Invariant tori in Hamiltonian systems with two degrees of freedom in a neighborhood of a resonance

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1 Introduction

The present note is based on the question, the first author listened from V.I.Arnold. The question is as follows. Consider a continuous family of invariant curves of an integrable two-dimensional exact symplectic map, and a resonant curve from this family. Take a perturbed map, which is also exact symplectic. Generically the resonant curve is destroyed, and in its neighborhood a domain of chaotic motions – the so called stochastic layer – appears. The boundary of the stochastic layer contains a pair of nonresonant curves which are a result of a deformation of curves from the initial family. What is the difference of frequencies on these boundary curves if the perturbation has the order $0 < \varepsilon \ll 1$?

Probably, both the question and the answer are known to specialists in KAM-theory. Nevertheless, we regard as useful to publish this note, since first, we do not know publications concerning this problem, and secondly, the answer to the raised question is not trivial, although it can be obtained as a combination of quite standard facts.

If the dynamical system in question is analytic, the difference of the frequencies is of order ε. This result is not obvious. At least, it cannot be obtained directly from the analysis of the Tailor expansion of the map in the perturbing parameter. It is well known that the the boundary invariant curves we discuss go essentially on the distance $\sim \sqrt{\varepsilon}$ from each other, but in some places this distance is exponentially small. Formally speaking, the answer is obtained as a result of the following calculation: $\sqrt{\varepsilon}/\log(e^{-c/\sqrt{\varepsilon}}) \sim \varepsilon$. If the system has a finite order of smoothness, the frequency difference is of order $\sqrt{\varepsilon}/\log\varepsilon$, because the argument of the logarithm in the previous formula is replaced in this case by ε^N, where N is the order of smoothness.

Note that we present only upper estimates for the frequency difference. However, we are sure that generically the lower estimates have the same order.

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2 Main theorems

2.1 The case of two degrees of freedom.

Consider the system with Hamiltonian

$$H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon). \tag{2.1}$$

Here $I \in \mathbf{R}^2$, $\varphi = (\varphi_1, \varphi_2) \bmod 2\pi$ are canonically conjugated coordinates, ε is a small parameter. The function H is assumed to be real-analytic in the phase variables and smooth in ε.

For $\varepsilon = 0$ the phase space of the system is foliated by two-dimensional invariant tori $\{(I, \varphi) | I = \mathrm{const}\}$. These tori are filled by quasi-periodic solutions:

$$\varphi = \omega(I)t + \varphi_0, \quad \omega(I) = \frac{\partial H_0}{\partial I}(I).$$

Recall that an invariant torus of the unperturbed ($\varepsilon = 0$) system is called resonant if $\langle k, \omega \rangle = 0$ for some nonzero vector $k \in \mathbf{Z}^2$. A nonresonant torus is called Diophantine if there are α, $\gamma > 0$ such that for any nonzero $k \in \mathbf{Z}^2$ the following inequality holds:

$$|\langle k, \omega \rangle| \geq 1/(\alpha |k|^\gamma). \tag{2.2}$$

Here $\langle \, , \rangle$ is the standard scalar product in $\mathbf{R}^2$. For small $\varepsilon > 0$ resonant tori are as a rule, destroyed, and, according to KAM-theory, Diophantine ones just slightly deform.

Suppose that the unperturbed system is isoenergetically non-degenerate i.e.,

$$\Delta = \det \begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2}(I^0) & \omega(I^0) \\ \omega^T(I^0) & 0 \end{pmatrix} \neq 0,$$

and on the energy level h there is a Diophantine torus

$$\mathbf{T}^2_{æ}(0, h) = \{(I, \varphi \,|\, I = I^0, \, H(I^0) = h, \, \omega_1(I^0)/\omega_2(I^0) = æ\}.$$

Then for small $\varepsilon > 0$ on the same energy level h there exists a torus $\mathbf{T}^2_{æ}(\varepsilon, h)$ with the same frequency ratio. We denote by

$$\Sigma(\varepsilon, h) = \{(I, \varphi) \in M | H(I, \varphi, \varepsilon) = h\}$$

the surface of energy level. The tori $\mathbf{T}^2_{æ}(\varepsilon, h) \subset \Sigma(\varepsilon, \, h)$ form a smooth in ε family.

Below all vectors are regarded as columns, and the superscript T denotes the operation of transposing.

Theorem 1 *Suppose that $H_0(I^0)=h$ and the following conditions hold*

1) $\omega(I^0)=\lambda(m,n)^T, \qquad \lambda\neq 0, n\neq 0$, *$m$ and n are either relatively prime integers, or $(m,n)=(0,1)$.*

2) *The function*

$$\nu_{\text{aut}}(q)=\frac{1}{2\pi\Delta}\int_0^{2\pi} H_1\big(I^0, mt+\frac{q}{n}, nt, 0\big)\,dt \tag{2.3}$$

has an unique global minimum on the circle $q \bmod 2\pi$, and this minimum is non-degenerate.

Then for small $\varepsilon>0$ there are two invariant tori $\mathbf{T}^2_{æ_j}(\varepsilon,h)\subset\Sigma(\varepsilon,h)$, $j=1,2$ such that on them

a) the action variable I satisfies the inequality $|I-I^0|\leq C\sqrt{\varepsilon}$;

b) $æ_1<m/n<æ_2$,

c) $|æ_1-æ_2|<c\varepsilon$.

The positive constants c and C do not depend on ε.

Remark 2.1 *The function ν_{aut} is 2π-periodic. The periodicity becomes obvious after the change $t\mapsto\tau-q\frac{\gamma}{n}$, where $\delta,\gamma\in\mathbf{Z}$ are such that $\delta n+\gamma m=1$.*

Remark 2.2 *The assumption on the analyticity of the Hamiltonian* (2.1) *is essential: in the case of a finite smoothness the estimate c) is much weaker:*

$$|æ_1-æ_2|<c\frac{\sqrt{\varepsilon}}{\log\varepsilon}.$$

In the following subsection we present a non-autonomous version of Theorem 1.

2.2 The case of one-and-a-half degrees of freedom.

Consider the system with Hamiltonian

$$H(p,q,t,\varepsilon)=H_0(p)+\varepsilon H_1(p,q,t,\varepsilon), \tag{2.4}$$

where the scalar variables $p\in\mathbf{R}$ and $q=q \bmod 2\pi$ are canonically conjugated, $t=t\bmod 2\pi$. The function H is assumed to be real-analytic on the extended phase space $M=D\times\mathbf{T}^2$ and smooth in ε.

For $\varepsilon=0$ the phase space M is foliated by two-dimensional invariant tori $\{(p,q,t)|p=\text{const}\}$. The motion on the tori is quasi-periodic:

$$q=æ(p)t+q_0; \quad æ(p)=\frac{\partial H_0}{\partial p}.$$

In the non-autonomous case the definitions of (non-)resonance and Diophantine properties are the same, but one should take as the frequency vector $(æ,1)^T$.

Theorem 2 *Suppose that for some $p = p^0$ the following conditions hold:*

1) $æ(p^0) = m/n, \quad n \neq 0,$ *m and n are either relatively prime integers, or* $m = 0$, $n = 1$.

2) The function

$$\nu_{\text{nonaut}}(q) = \frac{1}{2\pi}\Big(\frac{\partial^2 H_0}{\partial p^2}(p^0)\Big)^{-1}\int_0^{2\pi} H_1(p^0, mt + \frac{q}{n}, nt, 0)\,dt \tag{2.5}$$

has an unique global minimum on the circle $q \bmod 2\pi$, and this minimum is non-degenerate.

Then for small $\varepsilon > 0$ there are two invariant tori with frequencies $æ_1$ and $æ_2$ such that on these tori

a) the variable p satisfies the inequality $|p - p^0| < C\sqrt{\varepsilon}$;

b) $æ_1 < m/n < æ_2$*;*

c) $|æ_1 - æ_2| < c\varepsilon$.

Remark 2.3 *Theorem 2 follows from Theorem 1. Indeed, let us put $p = I_1$, $q = \varphi_1$, $t = \varphi_2$. Then trajectories of the system with Hamiltonian*

$$\hat{H} = H_0(I_1) + I_2 + \varepsilon H_1(I_1, \varphi_1, \varphi_2, \varepsilon), \tag{2.6}$$

are mapped by the projection $(I_1, I_2, \varphi_1, \varphi_2) \mapsto (I_1, \varphi_1, \varphi_2) = (p, q, t)$ to solutions of the system with Hamiltonian (2.4)*. The function* (2.6) *satisfies conditions of Theorem 1.*

2.3 Discrete case.

Consider a family of canonical self-maps P_ε of the two-dimensional cylinder $Z = \{(I, \varphi) \,|\, I \in \mathbf{R};\ \varphi \in \mathbf{T}^1\}$. The maps P_ε are determined by the generating function $S(J, \varphi, \varepsilon) = f(J) + \varepsilon W(J, \varphi, \varepsilon)$:

$$P_\varepsilon(I, \varphi) = (J, \psi), \quad J = I - S_\varphi(J, \varphi, \varepsilon), \quad \psi = \varphi + S_J(J, \varphi, \varepsilon).$$

The function S is real-analytic in J, φ, 2π-periodic in φ, and smooth in $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The map P_0 has the form:

$$J = I, \quad \psi = \varphi + f'(I).$$

The variable I enumerates circles (one-dimensional tori) invariant with respect to the action of P_0. We put

$$\mathbf{T}_æ = \{(I, \varphi) | \, I = I^0; \quad \frac{1}{2\pi} f'(I^0) = æ; \quad \varphi \in \mathbf{T}\}$$

Recall that in the discrete case the resonant and Diophantine conditions should be considered for the vector $(æ,\, 1)^T$.

Theorem 3 *Suppose that for $I = I^0$ the following conditions hold:*

1\. $æ(I^0) = \frac{m}{n}$*; m and n are either relatively prime integers, or $m = 0$, $n = 1$.*

2\. *The function*

$$\nu_{\text{discr}}(q) = \frac{1}{f''(I^0)} \sum_{k=0}^{n-1} W\big(I^0;\, \frac{2\pi k m + q}{n};\, 0\big)$$

has an unique minimum on the circle $q \bmod 2\pi$.

Then for small $\varepsilon > 0$ there exist two invariant circles with frequencies $æ_1$ and $æ_2$ such that

1) *The variable I on these circles satisfies the inequality $|I - I^0| < C\sqrt{\varepsilon}$; $C > 0$;*

2) $æ_1 < \frac{m}{n} < æ_2$*;*

3) $|æ_1 - æ_2| < c\varepsilon$*; $c > 0$.*

Proof of Theorem 3 can be obtained in the following way. According to the theorem on the inclusion of a diffeomorphism into a flow [3, 4, 8], P_ε can be regarded as a time-2π map in some non-autonomous Hamiltonian system with real-analytic Hamiltonian $H = H(J,\, \psi,\, t,\, \varepsilon)$, 2π-periodic in ψ and t. Moreover, we can assume that the function $H(J,\, \psi,\, t,\, 0)$ equals $\frac{1}{2\pi} f(J)$.

To reduce Theorem 3 to Theorem 2, it is sufficient to check conditions 1) and 2) for the Hamiltonian H.

Let us present H in the form

$$H(J,\, \psi,\, t,\, \varepsilon) = \frac{1}{2\pi} f(J) + \varepsilon H_1(J,\, \psi,\, t) + O(\varepsilon^2).$$

The time-2π map in this system is as follows:

$$\begin{aligned} \psi &= \varphi + f'(J) + O(\varepsilon), \\ J &= I - \varepsilon \int_0^{2\pi} \frac{\partial H_1}{\partial \psi}\big(I,\, \frac{f'(I)}{2\pi}\xi + \varphi,\, \xi\big)\, d\xi + O(\varepsilon^2), \end{aligned}$$

Here $O(\varepsilon)$, $O(\varepsilon^2)$ are 2π-periodic in φ, real-analytic in φ, I, and smooth in ε functions. Comparing this map with the one generated by the function S:

$$\begin{aligned} &\psi = \varphi + f'(I) + O(\varepsilon), \\ &J = I - \varepsilon W_\varphi(I,\, \varphi,\, 0) + O(\varepsilon^2), \end{aligned}$$

we get:

$W_\varphi(J,\, \varphi,\, 0) = \int_0^{2\pi} \frac{\partial H_1}{\partial \psi}\big(I,\, \frac{f'(I)}{2\pi}\xi + \varphi,\, \xi\big)\, d\xi + C(J)$.

Here $C(J)$ is a real-analytic function. Condition 1 of Theorem 2 obviously holds. Condition 2 holds due to the following equality:

$$\nu_{\text{nonaut}}(q) = \frac{1}{2\pi n} \nu_{\text{discr}}(q).$$

3 Proof of Theorem 1

Without loss of generality we assume that $H_0(I^0) = 0$. Let us perform the canonical change of the variables $(I, \varphi) \mapsto (J, \psi)$; $I = I^0 + \Lambda^T J$; $\varphi = \Lambda^{-1}\psi$, where

$$\Lambda = \begin{pmatrix} n & -m \\ \gamma & \delta \end{pmatrix},$$

$$\det \Lambda = n\delta + m\gamma = 1; \quad \delta, \gamma \in \mathbf{Z}, \tag{3.1}$$

and reduce the order on the level of the energy integral

$$H(I^0 + \Lambda^T J, \Lambda^{-1}\psi, \varepsilon) = 0. \tag{3.2}$$

Solving the equation (3.2) with respect to J_2, we get:

$$-J_2 = \frac{\Delta}{2\lambda} J_1^2 + \frac{\varepsilon}{\lambda} H_1(I^0, \Lambda^{-1}\psi, 0) + f(J_1, \psi, \varepsilon). \tag{3.3}$$

The function f is real-analytic in J_1, ψ and 2π-periodic in ψ. Moreover, $f = O(J_1^3, \varepsilon^2, \varepsilon J_1)$ as $J_1 \to 0$; $\varepsilon \to 0$. The function $-J_2 = -J_2(J_1, \psi_1, \psi_2, \varepsilon)$ is the Hamiltonian of the system on the zero energy level, where ψ_2 should be regarded by the new time, and the variables J_1 and ψ_1 are canonically conjugated. After the change of the variables $J_1 = \sqrt{\varepsilon}p$, $\psi_1 = q$, $\psi_2 = t$, $J_2 = -\sqrt{\varepsilon}\hat{H}$, the Hamiltonian (3.3) takes the form

$$\hat{H}(p, q, t, \varepsilon) = \frac{\Delta\sqrt{\varepsilon}}{\lambda}(p^2/2 + V_1(q, t)) + \varepsilon V_2(p, q, t, \varepsilon), \tag{3.4}$$

where $V_1(q, t) = \frac{1}{\Delta} H_1(I^0, \delta q + mt, nt - \gamma q, 0)$. The function V_2 is real-analytic and 2π-periodic in q and t, and smooth in $\sqrt{\varepsilon}$.

In the new variables the resonance we study takes the simplest form: the first frequency vanishes and the second one (corresponding to time) equals 1. Hence, now it is sufficient to find two invariant tori whose first frequencies are of order ε and have different signs.

Theorem 4 *[6] There exists a canonical real-analytic, 2π-periodic in $\tilde{x}$ and t, near-identical change of the coordinates $(p, q) \mapsto (\tilde{y}, \tilde{x})$ such that in the new coordinates $(\tilde{y}, \tilde{x})$ the Hamiltonian (3.4) has the form:*

$$H^*(\tilde{y}, \tilde{x}, t) = \frac{\Delta\sqrt{\varepsilon}}{\lambda}\Big(\frac{\tilde{y}^2}{2} + \nu_{\text{aut}}(\tilde{x})\Big) + \varepsilon U_2(\tilde{y}, \tilde{x}, \varepsilon) + \tilde{\psi}(\tilde{y}, \tilde{x}, \varepsilon, t). \tag{3.5}$$

The function H^* is 2π-periodic in $\tilde{x}$, $|\tilde{\psi}| \le c\exp(-\frac{\alpha}{\sqrt{\varepsilon}})$; $c, \alpha > 0$. The function U_2 is analytic in $\tilde{y}$, $\tilde{x}$.

The function (3.5) is 2π-periodic in t. If we put formally $\tilde{\psi} = 0$ then according to assumption 2) of Theorem 1 the remaining integrable part of the Hamiltonian (3.5) for small $\varepsilon > 0$ has the hyperbolic 2π-periodic solution $z(t) = (\hat{y},\, \hat{x},\, t)$ with two asymptotic surfaces. Here $(\hat{y},\, \hat{x}) = (0, x_0) + O(\sqrt{\varepsilon})$, where x_0 is the global minimum of the function $\nu_{\rm aut}$ on the circle $\mathbf{T}^1$. The domain in the space $\{\tilde{y},\, \tilde{x},\, t\}$, bounded by these surfaces, corresponds to the resonance in question. The solution $z(t)$ projects from the extended phase space $\{\tilde{y}, \tilde{x}, t\}$ to the space $\{\tilde{y}, \tilde{x}\}$ into a hyperbolic equilibrium, and the asymptotic surfaces to the corresponding separatrices. When the perturbation $\tilde{\psi}$ is taken into account, the solution $z(t)$ deforms slightly, and the asymptotic surfaces split [7].

Our following argument is shortly as follows. In the unperturbed ($\psi = 0$) system it is possible to introduce action-angle variables $(I,\, \varphi)$ so that $I \to 0$ when a point approaches the separatrices. Since the perturbation is exponentially small, from both sides of the resonance domain there exist invariant tori on which the action is exponentially small: $|I| < e^{-c/\sqrt{\varepsilon}}$. Unfortunately, this almost obvious fact is not a direct consequence of standard theorems from KAM-theory. The reason is that frequencies on these tori are not quite usual: the frequency corresponding to the angle variable φ is small and tends to zero together with ε. However, the situation can be reduced to standard by passing to the separatrix map (see below). When it is proven that on some pair of invariant tori $|I| < \mathrm{e}^{-\frac{c}{\sqrt{\varepsilon}}}$, it remains to use the fact that the frequency ratio on invariant tori close to separatrices is of order $-1/\log|I|$ (see Lemma 3.1 below).

Consider the time-2π map in the system with Hamiltonian (3.5):

$$T_\varepsilon : N \to N, \qquad N = D_{\tilde{y}} \times T^1_{\tilde{x}}, \qquad \varepsilon > 0,$$

where $T^1_{\tilde{x}}$ is the circle parametrized by the variable $\tilde{x}$.

On Fig. 1 the fixed point $\hat{z}_\varepsilon = z_\varepsilon(0)$ and the corresponding separatrices of the map T_ε are presented. Below we present the construction of the separatrix map. Details can be found in [9].

For any small $\varepsilon > 0$ near the point $\hat{z}_\varepsilon$ there exist normal coordinates (x, y) [5]. Their basic properties are as follows:

1) the coordinates of the point $\hat{z}_\varepsilon$ vanish: $\hat{z}_\varepsilon = (0, 0)$;

2) the coordinates (x, y) are symplectic: $d\tilde{x} \wedge d\tilde{y} = dx \wedge dy$;

3) the map T_ε has the form $L_\varepsilon(x, y) = (x/\Lambda, y\Lambda)$; the function $\Lambda(xy, \varepsilon)$ is real-analytic in the first argument at the point (0,0), and $\Lambda(0, \varepsilon) = 1 + O(\sqrt{\varepsilon})$ is the larger multiplier of the solution $z_\varepsilon(t)$.

The map L_ε has the first integral xy.

The normal coordinates can be continued along the separatrices as follows. Let the coordinates of a point $z \in N$ equal (x, y). Then the points $T^n_\varepsilon(z)$, $n \in \mathbf{Z}$ by definition have the coordinates $L^n_\varepsilon(x, y)$. Far from the point $\hat{z}$ we have at least two coordinate systems (defined by means of iteraions of T_ε and T^{-1}_ε respectively). The correspondence between these coordinate systems is given by the so called gluing maps (see Fig. 1):

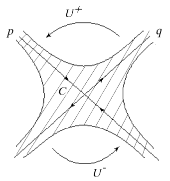

Figure 1:

$$
\begin{aligned}
U_\varepsilon^+ &: \{(x,y)|\, x \,\text{is small},\, y>0, y\sim 1\} &\to& \quad \{(x,y)|x>0,\, x\sim 1,\, y\,\text{is small}\},\\
U_\varepsilon^- &: \{(x,y)|\, x \,\text{is small},\, y<0, y\sim 1\} &\to& \quad \{(x,y)|x<0,\, x\sim 1,\, y\,\text{is small}\}.
\end{aligned}
$$

The maps L_ε and $U_\varepsilon^\pm$ commute: $L_\varepsilon U_\varepsilon^\pm = U_\varepsilon^\pm L_\varepsilon$.

Theorem 5 *[9] The gluing maps $U_\varepsilon^\pm$ are real-analytic in the domain $\{|x| < x_0,\quad y_0 \le \pm y \le Y_0\}$ and have the form*

$$U_\varepsilon^\pm(x,y) = \Big(\frac{\alpha_\pm^2}{y} + xk_\pm(xy), xy\Big(\frac{\alpha_\pm^2}{y} + xk_\pm(xy)\Big)^{-1}\Big) + (xf_\pm(x,y,\varepsilon), yg_\pm(x,y,\varepsilon)),$$

where $f_\pm$ and $g_\pm$ are such that

$$f_\pm(x/\Lambda, y\Lambda, \varepsilon) = f_\pm(x,y,\varepsilon), g_\pm(x/\Lambda, y\Lambda, \varepsilon) = g_\pm(x,y,\varepsilon),$$

and

$$|xf_\pm| + |yg_\pm| \le \hat{c}\exp\Big(-\frac{\alpha_\pm}{\sqrt{\varepsilon}}\Big), \quad \hat{\alpha}, \hat{c} > 0.$$

The dynamics of $T_\varepsilon : (x,y) \mapsto (X,Y)$ in a neighborhood of the separatrices can be explicitly studied if the maps $U_\varepsilon^\pm$ and L_ε are known. Indeed, in the normal coordinates T_ε acts as L, until the trajectory enters the domain of definition of the map U_ε^+ or U_ε^-. After application of a gluing map we again can use L_ε, and so on.

To get rid of the map L_ε, we perform the identification $(x,y) \sim (\Lambda^{-1}x, \Lambda y)$. Now all the information about the dynamics is contained in gluing maps. Below it is more convenient to use other coordinates:

$$I = xy\log\Lambda; \quad \varphi = \frac{\log|y|}{\log\Lambda}; \quad J = XY\log\Lambda; \quad \psi = \frac{\log|Y|}{\log\Lambda}, \quad \lambda = \log\Lambda(0). \tag{3.6}$$

To restore the information on the sign of y lost in the change (3.6), we introduce the variable $\sigma = \operatorname{sign} y \in \{+,-\}$. The map $S_\varepsilon : (I,\varphi,\sigma) \mapsto (J,\psi,\rho)$, which is obtained from the gluing maps after the change (3.6) is called the separatrix map, corresponding to T_ε. This map first appeared in [2, 10]. Here we use another construction which leads however to the same result. The explicit formulas for S_ε are as follows:

$$\begin{aligned} J &= I + r_1^\sigma(I,\varphi,\varepsilon), \\ \psi &= \varphi + \frac{1}{\lambda}\log\left|\frac{I}{\alpha_\sigma^2\lambda}\right| + O(I) + r_2^\sigma(I,\varphi,\varepsilon), \\ \rho &= \sigma\,\operatorname{sign}(J). \end{aligned} \tag{3.7}$$

Here $\lambda = \log\Lambda(0)$, the functions r_i^σ are 1-periodic in φ, real-analytic in I and φ, and $|r_1^\sigma| + |r_2^\sigma| = O(e^{-\alpha/\sqrt{\varepsilon}})$. We look for invariant KAM curves of the separatrix map. To this end we perform in (3.7) the change:

$$I = e^{-\alpha/(2\sqrt{\varepsilon})}(1+u\lambda), \quad J = e^{-\alpha/(2\sqrt{\varepsilon})}(1+v\lambda).$$

Here we assume that $u, v \in [-1/2; 1/2]$. The map (3.7) takes the form

$$v = u + a_1(\varphi,u,\varepsilon); \quad \psi = \varphi + æ(u,\varepsilon) + a_2(\varphi,u,\varepsilon); \quad \rho = \sigma = \sigma_0, \tag{3.8}$$

$$æ = -\tfrac{\alpha}{2\lambda\sqrt{\varepsilon}} - \tfrac{1}{\lambda}\log(\alpha^2\lambda) + \tfrac{1}{\lambda}\log(1+\lambda u). \tag{3.9}$$

The functions a_i are 1-periodic in φ real-analytic in u, φ, and satisfy the estimate $|a_1| + |a_2| = O(e^{-\alpha/(2\sqrt{\varepsilon})})$,

Theorem 6 *[5] Suppose that for some $u^0 \in [-1/2; 1/2]$ the frequency vector $\omega = (æ^0, 1)^T$, $æ^0 = æ(u^0,\varepsilon)$ is Diophantine, see (2.2). Then for small $\varepsilon > 0$ there exist real-analytic, 1-periodic functions $P(s)$, $Q(s)$ such that the curve*

$$(\varphi, u)(s) = (s + P(s), u^0 + Q(s)) \tag{3.10}$$

is invariant with respect to the map (3.8), and on it the map is a rigid shift: $s \mapsto s + æ^0$. Moreover, the following estimates hold: $|P| + |Q| \le \delta e^{-\alpha/(2\sqrt{\varepsilon})}$, $\delta = \text{const} > 0$.

It is easy to show that for some $u_0 \in [-1/2; 1/2]$ the frequency (3.9) is Diophantine. The curve (3.10) corresponds to a closed invariant curve of the map (3.7), where the variable I satisfies the inequality $|I| < c\,e^{-\alpha/(2\sqrt{\varepsilon})}$, and $\sigma = \rho$. This implies that the map T_ε has a closed invariant curve l, on which according to (3.6)

$$|xy| \le \frac{c}{\sqrt{\varepsilon}} e^{-\alpha/(2\sqrt{\varepsilon})}. \tag{3.11}$$

The equality $\sigma = \rho$ means that the curve l, passes the hyperbolic fixed point $\hat{z}$ only once, in other words, it is outside the resonant domain.

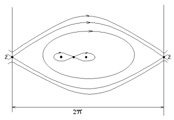

Figure 2:

Now it remains to estimate the frequency, corresponding to l in terms of the variables $(\tilde{x},\tilde{y})$. Putting in the system (3.5) formally $e^{-\alpha/\sqrt{\varepsilon}}=0$, we introduce the action-angle variables (K,Θ). Then the full system (3.5) takes the form:

$$\mathcal{H}=\sqrt{\varepsilon}\mathcal{H}_0(K,\varepsilon)+e^{-\alpha/\sqrt{\varepsilon}}\mathcal{H}_1(K,\Theta,\varepsilon,t) \tag{3.12}$$

The variables (K,Θ) are introduced in domains not containing separatrices of the system $(3.5)|_{\tilde{\psi}=0}$. We can assume that $K\to 0$ when a point approaches separatrices. The curve l corresponds to an invariant torus in the system (3.12).

The functions xy and $K(x,y)$ are first integrals of the system (3.12) provided $e^{-\alpha/\sqrt{\varepsilon}}=0$. The function $K(x,y)$ is continuous and is real-analytic outside the separatrices $\{xy=0\}$. According to Lemma 3.1, $K=O(xy\log xy)$ as $xy\to 0$. Therefore, due to (3.11), the invariant torus is situated in the domain $|K|<K_0\varepsilon^{-1}e^{-\alpha/(2\sqrt{\varepsilon})}$. The frequency on this torus is as follows: $|æ|=|\sqrt{\varepsilon}\frac{\partial H_0}{\partial K}|$. According to Corollary 3.1, $|\omega|\le\alpha\varepsilon/4$, where K_0 are positive constants. Theorem 1 is proved.

3.1 Frequencies on invariant curves, close to separatrices

Consider an autonomous Hamiltonian system with one degree of freedom in the domain $D\subset\mathbf{R}\times\mathbf{T}$. The Hamiltonian $H:D\to\mathbf{R}$ is assumed to be real-analytic. Suppose that there is a hyperbolic equilibrium $\hat{z}\in D$, and the corresponding separatrices are doubled (see Fig. 2). Three families of invariant closed curves have the separatrix loops of the point $\hat{z}$ as limit curves. For one of these families both loops form a limit curve, and for other two only one of the loops is limit. Consider one of the last two families. Following the standard procedure, we assign to each curve a value of the action variable I, where $I\to 0$ as a curve approach the separatrix.

Lemma 3.1 *The following relations hold true:*

$$H=\frac{\lambda I}{\log|I|}F\Big(\frac{1}{\log|I|},\frac{\log|\log|I||}{\log|I|},I\Big). \tag{3.13}$$

The function F is real-analytic at the point $(0,0,0)$ and $F(0,0,0)=1$. By $\pm\lambda$ we denote characteristic exponents of the equilibrium $\hat{z}$.

Differentiating the equality (3.13), we get:

Corollary 3.1

$$\omega = \frac{\partial H}{\partial I} = \frac{\lambda}{\log|I|} R\Big(\frac{1}{\log|I|}, \frac{\log|\log|I||}{\log|I|}, I\Big).$$

where R is real-analytic at the point $(0,0,0)$, and $R(0,0,0)=1$.

Proof. Let (x,y) be normal coordinates in a neighborhood of the hyperbolic point $\hat{z}$. The function H has the form $\tilde{H} = \tilde{H}(\xi) = \lambda\xi + O(\xi^2)$, $\xi = xy$. Let us find ξ from the equation $h = \tilde{H}(\xi)$. We have: $\xi = \frac{1}{\lambda} h f(h)$, where f is real-analytic at the point $h=0$, and $f(0)=1$. We can assume that $x>0, y>0$ on the family of invariant curves under consideration (other cases can be considered analogously). We denote

$$K_\tau = \{(x,y) \in \mathbf{R}^2 | x,y \in [0,\tau]\}; \quad l_h = \{(p,q) \in D | H(p,q) = h\},$$

The closed curve l_h can be broken into two parts: $l_1 = l_h \cap K_h$ and $l_2 = l_h \backslash K_h$.

As is well-known [1], $2\pi I = \int_{l_h} p\,dq + \text{const}$. The canonical coordinates (p,q) and (x,y) are related as follows: $pdq = ydx + dS$, where the function $S = S(x,y)$ is real-analytic in x,y at zero. Therefore,

$$2\pi I = \int_{l_h} p\,dq = \int_{l_1} y\,dx + \int_{l_1} dS + \int_{l_2} p\,dq.$$

We have:

$$\int_{l_1} y\,dx = \int_{\lambda^{-1}hf(h)/\tau}^{\tau} \frac{hf(h)}{\lambda x}\,dx = \frac{hf(h)}{\lambda}\big(2\log\tau + \log\lambda - \log|h| f(h)\big).$$

The family of curves $l_2 = l_2(h)$ is real-analytic in h at zero. Therefore, $\int_{l_2} pdq$ is real-analytic at zero. Hence, to find the function $H(I) = h$, we should solve the equation

$$I = -\frac{1}{\lambda} h f(h) \log|h| + h g(h), \tag{3.14}$$

with respect to h, where f and g are real-analytic at zero. We put $h = -\frac{\lambda I}{\log|I|}\chi$, where $\chi(I)$ is a new unknown function. Substituting the last equality into (3.14), we get:

$$G(\chi, I, 1/\log|I|, \log|\log|I||/\log|I|) = 0,$$

where G is real-analytic at the point $(1,0,0,0)$, $G(1,0,0,0)=0$, and $\frac{\partial G}{\partial \chi}(1,0,0,0) \neq 0$. Now Lemma 3.1 follows from the implicit function theorem.

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