$ is one of monomials (5.3).

Suppose that (B.6) holds for $j = s$. Then it is sufficient to prove that $z_s = \pi(\widehat{p}^{\beta_s} \circ \widehat{x}^{\alpha_s} \circ \widehat{p}^{\beta_{s+1}})$ is composite. Hence, we have reduced the general case to the one considered above. $\square$

B.3. Proof of Proposition 5.3. For any monomial $z \in \mathbf{F}^{\mu,\nu}$, its projection $\pi(z)$ is a linear combination of primitive monomials. Therefore, $\mathbf{F}^{\mu,\nu}_{\mathrm{prim}}$ generates the vector space $\pi(\mathbf{F}^{\mu,\nu})$. In particular, the number of elements

$$\#\mathbf{F}^{\mu,\nu}_{\mathrm{prim}} \ge \dim \pi(\mathbf{F}^{\mu,\nu}).$$

To prove that vectors from $\mathbf{F}^{\mu,\nu}_{\mathrm{prim}}$ are linearly independent, we note that

$$\#\mathbf{F}^{\mu,\nu}_{\mathrm{prim}} \le \text{The number of monomials (5.5)} = m_1 \cdot \ldots \cdot m_n,$$

where $m_j = \max\{\mu_j, \nu_j\} + 1$. Hence, by (5.2),

$$\dim \pi(\mathbf{F}^{\mu,\nu}) = m_1 \cdot \ldots \cdot m_n = \#\mathbf{F}^{\mu,\nu}_{\mathrm{prim}}.$$

This implies that $\mathbf{F}^{\mu,\nu}_{\mathrm{prim}}$ contains all monomials (5.5) and forms a basis in $\pi(\mathbf{F}^{\mu,\nu})$. $\square$

C. SOME MAJORANT INEQUALITIES

In this appendix, ζ and ξ are one-dimensional complex variables.

Proposition C.1. *For any* $0 < a < b$,

$$\frac{1}{(1-a\zeta)(1-b\zeta)} \ll \frac{b}{(b-a)(1-b\zeta)}. \tag{C.1}$$

Proof. The proof follows immediately from the equation

$$\frac{b}{(b-a)(1-b\zeta)} = \frac{1}{(1-a\zeta)(1-b\zeta)} + \frac{a}{(b-a)(1-a\zeta)}. \quad \square$$

Proposition C.2. *For any* $0 < \beta < \alpha$,

$$\frac{1-\alpha\xi-\sqrt{(1-\alpha\xi)^2-\beta^2\xi^2}}{\xi^2} \ll \frac{\beta(\alpha+\beta)}{4(1-(\alpha+\beta)\xi)}.$$

Proof. We start with the equation

$$1-\sqrt{1-\zeta} = \sum_{k=1}^{\infty} \frac{(2k-3)!!}{(2k)!!}\zeta^k.$$

Now we see that

$$\frac{1-\alpha\xi-\sqrt{(1-\alpha\xi)^2-\beta^2\xi^2}}{\xi^2} = \frac{1-\alpha\xi}{\xi^2}\left(1-\sqrt{1-\frac{\beta^2\xi^2}{(1-\alpha\xi)^2}}\right) = \frac{1-\alpha\xi}{\xi^2}\sum_{k=1}^{\infty}\frac{(2k-3)!!}{(2k)!!}\frac{\beta^{2k}\xi^{2k}}{(1-\alpha\xi)^{2k}}$$

$$\ll \frac{\beta^2}{2(1-\alpha\xi)}\sum_{k=0}^{\infty}\frac{\beta^{2k}\xi^{2k}}{(1-\alpha\xi)^{2k}} = \frac{\beta^2(1-\alpha\xi)}{2(1-(\alpha+\beta)\xi)(1-(\alpha-\beta)\xi)}$$

$$\ll \frac{\beta^2}{2(1-(\alpha+\beta)\xi)(1-(\alpha-\beta)\xi)}.$$

It remains to use (C.1). $\square$

Proposition C.3. *For any* $k \in \mathbb{N}$ *and* $\lambda > 0$,

$$\frac{\lambda^k\zeta^k}{1-\lambda\zeta} \ll \frac{1}{1-\lambda\zeta}. \tag{C.2}$$

Proof. The proof follows from the equation $1/(1-\lambda\zeta) = 1+\lambda\zeta+\lambda^2\zeta^2+\dots$. $\square$

Proposition C.4. *For any* $\alpha, k \in \mathbb{N}$ *and* $\lambda > 0$,

$$\frac{d}{d\zeta}\frac{\zeta^k}{(1-\lambda\zeta)^\alpha} \ll (k+\alpha)\frac{\zeta^{k-1}}{(1-\lambda\zeta)^{\alpha+1}}. \tag{C.3}$$

Proof. Explicit computation yields

$$\frac{d}{d\zeta}\frac{\zeta^k}{(1-\lambda\zeta)^\alpha} = \frac{k\zeta^{k-1}}{(1-\lambda\zeta)^\alpha} + \frac{\alpha\lambda\zeta^k}{(1-\lambda\zeta)^{\alpha+1}} \ll (k+\alpha)\frac{\zeta^{k-1}}{(1-\lambda\zeta)^{\alpha+1}}. \quad \square$$

Proposition C.5. *For any $l \in \mathbb{Z}_+$ and constant $a, \lambda > 0$, consider the sequence*

$$b_0 = \frac{a_0 \zeta^{l+1}}{1 - \lambda\zeta}, \qquad b_{s+1} = a \frac{d}{d\zeta}\left(\frac{\zeta^2}{1-\lambda\zeta}\right)\frac{db_s}{d\zeta}. \tag{C.4}$$

Then

$$b_s \ll \frac{a_0 s!\, \zeta^{l+1}}{1-\lambda\zeta}\left(\frac{3a(l+3)}{(1-\lambda\zeta)^3}\right)^s. \tag{C.5}$$

Moreover, let

$$A = 3a(l+3)\frac{(1+\sigma)^2}{\sigma^2}$$

for some $\sigma > 0$. Suppose that $a = a(\sigma)$ is so small that $A < 1$. Then

$$\sum_{s=0}^{\infty} \frac{b_s}{s!} \ll \frac{a_0 \zeta^{l+1}}{1 - A - (1+\sigma)\lambda\zeta}. \tag{C.6}$$

Proof. First, we prove (C.5) by induction. The case $s = 0$ is obvious. If (C.5) holds for some $s \geq 0$, we have

$$b_{s+1} = a_0\, s!\, a^{s+1} \cdot 3^s (l+3)^s \frac{d}{d\zeta}\left(\frac{\zeta^2}{1-\lambda\zeta}\right)\frac{d}{d\zeta}\left(\frac{\zeta^{l+1}}{(1-\lambda\zeta)^{3s+1}}\right).$$

Then, by (C.3),

$$b_{s+1} \ll a_0\, s!\, a^{s+1} \cdot 3^{s+1}(l+3)^s(l+3s+2)\frac{\zeta^{l+1}}{(1-\lambda\zeta)^{3s+4}}.$$

It remains to apply the obvious inequality $l + 3s + 2 < (s+1)(l+3)$.

Now we prove (C.6). By Proposition C.1, we have

$$\frac{1}{(1-\lambda\zeta)^3} \ll \frac{1}{1-(1+\sigma)\lambda\zeta}\,\frac{1}{(1-\lambda\zeta)^2} \ll \frac{(1+\sigma)^2}{\sigma^2}\,\frac{1}{(1-(1+\sigma)\lambda\zeta)^3}.$$

Therefore,

$$\sum_{s=0}^{\infty}\frac{b_s}{s!} \ll \frac{a_0\zeta^{l+1}}{1-\lambda\zeta}\sum_{s=0}^{\infty}\left(\frac{A}{1-(1+\sigma)\lambda\zeta}\right)^s \ll \frac{a_0\zeta^{l+1}}{1-A-(1+\sigma)\lambda\zeta}. \qquad \square$$

D. OPERATORS $[\widehat{p}_j, \cdot\,]$ AND THEIR RIGHT INVERSE

For $j \in \{1, \dots, n\}$, consider the operator $[\widehat{p}_j, \cdot\,]\colon \mathcal{QO} \to \mathcal{QO}$.

Proposition D.1. *The operator $[\widehat{p}_j, \cdot\,]$ has a right inverse $\mathbf{I}_j\colon \mathcal{QO} \to \mathcal{QO}$,*

$$[\widehat{p}_j, \mathbf{I}_j(\cdot)] = \mathrm{id}_{\mathcal{QO}},$$

such that

$$\operatorname{Aver} \mathbf{I}_j(\pi(z)) \ll x_j \cdot x^\alpha p^\beta$$

for any monomial $z \in \mathbf{F}^{\alpha,\beta}$, $(\alpha, \beta) \in \mathbb{Z}_+^{2n}$.

Corollary D.1. *For any $F \in \mathcal{QO}$ and $f \in \mathcal{CO}$ such that $F(\widehat{x}, \widehat{p}) \ll f(x,p)$, we have*

$$\mathbf{I}_j F(\widehat{x}, \widehat{p}) \ll x_j \cdot f(x,p).$$

Proof of Proposition D.1. It is sufficient to consider the case when z contains only multipliers $\widehat{x}_j$ and $\widehat{p}_j$. Hence, we can set $n = 1$ and $(\alpha, \beta) \in \mathbb{Z}_+^2$. We denote, for short,

$$\widehat{x}_j = \widehat{x}, \qquad \widehat{p}_j = \widehat{p}, \qquad \mathbf{I}_j = \mathbf{I}.$$

We can also assume that $z \in \mathbf{F}_{\mathrm{prim}}^{\alpha,\beta}$.

Consider the case $\alpha \geq \beta$. Then, for some $l \in \{0, \dots, \beta\}$, we have $z = \widehat{p}^l \circ \widehat{x}^\alpha \circ \widehat{p}^{\beta - l}$. We set

$$\mathbf{I}(\pi(z)) = \frac{1}{\alpha + 1}\pi\big(\widehat{p}^l \circ \widehat{x}^{\alpha+1} \circ \widehat{p}^{\beta-l}\big).$$

Then the assertion of Proposition D.1 obviously holds.

The case $\alpha < \beta$ is more complicated. For some $l \in \{0, \dots, \alpha\}$, we have $z = \widehat{x}^l \circ \widehat{p}^\beta \circ \widehat{x}^{\alpha-l}$. Suppose, for definiteness, that $l \geq \alpha - l$ (the case $l < \alpha - l$ is analogous). We set

$$\begin{aligned}\mathbf{I}_j(\pi(z)) = \frac{1}{l+1}\pi(\widehat{x}^{l+1} \circ \widehat{p}^\beta \circ \widehat{x}^{\alpha-l}) - \frac{\alpha - l}{(l+1)(l+2)}\pi(\widehat{x}^{l+2} \circ \widehat{p}^\beta \circ \widehat{x}^{\alpha-l-1}) \\ + \frac{(\alpha-l)(\alpha-l-1)}{(l+1)(l+2)(l+3)}\pi(\widehat{x}^{l+3} \circ \widehat{p}^\beta \circ \widehat{x}^{\alpha-l-2}) - \ldots + \frac{(-1)^{\alpha-l}(\alpha-l)!}{(l+1)(l+2)\dots(\alpha+1)}\pi(\widehat{x}^{\alpha+1} \circ \widehat{p}^\beta).\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{Aver}\mathbf{I}(\pi(z)) &= \left(1 + \frac{\alpha - l}{l+2} + \frac{(\alpha-l)(\alpha-l-1)}{(l+2)(l+3)} + \ldots + \frac{(\alpha-l)!}{(l+2)\dots(\alpha+1)}\right)\frac{x^{\alpha+1}p^\beta}{l+1} \\ &\ll x^{\alpha+1}p^\beta. \qquad \square\end{aligned}$$

E. PROOF OF LEMMA 17.1

Estimate (17.3) follows from the definition of $Q_j^{(m)}$. Now let us construct a solution of (17.2). The equations $\big[\widehat{p}_s + P_s^{(m)}, \widehat{p}_l + P_l^{(m)}\big] = 0$ imply that

$$\big[\widehat{p}_s, P_l^{(m)}\big] - \big[\widehat{p}_l, P_s^{(m)}\big] = -\big[P_s^{(m)}, P_l^{(m)}\big] = O_{2^{m+1}}(\widehat{x}, \widehat{p}), \qquad s, l = 1, \dots, n.$$

Substituting $\check{P}_j^{(m)} + Q_j^{(m)}$ for $P_j^{(m)}$ in this equation, we obtain

$$\big[\widehat{p}_s, \check{P}_l^{(m)}\big] - \big[\widehat{p}_l, \check{P}_s^{(m)}\big] = O_{2^m+1}(\widehat{x}, \widehat{p}) = 0$$

because $\deg\big[\widehat{p}_s, \check{P}_l^{(m)}\big]$ and $\deg\big[\widehat{p}_l, \check{P}_s^{(m)}\big]$ do not exceed 2^m.

We set $\chi^{(1)} = \mathbf{I}_1(\check{P}_1^{(m)})$, where the operator $\mathbf{I}_1$ (a right inverse to $[\widehat{p}_1, \cdot\,]$) is defined in Appendix D. Then

$$\big[\widehat{p}_1, \chi^{(1)}\big] = \check{P}_1^{(m)}, \qquad \chi^{(1)} \ll \frac{\mu_m \xi^{2^m+2}}{1 - \lambda_m \xi}.$$

For any $j = 2, \dots, n$, the observable

$$\Phi_j^{(1)} := \check{P}_j^{(m)} - \big[\widehat{p}_j, \chi^{(1)}\big]$$

is a polynomial that contains only terms of degrees $2^m + 1, 2^m + 2, \dots, 2^{m+1}$. Moreover, it does not depend on $\widehat{x}_1$. Indeed,

$$\big[\widehat{p}_1, \Phi_j^{(1)}\big] = \big[\widehat{p}_1, \check{P}_j^{(m)}\big] - \big[\widehat{p}_j, \check{P}_1^{(m)}\big] = 0.$$

In fact, $\Phi_j^{(1)}$ equals the sum of all terms of $\check{P}_j^{(m)}$ that are independent of $\widehat{x}_1$.

We set $\chi^{(2)} = \mathbf{I}_2(\Phi_2^{(1)})$. Then

$$[\widehat{p}_2, \chi^{(1)} + \chi^{(2)}] = \check{P}_2^{(m)}, \qquad \chi^{(2)} \ll \frac{\mu_m \xi^{2^m+2}}{1-\lambda_m\xi},$$

and for any $j = 3, \dots, n$, the observable

$$\Phi_j^{(2)} := \check{P}_j^{(m)} - [\widehat{p}_j, \chi^{(1)} + \chi^{(2)}]$$

is a polynomial that contains only terms of degrees $2^m+1, 2^m+2, \dots, 2^{m+1}$. Moreover, it does not depend on $\widehat{x}_1$ and $\widehat{x}_2$. We set $\chi^{(3)} = \mathbf{I}_3(\Phi_3^{(2)})$.

Continuing similarly, we define $\chi^{(4)}, \dots, \chi^{(n)}$. Finally, we set $\chi_m = \chi^{(1)} + \dots + \chi^{(n)}$. $\square$

F. PROOF OF LEMMA 17.2

We have

$$Q_j^{(m)} \ll \frac{\mu_m \lambda_m^{2^m} \xi^{2^{m+1}+1}}{1-\lambda_m\xi} = b_0,$$

where b_0 is defined by (C.4) with

$$\zeta = \xi, \qquad a_0 = \mu_m \lambda_m^{2^m}, \qquad \lambda = \lambda_m, \qquad l = 2^{m+1}.$$

By Proposition C.3,

$$\chi_m \ll \frac{n\mu_m \xi^{2^m+2}}{1-\lambda_m\xi} \ll \frac{n\mu_m \lambda_m^{-2^m} \xi^2}{1-\lambda_m\xi}.$$

Therefore,

$$[\chi_m, Q_j^{(m)}] \ll \left\{\!\!\left\{ \frac{n\mu_m\lambda_m^{-2^m}\xi^2}{1-\lambda_m\xi}, \frac{\mu_m\lambda_m^{2^m}\xi^{2^{m+1}+1}}{1-\lambda_m\xi} \right\}\!\!\right\} \ll 2n\frac{d}{d\xi}\left(\frac{n\mu_m\lambda^{-2^m}\xi^2}{1-\lambda_m\xi}\right)\frac{db_0}{d\xi} \ll b_1,$$

where b_1 is defined by (C.4) with

$$a = 2n^2\mu_m\lambda_m^{-2^m}, \qquad k = 2^m. \tag{F.1}$$

Similarly, $[\chi_m, [\chi_m, Q_j^{(m)}]] \ll b_2$, and so on. Hence, by Proposition C.5,

$$U_j^{(m+1)} \ll \sum_{s=0}^{\infty} \frac{b_s}{s!} \ll \frac{\mu_m \lambda_m^{2^m} \xi^{2^{m+1}+1}}{1 - A_m - (1+\sigma)\lambda_m\xi}.$$

Now we estimate $V_j^{(m+1)}$. We have

$$[\chi_m, \check{P}_j^{(m)}] \ll \left\{\!\!\left\{ \frac{n\mu_m\xi^{2^m+2}}{1-\lambda_m\xi}, \frac{\mu_m\xi^{2^m+1}}{1-\lambda_m\xi} \right\}\!\!\right\} \ll 2n^2\mu_m^2(2^m+3)(2^m+2)\frac{\xi^{2^m+1}}{(1-\lambda_m\xi)^4}$$
$$\ll 2n^2\mu_m^2(2^m+3)^2\frac{(1+\sigma)^3}{\sigma^3}\frac{\xi^{2^m+1}}{1-(1+\sigma)\lambda_m\xi}.$$

(In the last estimate, we used Proposition C.1.) Therefore, $[\chi_m, \check{P}_j^{(m)}] \ll b_0$, where b_0 is defined by (C.4) with

$$\zeta = \xi, \qquad a_0 = 2n^2\mu_m^2(2^m+3)^2\frac{(1+\sigma)^3}{\sigma^3}, \qquad \lambda = (1+\sigma)\lambda_m, \qquad l = 2^{m+1}.$$

We obtain $\big[\chi_m\big[\chi_m, \check{P}_j^{(m)}\big]\big] \ll b_1$, where b_1 is defined by (C.4) with a as in (F.1). Repeating the arguments, we obtain

$$V_j^{(m)} \ll \sum_{s=0}^{\infty} \frac{b_s}{s!} \ll 2n^2\mu_m^2(2^m+3)^2 \frac{(1+\sigma)^3}{\sigma^3} \frac{\xi^{2^{m+1}+1}}{1-A_m-(1+\sigma)\lambda_m\xi}. \quad \square$$

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