STABILITY LOSS DELAY FOR A PERIODIC TRAJECTORY IN A SYSTEM WITH A SLOWLY VARYING PARAMETER

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1. Introduction

Stability loss delay is an interesting, important and so far not completely clear phenomenon. Its essence is as follows. Consider a system of differential equations depending on a slowly varying parameter. Suppose that the system has an equilibrium position or a periodic trajectory for any fixed value of the parameter. Suppose also that the parameter passes through a bifurcational value: the equilibrium (periodic trajectory) losses stability but remains nondegenerate. In the case of equilibrium a pair of conjugate eigenvalues leaves the left half-plane not passing through zero. For periodic trajectory either a pair of conjugate multipliers leaves the unit circle not passing through the point 1, or one real multiplier goes away from the unit circle through the point -1. If the system is analytic, a delay of stability loss takes place: phase points attracted to the equilibrium (periodic trajectory) long before the moment of the bifurcation remain close to the unstable equilibrium (periodic trajectory) until change of the parameter is of order unit. Velocity of the parameter changing can be arbitrary small. In non-analytic systems (even in the C^∞ case) in general there is no such a delay of stability loss.

In the present paper stability loss delay for periodic trajectories is considered. We obtain a lower estimate for the time of the delay. Under some assumptions this estimate gives also an asymptotics of the escape time. We illustrate analytic results by results of numerical experiments.

The phenomenon of stability loss delay for equilibriums was discovered in [1] for one model system of ODE. In [2–4] it was shown that this phenomenon is unavoidable in analytic systems of differential equations and in analytic maps when a parameter passes slowly through a bifurcational value as was described above. For the case of equilibrium an asymptotic formula for the moment of departure from an unstable equilibrium is known under rather general conditions [4, 5]. Some aspects of stability loss delay for periodic trajectories are considered in [6]. There are examples of stability loss delay in some problems of laser physics, biophysics and chemical kinetics (references can be found in [7]).

2. Basic equations

We consider systems of the following form:

$$\dot{x} = f(x, \tau, \varepsilon), \quad \dot{\tau} = \varepsilon. \tag{2.1}$$

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Here ε is a small nonnegative parameter, and x belongs to an open subset of $\mathbb{R}^{n+1}$. It is useful to consider the system (2.1) for $\varepsilon = 0$ separately. It is as follows:

$$\dot{x} = f(x, \tau, 0), \quad \tau = \text{const.} \tag{2.2}$$

Suppose that for any fixed $\tau \in [\tau_1, \tau_2]$ the system (2.2) has a periodic solution continuously depending on τ. Let L_τ be the trajectory of this solution and let $T(\tau)$, $\omega(\tau) = 2\pi/T$, $\rho_1(\tau), \dots, \rho_n(\tau)$ be period, frequency and multipliers of L_τ respectively. Later we consider L_τ as sets in the space of the variables x, τ as well as in the space of the variables x.

We assume that for $\tau < \tau_* \in [\tau_1, \tau_2]$ the periodic solution L_τ is linearly asymptotically stable (all the multipliers are inside the unit circle) and it losses the stability at $\tau = \tau_*$. Either a pair of complex conjugate multipliers ρ_1 and $\rho_2 = \bar{\rho}_1$ leaves the unit circle and $\rho_{1,2}(\tau_*) \neq 1$, or one real multiplier ρ_1 leaves the unit circle through the point -1. The other multipliers remain inside the unit circle.

If the right-hand sides of the equations (2.1) can be continued analytically in x, τ and smoothly in ε to a complex neighborhood N of the periodic trajectory L_{τ_*} then the stability loss delay occur provided the neighborhood N can be chosen independent of ε [3].

3. Basic assumptions

Later on we assume that the periodic solutions L_τ exist for $\tau \in U$, where $U \subset \mathbb{C}$ is an open neighborhood of the point τ_*. Moreover, we need the following assumptions to hold:

*C*1. *In the domain U all the multipliers $\rho_l(\tau)$ are analytic in τ and distinct from unit. The frequency ω is analytic and does not vanish in U.*

*C*2. *In the domain U the multipliers $\rho_l(\tau)$ are parewise distinct.*

a. Escape of a pair of conjugate multipliers from the unit circle. Suppose that a pair of complex conjugate multipliers ρ_1 and $\rho_2 = \bar{\rho}_1$ leaves the unit circle at $\tau = \tau_*$ and $\rho_{1,2}(\tau_*) \neq 1$. In the stationary bifurcation theory such a loss of stability of a periodic solution is called the Poincaré-Andronov-Hopf bifurcation.

Without loss of generality we may assume that

$$\operatorname{Im} \rho_1(\tau_*) = -\operatorname{Im} \rho_2(\tau_*) < 0.$$

We put

$$\begin{gathered} \lambda_l^0(\tau) = \frac{\log \rho_l(\tau)}{T(\tau)}, \quad \operatorname{Im} \lambda_l^0(\tau_*) T(\tau_*) \in (-\pi, \pi), \\ \lambda_l^k(\tau) = \lambda_l^0(\tau) - ik\omega(\tau), \quad k \in \mathbb{Z}. \end{gathered} \tag{3.1}$$

Obviously, for any $k \in \mathbb{Z}$ the equality $\rho_l = \exp(T\lambda_l^k)$ holds. We have the following relations:

$$\lambda_1^k(\tau) = \bar{\lambda}_2^{-k}(\bar{\tau}), \qquad \tau \in U, \quad k \in \mathbb{Z}, \tag{3.2}$$

$$\operatorname{Im} \lambda_1^0(\tau_*) T(\tau_*) \in (-\pi, 0), \quad \operatorname{Re} \lambda_1^k(\tau_*) = \operatorname{Re} \lambda_2^k(\tau_*) = 0. \tag{3.3}$$

Below an important role is played by the functions

$$\psi_l^k(\tau) = \int_{\tau_*}^{\tau} \lambda_l^k(s)\,ds, \quad \phi_l^k(\tau) = \operatorname{Re}\psi_l^k(\tau), \quad \phi^{\pm}(\tau) = \mp\operatorname{Re}\int_{\tau_*}^{\tau} i\omega(s)\,ds, \quad \tau \in U. \tag{3.4}$$

Obviously, $\phi^{\pm}(\tau) = 0$ for real τ. Later on we need the following assumption

*C*3. *The equation* $\phi^{\pm}(\tau) = 0$, $\tau \in U$ *has only real solutions.*

Let $\gamma_l^k(\sigma)$ be level lines of the functions $\phi_l^k(\tau)$: $\gamma_l^k(\sigma) = \{\tau \in \mathbb{C} : \phi_l^k(\tau) = \sigma\}$. According to (3.3), the functions ϕ_1^k and ϕ_2^k, restricted to the real axis, have a critical point $\tau = \tau_*$. Consequently, the curves $\gamma_l^k(0)$, $l = 1, 2$ are tangent to the real axis at the point τ_*. Since the functions $\operatorname{Re}\lambda_l^k(\tau)$, $l = 1, 2$, $\tau \in \mathbb{R}$ change sign from $-$ to $+$ when τ passes through the point τ_*, the point τ_* is a local minimum of the functions ϕ_1^k and ϕ_2^k for real τ.

The relations (3.1)–(3.2) imply that for fixed $\sigma > 0$ the curves $\gamma_l^k(\sigma)$, $l = 1, 2$ intersect at the real points $\tau_{\pm}(\sigma)$, where

$$\tau_-(\sigma) < \tau_* < \tau_+(\sigma), \qquad \int_{\tau_*}^{\tau_{\pm}(\sigma)} \operatorname{Re}\lambda_l^k(s)\,ds = \sigma, \qquad s \in \mathbb{R}. \tag{3.5}$$

Note that due to (3.2) the curves $\gamma_1^k(\sigma)$ and $\gamma_2^{-k}(\sigma)$ are symmetric in the real axis.

Now consider the family $\gamma_1^k(\sigma)$, $\sigma > 0$ in more detail. Let $\Gamma^k(\sigma)$, $\sigma \geqslant 0$ be continuous in σ families of segments of the curves $\gamma^k(\sigma)$ satisfying the following conditions:

(1) endpoints of $\Gamma^k(\sigma)$ are $\tau_{\pm}(\sigma)$,
(2) the segments $\Gamma^k(0)$ degenerate to the point τ_*,
(3) $\Gamma^k(\sigma) \subset U$.

Proposition 3.1. *The families* $\Gamma^k(\sigma)$ *are defined on an interval* $[0, \sigma_0]$, $\sigma_0 > 0$.

Geometry of the curves $\Gamma^k(\sigma)$ is described by the following

Proposition 3.2. *Suppose that the equation* $|\rho_1(\tau)| = 1$ *has on the line segment* $[\tau_-(\sigma_0), \tau_+(\sigma_0)] \subset \mathbb{R}$ *the unique solution* $\tau = \tau_*$. *Let the families* $\Gamma^0(\sigma)$ *and* $\Gamma^{-1}(\sigma)$ *be defined for* $0 \leqslant \sigma \leqslant \sigma_0$. *Then the following assertions hold.*

(*a*) *The curves* $\Gamma^0(\sigma)$ (*respectively,* $\Gamma^{-1}(\sigma)$), $0 \leqslant \sigma \leqslant \sigma_0$ *lie in the upper* (*respectively, in the lower*) *half-plane.*

(*b*) *The families* $\Gamma^k(\sigma)$ *are also defined for* $0 \leqslant \sigma \leqslant \sigma_0$.

(*c*) *All the curves* $\Gamma^k(\sigma)$, $0 \leqslant \sigma \leqslant \sigma_0$ *lie in the closed domain* $\hat{D} \subset U$ *bounded by* $\Gamma^{-1}(\sigma_0)$ *and* $\Gamma^0(\sigma_0)$.

Let us fix σ_0 satisfying conditions of Proposition 3.2. We denote by D^{-1} (respectively, D^0) the domain bounded on the complex plane $\mathbb{C}_\tau$ by the curve $\Gamma^{-1}(\sigma_0)$ (respectively, $\Gamma^0(\sigma_0)$) and the line segment $[\tau_-(\sigma_0), \tau_+(\sigma_0)]$. Let $D_s^{-1}, D_s^0 \subset \mathbb{C}_\tau$ be the domains symmetric in the real axis to D^{-1} and D^0 respectively. Our basic assumptions are as follows:

*A*1. *Either* $D_s^{-1} \subset D^0$ *or* $D_s^0 \subset D^{-1}$.

We put $D = D^0 \cup D_s^0$ in the first case and $D = D^{-1} \cup D_s^{-1}$ in the second one. Without loss of generality we may assume that the first case takes place.

*A*2. *For any* $3 \leqslant l \leqslant n$ *and* $\tau \in D \cap \mathbb{R}$ *the inequalities* $|\rho_l(\tau)| < 1$ *hold.*

*A*3. *Each of the functions* $\phi_l^k(\tau), \phi^{\pm}(\tau)$ (*except* $\phi_1^0(\tau)$ *and* $\phi_2^0(\tau)$) *restricted to the boundary* ∂D *of the domain* D *has a unique maximum and maximums of the functions* $\phi^{\pm}|_{\partial D}$ *are nondegenerate.*

Remark. Since $\phi^{\pm}(\tau)$ vanish for real τ, the curve ∂D is smooth at the points of maximum of the functions $\phi^{\pm}|_{\partial D}$.

Proposition 3.3. *If the multipliers* $\rho_1(\tau), \rho_2(\tau)$, $\tau \in \mathbb{R}$ *cross the unit circle at the moment* $\tau = \tau_*$ *with a nonzero velocity then conditions* $A1$–$A3$ *are valid for small* $\sigma_0 > 0$.

b. Escape of a real multiplier from the unit circle through the point -1**.** In this subsection we assume that stability loss of the periodic solution L_τ occurs as a result of escape of the real multiplier ρ_1 from the unit circle through -1. This case turns out to be very similar to that considered in the previous subsection and all the constructions are almost the same. Here we again assume that the conditions $C1$–$C2$ hold.

First, we introduce the characteristic exponents λ_l^k by using relations (3.1), where the condition $\operatorname{Im} \lambda_1^0(\tau_*)T(\tau_*) \in (-\pi, \pi)$ is replaced by the following one: $\lambda_1^0(\tau_*)T(\tau_*) = -i\pi$.

Second, we introduce the functions $\psi_l^k(\tau), \phi_l^k(\tau), \phi^{\pm}(\tau)$ and the curves $\gamma_l^k(\sigma)$ in the same way as in subsection **a.** We assume that the functions $\phi^{\pm}(\tau)$ satisfy condition $C3$. The point τ_* is a local minimum of the functions ϕ_1^k for real τ and the curves $\gamma_1^k(0)$ are tangent to the real axis at the point τ_*. For fixed $\sigma > 0$ the curves $\gamma_1^k(\sigma)$ intersect at the real points $\tau_{\pm}(\sigma)$ satisfying (3.5) for $l = 1$.

Now we can consider families of the segments $\Gamma^k(\sigma)$, $\sigma \geqslant 0$ defined by conditions (1)–(3). Propositions 3.1–3.2 obviously, remain valid. Let σ_0 satisfy conditions of Proposition 3.2 and let $D \subset \mathbb{C}_\tau$ be the domain bounded by the curves $\Gamma^0(\sigma_0)$ and $\Gamma^{-1}(\sigma_0)$. We assume that the following conditions are valid:

*B*1. *For any* $2 \leqslant l \leqslant n$ *and* $\tau \in D$ *the inequalities* $|\rho_l(\tau)| < 1$ *hold.*

*B*2. *Each of the functions* $\phi_l^k(\tau), \phi^{\pm}(\tau)$ (*except* $\phi_1^0(\tau)$ *and* $\phi_1^{-1}(\tau)$) *restricted to the boundary* ∂D *of the domain* D *has a unique maximum and maximums of the functions* $\phi^{\pm}|_{\partial D}$ *are nondegenerate.*

We have the following analog of Proposition 3.3:

Proposition 3.4. *If the multiplier* $\rho_1(\tau)$, $\tau \in \mathbb{R}$ *cross the unit circle at the moment* $\tau = \tau_*$ *with a nonzero velocity then conditions* $B1$–$B2$ *hold for small* $\sigma_0 > 0$.

4. Estimates for time of the stability loss delay

In this section we present theorems estimating time of the stability loss delay in the both cases we deal with: when two complex conjugate multipliers leave unit circle and when one real multiplier does the same. We call these two cases **A** and **B** respectively.

We define the functions

$$\Pi\colon [\tau_-(\sigma_0), \tau_*] \to [\tau_*, \tau_+(\sigma_0)], \quad \hat{\Pi}\colon [\tau_-(\sigma_0), \tau_*] \to [\tau_*, \tau_+(\sigma_0)]$$

by the following relations

$$\int_{\tau_*}^{\tau} \operatorname{Re}\lambda_1^0(s)\,ds = \int_{\tau_*}^{\Pi(\tau)} \operatorname{Re}\lambda_1^0(s)\,ds, \quad \int_{\tau_*}^{\tau} \mu(s)\,ds = \int_{\tau_*}^{\hat{\Pi}(\tau)} \operatorname{Re}\lambda_1^0(s)\,ds,$$
$$\mu(\tau) = \max_{1\leqslant j\leqslant n} \operatorname{Re}\lambda_j^0(\tau).$$

The functions $\Pi(\tau)$ and $\hat{\Pi}(\tau)$ evidently coincide for τ close to τ_*. Consider a solution $x(t)$ of the system (2.1) with initial conditions at $t = t_0 = \tau_0/\varepsilon$, $\tau_0 \in [\tau_-(\sigma_0), \tau_*)$.

Theorem 4.1. *Suppose that the point $x(t_0)$ is inside c_1^{-1}-neighborhood of the periodic trajectory L_{τ_0}. Then for $\tau_0 + c_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \hat{\Pi}(\tau_0) - c_3\varepsilon|\log\varepsilon|$ the solution $x(t)$ is inside $c_4\varepsilon$-neighborhood of $L_{\varepsilon t}$.*

Here and below c_j are positive constants. Appearance of a constant c_j in some assertion means that there exists a constant c_j satisfying this assertion.

According to Theorem 4.1 for any solution attracted to the family L_τ at $\tau \approx \tau_0$, the loss of stability delays at least up to the moment $\tau \approx \hat{\Pi}(\tau_0)$.

To calculate an asymptotics of the escape moment we need additionary assumptions and notations. Let the following condition hold:

*A*4. $\operatorname{Re}\lambda_j^0(\tau) < \operatorname{Re}\lambda_1^0(\tau)$ *for all* $j = 3, \dots, n$ *and* $\tau \in [\tau_-(\sigma_0), \tau_*]$
for the case **A** and

*B*3. $\operatorname{Re}\lambda_j^0(\tau) < \operatorname{Re}\lambda_1^0(\tau)$ *for all* $j = 2, \dots, n$ *and* $\tau \in [\tau_-(\sigma_0), \tau_*]$
for the case **B**.

Obviously, *A*4 implies the identity $\hat{\Pi}(\tau) \equiv \Pi(\tau)$ in the case **A** and *B*3 implies the same identity in the case **B**.

If the point $x(t_0)$ is close to the trajectory L_{τ_0} then there exists unique point $x_{00} \in L_{\tau_0}$ which is the closest to $x(t_0)$. Consider a hyperplane $\Theta_0 \subset \mathbb{R}^{n+1}$ containing the point x_{00} and transversal to L_{τ_0} at x_{00}. In a sufficiently small neighborhood of the point x_{00} on the hyperplane Θ_0 the first return map of the system (2.2) with $\tau = \tau_0$ can be defined. The multipliers $\rho_1(\tau_0), \dots, \rho_n(\tau_0)$ are eigenvalues of the linearized at x_{00} first return map. Let Θ' and Θ'' be eigenspaces of the linearized first return map, which correspond to the multipliers ρ_1, ρ_2 and $\rho_3, \dots, \rho_n$ respectively in case **A**. In case **B** they correspond to ρ_1 and $\rho_2, \dots, \rho_n$. We denote by β' and β'' projections of the vector $x(t_0) - x_{00}$ to Θ' and Θ''. If $n = 2$ we put $\beta'' = 0$.

Theorem 4.2. *Suppose that the point $x(t_0)$ is inside c_5^{-1}-neighborhood of the periodic trajectory L_{τ_0}. Let $|\beta'| > c_6\varepsilon$ and $|\beta''| < c_7^{-1}\varepsilon$. Then*

1. *for $\tau_0 + c_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \Pi(\tau_0) - c_3\varepsilon|\log\varepsilon|$ the solution $x(t)$ is inside $c_4\varepsilon$-neighborhood of $L_{\varepsilon t}$,*
2. *the point $x(t_d)$ is outside c_8^{-1}-neighborhood of the trajectory $L_{\varepsilon t_d}$ for some t_d, where $|\varepsilon t_d - \Pi(\tau_0)| < c_9\varepsilon|\log\varepsilon|$.*

Assertion (1) of Theorem 4.2 is just a repetition of Theorem 4.1. According to assertion (2) , a solution attracted to the family L_τ at $\tau \approx \tau_0$ escapes from L_τ at $\tau \approx \Pi(\tau_0)$.

Introducing assumptions additionary to *A*1–*A*3 or to *B*1–*B*2 one can get lower estimates for the delay time, which are better than ones in Theorem 4.1 (see e.g.

assumptions $A4, B3$ and Theorem 4.2). Below we formulate conditions $A5$ and $B4$, which as a rule are less restrictive compared with $A4$ and $B3$. Under condition $A5, B4$ the delay time turns out to be greater than in Theorem 4.1 but less than in Theorem 4.2 (see Theorem 4.3). We denote

$$\nu_j^k(\tau) = \lambda_j^k(\tau) - \lambda_1^0(\tau), \quad \kappa_j^k(\tau) = \mathrm{Re}\int_{\tau_*}^{\tau} \nu_j^k(s)\,ds, \qquad k \in \mathbb{Z}.$$

Here $j = 3, \dots, n$ in case **A** and $j = 2, \dots, n$ in case **B**. Let $\delta_j^k(\sigma) \subset \mathbb{C}_\tau$ be level lines of the functions $\kappa_j^k(\tau)$. Assume that the following condition holds.

$A5, B4$. There exists a domain $\tilde{D} \subset D$ such that
(a) the set $\tilde{D} \cap \mathbb{R} \subset \mathbb{C}$ coincides with the line segment $[\tau_-(\sigma_0), \tau_+(\sigma_0)]$,
(b) any curve $\delta_j^k(\sigma)$ has at most two points in common with the boundary of the domain $\tilde{D}$.

We define the function $\tilde{\Pi}\colon [\tau_-(\sigma_0), \tau_*] \to [\tau_*, \tau_+(\sigma_0)]$ by the equality

$$\int_{\tau_*}^{\tau} \tilde{\mu}(s)\,ds = \int_{\tau_*}^{\tilde{\Pi}(\tau)} \mathrm{Re}\,\lambda_1^0(s)\,ds,$$

$$\tilde{\mu}(\tau) = \begin{cases} \max\{\mathrm{Re}\,\lambda_1^0(\tau), 2\mathrm{Re}\,\lambda_3^0(\tau), \dots, 2\mathrm{Re}\,\lambda_n^0(\tau)\} & \text{in case } \mathbf{A}, \\ \max\{\mathrm{Re}\,\lambda_1^0(\tau), 2\mathrm{Re}\,\lambda_2^0(\tau), \dots, 2\mathrm{Re}\,\lambda_n^0(\tau)\} & \text{in case } \mathbf{B}. \end{cases}$$

Obviously, $\hat{\Pi}(\tau) \leqslant \tilde{\Pi}(\tau) \leqslant \Pi(\tau)$. Under conditions $A5, B4$ we can improve estimate of Theorem 4.1 in the following way.

Theorem 4.3. *Suppose that the point $x(t_0)$ is inside c_1^{-1}-neighborhood of the trajectory L_{τ_0}. Then for $\tau_0 + c_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \tilde{\Pi}(\tau_0) - c_3\varepsilon|\log\varepsilon|$ the solution $x(t)$ is inside $c_4\varepsilon$-neighborhood of $L_{\varepsilon t}$.*

5. Discussion of results and conditions. Examples

When $\varepsilon \to 0$ the attraction of a solution to the family L_τ and the escape take place instantly in the slow time. The corresponding limit values of the slow time are called the moments of fall and escape. Dependence of the escape moment on the moment of fall, if it exists for a majority of initial conditions with the given moment of fall, determines an in-out function. In this definition one should take the initial conditions from a small $O(1)$-neighborhood of the periodic trajectory L_{τ_0}, where τ_0 is the moment of fall.

According to Theorem 4.2, under certain assumptions the in-out function coincides with the function Π introduced in Section 4. Theorems 4.1 and 4.3 give lower estimates for the moment of escape. Apparently, the estimate given in Theorem 4.1 is not sharp. (This estimate would be sharp if we did not assume that condition C2 holds, see Example 5.2, but without this condition the estimate has not been proved.) The estimate given by Theorem 4.3 is sharp (see Example 5.3).

Theorems 4.1 and 4.2 would be rather simple assertions if the surface $M = \bigcup_{\tau_1 \leqslant \tau \leqslant \tau_2} L_\tau$, formed by the periodic solutions L_τ, were invariant with respect to the phase flow of the system (2.1). This case can be called the case of a "trivial" delay of stability loss. In this case the solution falls by the moment $\tau = \tau_*$ to an exponentially narrow neighborhood of the surface M. Hence, the solution needs a long time ($\sim 1/\varepsilon$) to go aside from the surfase M. Theorems 4.1 and 4.2 give an estimate of this time and asymptotics of the moment of escape.

As a rule the surface M is not invariant. In proofs of Theorems 4.1-4.2 a surface M_ε is constructed which is invariant with respect to the phase flow of the system (2.1) and $O(\varepsilon)$-close to the surface M for $\tau \in I$, where $I \subset \mathbb{R}$ is an interval containing τ_*. It is important that in general there exists a maximal interval I_* for which a surface M_ε with such properties can be constructed (see Example 5.1 below). This interval is determined by behaviour of the system for complex values of τ. When the surface M_ε is constructed the original problem is reduced to the problem of analysis of a "trivial" stability loss delay.

Remarks. **1**. In general the surface M is not analytic in ε. The formal series in ε of the perturbation theory for M_ε diverges at $\tau = \tau_*$.

2. In the paper [3] a formal series of the perturbation theory for M_ε is cutted at terms of order $\varepsilon^{\mathrm{const}/\varepsilon}$. By this method the existence of a surface invariant with a precision $\sim \exp(-c_1^{-1}/\varepsilon)$ has been proved but the constant c_1 has not been controlled. The existence of an invariant surface M_ε can be easily derived from this. But for the interval $I = [a, b]$ only the estimate $a < \tau_* - c_2^{-1}, b > \tau_* + c_3^{-1}$ with constants c_2, c_3 we can not control is obtained.

To prove Theorem 4.3 an additional analysis is needed after constructing the surface M_ε and reducing the problem to the case of "trivial" delay. We introduce for the system (2.1) new coordinates such that the matrix of the system linearized near M_ε has a nill corner: the equations corresponding to the multipliers $\rho_{1,2}$ separate (recall that the multipliers $\rho_{1,2}$ are responsible for the stability loss). To construct these coordinates it is necessary again to deal with complex values of τ. The idea is similar to that of constructing of the surface M_ε. Estimates in the new variables lead to Theorem 4.3.

Below we present examples illustrating the phenomenon of stability loss delay and behavior of in-out function.

Example 5.1. (*Unit multiplier*). Consider the following linear nonhomogeneous system written in a complex form:

$$\dot z = (\tau - \alpha i)z + \varepsilon h(\vartheta), \quad \dot\vartheta = 1, \quad \dot\tau = \varepsilon, \qquad z = x_1 + i x_2, \tag{5.1}$$
$$h = \sum_{k=-\infty}^{\infty} h^k e^{ik\vartheta}, \quad h^k = e^{-|k|}, \quad 0 < \alpha < 1/2.$$

The variable ϑ is defined on the circle $\mathbb{R}/(2\pi\mathbb{Z})$. For $\varepsilon = 0$ the system has the periodic solution $z = 0$, $\dot\vartheta = 1$, $\tau = \mathrm{const}$. Its multipliers are $\rho_{1,2} = e^{2\pi(\tau\mp\alpha i)}$. The periodic solution is stable for $\tau < 0$ and unstable for $\tau > 0$. At $\tau = \tau_* = 0$ there are two conjugate multipliers on the unit circle. The eigenvalues introduced in Section 3 are $\lambda_{1,2}^k = \tau \mp i\alpha - ik$. The functions $\psi_{1,2}^k$ and $\phi_{1,2}^k$ are as follows:

$$\psi_{1,2}^k = (\tau \mp i\alpha - ik)^2/2 - (\mp i\alpha - ik)^2/2, \quad \phi_{1,2}^k = \mathrm{Re}\,\psi_{1,2}^k.$$

The eigenvalue λ_1^0 vanishes (and the multiplier ρ_1 becomes equal to unit) for $\tau = i\alpha$. The level lines $\phi_1^0 = \mathrm{const}$ passing through the point $i\alpha$ are two straight lines intersecting the real axis at the points $-\alpha$ and α. The other level lines of ϕ_1^0 are hyperbolas (see Fig. 1). Level lines of function ϕ_1^k can be obtained from level lines of ϕ_1^0 by the shift on k along the imaginary axis. Level lines of ϕ_2^k are symmetric to those of ϕ_1^{-k} in the real axis. Let D_* be the square bounded by the level (straight) lines of $\phi_{1,2}^0 = \text{const}$ passing through the points $\pm i$. We can choose as a domain D (see Section 3) any subdomain of D_* bounded by the hyperbolas $\phi_1^0 = \phi_1^0(\sigma)$ and $\phi_2^0 = \phi_2^0(\sigma)$ which intersect the real axis at the points $\pm\sigma$, $0 < \sigma < \alpha$.

To investigate stability loss delay in the system (5.1) we construct an integral surface M_ε of the form

$$z = \hat{z}(\tau, \vartheta, \varepsilon) = \sum_{k=-\infty}^{\infty} z^k(\tau, \varepsilon) e^{ik\vartheta}. \tag{5.2}$$

Coefficients z^k satisfy the differential equations

$$\dot{z}^k = (\tau - \alpha i - ik) z^k + \varepsilon h^k. \tag{5.3}$$

Hence, we reduce the problem on stability loss of a periodic solution to the problem on stability loss of an equilibrium.

Behavior of solutions of equations (5.3) is well-known (see e.g. [4]). In particular let us choose initial conditions at $\tau = \tau_0 = -\alpha - 1/2$ in the following way: $z^k(\tau_0, \varepsilon) = h^k$. Then for $k \neq 0$ we have $|z^k| < c_1 \varepsilon h^k$ for $\tau_0 + c_2 \varepsilon |\log \varepsilon| \leqslant \tau \leqslant -\tau_0 - c_2 \varepsilon |\log \varepsilon|$. The function z^0 satisfies the estimates

$$\begin{aligned} |z^0| &\leqslant c_1 \varepsilon \quad \text{for} \quad \tau_0 + c_2 \varepsilon |\log \varepsilon| \leqslant \tau \leqslant \alpha - c_2 \varepsilon |\log \varepsilon|, \\ |z^0| &> 1 \qquad \text{for} \quad \tau \geqslant \alpha + c_2 \varepsilon |\log \varepsilon|. \end{aligned}$$

Hence for the surface (5.2) we have:

$$\begin{aligned} |\hat{z}| &= O(\varepsilon) \quad \text{for} \quad \tau_0 + c_2 \varepsilon |\log \varepsilon| \leqslant \tau \leqslant \alpha - c_2 \varepsilon |\log \varepsilon|, \\ |\hat{z}| &> 1/2 \quad \text{for} \quad \tau = \alpha + c_2 \varepsilon |\log \varepsilon|. \end{aligned}$$

The magnitude $\xi = z - \hat{z}$ satisfies the homogeneous equation $\dot{\xi} = (\tau - \alpha i)\xi$.

Now it is easy to construct in-out function for the system (5.1). Phase points attracted to the family L_τ at $\tau \approx \tau_- \in (-\alpha, 0)$ escape at $\tau \approx \tau_+ = -\tau_-$. Therefore, for $\tau \in (-\alpha, 0)$ the in-out function coincides with the function Π introduced in Section 4 (cf. Theorem 4.2). Phase points attracted to L_τ at $\tau \approx \tau_- < -\alpha$ escape at $\tau \approx \alpha$. Such a metamorphose of the in-out function at the point $\tau = -\alpha$ happens because the Stokes line passes through this point. Here we call by the Stokes line the curve $\phi_1^0 = \text{const}$ going from the point αi, where the multiplier ρ_1 becomes equal to unit (cf. assumption C1).

Remark. In the example we have considered above the periodic solution L_τ and its multipliers are regular at the point, where a multiplier becomes equal to unit. This case is degenerate. In a typical system the periodic solution and its multipliers branch at the point, where unit multiplier appear. Such a situation occur in numerical example... (see Section).

Example 5.2. (*Double multiplier*). Consider the system

$$\begin{aligned} &\dot{z} = (\tau - \alpha i) z + \varepsilon w + \varepsilon h(\vartheta), \qquad && z = x_1 + i x_2, \\ &\dot{w} = (-\mu - \beta i) w, && w = x_3 + i x_4, \\ &\dot{\vartheta} = 1, \qquad \dot{\tau} = \varepsilon. \end{aligned} \tag{5.4}$$

Here h and α are the same as in (5.1), $0 < \beta < 1$ and $0 < \mu < \alpha$. Multipliers in the system (5.4) are $\rho_{1,2} = e^{2\pi(\tau \mp \alpha i)}$ and $\rho_{3,4} = e^{2\pi(-\mu \mp \beta i)}$. The system (5.4) has a double multiplier at $\tau = -\mu \pm (\alpha - \beta)i + ik$ and $\tau = -\mu \pm (\alpha + \beta)i + ik$, $k \in \mathbb{Z}$. In the example in question only the pair with $k = 0$ is essential. If the condition

$$\mu + |\alpha - \beta| < \alpha \tag{5.5}$$

holds, the double multiplier is inside the square D_*, see Fig 1. In this example the in-out function F can be calculated by using an explicit solution of (5.4).

Proposition 5.1. (a). *Suppose that the condition* (5.5) *is valid. Then*

$$F(\tau) = \begin{cases} -\tau, & -\mu - |\alpha - \beta| \leqslant \tau < 0, \\ \sqrt{(\alpha - \beta)^2 - \mu^2 - 2\tau\mu}, & -\frac{\alpha^2 + \mu^2 - (\alpha - \beta)^2}{2\mu} \leqslant \tau < -\mu - |\alpha - \beta|, \\ \alpha, & \tau < -\frac{\alpha^2 + \mu^2 - (\alpha - \beta)^2}{2\mu}. \end{cases}$$

(b). *If the condition* (5.5) *does not hold then*

$$F(\tau) = \begin{cases} -\tau, & -\alpha \leqslant \tau < 0, \\ \alpha & \tau < -\alpha. \end{cases}$$

Let condition (5.5) be satisfied. Theorem 4.1 gives us an estimate from below for the in-out function on the interval $(-\sqrt{\mu^2 + 2\alpha|\alpha - \beta| - (\alpha - \beta)^2}, 0)$ only. (The left endpoint of this interval is connected with the point τ where two multipliers coincide by a level line of the function ϕ_1^0). Condition A2 does not allow us to enlarge the interval. The functions Π and $\hat{\Pi}$, introduced in the Section 4, are as follows:

$$\Pi(\tau) = -\tau, \qquad \hat{\Pi}(\tau) = \begin{cases} -\tau, & -\mu \leqslant \tau < 0, \\ \sqrt{-2\mu\tau - \mu^2}, & \tau < -\mu. \end{cases}$$

We see, that $\hat{\Pi} < F < \Pi$ for $-\alpha < \tau < -\mu - |\alpha - \beta|$ and $\alpha \neq \beta$. If $\alpha = \beta$ (the double multiplier is on the real axis), $F(\tau) \equiv \hat{\Pi}$. This example shows that i) one can not replace $\hat{\Pi}$ by Π in the Theorem 4.1 and ii) the assertion of Theorem 4.1 may remain valid without assumption A2 about absence of double multipliers. We believe, that also in general case one can skip assumption A2 in the Theorem 4.1. (See also numerical example in the Section , where there is a double multiplier on the real axis).

Remark. In Example 5.2 the multipliers remain regular at the point τ, where a double multiplier appears. In a general case the multipliers branch at this point. Such a situation takes place in the numerical example ... .

Example 5.3. (*Resonance.*) Consider the system

$$\begin{aligned} &\dot{z} = (\tau - \alpha i)z + \varepsilon w^2 + \varepsilon h(\vartheta), \quad \dot{w} = (-\mu - \beta i)w, \\ &\dot{\vartheta} = 1, \quad \dot{\tau} = \varepsilon. \end{aligned} \tag{5.6}$$

Here $h, \alpha, \mu, \beta, z, w$ and multipliers $\rho_{1,2,3,4}$ are the same as in Example 5.2. There is the resonance $\rho_1 = \rho_2^2$ in the system (5.6) at $\tau = -2\mu \pm i(\alpha - 2\beta) + ik$, $\tau = -2\mu \pm i(\alpha + 2\beta) + ik$, $k \in \mathbb{Z}$. In Theorem 4.3 the in-out function is estimated on the interval $(-\alpha, 0)$ from below by the function $\tilde{\Pi}$. The exact in-out function for the system (5.6) is given by Proposition 5.1, where μ and β should be replaced by 2μ and 2β. In particular, if $\alpha = 2\beta$ (resonance on the real axis) the in-out function for $-\alpha < \tau < 0$ coincides with the function $\tilde{\Pi}$. This example shows, that the estimate of the Theorem 4.3 cannot be improved.

7. Proof of Theorems 4.1–4.3

In this section we prove Theorems 4.1–4.3. We consider only case **A**. The proof in case **B** is the same. Below we assume that conditions $C1$–$C3$ and $A1$–$A3$ hold.

a. Basic Lemma. Since for any $\tau \in D$ all the multipliers $\rho_l(\tau)$ are parewise distinct and distinct from unit (see assumptions C1–C2), in the vicinity of the trajectories L_τ, $\tau \in D$ there exists an analytic change of the variables $(x_1, \dots, x_{n+1}) \to (\hat\xi_1, \dots, \hat\xi_n, \hat\varphi)$ reducing the system (2.2) to the form

$$d\hat\xi/dt = \Lambda(\tau)\hat\xi + \Phi(\hat\xi, \hat\varphi, \tau), \quad d\hat\varphi/dt = \omega(\tau) + \Phi'(\hat\xi, \hat\varphi, \tau),$$
$$\Lambda = \operatorname{diag}(\lambda_1^0, \dots, \lambda_n^0), \quad \Phi = (\Phi_1, \dots, \Phi_n)^T = O(|\hat\xi|^2), \quad \Phi' = O(|\hat\xi|).$$

All the functions $\Phi', \Phi_1, \dots, \Phi_n$ are 2π-periodic in $\hat\varphi$. The variables $(\hat\xi_1, \dots, \hat\xi_n, \hat\varphi)$ are usually called the Floquet coordinates.

The system (2.1) in the variables $\hat\xi, \hat\varphi, \tau$ has the form

$$\begin{aligned} d\hat\xi/dt &= [\Lambda(\tau) + \varepsilon\hat B(\hat\varphi, \tau, \varepsilon)]\hat\xi + \varepsilon\hat h(\hat\varphi, \tau, \varepsilon) + \hat F(\hat\xi, \hat\varphi, \tau, \varepsilon), \\ d\hat\varphi/dt &= \omega(\tau) + \varepsilon\hat a(\hat\varphi, \tau, \varepsilon) + \hat F'(\hat\xi, \hat\varphi, \tau, \varepsilon), \\ d\tau/dt &= \varepsilon. \end{aligned} \tag{7.1}$$

Here the functions $\hat B = (\hat B_{lk})$, $\hat h = (\hat h_1, \dots, \hat h_n)^T$, $\hat F = (\hat F_1, \dots, \hat F_n)^T$, $\hat a$, and $\hat F'$ are 2π-periodic in $\hat\varphi$ and the following conditions hold:

$$\hat F = O(|\hat\xi|^2), \quad \hat F' = O(|\hat\xi|).$$

By a 2π-periodic in φ analytic change of variables $\xi = \hat\xi + O(\varepsilon)$, $\varphi = \hat\varphi + O(\varepsilon)$ we can reduce the system (7.1) to the following form:

$$\begin{aligned} \dot\xi &= [\tilde\Lambda(\tau, \varepsilon) + \varepsilon^2 B_0(\varphi, \tau, \varepsilon)]\xi + \varepsilon^3 h_0(\varphi, \tau, \varepsilon) + F_0(\xi, \varphi, \tau, \varepsilon), \\ \dot\varphi &= \tilde\omega(\tau, \varepsilon) + \varepsilon^3 a_0(\varphi, \tau, \varepsilon) + F_0'(\xi, \varphi, \tau, \varepsilon), \\ \dot\tau &= \varepsilon, \end{aligned} \tag{7.2}$$

where for any $\tau \in D$

$$\tilde\Lambda(\tau, \varepsilon) = \operatorname{diag}(\tilde\lambda_l(\tau, \varepsilon)), \quad \tilde\lambda_l(\tau, \varepsilon) = \lambda_l^0(\tau) + O(\varepsilon), \tag{7.3}$$
$$\tilde\omega(\tau, \varepsilon) = \omega(\tau) + O(\varepsilon), \quad F_0 = O(|\xi|^2), \quad F_0' = O(|\xi|).$$

We call a function bounded if its absolute value is bounded by a constant not depending on ε.

Basic Lemma. *There exists a change of the variables*

$$\xi = \eta + \vartheta(\psi, \tau), \quad \varphi = \psi + \varkappa(\psi, \tau), \quad \vartheta = (\vartheta_1, \dots, \vartheta_n)^T \tag{7.4}$$

such that the functions

$$\vartheta(\varphi, \tau)/\varepsilon^2, \quad \varkappa(\varphi, \tau)/\varepsilon \qquad \tau \in D, \quad |\operatorname{Im}\psi| \leqslant \beta_0/2$$

are bounded and the system (7.2) *in the variables* η, ψ, τ *has the form*

$$\begin{aligned} \dot{\eta} &= [\tilde{\Lambda}(\tau, \varepsilon) + \varepsilon^2 B(\psi, \tau, \varepsilon)]\eta + F(\eta, \psi, \tau, \varepsilon), \\ \dot{\psi} &= \tilde{\omega}(\tau, \varepsilon) + F'(\eta, \psi, \tau, \varepsilon), \\ \dot{\tau} &= \varepsilon, \end{aligned} \tag{7.5}$$

Here the functions $B, \tilde{\omega}, F, F'$ *are analytic and bounded for*

$$|\eta| \leqslant \alpha_0/2, \quad \tau \in D, \quad |\mathrm{Im}\,\psi| \leqslant \beta_0/2,$$

$F = O(|\eta|^2)$*,* $F' = O(|\eta|)$ *and the constants* α_0*,* β_0 *are positive.*

We prove Basic Lemma in Section 8.

b. Separation Lemma. To prove Theorem 4.3 we need one more lemma. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the following linear subspaces of $L(n, \mathbb{R})$:

$$\mathfrak{M} = \{M = (M_{jk}) \in L(n, \mathbb{R})\colon M_{jk} = M_{kj} = 0, \quad \text{for } j = 1, 2,\ k = 3, \dots, n\}$$
$$\mathfrak{N} = \{N = (N_{jk}) \in L(n, \mathbb{R})\colon N_{jk} = N_{kj} = 0, \quad \text{for } j, k = 1, 2, \text{ or } j, k = 3, \dots, n\}$$

Obviously, $L(n, \mathbb{R}) = \mathfrak{N} \oplus \mathfrak{M}$ and for any $N_1, N_2 \in \mathfrak{N}$, $M \in \mathfrak{M}$ we have:

$$N_1 N_2 \in \mathfrak{M}, \quad N_1 M \in \mathfrak{N}, \quad M N_1 \in \mathfrak{N}.$$

Separation Lemma. *Suppose that condition* $A5$ *holds. Then there exists a linear change of the variables*

$$\eta = (I + \varepsilon N(\psi, \tau, \varepsilon))\zeta, \qquad I = diag(1, \dots, 1) \tag{7.6}$$

such that the matrix function $N(\psi, \tau, \varepsilon) \in \mathfrak{N}$ *is bounded in the complex domain* $|Im\,\psi| \leqslant \beta_0/4$*,* $\tau \in D$ *and the system* (7.5) *in the variables* ζ, ψ, τ *takes the form*

$$\begin{aligned} \dot{\zeta} &= [\tilde{\Lambda}(\tau, \varepsilon) + \varepsilon^2 \tilde{B}(\psi, \tau, \varepsilon)]\zeta + \tilde{F}(\zeta, \psi, \tau, \varepsilon), \\ \dot{\psi} &= \tilde{\omega}(\tau, \varepsilon) + \tilde{F}'(\zeta, \psi, \tau, \varepsilon), \\ \dot{\tau} &= \varepsilon, \end{aligned} \tag{7.7}$$

where $\tilde{B}(\psi, \tau, \varepsilon) \in \mathfrak{M}$*, the functions* $\tilde{B}, \tilde{F}, \tilde{F}'$ *are analytic and bounded for*

$$|\eta| \leqslant \alpha_0/4, \quad \tau \in D, \quad |\mathrm{Im}\,\psi| \leqslant \beta_0/4,$$

$\tilde{F} = O(|\eta|^2)$ *and* $\tilde{F}' = O(|\eta|)$*.*

Separation Lemma shows that condition $A5$ implies the possibility of linear separation of the variables η_1, η_2 from $\eta_3, \dots, \eta_n$ by a linear change. In case **B** the corresponding statement asserts that the variable η_1 can be separated in linear terms from $\eta_2, \dots, \eta_n$ provided condition $B4$ holds.

c. Proof of Theorem 4.1. Consider the solution $\eta(t), \psi(t)$ of the system (7.5) with initial conditions at $t = t_0 = \tau_0/\varepsilon$.

Lemma 7.1. *Suppose that* $|\eta(t_0)| < d_1^{-1}$. *Then for* $\tau_0 + d_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \hat{\Pi}(\tau_0) - d_3\varepsilon|\log\varepsilon|$ *we have* $|\eta(t)| < d_4\varepsilon$.

Theorem 4.1 is an obvious corollary of Lemma 7.1. One should consider the solution $x(t)$ in the variables η, ψ and use Lemma 7.1.

Proof of Lemma 7.1. Introduce the Lyapunov function

$$W(\eta) = \sum_{j=1}^{n} \eta_j \bar{\eta}_j.$$

Denote $w(t) = \sqrt{W(\eta(t))} \geqslant 0$. Then (7.2), (7.3) and the definition of $\mu(\tau)$ (Section 4.**a**) imply:

$$\dot{w} \leqslant [\mu(\tau) + O(\varepsilon) + O(w)]w. \tag{7.8}$$

If $w < a_1^{-1}$ then the expression in square brackets in (7.8) for $\tau = \tau_0$ is smaller than $\mu(\tau_0)/2$. Let $w(t_0) < a_1^{-1}/2$. Then (7.8) implies the estimate $w(t_1) < \varepsilon$ for $t_1 = t_0 + d_2|\log\varepsilon|$. On time interal $[t_0, t_1]$ function $w(t)$ decay exponentialy: $w(t) < w(t_0)\exp(\mu(\tau_0)t/4)$. Denote by $\hat{t}_2$ supremum of time moments t_2, $\varepsilon t_2 < \hat{\Pi}(\tau_0) + 1$ such that for $t_1 \leqslant t \leqslant t_2$ the inequality $w(t) < 2\varepsilon$ is satisfied. Obviously, $\int_{t_0}^{\hat{t}_2} w(t)dt = O(1)$. Therefore, for $t_1 \leqslant t \leqslant \hat{t}_2$ equation (7.8) implies the estimate

$$w(t) \leqslant a_2 w(t_1) \exp\Big(\frac{1}{\varepsilon}\int_{\varepsilon t_1}^{\varepsilon t} \mu(s)ds\Big). \tag{7.9}$$

By using (7.9) we get, that $\varepsilon\hat{t}_2 > \hat{\Pi}(\tau_0) - d_3\varepsilon|\log\varepsilon|$. We see also that $w(t) < \varepsilon, \eta(t) < d_4\varepsilon$ for $\tau_0 + d_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \hat{\Pi}(\tau_0) - d_3\varepsilon|\log\varepsilon|$. Lemma 7.1 is proved.

d. Proof of Theorem 4.2. Denote for the system (7.5) by $\eta^{(1)}$ and $\eta^{(2)}$ the vectors, formed by the first two and the last $(n-2)$ components of the vector η respectively. Consider the solution $\eta(t), \psi(t)$ of the system (7.5) with initial conditions at $t = t_0 = \tau_0/\varepsilon$. Suppose, that assumption A4 is satisfied.

Lemma 7.2. *Suppose that* $|\eta(t_0)| < d_5^{-1}$. *Let* $|\eta^{(2)}(t_0)| < d_6\varepsilon$ *and* $|\eta^{(1)}(t_0)| > d_7 d_6\varepsilon$. *Then*

(1) *for* $\tau_0 + d_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \Pi(\tau_0) - d_3\varepsilon|\log\varepsilon|$ *we have* $|\eta(t)| < d_4\varepsilon$.
(2) $|\eta(t_d)| > d_8^{-1}$ *for some* t_d, *where* $|\varepsilon t_d - \Pi(\tau_0)| < d_9\varepsilon|\log\varepsilon|$.

Theorem 4.2 is an obvious corollary of Lemma 7.2. One should consider the solution $x(t)$ in the variables η, ψ and use Lemma 7.1.

Proof of Lemma 7.2. Item (1) of Lemma 7.2 is just a repetition of Lemma 7.1. To prove item (2) introduce functions

$$W_1(\eta) = \sum_{j=1}^{2} \eta_j\bar{\eta}_j, \quad W_2(\eta) = \sum_{j=3}^{n} \eta_j\bar{\eta}_j.$$

Denote $w_l(t) = \sqrt{W_l(\eta(t))} \geqslant 0, \quad l = 1, 2$. The constant d_6 in Lemma 7.2 can be defined arbitrarily; for example, $d_6 = 1$. Define d_7 in such a way, that $w_1(t_0) > 2w_2(t_0)$.

For any time moment t such that $w_1(t) \neq 0, w_2(t) \neq 0$ and $w_1(t) \geqslant w_2(t)$ the system (7.7) implies

$$\dot{w}_1 = [\mathrm{Re}\lambda_1^0(\tau) + O(\varepsilon) + O(w_1)]w_1, \tag{7.10}$$
$$\dot{w}_2 < [-\gamma(\tau) + O(\varepsilon) + O(w_1)]w_2 + O(w_1^2). \tag{7.11}$$

Here $-\gamma(\tau) = \max \mathrm{Re}\lambda_j^0$, where $3 \leqslant j \leqslant n$. Assumption A4 implies the estimate $\gamma(\tau) < \mathrm{Re}\lambda_1^0$.

Due to (7.10), (7.11) for $w_1 < a_1^{-1}$ the following implication is valid: if $w_1 = w_2 \neq 0$ then $\dot{w}_1 > \dot{w}_2$. Also for $w_1 < a_1^{-1}$ the expressions in square brackets in (7.10) and (7.11) for $\tau = \tau_0$ are smaller, than $\mathrm{Re}\lambda_1^0(\tau)/2$ and $-\gamma(\tau)/2$ respectively, and the expression in square brackets in (7.10) for $\tau = \Pi(\tau_0)$ is bigger than $\mathrm{Re}\lambda_1^0(\tau)/2$. Let $w_1(t_0) < a_1^{-1}/2$. Denote by t_d supremum of time moments t_1, $\varepsilon t_1 < \Pi(\tau_0) + 1$ such that for $t_0 \leqslant t \leqslant t_1$ the inequality $w_1(t) < a_1^{-1}$ is satisfied. Then for $t_0 \leqslant t \leqslant t_d$ the inequality $w_1 \geqslant w_2$ is satisfied and, therefore, the estimates (7.10)–(7.11) are valid. For $t_0 \leqslant t \leqslant t_d$ equation (7.10) implies the estimate

$$a_2^{-1} w_1(t_0) \exp\Big(\frac{1}{\varepsilon}\int_{\tau_0}^{\varepsilon t} \mathrm{Re}\lambda_1^0(s)ds\Big) < w_1(t) < a_2 w_1(t_0) \exp\Big(\frac{1}{\varepsilon}\int_{\tau_0}^{\varepsilon t} \mathrm{Re}\lambda_1^0(s)ds\Big). \tag{7.12}$$

Inequalities (7.12) show that $|\varepsilon t_d - \Pi(\tau_0)| < d_9 \varepsilon |\log \varepsilon|$. Lemma 7.2 is proved.

e. Proof of Theorem 4.3. Consider the solution $\eta(t), \psi(t)$ of the system (7.5) with initial conditions at $t = t_0 = \tau_0/\varepsilon$.

Lemma 7.3. *Suppose that* $|\eta(t_0)| < d_1^{-1}$. *Then for*

$$\tau_0 + d_2\varepsilon|\log\varepsilon| \leqslant \varepsilon t \leqslant \tilde{\Pi}(\tau_0) - d_3\varepsilon|\log\varepsilon|$$

we have $|\eta(t)| < d_4\varepsilon$.

Lemma 7.3 implies Theorem 4.3 when we consider the solution $x(t)$ in the variables η, ψ.

Proof of Lemma 7.3. Consider the functions

$$W_1(\eta) = \sum_{j=1}^{2} \eta_j \bar{\eta}_j, \quad W_2(\eta) = \sum_{j=3}^{n} \eta_j \bar{\eta}_j, \quad W(\eta) = W_1(\eta) + W_2(\eta)$$

and denote $w_l(t) = \sqrt{W_l(\eta(t))} \geqslant 0, \quad l = 1, 2, \quad w(t) = \sqrt{W(\eta(t))} \geqslant 0$.

For any time moment t such that $w_1(t) \neq 0, w_2(t) \neq 0$ the system (7.5) implies

$$\begin{aligned} \dot{w}_1 &= [\mathrm{Re}\lambda_1^0(\tau) + O(\varepsilon) + O(w)]w_1 + O(w_2^2), \\ \dot{w}_2 &< [-\gamma(\tau) + O(\varepsilon) + O(w)]w_2 + \varepsilon^2 O(w_1). \end{aligned} \tag{7.13}$$

Here $-\gamma(\tau) = \max \mathrm{Re}\lambda_j^0$, where $3 \leqslant j \leqslant n$. Note that the first equation (7.13) does not contain terms of first order in w_2.

If $|\eta(\tau_0)| < d_1^{-1}$ then (7.13) implies the estimate $w(t_1) < \varepsilon^3$ for $t = t_1 = t_0 + d_2|\log\varepsilon|$. Let the functions $\tilde{w}_1(t)$ and $\tilde{w}_2(t)$ satisfy the relations

$$w_1(t) = \tilde{w}_1(t)\exp\Big(\frac{1}{\varepsilon}\int_{\varepsilon t_1}^{\tau}\tilde{\mu}(s)\,ds\Big), \quad w_2(t) = \tilde{w}_2(t)\exp\Big(\frac{1}{\varepsilon}\int_{\varepsilon t_1}^{\tau}\frac{\tilde{\mu}(s)}{2}\,ds\Big), \tag{7.14}$$
$$\tilde{\mu}(\tau) = \max\{\operatorname{Re}\lambda_1^0(\tau), -2\gamma(\tau)\}.$$

Equations (7.13) can be transformed as follows:

$$\begin{aligned} d\tilde{w}_1/dt &= [\operatorname{Re}\lambda_1^0(\tau) - \tilde{\mu}(\tau) + O(\varepsilon) + O(w)]\tilde{w}_1 + O(\tilde{w}_2^2),\\ d\tilde{w}_2/dt &< [-\gamma(\tau) - \tilde{\mu}(\tau)/2 + O(\varepsilon) + O(w)]\tilde{w}_2 + \varepsilon^2 O(\sqrt{w_1\tilde{w}_1}). \end{aligned} \tag{7.15}$$

Denote by $\hat{t}_2$ supremum of time moments t_2, $\varepsilon t_2 < \tilde{\Pi}(\tau_0) + 1$ such that for $t_1 \leqslant t \leqslant t_2$ the following inequalities hold:

$$w(t) < \varepsilon^2, \quad \tilde{w}_1(t) < \varepsilon^2, \quad \tilde{w}_2(t) < \varepsilon^2.$$

Now the relations (7.14)–(7.15) imply the inequality $\varepsilon\hat{t}_2 \geqslant \tilde{\Pi}(\tau_0) - d_3\varepsilon|\log\varepsilon|$. For $t_0 + d_2|\log\varepsilon| \leqslant t \leqslant \hat{t}_2$ we have $|\eta(t)| < d_4\varepsilon$. Lemma 7.3 is proved.

8. Proof of the Basic Lemma

We obtain the change (7.4) going to the limit in the following sequence of changes:

$$\xi^{(m)} = \xi^{(m+1)} + \vartheta^{(m)}(\varphi^{(m+1)},\tau), \quad \varphi^{(m)} = \varphi^{(m+1)} + \varkappa^{(m)}(\varphi^{(m+1)},\tau), \tag{8.1}$$
$$\xi^{(0)} = \xi, \quad \varphi^{(0)} = \varphi, \qquad \xi^{(\infty)} = \eta, \quad \varphi^{(\infty)} = \psi.$$

In the variables $\xi^{(m)}, \varphi^{(m)}, \tau$ the system (7.2) has the form

$$\begin{aligned} \dot{\xi}^{(m)} &= [\tilde{\Lambda}(\tau) + \varepsilon^2 B_m(\varphi^{(m)},\tau)]\xi^{(m)} + \varepsilon^3 h_m(\varphi^{(m)},\tau) + F_m(\xi^{(m)},\varphi^{(m)},\tau),\\ \dot{\varphi}^{(m)} &= \tilde{\omega}(\tau) + \varepsilon^3 a_m(\varphi^{(m)},\tau) + F'_m(\xi^{(m)},\varphi^{(m)},\tau), \quad \dot{\tau} = \varepsilon, \end{aligned} \tag{8.2}$$
$$F_m = O(|\xi^{(m)}|^2), \quad F'_m = O(|\xi^{(m)}|).$$

Here and henceforth for brief we do not write ε among arguments of the functions $\tilde{\Lambda}, B_m, h_m$ etc. We assume that the functions $\vartheta^{(m)}$ and $\varkappa^{(m)}$ satisfy the equations

$$\begin{aligned} &\varepsilon\vartheta_\tau^{(m)}(\varphi,\tau) + \tilde{\omega}(\tau)\vartheta_\varphi^{(m)}(\varphi,\tau) - \tilde{\Lambda}(\tau)\vartheta^{(m)}(\varphi,\tau)\\ &\qquad\qquad = \varepsilon^2 B_m(\varphi,\tau)\vartheta^{(m)}(\varphi,\tau) + \varepsilon^3 h_m(\varphi,\tau),\\ &\varepsilon\varkappa_\tau^{(m)}(\varphi,\tau) + \tilde{\omega}(\tau)\varkappa_\varphi^{(m)}(\varphi,\tau) = \varepsilon^3 a_m(\varphi,\tau) + (F'_m)_\xi(0,\varphi,\tau)\vartheta^{(m)}(\varphi,\tau), \end{aligned} \tag{8.3}$$

where the subscripts τ, φ, ξ denote partial derivatives and

$$(F'_m)_\xi\vartheta^{(m)} = \sum_{l=1}^{n}(F'_m)_{\xi_l}\vartheta_l^{(m)}.$$

We introduce the following notations:

(1) If $f(\varphi,\tau)=\sum_{k\in\mathbb{Z}} f^k(\tau)e^{ik\varphi}$ is an analytic scalar function and $\beta>0$ then

$$||f||_\beta = \sup_{k\in\mathbb{Z},\tau\in D} |f^k(\tau)|e^{\beta|k|}.$$

(2) For a vector $v=(v_1(\varphi,\tau),\dots,v_n(\varphi,\tau))^T$ and a matrix $V=(V_{lj}(\varphi,\tau))$, $1\leqslant l,j\leqslant n$ we put

$$||v||_\beta = \max_{1\leqslant l\leqslant n} ||v_l||_\beta, \quad ||V||_\beta = \max_{1\leqslant l,j\leqslant n} ||V_{l,j}||_\beta.$$

(3) For positive α and β let

$$E_{\alpha,\beta} = \{x\in\mathbb{C}^n, \tau\in D, \varphi\in\mathbb{C}/(2\pi\mathbb{Z})\colon |x|\leqslant\alpha, |\operatorname{Im}\varphi|\leqslant\beta\}.$$

Then for any function $F(x,\varphi,\tau)$ analytic in $E_{\alpha,\beta}$ we put

$$|F|_{\alpha,\beta} = \max_{E_{\alpha,\beta}} |F|.$$

Obviously, for some positive $\alpha_0,\beta_0,\chi_0,b_0$ and f_0 the following relations hold:

$$\begin{gathered} ||h_0||_{\beta_0}\leqslant\chi_0, \quad ||B_0||_{\beta_0}\leqslant b_0, \quad ||a_0||_{\beta_0}\leqslant\chi_0,\\ ||F_0||_{\alpha_0,\beta_0}\leqslant f_0, \quad ||(F_0)_{\xi\xi}||_{\alpha_0,\beta_0}\leqslant f_0,\\ ||F_0'||_{\alpha_0,\beta_0}\leqslant f_0, \quad ||(F_0')_{\xi}||_{\alpha_0,\beta_0}\leqslant f_0. \end{gathered} \tag{8.4}$$

For any $m\in\mathbb{N}$ we put

$$\begin{gathered} \alpha_m=\alpha_0(1+2^{-m})/2, \quad \beta_m=\beta_0(1+2^{-m})/2, \quad \beta_m'=(\beta_m+\beta_{m+1})/2,\\ \chi_{m+1}=(\sqrt{\varepsilon}\chi_m)^{3/2}, \quad b_m=2^m\chi_m, \quad f_m=f_0(2-2^{-m}). \end{gathered}$$

Lemma 8.1. *Suppose that the following relations hold:*

$$\begin{gathered} B_m=\tilde B_0+\tilde B_1+\dots+\tilde B_m, \quad ||\tilde B_j||_{\beta_j}\leqslant b_j, \quad \tilde B_0=B_0,\\ ||h_m||_{\beta_m}\leqslant\chi_m, \quad ||a_m||_{\beta_m}\leqslant\chi_m, \quad ||(F_m')_\xi(0,\varphi,\tau)||_{\beta_m}\leqslant f_m. \end{gathered}$$

Then the equations (8.3) *have a solution* $\vartheta^{(m)},\varkappa^{(m)}$ *satisfying the estimates*

$$||\vartheta^{(m)}||_{\beta_m'}\leqslant P\varepsilon^2\chi_m, \quad ||\varkappa^{(m)}||_{\beta_m'}\leqslant Q2^m\varepsilon\chi_m, \qquad P,Q>0. \tag{8.5}$$

We prove Lemma 8.1 in Section 9. Substituting into (8.2) the relations (8.1) and using (8.3) we get:

$$\begin{aligned} \dot\xi^{(m+1)} &= (\tilde\Lambda(\tau)+\varepsilon^2B_m(\varphi^{(m)},\tau))\xi^{(m+1)}+U_m(\xi^{(m+1)},\varphi^{(m+1)},\tau),\\ \dot\varphi^{(m+1)} &= \tilde\omega(\tau)+V_m(\xi^{(m+1)},\varphi^{(m+1)},\tau), \end{aligned}$$

where

$$\begin{aligned} V_m &= [1 + \varkappa_\varphi^{(m)}(\varphi^{(m+1)}, \tau)]^{-1} \big[\varepsilon^3 (a_m(\varphi^{(m)}, \tau) - a_m(\varphi^{(m+1)}, \tau)) \\ &\quad + F'_m(\xi^{(m)}, \varphi^{(m)}, \tau) - (F'_m)_\xi(0, \varphi^{(m+1)}, \tau)\, \vartheta^{(m)}(\varphi^{(m+1)}, \tau)\big], \\ U_m &= -V_m \vartheta_\varphi^{(m)}(\varphi^{(m+1)}, \tau) + \varepsilon^2 \big(B_m(\varphi^{(m)}, \tau) - B_m(\varphi^{(m+1)}, \tau)\big)\, \vartheta^{(m)}(\varphi^{(m+1)}, \tau) \\ &\quad + \varepsilon^3 (h_m(\varphi^{(m)}, \tau) - h_m(\varphi^{(m+1)}, \tau)) + F_m(\xi^{(m)}, \varphi^{(m)}, \tau). \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon^3 h_{m+1}(\varphi^{(m+1)}, \tau) &= U_m(0, \varphi^{(m+1)}, \tau), \\ \varepsilon^2 \tilde{B}_{m+1}(\varphi^{(m+1)}, \tau) &= \varepsilon^2 B_m(\varphi^{(m)}, \tau) - \varepsilon^2 B_m(\varphi^{(m+1)}, \tau) \\ &\quad + (U_m)_\xi(0, \varphi^{(m+1)}, \tau), \\ F_{m+1}(\xi^{(m+1)}, \varphi^{(m+1)}, \tau) &= U_m(\xi(m+1), \varphi^{(m+1)}, \tau) \\ &\quad - U_m(0, \varphi^{(m+1)}, \tau) - (U_m)_\xi(\vartheta^{(m)}(0, \varphi^{(m+1)}, \tau)\, \xi^{(m+1)}, \\ \varepsilon^3 a_{m+1}(\varphi^{(m+1)}, \tau) &= V_m(0, \varphi^{(m+1)}, \tau), \\ F'_{m+1}(\xi^{(m+1)}, \varphi^{(m+1)}, \tau) &= V_m(\xi^{(m+1)}, \varphi^{(m+1)}, \tau) - V_m(0, \varphi^{(m+1)}, \tau). \end{aligned}$$

Now Basic Lemma follows from

Lemma 8.2. *For all integer $m \geqslant 0$ the following estimates hold:*

$$\begin{gathered} ||h_m||_{\beta_m} \leqslant \chi_m, \quad ||\tilde{B}_m||_{\beta_m} \leqslant b_m, \quad ||a_m||_{\beta_m} \leqslant \chi_m, \\ ||F_m||_{\alpha_m, \beta_m} \leqslant f_m, \quad ||(F_m)_{\xi\xi}||_{\alpha_m, \beta_m} \leqslant f_m, \\ ||F'_m||_{\alpha_m, \beta_m} \leqslant f_m, \quad ||(F'_m)_\xi||_{\alpha_m, \beta_m} \leqslant f_m. \end{gathered}$$

Lemma 8.2 is analogous to some technical assertions in KAM-theory [8–9]. Proof the lemma is quite standard. It is based on Lemma 8.1 and on quadratic convergence of our sequence of the variable changes. We use induction in m. For $m = 0$ assertions of the lemma hold (see (8.4)). Suppose that for $m \leqslant l$ Lemma 8.2 is valid. Then by using Lemma 8.1 and Cauchy estimates we obtain the following relations:

$$||V_l(0, \varphi^{(l+1)}, \tau)||_{\beta'_l} \leqslant c'_1 4^l \varepsilon^3 \chi_l^2, \quad ||U_l(0, \varphi^{(l+1)}, \tau)||_{\beta'_l} \leqslant c'_2 4^l \varepsilon^4 \chi_l^2.$$

Now we obtain the following assertions:

(i) $||\varepsilon^3 h_{l+1}(\varphi^{(l+1)}, \tau)||_{\beta_{l+1}} \leqslant c'_2 4^l \varepsilon^4 \chi_l^2$.

(ii) $||\varepsilon^2 \tilde{B}_{l+1}(\varphi^{(l+1)}, \tau)||_{\beta_{l+1}} \leqslant 2^l \varepsilon^2 \chi_l$.

(iii) $||\varepsilon^3 a_{l+1}(\varphi^{(l+1)}, \tau)||_{\beta_{l+1}} \leqslant c'_1 4^l \varepsilon^3 \chi_l^2$.

(iv) The function $F'_{l+1}(\xi^{(l+1)}, \varphi^{(l+1)}, \tau)$ satisfies the relation

$$F'_{l+1} = F'_l(\xi^{(l+1)} + \vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau) - F'_l(\vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau).$$

Hence the derivatives of the functions F_l and F_{l+1} with respect to ξ^{m+1} have the same estimates. Since $F'_l = O(\xi^{(m)}$, the following estimate holds:

$$||F'_{l+1}||_{\alpha_{l+1}, \beta_{l+1}} \leqslant f_l(1 + c'_3 2^l \varepsilon^2 \chi_l.$$

The function $F_{l+1}(\xi^{(l+1)}, \varphi^{(l+1)}, \tau)$ and its derivatives can be estimated analogously with the help of the equality

$$\begin{aligned} F_{l+1} = -\vartheta_\varphi^{(l)}(\varphi^{(l+1)}, \tau)\big[&F_l'(\xi^{(l+1)} + \vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau) \\ &- F_l'(\vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau) - (F_l')_\xi(\vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau)\xi^{(l+1)}\big] \\ &+ F_l(\xi^{(l+1)} + \vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau) \\ &- F_l'(\vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau) - (F_l')_\xi(\vartheta^{(l)}(\varphi^{(l+1)}, \tau), \varphi^{(l)}, \tau)\xi^{(l+1)} \end{aligned}$$

Lemma 8.2, obviously, follows from assertions (i)–(iv).

9. Proof of Lemma 8.1

In this section we write for brief $\vartheta, \varkappa$ instead of $\vartheta^{(m)}, \varkappa^{(m)}$. First, we expand the equations (8.3) in the Fourier series in φ. Fourier coefficients $\vartheta^k(\tau), \varkappa^k(\tau)$ satisfy the system

$$\begin{aligned} \varepsilon\vartheta^k_\tau + (ik\tilde\omega(\tau) - \tilde\Lambda)\vartheta^k &= \sum_{j\in\mathbb{Z}} \varepsilon^2 B_m^{k-j}\vartheta^j + \varepsilon^3 h_m^k, \\ \varepsilon\varkappa^k_\tau + ik\tilde\omega\varkappa^k &= \varepsilon^3 a_m^k + \sum_{j\in\mathbb{Z}} (F_m')_\xi^{k-j}(0,\tau)\vartheta^j, \end{aligned} \tag{9.1}$$

where B_m^k, h_m^k, a_m^k, and $(F_m')_\xi^k$ are Fourier coefficients of the functions B_m, h_m, a_m, and $(F_m')_\xi$ respectively.

Let the function ϕ^+ (respectively ϕ^-), see (3.4), restricted to the domain D has a local maximum at the point τ_+ (respectively τ_-). Since $\phi^\pm$ are harmonic, $\tau_\pm$ lie on the boundary of D. According to A3 the maximums $\tau_\pm$ are unique (and consequently, are global on D). Since the function $\omega(\tau)$ is real and positive for real $\tau \in D$, the points $\tau_\pm$ are outside the real axis. Therefore, the boundary ∂D of the domain D is smooth at these points.

Proposition 9.1. *Suppose that the maximums $\tau_\pm$ of the functions $\phi^\pm$ restricted to ∂D are nondegenerate. Then there exist points τ_l^k and τ^k, where $k \in \mathbb{Z}$, $l = 1, \dots, n$ such that for any $\tau \in D$*

$$\tilde\varphi_l^k(\tau) \leqslant 0, \quad \tilde\varphi^k(\tau) \leqslant 0,$$

where

$$\tilde\varphi_l^k(\tau) = \mathrm{Re}\int_{\tau_l^k}^{\tau} (\tilde\lambda_l^0(s) - ik\tilde\omega(s))\, ds, \quad \tilde\varphi^k(\tau) = -\mathrm{Re}\int_{\tau^k}^{\tau} ik\tilde\omega(s)\, ds.$$

Furthermore, there exists a system of curves $\mathfrak{C}_{kl}^\tau(\nu) \subset D$, $\mathfrak{C}_k^\tau(\nu) \subset D$, $\tau \in D$, $\nu \in [0,1]$ (here we regard ν as a parameter on a curve and τ, k, l as parameters labeling the curves) with the following properties.

(i) $\mathfrak{C}_{kl}^\tau(0) = \tau_l^k$, $\mathfrak{C}_k^\tau(0) = \tau^k$; $\mathfrak{C}_{kl}^\tau(1) = \tau$, $\mathfrak{C}_k^\tau(1) = \tau$.

(ii) *There exists a constant K such that for any $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant 1$*

$$\tilde\varphi_l^k(\mathfrak{C}_{kl}^\tau(\nu_2)) - \tilde\varphi_l^k(\mathfrak{C}_{kl}^\tau(\nu_1)) \leqslant K\varepsilon, \quad \tilde\varphi^k(\mathfrak{C}_k^\tau(\nu_2)) - \tilde\varphi^k(\mathfrak{C}_k^\tau(\nu_1)) \leqslant K\varepsilon.$$

(iii) *Lengths of the curves $\mathfrak{C}_{kl}^\tau$, $\mathfrak{C}_k^\tau$ are bounded from above by a constant L not depending on ε.*

It is easy to check that solutions of the following integral equation

$$(\vartheta, \varkappa) = P_m(\vartheta), \tag{9.2}$$

satisfy the system (9.1). In more detail,

$$\vartheta_l^k = [P_m^{(\vartheta)}(\vartheta)]_l^k, \quad \varkappa^k = [P_m^{(\varkappa)}(\vartheta)]^k,$$

where

$$[P_m^{(\vartheta)}(\vartheta)]_l^k = \frac{1}{\varepsilon}\int_{\tau_l^k}^{\tau} e^{\frac{1}{\varepsilon}\int_\nu^\tau (\tilde\lambda_l(s) - ik\tilde\omega(s))\,ds}\Big[\sum_{j\in\mathbb{Z}} \varepsilon^2 (B_m^{k-j}\vartheta^j)_l(\nu) + \varepsilon^3 (h_m^k)_l(\nu)\Big]\,d\nu, \tag{9.3}$$

$$[P_m^{(\varkappa)}(\vartheta)]^k = \frac{1}{\varepsilon}\int_{\tau^k}^{\tau} e^{-\frac{1}{\varepsilon}\int_\nu^\tau ik\tilde\omega(s)\,ds}\Big[\varepsilon^3 a_m^k(\nu) + \sum_{j\in\mathbb{Z}} (F_m')_\xi^{k-j}(0,\nu)\vartheta^j(\nu)\Big]\,d\nu, \tag{9.4}$$

$$(B_m^{k-j}\vartheta^j)_l = \sum_{r=1}^n B_{m,l,r}^{k-j}\vartheta_r^j, \qquad (F_m')_\xi^{k-j}(0,\nu)\vartheta^j = \sum_{r=1}^n (F_m')_{\xi_r}^{k-j}\vartheta_r^j,$$

$$P_m(\vartheta) = \sum_{k\in\mathbb{Z}} \Big(\big([P_m^{(\vartheta)}(\vartheta)]_1^k, \dots, [P_m^{(\vartheta)}(\vartheta)]_n^k\big)^T,\ [P_m^{(\varkappa)}(\vartheta)]^k\Big)e^{ik\varphi}.$$

Here we assume that the points τ_l^k and τ^k are determined by Proposition 9.1. Integrals in the right-hand sides of (9.3)–(9.4) can be taken along any paths (with the indicated endpoints) lying in the domain D. Since the integrands are holomorphic functions and D is simple-connected, values of the integrals do not depend on choice of the paths.

To prove that the system (9.3)–(9.4) has a solution we show that the operator

$$P_m^{(\vartheta)}\colon S_{\beta_m'} \to S_{\beta_m'}, \qquad S_{\beta_m'} = \{\vartheta\colon ||\vartheta||_{\beta_m'} < \infty\}$$

is contracting. We regard $S_{\beta_m'}$ as a Banach space with the norm $||\vartheta||_m = ||\vartheta||_{\beta_m'}$.

We can assume that in equations (9.3) and (9.4) the integration takes place along the paths $\mathfrak{C}_{kl}^\mu$ and $\mathfrak{C}_k^\mu$ respectively. Then according to Proposition 9.1,

$$\mathrm{Re}\Big(\frac{1}{\varepsilon}\int_\nu^\tau (\tilde\lambda_l^0(s) - ik\tilde\omega(s))\,ds\Big) \leqslant 2K, \quad -\mathrm{Re}\Big(\frac{1}{\varepsilon}\int_\nu^\tau ik\tilde\omega(s)\,ds\Big) \leqslant 2K.$$

Hence the exponentials in (9.3)–(9.4) are uniformly bounded by the magnitude e^{2K}. Let the constant L be defined by Proposition 9.1 and $M = Le^{2K}$.

For any two functions $\vartheta, \vartheta_* \in S_{\beta_m'}$ we have:

$$\begin{aligned}
\big|[P_m^{(\vartheta)}(\vartheta)]_l^k - [P_m^{(\vartheta)}(\vartheta_*)]_l^k\big| &\leqslant M\varepsilon \max_{\tau\in D}\Big|\sum_{j\in\mathbb{Z}} \big(B_m^{k-j}(\tau)(\vartheta^j(\tau) - \vartheta_*^j(\tau))\big)_l\Big| \\
&\leqslant nM\varepsilon \sum_{r=0}^m \sum_{j\in\mathbb{Z}} ||\tilde B_r||_{\beta_r} ||\vartheta - \vartheta_*||_m e^{-|k-j|\beta_r - |j|\beta_m'} \\
&\leqslant nM\varepsilon \sum_{r=0}^m b_r \Big(\frac{1}{1 - e^{\beta_m' - \beta_r}} + \frac{2}{1 - e^{-\beta_m'}}\Big) ||\vartheta - \vartheta_*||_m e^{-|k|\beta_m'} \\
&\leqslant ||\vartheta - \vartheta_*||_m e^{-|k|\beta_m'}/2.
\end{aligned}$$

Therefore,

$$||P_m^{(\vartheta)}(\vartheta) - P_m^{(\vartheta)}(\vartheta_*)||_m \leqslant ||\vartheta - \vartheta_*||_m/2.$$

and the operator $P_m^{(\vartheta)}$ is contracting with the contraction coefficient not exceeding $1/2$. The existence of the solution $(\vartheta, \varkappa)$ is established.

Now we prove the estimate (8.5). The triangle inequality implies:

$$||\vartheta||_m = ||P_m^{(\vartheta)}(\vartheta)||_m \leqslant ||P_m^{(\vartheta)}(\vartheta) - P_m^{(\vartheta)}(0)||_m + ||P_m^{(\vartheta)}(0)||_m \tag{9.5}$$

Since $||P_m^{(\vartheta)}(\vartheta) - P_m^{(\vartheta)}(0)||_m \leqslant ||\vartheta||_m/2$, by using (9.5) we obtain the following estimate:

$$||\vartheta||_m \leqslant 2||P_m^{(\vartheta)}(0)||_m.$$

The magnitude $||P_m^{(\vartheta)}(0)||_m$ can be estimated as follows:

$$\left|\left(P_m^{(\vartheta)}(0)\right)_l^k\right| \leqslant M\varepsilon^2||h_m||_{\beta_m} e^{-\beta_m|k|} \leqslant M\varepsilon^2\chi_m e^{-\beta'_m|k|}.$$

These inequalities imply the first estimate (8.5) with $P = 2M$. Now let us verify the second one:

$$\begin{aligned}
\left|\left(P_m^{(\varkappa)}(\vartheta)\right)^k\right| &\leqslant \varepsilon^{-1}M\Big[\varepsilon^3||a_m||_{\beta_m}e^{-\beta_m|k|} + \max_{\tau\in D}\Big|\sum_{j\in\mathbb{Z}}(F'_m)_\xi^{k-j}(0,\tau)\vartheta^j(\tau)\Big|\Big] \\
&\leqslant \varepsilon^{-1}M\Big[\varepsilon^3\chi_m e^{-\beta_m|k|} + n\sum_{j\in\mathbb{Z}} f_m||\vartheta||_m e^{-|k-j|\beta_m - |j|\beta'_m}\Big] \\
&\leqslant \varepsilon^{-1}M\Big[\varepsilon^3\chi_m + \frac{3nf_m}{1-e^{\beta'_m-\beta_m}}P\varepsilon^2\chi_m\Big]e^{-|k|\beta'_m}.
\end{aligned}$$

Hence, we can put $Q = 48Mnf_0P/\beta$. Lemma 8.1 is proved.

10. Proof of the Separation Lemma

The change (7.6) transforms the first equation (7.5) to

$$\dot\zeta = M\zeta + O(|\zeta|^2), \qquad M = (I+\varepsilon N)^{-1}(-\varepsilon N_\psi\tilde\omega - \varepsilon^2 N_\tau + (\tilde\Lambda + \varepsilon^2 B)(I+\varepsilon N)), \tag{10.1}$$

where we denote by $N_\psi\tilde\omega$ the sum $\sum_{k=1}^n N_{\psi_k}\tilde\omega_k$. The lemma asserts that we can take the function N with values in $\mathfrak{N}$ in such a way that $M \in \mathfrak{M}$. To prove this we study the equation

$$\varepsilon^2 N_\tau = -\varepsilon N_\psi\tilde\omega - (I+\varepsilon N)M + (\tilde\Lambda + \varepsilon^2 B)(I+\varepsilon N) \tag{10.2}$$

which is a consequence of (10.1). We put

$$\begin{gathered}
B = B_{\mathfrak{N}} + B_{\mathfrak{M}}, \qquad B_{\mathfrak{N}} \in \mathfrak{N}, \quad B_{\mathfrak{M}} \in \mathfrak{M}, \\
M = \tilde\Lambda + \varepsilon^2 B_{\mathfrak{M}} + \varepsilon^3 B_{\mathfrak{N}} N \in \mathfrak{M}.
\end{gathered}$$

Equality (10.2) takes the form

$$\varepsilon N_\tau + N_\psi\tilde\omega - [\tilde\Lambda, N] = \varepsilon B_{\mathfrak{N}} + \varepsilon^2[B_{\mathfrak{M}}, N] - \varepsilon^3 N B_{\mathfrak{N}} N. \tag{10.3}$$

Here the brackets $[\,,]$ denote the matrix commutator. Since all terms in (10.3) belong to $\mathfrak{N}$, we can regard this equation as a system for the coefficients N_{kl}, where $k = 1, 2$, $l = 3, \dots, n$ or $l = 1, 2$, $k = 3, \dots, n$:

$$\varepsilon(N_{kl})_\tau + (N_{kl})_\psi \tilde{\omega} - (\tilde{\lambda}_k - \tilde{\lambda}_l)N_{kl} = \varepsilon B_{kl} + \varepsilon^2\big([B_{\mathfrak{M}}, N]\big)_{kl} - \varepsilon^3(NB_{\mathfrak{N}}N)_{kl}. \tag{10.4}$$

This system is analogous to (8.2). It can be solved following the same line (see §10), i.e.

(1) We expand the system (10.4) into the Fourier series in ϕ.

(2) We search for a solution as the sum of a rapidly converging series (Newton method), where each term of the series satisfy certain linear system.

(3) We construct solutions of these linear systems by using the method of contracting mappings.

The details are omitted.

11. Proof of propositions

1. Proof of Proposition 3.1. Suppose for simplicity that $\tau_* = 0$ and expand the functions λ_1^k into Maclaurin series:

$$\lambda_1^k(\tau) = \lambda_1^0(0) + p\tau^l + O(\tau^{l+1}) - ik\omega(\tau), \quad l \geqslant 1. \tag{11.1}$$

The magnitude $\lambda_1^0(0)$ is purely imaginary. If $\tau \in \mathbb{R}$ increases and passes through the origin, $\lambda_1^0(\tau)$ crosses the imaginary axis from the left to the right. Consequently, $\operatorname{Re} p > 0$ and l is odd. In a typical situation $l = 1$. The function $\omega(\tau)$ is real-analytic. We put $\tau = x + iy$. The functions ϕ_1^k have the form:

$$\phi_1^k(\tau) = \int_0^x \operatorname{Re}\lambda_1^0(s)\,ds + \big[-\operatorname{Im}\lambda_1^0(x) + k\omega(x)\big]y + O((|k|+1)y^2). \tag{11.2}$$

The equations $\phi_1^k(\tau) = \sigma$, with small $\sigma > 0$ has the following solutions:

$$\begin{aligned} &\text{If } \ y = 0 \ \text{ then } \ x = \tau_\pm(\sigma), \quad \tau_\pm(\sigma) = \pm\Big(\frac{(l+1)\sigma}{\operatorname{Re} p}\Big)^{1/(l+1)} + O(\sigma^{2/(l+1)}), \\ &\text{If } x \in [\tau_-(\sigma), \tau_+(\sigma)] \text{ then } \ y = \frac{\sigma - \operatorname{Re} p\, x^{l+1}/(l+1) + O(\sigma x) + O(x^{l+2})}{k\omega(0) + i\lambda_1^0(0)}. \end{aligned} \tag{11.3}$$

Proposition 3.1 is proved.

2. Proof of Proposition 3.2.

(a) Equations (11.3) show that for small $\sigma > 0$ the segments $\Gamma^0(\sigma)$ lie in the upper half-plane. Suppose that some segment $\Gamma^0(\sigma)$, $0 \leqslant \sigma \leqslant \sigma_0$ has a point with negative imaginary part. Then for some $\sigma = \sigma' \in (0, \sigma_0)$ the equation $\phi_1^0(x) = \sigma'$, $x \in U \cap \mathbb{R}$ has a solution distinct from $\tau_\pm(\sigma')$. But this is impossible since the function $\phi_1^0(x)$ is monotonic on the intervals $U \cap \mathbb{R}^+$, $U \cap \mathbb{R}^-$, where $\mathbb{R}^\pm = \{x \in \mathbb{R} : \pm x \geqslant 0\}$. Arguments concerning the curves $\Gamma^{-1}(\sigma)$ are analogous.

(b) There are two possible reasons because of which a family $\Gamma^k(\sigma)$ may be not defined for $0 \leqslant \sigma \leqslant \sigma_0$. Either the curves $\Gamma^k(\sigma)$ go out from the domain U, or there is a singularity for these curves inside U. The first possibility does not take place due to assertion (c). The second one is equivalent to the condition

$\partial\phi_l^k/\partial x = \partial\phi_l^k/\partial y = 0$, for some $\tau = x + iy \in U$. This condition can be rewritten in the form $\lambda_l^k(\tau) = 0$ or $\rho_l(\tau) = 1$. The last equality holds nowhere in U due to assumption $C1$.

(c) First, note that the magnitude $-\mathrm{Im}\,\lambda_1^0(x)$ (respectively, $-\mathrm{Im}\,\lambda_1^0(x) - \omega(x)$) is positive (respectively, negative) for $x \in U \cap \mathbb{R}$. Indeed, for $x = 0$ (recall that we assume that $\tau_* = 0$) this assertion follows from definition of λ_1^0. If $-\mathrm{Im}\,\lambda_1^0(x)$ is negative for some $x \in U \cap \mathbb{R}$ then there exists $x' \in U \cap \mathbb{R}$ such that $-\mathrm{Im}\,\lambda_1^0(x') = 0$. Let us choose the closest to zero point x' satisfying this equation. Then at this point the relation $\rho_1 = \rho_2$ holds which contradicts to assumption $C2$.

The following inequalities hold for $x \in U \cap \mathbb{R}$:

$$\omega(x) > 0, \quad \mathrm{Re}\,\lambda_1^0(x) > 0 \text{ for } x > 0, \quad \mathrm{Re}\,\lambda_1^0(x) < 0 \text{ for } x < 0.$$

Using these remarks and the equality (11.2) we see that for small $\beta \geqslant 0$ the curves $\Gamma^k(\sigma_0)$ can be partially determined by the relations

$$x = \tau_-(\sigma_0) + \beta, \quad y = \frac{-\mathrm{Re}\,\lambda_1^0(\tau_-(\sigma_0))\beta}{-\mathrm{Im}\,\lambda_1^0(\tau_-(\sigma_0)) + k\omega(\tau_-(\sigma_0))} + O(\beta^2),$$
$$x = \tau_+(\sigma_0) - \beta, \quad y = \frac{\mathrm{Re}\,\lambda_1^0(\tau_+(\sigma_0))\beta}{-\mathrm{Im}\,\lambda_1^0(\tau_+(\sigma_0)) + k\omega(\tau_+(\sigma_0))} + O(\beta^2)$$

which imply in particular, that for $k \geqslant 0$ ($k < 0$) the segments $\Gamma^k(\sigma_0)$ lie at least near the points $\tau_\pm(\sigma_0)$ in the upper (lower) half-plane. Moreover, these relations show that for any integer k distinct from 0 and -1 the curve $\Gamma^k(\sigma_0)$ lies between $\Gamma^0(\sigma_0)$ and $\Gamma^{-1}(\sigma_0)$ at least near the points $\tau_\pm(\sigma_0)$. Hence, to complete the proof of assertion (c) it is sufficient to verify that the curves $\Gamma^k(\sigma_0)$ and $\Gamma^l(\sigma_0)$, $l = 0, -1$ intersect only at the points $\tau_\pm(\sigma_0)$. The existence of other intersection point τ is equivalent to the equality $\phi_1^k(\tau) = \phi_1^l(\tau) = \sigma_0$ which cannot hold for real $\tau \neq \tau_\pm(\sigma_0)$. For $\tau \notin \mathbb{R}$ this equality contradicts to assumption $C3$. Proposition 3.2 is proved.

3. Proof of Proposition 3.3. Note that two positive numbers

$$i\lambda_1^0(0) \quad \text{and} \quad \omega(0) - i\lambda_1^0(0) \quad \text{are distinct} \tag{11.5}$$

because otherwise $i\lambda_1^0(0) = -i\omega(0)/2$ and $\rho_1(0) = \rho_2(0) = -1$ (the last relation contradicts to assumption $C2$). Now assumption $A1$ follows for small $\sigma_0 > 0$ from (11.3) and (11.5).

Assumption $A2$ holds for small $\sigma_0 > 0$ because the corresponding domain $D = D(\sigma_0)$ is a small neighborhood of the point $\tau = \tau_*$.

Now we verify assumption $A3$. According to condition of Proposition 3.3 we have $l = 2$ in the relations (11.1) and (11.3). Putting $\tau_* = 0$ and $\tau = x + iy$, we have the following expansions:

$$\phi^\pm(\tau) = \omega(x)y + O(y^2),$$
$$\phi_j^k(\tau) = \mathrm{Re}\,\lambda_j^0(0)x + (-\mathrm{Im}\,\lambda_j^0(x) + k\omega(x))y + O(x^2) + O((|k|+1)y^2).$$

First, take the function $\phi^+|_{\partial D}$. It has a maximum on the curve Γ^0. Since for small $\sigma_0 > 0$ the curve $\Gamma^0(\sigma_0)$ can be defined by the relations (11.3), the function $\phi^+|_{\Gamma^0}$ has the form

$$\frac{\omega(0)(\sigma - \mathrm{Re}p\,x^2/2)}{i\lambda_1^0(0)} + O(\sigma x) + O(x^3), \qquad x \in [\tau_-(\sigma_0), \tau_+(\sigma_0)].$$

This function has a maximum at a point $x = O(\sigma_0)$. This maximum is nondegenerate since $\omega(0)\mathrm{Re}p/(i\lambda_1^0(0)) > 0$. The function ϕ^- can be analysed analogously.

For $x \in [\tau_-(\sigma_0), \tau_+(\sigma_0)]$ the functions $\phi_j^k|_{\Gamma_0}$, $j \geqslant 3$ can be presented in the following form:

$$\mathrm{Re}\,\lambda_j^0(0)x + (-\mathrm{Im}\,\lambda_j^0(0) + k\omega(x))\frac{\sigma - \mathrm{Re}p\,x^2/2}{i\lambda_1^0(0)} + O(\sigma x) + O(x^2). \tag{11.6}$$

Formulas determining the functions $\phi_j^k|_{\Gamma_{-1}}$, $j \geqslant 3$ are analogous.

Since $\tau_\pm(\sigma) = O(\sqrt{\sigma})$ and $\mathrm{Re}\,\lambda_j^k(0) < 0$, it is clear that for $|k| < (c\sqrt{\sigma}_0)^{-1}$ (the constant c is sufficiently large) the functions $\phi_j^k|_{\partial D}$, $j \geqslant 3$ have maximums only at the point $\tau_-(\sigma_0)$. If $|k| \geqslant (c\sqrt{\sigma_0})^{-1}$, the functions $\phi_j^k(\tau)/k$ are small perturbations of the function $\phi^+(\tau)$:

$$\phi_j^k(\tau)/k = \phi^+(\tau) + \frac{1}{k}\int_0^\tau \lambda_j^0(s)\,ds.$$

Hence, assumption $A3$ in this case holds because it holds for the functions $\phi^\pm$.

Assumption $A3$ for the functions $\phi_j^k|_{\partial D}$, $j = 1, 2$ and small σ_0 can be verified analogously by using (11.5).

4. Proof of Proposition 9.1. Let τ^k be maximums of the functions

$$-\mathrm{Re}\int_0^\tau ik\tilde{\omega}(s)\,ds \tag{11.7}$$

restricted to D. For small values of ε and $k \neq 0$ these maximums are unique according to A3 and due to the closeness of the functions $\tilde{\omega}$ and ω on D. (The point $\tau^0 \in D$ is arbitrary.) Note that $\tau^k \in \partial D$ for $k \neq 0$ and $\tau^k = \tau^l$ if k and l have the same sign. Moreover, for $k \neq 0$ restrictions of the functions (11.7) to the boundary ∂D have nondegenerate maximums at the points τ^k.

Analogously, let τ_l^k be global maximums of the functions

$$-\mathrm{Re}\int_0^\tau \big(\tilde{\lambda}_l^0 - ik\tilde{\omega}(s)\big)\,ds \tag{11.8}$$

restricted to D. Note that if $|k|$ is not very large, the functions (11.8) restricted to D can have other (local) maximums.

Inequalities $\tilde{\phi}_l^k(\tau) \leqslant 0$, $\tilde{\phi}^k(\tau) \leqslant 0$, $\tau \in D$ hold according to the definition of τ_l^k and τ^k. Now we construct the curves $\mathfrak{C}_{kl}^\tau$, $\mathfrak{C}_k^\tau$.

Let any of the curves $\mathfrak{C}_{kl}^\tau$ ($\mathfrak{C}_k^\tau$) consist of two parts. The second part links the boundary ∂D with the point τ along a level line of the function $\tilde{\phi}_l^k$ ($\tilde{\phi}^k$). Note that since the functions $\tilde{\phi}_l^k, \tilde{\phi}^k$ are harmonic, their level lines having common points with D, reach ∂D. The first part goes along the curve ∂D and links the point τ_l^k (τ^k) with the beginning of the second part. Obviously, these curve satisfies the properties (i)–(iii).

Appendix 1. Geometry of the curves ϕ_l^k = const near the branching points

Curve ϕ_l^k = const (Section 3) is tangent to the vector $i\overline{\lambda}_l^k(\tau)$ at the point τ of the complex plane. Tangent vector field $i\overline{\lambda}_l^k$ has in general branching at the points, where either $\lambda_l^k = 0$ (there is multiplier 1) or $\lambda_l^k = \lambda_j^r$, $l \neq j$ (there is a double multiplier). In this appendix geometry of the curves ϕ_l^k = const near such points is described.

A neighborhood of a point with multiplier 1. Let at $\tau = \tau_c$ the system (2.2) has a periodic solution with a simple multiplier 1. Then at the point τ_c the periodic solution as a function of the parameter τ has in general a second order branching. Correspondingly, the multipliers and the eigenvalues have branching also. Enumerate eigenvalues in such a way that λ_1^k correspond to multiplier 1. Then

$$\lambda_1^k(\tau) = ki\omega(\tau_c) + \alpha\sqrt{\tau-\tau_c} + \beta_k(\tau-\tau_c) + O(|\tau-\tau_c|^{3/2}), \tag{A.1}$$
$$\lambda_l^k(\tau) = \lambda_l^k(\tau_c) + \alpha_l\sqrt{\tau-\tau_c} + \beta_{l,k}(\tau-\tau_c) + O(|\tau-\tau_c|), \quad l>1.$$

The functions $O(\cdot)$ in (A.1) are analytic on the Riemann surface of square root near τ_c. Such a representation of eigenvalues can be obtained by an analysis of unfolding for the system having a periodic solution with multiplier 1.

Proposition A1. *Let $\alpha \neq 0, \beta_0 \neq 0$, and the ratio β_0^3/α^4 is not purely imaginary. Then the family of the curves ϕ_1^0 = const looks like it is shown in Fig. a.*

Angles between the curves ϕ_1^0 = const at the point τ_c are equal either to $2\pi/3$ or to 0. Tangency of the curves at the point τ_c is quadratic. There are two curves passing through any point near τ_c because due to branching of the eigenvalues in (A.1) the tangent field is double valued. (V. V. Sidorenko attracted our attention to this effect of branching of the tangency field and the family of the curves ϕ_1^0 = const). For the problem of stability loss delay the existence of two curves ϕ_1^0 = const passing throughg the same point is, probably, not essential. These curves correspond to different periodic solutions which transform into each other as a result of circuit around the point τ_c.

Behavior of the family of the curves ϕ_1^k = const, $k \neq 0$ for $\alpha \neq 0$, and of the family ϕ_l^k = const, $l \neq 0$ for $\alpha_l \neq 0$, is exactly the same as the behavior of the family of the curves ϕ_1^0 = const in the case of a double multiplier (see next subsection).

There is angle $2\pi/3$ between curves ϕ_1^0 = const at the point corresponding to a periodic solution with multiplier 1 in Fig. (see numerical example, Section).

Proof of Proposition A1. We consider the family of curves

$$\phi_1^0 = \mathrm{Re}\Big[\frac{2}{3}\alpha(\tau-\tau_c)^{3/2} + \frac{1}{2}\beta_0(\tau-\tau_c)^2 + O(|\tau-\tau_c|^{5/2})\Big] = \mathrm{const}$$

on the covering surface, i.e. for $\tau - \tau_c = \vartheta^2$. By scaling, rotation, and reflections one can achieve that $2\alpha/3 = 1$, $\beta_0/2 = a + bi$ and $a > |b|\sqrt{3}$. Now we have:

$$\phi_1^0 = \mathrm{Re}[\vartheta^3 + (a+bi)\vartheta^4 + O(|\vartheta|^5)] = \mathrm{const}. \tag{A.2}$$

Portrait of the family (A.2) in the plane ϑ near the point $\vartheta = 0$ is shown in Fig. .b. Returning to the varible τ we obtain Fig. .a. In Figs. .a,b, we denote with the same numbers the curves which are mapped to each other under the transformation $\tau - \tau_c = \vartheta^2$.

A neighborhood of a point with double multiplier. Let at the point $\tau = \tau_c$ two multipliers of the periodic solution L_τ coincide (and are different from 1). Then at τ_c these multipliers as functions of the parameter τ have in general a second order branching: they transform into each other as a result of a circuit around the point τ_c. Let the eigenvalues λ_1^k and λ_2^k correspond to these multipliers. Then

$$\begin{gathered}\lambda_1^k(\tau) = ki\omega(\tau_c) + \alpha + \beta\sqrt{\tau - \tau_c} + O(|\tau - \tau_c|),\\ \lambda_2^k(\tau) = ki\omega(\tau_c) + \alpha - \beta\sqrt{\tau - \tau_c} + O(|\tau - \tau_c|),\\ ki\omega(\tau_c) + \alpha \neq 0.\end{gathered} \tag{A.3}$$

The functions $O(\cdot)$ in (A.3) are analytic on the Riemann surface of square root near τ_c. Such a representation of eigenvalues can be obtained by an analysis of unfolding for the system having a periodic solution with double multiplier.

Proposition A2. *Suppose that $\beta \neq 0$. Then the family of the curves $\phi_{1,2}^0 = const$ looks like it is shown in Fig. .a.*

Tangency of the curves at the point τ_c is quadratic.

Behavior of the curves $\phi_{1,2}^0 = \text{const}$ in Fig. (see numeric example, Section) near the point with double multiplier is the same as in Fig. .a.

*Proof of Proposition A*2. Consider the family of curves

$$\phi_1^0 = \mathrm{Re}\big[\alpha(\tau - \tau_c) + \frac{2}{3}\beta(\tau - \tau_c)^{3/2} + O(|\tau - \tau_c|^2)\big] = \text{const}$$

on the covering surface, i.e. for $\tau - \tau_c = \vartheta^2$. By scaling, rotation, and reflections one can achieve that $\alpha = 1$, $2\beta/3 = a + bi$ and $a > |b|$. Now we have

$$\phi_1^0 = \mathrm{Re}\left[\vartheta^2 + (a + bi)\vartheta^3 + O\left(|\vartheta|^4\right)\right] = \text{const}. \tag{A.4}$$

The portrait of the family (A.4) in the plane ϑ near the point $\vartheta = 0$ is shown in Fig. .b. Returning to the varible τ we obtain Fig. .a. In Figs. .a,b, we denote by the same numbers the curves which are mapped to each other under the transformation $\tau - \tau_c = \vartheta^2$.

Appendix 2. Effects of computational errors in numerical simulation of stability loss delay.

Consider a problem with stability loss delay. Phase point, while moving during the time $\sim 1/\varepsilon$ in the region of stability ($\tau < \tau_*$) is attracted exponentially close (the distance is $\sim \exp(-\text{const}/\varepsilon)$) to the invariant manifold M_ε (Section 5). Thus, the phase point needs time $\sim 1/\varepsilon$ to go away at the distance $\sim \varepsilon$ from M_ε while moving in the instability region ($\tau > \tau_*$). In numerical simulations error of determination of the phase point position at $\tau = \tau_*$ should be smaller then the distance of the phase point from M_ε. Otherwise the delay time, which is observed in simulation, is determined by the error of simulation. Below we estimate delay time observed in the last case. To be definite we consider case **A**, Section 4. We suppose that the number of variables for frozen τ is equal to 3 (i.e. $n = 2$), and that at the moment τ_* real part of the eigenvalue passes through 0 with a velocity $\varepsilon p \neq 0$.

Consider a phase point, which is attracted to M_ε at $\tau = \tau_0$. According to Theorem 4.2 this point departs from M_ε at $\tau = \Pi(\tau_0)$. It follows from the proof of Theorem 4.2 that at the moment $\tau = \tau_*$ the distance of this phase point from M_ε should be of the order $\sim \exp(-\Phi_1^0(\tau_0)/\varepsilon)$ (the function Φ_1^0 is defined in Section 3). Let us suppose that in a numerical simulation position of this phase point is defined sequentially at discrete time moments with the step Δ, and that the error of numerical method for one step is a value of order R. The essential influence on the delay time is exerted by errors of the numerical method on a time interval where $|\tau - \tau_*|$ has the order $\sqrt{\varepsilon}$ or smaller. Indeed, for τ close to τ_* the distance $r(\tau)$ of the phase point from M_ε is in the main approximation $r(\tau) = r(\tau_*)\exp(-\frac{p(\tau-\tau_*)^2}{2\varepsilon})$. Therefore, the effect of the errors in the region $\tau_* - \tau \gg \sqrt{\varepsilon}$ is suppressed during the motion in the stability region, and the effect of the errors in the region $\tau - \tau_* \gg \sqrt{\varepsilon}$ has no time to grow enough till the moment of departure.

The region $|\tau - \tau_*| < \sqrt{\varepsilon}$ contains $N \sim 1/(\sqrt{\varepsilon}\Delta)$ steps of the numerical method. Suppose that the total error on these steps is of order $\sqrt{N}R \sim R/(\varepsilon^{1/4}\Delta^{1/2})$ (i.e. the errors accumulate as random independent values with zero average). If $R/(\varepsilon^{1/4}\Delta^{1/2}) > \exp(-\Phi_1^0(\tau_0)/\varepsilon)$, then the observed departure moment is determined by errors of the numerical simulation. There is some ambiguity in definition of the observable departure moment. We will consider $\tau = \tau_r$ as the departure moment, if at this moment the distance of the phase point from M_ε is equal to $\sqrt{\varepsilon}$. This means that at this moment $(R/(\varepsilon^{1/4}\Delta^{1/2}))\exp(\Phi_1^0(\tau_r)/\varepsilon) \sim \sqrt{\varepsilon}$ and, therefore, $\Phi_1^0(\tau_r) \approx \varepsilon\log(\varepsilon^{3/4}\Delta^{1/2}/R)$. Usually $R \ll \varepsilon^{3/4}\Delta^{1/2}$. Consequently, $\Phi_1^0(\tau_r) \approx \varepsilon\log(1/R)$. If τ_r is close to τ_* then $\Phi_1^0(\tau_r) \approx p(\tau_r - \tau_*)^2/2$. In this case $\tau_r \approx \tau_* + \sqrt{(2\varepsilon/p)\log(1/R)}$. The accuracy of the last formula is $O\Big(\sqrt{\frac{\varepsilon}{p\log(1/R)}}\Big)$.

Suppose the error of numerical method be the rounding error due to finiteness of number of digits in a computer. Let the system under consideration is time periodic (as in numerical examples of Section...). Then the following interesting computational phenomenon appear. Let us start to compute the motion of several phase points from the same time moment $t = t_0 = \tau_0/\varepsilon$ but from different initial values of coordinats in $O(1)$-neighborhood of M_ε. Then for these phase points the observed departure moments coincide identically. The reason is that in the region of stability the computed trajectories of these phase points approach to M_ε and to each other so close that starting from a certain step these trajectories coincide identically due to finiteness of number of digits in computer.

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