SEPARATRIX SPLITTING FOR A PENDULUM WITH RAPIDLY OSCILLATING SUSPENSION POINT

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Abstract. We study exponentially small separatrix splitting in pendulum with rapidly oscillating suspension point. We use a new averaging method which is a continuous version of the Neishtadt averaging procedure. An asymptotic formula for the rate of the separatrix splitting is obtained. In the problem under consideration the ordinary Poincaré-Melnikov integral gives a wrong answer. The correct asymptotics has a form of an infinite convergent series in powers of some (not small) parameter of the problem. The Melnikov integral is the zero term of this series.

§1. Introduction. Main result

Exponentially small effects in analytic dynamical systems were discovered by Poincaré [1], but their detailed studying began much later. These effects play an essential role in a number of chaotic phenomena, e.g. in the problem of the Arnold diffusion in Hamiltonian systems close to integrable ones [2,3]. Diffusion and the width of a stochastic layer in degenerate Hamiltonian systems were studied by Zaslavsky et al [4–7]. For quasiconvex Hamiltonians apparently optimal estimates (from above) for Arnold diffusion were obtained by Lochak, Neishtadt [8–9] and Pöschel [10].

The problem of exponentially small separatrix splitting became popular after works of Lazutkin et al [11–15] and also of Holmes, Marsden and Scheurle [16]. In papers of Gelfreich [17], Delshams, Seara [18] and others a number of systems with exponentially small separatrix splitting were studied. In the most of the results the ordinary Poincaré-Melnikov integral predicts correct asymptotics of splitting. Actually the Poincaré-Melnikov method is not obliged to work correctly in the case of exponentially small splitting. Furthermore, in a general case it gives a wrong answer.

In this paper we study pendulum with rapidly oscillating suspension point. Using some special averaging method (which is a continuous version of the Neishtadt averaging procedure [19]) we obtain an asymptotic formula for the separatrix splitting. This formula differs from that presented by the ordinary Poincaré-Melnikov method.

Consider the real Hamiltonian system

$$\dot x = \partial \hat H/\partial y, \quad \dot y = -\partial \hat H/\partial x, \tag{1.1}$$

$$\hat H = \hat H^0(x,y) + e^{it/\varepsilon}\hat H^+(x,y) + e^{-it/\varepsilon}\hat H^-(x,y). \tag{1.2}$$

Here $y \in \mathbb{R}$ and $x \in \mathbb{R}/2\pi\mathbb{Z}$. The parameter ε is small. We take

$$\hat{H}^0 = y^2/2 - \cos x - 1, \quad \hat{H}^\pm = B_+^\pm e^{ix} + B_-^\pm e^{-ix}, \quad B_\pm^\pm \in \mathbb{C}. \tag{1.3}$$

Hence, we have a pendulum with rapidly oscillating suspension point, ε^{-1} being the frequency of oscillations. The relations $\bar{B}_+^+ = B_-^-$, $\bar{B}_-^+ = B_+^-$ hold where overbars denote complex conjugation. Note that an analysis of a particular case of the system (1.1)–(3.1) led Poincaré to the discovery of exponentially small effects [1].

For small values of ε trajectories of the system (1.1)–(1.3) are close to those of the usual pendulum:

$$\dot{x} = \partial\hat{H}^0/\partial y, \quad \dot{y} = -\partial\hat{H}^0/\partial x. \tag{1.4}$$

The hyperbolic equilibrium $(x, y) = (\pi, 0)$ of the pendulum (1.4) has a separatrix loop γ in the domain $\{y > 0\}$. For small ε the system (1.1)–(1.3) has a hyperbolic $2\pi\varepsilon$-periodic motion $p_\varepsilon(t)$ which is close to the equilibrium $(\pi, 0)$. There is a pair of asymptotic to $p_\varepsilon(t)$ surfaces $\Gamma_\varepsilon^\pm$ in the extended phase space

$$\widetilde{P} = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \times (\mathbb{R}/(2\pi\varepsilon\mathbb{Z})) = \{x, y, t\}.$$

These surfaces are close to $\gamma \times (\mathbb{R}/(2\pi\varepsilon\mathbb{Z}))$, but in general they do not coincide (see Fig. 1 that presents the section Π_c of the phase space $\widetilde{P}$ by a plane $t = c = \text{const}$). The rate of their splitting is exponentially small in ε. We measure this rate by the area $\mathcal{A}$ of a domain $D \subset \Pi_c$ bounded by the stable and unstable separatrices. Note that $\mathcal{A}$ is a symplectic invariant, i.e. $\mathcal{A}$ does not depend on coordinates on the symplectic manifold $(\Pi_c, dx \wedge dy)$. Moreover, it remains constant when c changes.

Theorem 1.1. *For small $\varepsilon > 0$ the area $\mathcal{A}$ can be estimated in the following way*:

$$\mathcal{A} = 4\varepsilon^{-1} e^{-\pi\varepsilon^{-1}/2} \big(|J(B_-^+, B_-^-)| + O(\varepsilon \log \varepsilon^{-1})\big), \tag{1.5}$$

where the function J is a solution of some nonlinear problem not containing the small parameter ε. The function J is defined in §2 (see (2.20)). The following equalities are satisfied:

$$J(B_-^+, B_-^-) = 2B_-^+ f(4B_-^+ B_-^-), \qquad f(0) = 4\pi.$$

The function $f(z)$ is an entire real-analytic function of the complex variable z.

Remark 1. The coefficient $\pi/2$ at ε^{-1} in the main exponential term of (1.5) has a clear analytic meaning. It is the distance from the real axis to the nearest singularity of the separatrix solution of the pendulum (1.4).

Remark 2. The relation (1.5) estimates splitting of the upper separatrix of the pendulum. The splitting of the lower separatrix can be estimated by the same formula where the constants B_-^+, B_-^- are replaced by B_+^+, B_+^- respectively. This follows from the change $x, y \to -x, -y$.

Remark 3. If $B_-^- = 0$ then $J(B_-^+, B_-^-) = 8\pi B_-^+$. Actually this means that the ordinary Poincaré-Melnikov integral gives correct asymptotics of separatrix splitting for $B_-^- = 0$. The function f is not a constant. Let $f = 2\pi \sum_0^\infty f_n z^n$. Then $f_0 = 2$. The coefficient f_1 can be calculated explicitly:

$$f_1 = -\frac{1}{9} + \frac{32}{15} \sum_{k,l=1}^{\infty} \frac{(8k^2 - 4k + 1)l(l+1)}{3^{2k+l-1}(2k-1)(2k+l-1)(2k+l)} = 0.6585673851875\dots$$

Together with Carles Simo we calculated numerically other Tailor coefficients of the function f. In particular,

$$f_2 = 6.651741\ldots \cdot 10^{-2}, \quad f_3 = 3.21010\ldots \cdot 10^{-3}, \quad f_4 = 9.03367\ldots \cdot 10^{-5},$$
$$f_5 = 1.6620\ldots \cdot 10^{-6}, \quad f_6 = 2.1534\ldots \cdot 10^{-8}, \quad f_7 = 2.070\ldots \cdot 10^{-10},$$
$$f_8 = 1.53\ldots \cdot 10^{-12}.$$

These estimates show that probably, all the coefficients f_n are positive and decrease like $(n!)^{-2}$.

Remark 4. The change $t \to \varepsilon t$ transforms the system (1.1)–(1.2) into the following one:

$$\dot{x} = \partial(\varepsilon\hat{H})/\partial y, \quad \dot{y} = -\partial(\varepsilon\hat{H})/\partial x, \tag{1.6}$$
$$\hat{H} = \hat{H}^0(x,y) + e^{it}\hat{H}^+(x,y) + e^{-it}\hat{H}^-(x,y). \tag{1.7}$$

For short the new time is denoted here again by t. All our further constructions deal with the system (1.6),(1.7),(1.3) instead of (1.1)–(1.3). Certainly the area $\mathcal{A}$ calculated in the system (1.6),(1.7),(1.3) coincides with that calculated in the original system.

§2. Problem of Evaluation the Function J.

a. Averaging Method.

The proof of Theorem 1.1 is based on a continuous version of the Neishtadt averaging procedure [19] reducing the rate of dependence of our Hamiltonian on time to an exponentially small order. After averaging we use the usual Poincaré-Melnikov method.

We search for a change of the variables weakening the dependence of the Hamiltonian $\varepsilon\hat{H}$ (see Remark 4 to Theorem 1.1) on time as a shift along trajectories of the Hamiltonian vector field

$$\partial x/\partial\delta = \partial F/\partial y, \quad \partial y/\partial\delta = -\partial F/\partial x, \qquad F = F(x,y,t,\varepsilon,\delta). \tag{2.1}$$

We choose $F = \xi H(x,y,t,\varepsilon,\delta)$, where

$$\xi H = i\varepsilon(e^{it}H^+ - e^{-it}H^-).$$

The shift along trajectories of (2.1) on the magnitude $\delta \geqslant 0$ transforms the original Hamiltonian $\varepsilon\hat{H}$ into εH

$$H(x,y,t,\varepsilon,\delta) = H^0(x,y,\varepsilon,\delta) + e^{it}H^+(x,y,\varepsilon,\delta) + e^{-it}H^-(x,y,\varepsilon,\delta),$$
$$H(x,y,t,\varepsilon,0) = \hat{H}(x,y,t,\varepsilon).$$

The function H satisfies the equation

$$H_\delta = (\xi H)_t - \{\xi H, H\}, \tag{2.2}$$

where the subindices δ, t denote partial derivatives, and $\{\ ,\ \}$ is the Poisson bracket: $\{f, g\} = f_y g_x - f_x g_y$ for any two functions f and g. Equation (2.2) is equivalent to the system

$$H^0_\delta = -2i\varepsilon\{H^+, H^-\}, \quad H^\pm_\delta + H^\pm = \pm i\varepsilon\{H^0, H^\pm\}, \tag{2.3}$$

$$H^0(x, y, \varepsilon, 0) = \hat{H}^0(x, y), \quad H^\pm(x, y, \varepsilon, 0) = \hat{H}^\pm(x, y). \tag{2.4}$$

Henceforth an important role is played by complex singularities of the separatrix solution of the system (1.4). An informal explanation of this phenomenon is as follows. Consider the system (2.3′),(2.4):

$$H^0_\delta = 0, \quad H^\pm_\delta + H^\pm = \pm i\varepsilon\{H^0, H^\pm\}, \tag{2.3′}$$

where the system (2.3′) differs from (2.3) only by the right hand side of the first equation. This simplified system can be solved explicitly:

$$H^0 = \hat{H}^0, \quad H^\pm = e^{-\delta}\hat{H}^\pm \circ g^{\pm i\varepsilon\delta},$$

where g^t is the phase flow g^t of the pendulum (1.4). The functions $e^{-\delta}\hat{H}^\pm \circ g^{\pm i\varepsilon\delta}$ decrease rapidly in absolute value as δ increases, but they can have singularities for real x and y when δ is of order $\sim \varepsilon^{-1}$ because the functions $g^t(x, y)$ have singulatities for complex t. Hence, values of the parameter δ cannot be chosen too large.

Although there exists an explicit solution of equations (2.3′), we cannot solve the complete system (2.3)–(2.4) precisely. Asymptotics of the separatrix splitting can be obtained as a result of the analysis of the system (2.3)–(2.4) in the vicinity of the most essential singularity.

b. Time-energy coordinates.

The separatrix solution $(x_\gamma(t), y_\gamma(t)) = g^t(0, 2)$ of the system (1.4) corresponds to the energy level $\hat{H}^0 = 0$ and can be obtained explicitly by using the quadratures

$$t = \int_0^{x_\gamma(t)} f(0, u)\, du, \quad f(h, u) = \frac{1}{\sqrt{2(1 + h + \cos u)}}.$$

Here the function f is supposed to be positive for real values of u close to zero.

If the function x_γ has a singularity at the point $t = \alpha$ then $\alpha = \int_0^{\pm i\infty} f(0, u)\, du$. The magnitude α in this relation depends on path of integration. The singularity with the minimal positive imaginary part corresponds to the ray $R = \{x \in \mathbb{C} : \operatorname{Re} x = 0,\ \operatorname{Im} x \geqslant 0\}$. It is easy to calculate: $\int_R f(0, u)\, du = i\pi/2$.

Solutions of the system (1.4), whose initial conditions are close to $(0, 2)$, have singularities close to the point $t = i\pi/2$. Let $(x(t), y(t))$ be such a solution and t_s a corresponding singularity. We put $\nu = \exp(ix/2)$. For large values of $\operatorname{Im} x$ we have $f(h, x) = \nu(1 + O(\nu))$. Thus for values of time t, close to t_s

$$t_s - t = \int_{x(t)}^{+i\infty} f(h, u)\, du = 2i\nu(t)\big[1 + O(\nu(t))\big],$$

where $\nu(t) = \exp(ix(t)/2)$. We see that

$$\nu(t) = \frac{t_s - t}{2i}(1 + O(t_s - t)), \quad y(t) = \dot{x}(t) = \frac{2}{i}\nu^{-1}\dot{\nu} = \frac{2i}{t_s - t}(1 + O(t_s - t)).$$

Therefore we have:

$$\lim_{t\to t_s} \nu(t) = 0, \quad \lim_{t\to t_s} y(t) = \infty, \quad \lim_{t\to t_s} y(t)\nu(t) = 1. \tag{2.5}$$

It is convenient to introduce coordinates τ, h of the type time–energy for the system (1.4). The variable h is defined by the equality

$$h = \hat{H}^0 = (y^2 - \nu^{-2} - \nu^2)/2 - 1. \tag{2.6}$$

The variable τ is canonically conjugate to h, and the function $\tau(x, y)$ vanishes for $\nu = 0$. According to relations (2.5) we claim that $\lim_{\tau\to 0} y(\tau, h)\nu(\tau, h) = 1$.

Proposition 2.1.

$$\nu^{-1} = -\sqrt{2h}i/\sin(\tau\sqrt{h/2}) + F_1(\tau, h\tau^2), \tag{2.7}$$

$$y = -\sqrt{2h}i/\tan(\tau\sqrt{h/2}) + F_2(\tau, h\tau^2), \tag{2.8}$$

where the functions $F_1(\tau,\sigma)$ and $F_2(\tau,\sigma)$ are analytic at the point $(\tau,\sigma) = (0,0)$.

Corollary. In the variables τ, h the functions $\hat{H}^0$ and $\hat{H}^\pm$ have the form

$$\hat{H}^0 = h, \quad \hat{H}^\pm = -4hB^\pm_-[1 - \cos(\tau\sqrt{2h})]^{-1} + \tau^{-1}F^\pm(\tau, h\tau^2), \tag{2.9}$$

where the functions $F^\pm(\tau,\sigma)$ are analytic at the point (0,0).

Proof of the corollary is a straightforward computation with the help of relations (1.3) and (2.7)–(2.8).

Remark. The functions $\tau\nu^{-1}$ and τy are obviously analytic in τ and $\sigma = h\tau^2$ at zero.

Proof of Proposition 2.1. Equality (2.6) can be solved with respect to y:

$$y = \pm\nu^{-1}\sqrt{1 + 2h\nu^2 + 2\nu^2 + \nu^4}. \tag{2.10}$$

We suppose that for small $|\tau|$ and $|h\tau^2|$ the value of $|\nu|$ is small, and the value of $y\nu$ is close to 1. Thus it is necessary to take in (2.10) the sign "+". We get:

$$y = \nu^{-1}\sqrt{1 + 2h\nu^2} + \varphi_1(\nu, h\nu^2), \tag{2.11}$$

where the function $\varphi_1(\nu,\sigma)$ is analytic at the point $(0,0)$.

Let $S = S(x,h)$ be a generating function of the canonical transformation $x, y \to \tau, h$. Equality (2.11) is equivalent to that $y = \partial S/\partial x$. Consequently,

$$S = \frac{2}{i}\int \nu^{-2}\sqrt{1 + 2h\nu^2}\,d\nu + \varphi_2(\nu, h\nu^2) + \varphi_3(h).$$

The function φ_2 is analytic at zero and φ_3 is arbitrary. We put $\varphi_3 = 0$ and have:

$$\tau = \frac{\partial S}{\partial h} = \frac{2}{i}\int_0^\nu \frac{d\nu}{\sqrt{1+2h\nu^2}} + \nu^2\varphi_4(\nu, h\nu^2) = \frac{2\operatorname{arcsinh}(\sqrt{2h}\nu)}{i\sqrt{2h}} + \nu^2\varphi_4(\nu, h\nu^2). \tag{2.12}$$

Here we use the notation arcsinh for the function inverse to the hyperbolic sinus. The function φ_4 is analytic at zero. According to the implicit function theorem equation (2.12) can be solved with respect to ν:

$$\nu = \tau\varphi_5(\tau, h\tau^2), \tag{2.13}$$

where the function φ_5 is analytic at zero.

To find $\varphi_5(0, h\tau^2)$ we put $\nu = \tilde\nu\beta$, $\tau = \tilde\tau\beta$, $h = \tilde h\beta^{-2}$, where β is some small parameter. We get from (2.12) and (2.13):

$$i\tilde\tau\sqrt{\tilde h/2} = \operatorname{arcsinh}\Big(\sqrt{2\tilde h}\tilde\nu\Big) + i\beta\tilde\nu^2\sqrt{\tilde h/2}\,\varphi_4(\beta\tilde\nu, \tilde h\tilde\nu^2), \quad \tilde\nu = \tilde\tau\varphi_5(\beta\tilde\tau, \tilde h\tilde\tau^2).$$

We have:

$$\tilde\nu = i\Big(\sqrt{2\tilde h}\Big)^{-1}\sin\Big(\tilde\tau\sqrt{\tilde h/2}\Big) + O(\beta) = \tilde\tau\varphi_5(\beta\tilde\tau, \tilde h\tilde\tau^2).$$

Consequently,

$$\tilde\nu = \Big(i\sqrt{2\tilde h}\Big)^{-1}\sin\Big(\tilde\tau\sqrt{\tilde h/2}\Big) + \beta\tilde\tau^2\varphi_6(\beta\tilde\tau, \tilde h\tilde\tau^2),$$

where the function $\varphi_6(\beta\tilde\tau, \tilde h\tilde\tau^2) = \big(\varphi_5(\beta\tilde\tau, \tilde h\tilde\tau^2) - \varphi_5(0, \tilde h\tilde\tau^2)\big)(\beta\tilde\tau)^{-1}$ is analytic at zero. Returning to the original variables τ, h, ν we obtain (2.7). Then we obtain (2.8) from (2.7) and (2.11). Proposition 2.1 is proved.

Different time-energy coordinates h, τ_* can be introduced in a neighborhood of the real part of the separatrix γ. We take small 1-dimensional complex-analytic surface $\Phi = \{(x,y) : x = 0,\ |y-2| < \text{const}\}$ (the real dimension of Φ is 2) in the complex phase space

$$P_{\mathbb C} = (\mathbb C/2\pi\mathbb Z)\times\mathbb C = \{x, y\}. \tag{2.14}$$

Let h satisfy as before equality (2.6), and $\tau_* = 0$ on the surface Φ. We can evidently continue the coordinates τ_*, h from Φ to an open subset of $P_{\mathbb C}$ by using the phase flow g^t.

The domains, where the coordinates τ, h and τ_*, h are defined, intersect. On the intersection we obviously have:

$$\tau = \tau_* - T(h), \quad T(0) = i\pi/2.$$

The function $T(h)$ is analytic for h close to 0.

We put

$$\begin{aligned} \hat H^0_*(\tau_*, h) &= \hat H^0(x,y), & \hat H^\pm_*(\tau_*, h) &= \hat H^\pm(x,y), \\ H^0_*(\tau_*, h, \delta) &= H^0(x,y,\delta), & H^\pm_*(\tau_*, h, \delta) &= H^\pm(x,y,\delta). \end{aligned} \tag{2.15}$$

c. Averaging in a neighborhood of the singularity.

We perform the change

$$\tau=\varepsilon\tilde\tau,\quad h=\varepsilon^{-2}\tilde h,\quad H^0=\varepsilon^{-2}\tilde H^0,\quad H^\pm=\varepsilon^{-2}\tilde H^\pm \tag{2.16}$$

and put $\{f,g\}^\sim=f_{\tilde h}g_{\tilde\tau}-f_{\tilde\tau}g_{\tilde h}$ for any smooth $f=f(\tilde\tau,\tilde h)$, $g=g(\tilde\tau,\tilde h)$. The system (2.3)–(2.4) takes the form

$$\tilde H^0_\delta=-2i\{\tilde H^+,\tilde H^-\}^\sim,\quad \tilde H^\pm_\delta+\tilde H^\pm=\pm i\{\tilde H^0,\tilde H^\pm\}^\sim. \tag{2.17}$$

The initial conditions for this system can be calculated from (2.9). Note that the terms $\tau^{-1}F^\pm$ of (2.9) have the order ε after the change (2.16). We drop these terms and obtain the problem (2.17) with the initial conditions

$$\tilde H^0(\tilde\tau,\tilde h,0)=\tilde h,\quad \tilde H^\pm(\tilde\tau,\tilde h,0)=-4\tilde hB^\pm_-\big[1-\cos\big(\tilde\tau\sqrt{2\tilde h}\,\big)\big]^{-1}. \tag{2.18}$$

This nonlinear Cauchy problem does not contain any small parameter. The functions (2.18) are analytic in the complex domain $\{\tilde\tau\neq0,\ |\tilde h\tilde\tau^2|<2\pi^2\}$.

Theorem 2.1. *Let the positive constants $\tilde c,\tilde c_1,\tilde c_2,\tilde c_3$ be large. Then the following assertions hold.*

(1) *For any $\hat\delta\geqslant0$ and $0\leqslant\delta\leqslant\hat\delta$ the system (2.17)–(2.18) has a solution $\tilde H^0(\tilde\tau,\tilde h,\delta)$, $\tilde H^\pm(\tilde\tau,\tilde h,\delta)$ analytic in $\tilde\tau$ and $\tilde h$ in the domain*

$$D_0^{\hat\delta}=\{\operatorname{Im}\tilde\tau\leqslant-\tilde c-\hat\delta,\ |\tilde h(\tilde\tau+i\hat\delta)^2|\leqslant\tilde c_1^{-1}\}.$$

(2) *For $(\tilde\tau,\tilde h)\in D_0^{\hat\delta}$ and $0\leqslant\delta\leqslant\hat\delta$ the following estimate holds*

$$|\tilde H^+(\tilde\tau,\tilde h,\delta)|\leqslant\tilde c_2e^{-\delta}|\tilde\tau+i\delta|^{-2}.$$

(3) *For arbitrary constants $c_3>0$, $c_5\geqslant\tilde c$ and any two functions $\tilde p_1(t)$, $\tilde p_2(t)$ such that*

$$|\tilde p_1(t)|\leqslant\tilde c'\log(1+|t|),\quad |\tilde p_2(t)|\leqslant\tilde c'',\quad -c_3^{-1}/\varepsilon\leqslant t\leqslant c_3^{-1}/\varepsilon$$

the following estimate holds:

$$\begin{aligned}\int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon}&e^{it}\big|\tilde H^+(t-i\hat\delta-ic_5,0,\hat\delta)\\&-\tilde H^+\big(t_\gamma+\varepsilon\tilde p_1(t)-\varepsilon^{-1}T(\varepsilon^2\tilde p_2(t)),\varepsilon^4\tilde p_2(t),\hat\delta\big)\big|dt\leqslant e^{-\hat\delta}\tilde c_3\varepsilon\log\varepsilon^{-1},\\&t_\gamma=t+i\pi\varepsilon^{-1}/2-i\hat\delta-ic_5\end{aligned} \tag{2.19}$$

*provided $\hat\delta\leqslant\text{const}/\varepsilon$. (The function T is defined in §2**b**.)*

(4) *The integral*

$$\tilde I(\delta)=\int_{-\infty}^{\infty}e^{\delta+i(t-i\Theta)}\tilde H^+(t-i\Theta-i\delta,0,\delta)\,dt,\qquad \Theta\geqslant\tilde c$$

exists and does not depend on $\Theta \in \mathbb{R}$. *Moreover,* $\tilde{I}(\delta) = J + O(\delta^{-1})$, *where*

$$J = \lim_{\delta \to +\infty} \tilde{I}(\delta), \qquad J = J(B_-^+, B_-^-). \tag{2.20}$$

Now we show that the function J can be expressed through a function of one complex argument. We put $G_0 = \tilde{H}^0$, $G_\pm = \tilde{H}^\pm/(2B_-^\pm)$, $B_* = 4B_-^+ B_-^-$. Then the system (2.17)–(2.18) takes the form

$$G_\delta^0 = -2iB_* \{G^+, G^-\}^\sim, \quad G_\delta^\pm + G^\pm = \pm i\{G^0, G^\pm\}^\sim,$$
$$G^0(\tilde{\tau}, \tilde{h}, 0) = \tilde{h}, \quad G^\pm(\tilde{\tau}, \tilde{h}, 0) = -2\tilde{h}\Big(1 - \cos\Big(\tilde{\tau}\sqrt{2\tilde{h}}\Big)\Big)^{-1}.$$

This system contains only one parameter B_*. Moreover, for $B_* = 0$ we have: $G^0 \equiv \tilde{h}$, $G^+(\tilde{\tau}, \tilde{h}, \delta) = e^{-\delta} G^+(\tilde{\tau} + i\delta, \tilde{h}, 0)$. Consequently, in this case

$$\tilde{I}(\delta) = -\int_{-\infty}^{+\infty} e^{i(t - i\Theta)} 4B_-^+ (t - i\Theta)^{-2}\, dt = 8\pi B_-^+.$$

Thus we obtain that $J(B_-^+, B_-^-) = 2B_-^+ f(B_*)$, $f(0) = 4\pi$. The function f is entire-analytic because it is an uniformly convergent series of functions depending analytically on B_*. This fact follows from the proof of Theorem 2.1.

§3. Poincaré Method

In this section we consider the Poincaré method of the analysis of asymptotic surface splitting. More precisely, we discuss the application of this method to our problem.

The averaging procedure of the previous section (subsection **a.**) enables us to reduce the explicit dependence of our Hamiltonian on time t to an exponentially small term. So we consider the following Hamiltonian system:

$$\dot{x} = \partial H/\partial y, \quad \dot{y} = -\partial H/\partial x, \tag{3.1}$$
$$H = \varepsilon(\mathcal{H}^0(x, y, \varepsilon) + \sigma \mathcal{H}^*(x, y, t, \varepsilon)), \tag{3.2}$$
$$\mathcal{H}^* = e^{it}\mathcal{H}^+(x, y, \varepsilon) + e^{-it}\mathcal{H}^-(x, y, \varepsilon). \tag{3.3}$$

Here σ is an additional small parameter, $\sigma \ll \varepsilon$. It is convenient to assume henceforth that the Hamiltonian (3.2)–(3.3) is obtained from $\varepsilon\hat{H}$ as a result of our averaging procedure, and $\sigma \sim \exp(-\pi\varepsilon^{-1}/2)$.

Suppose that for small values of ε the system

$$\dot{x} = \partial(\varepsilon\mathcal{H}^0)/\partial y, \quad \dot{y} = -\partial(\varepsilon\mathcal{H}^0)/\partial x \tag{3.4}$$

has hyperbolic equilibrium positions $(x_\varepsilon^0, y_\varepsilon^0)$ with the doubly asymptotic to them solutions (separatrices)

$$\gamma_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t)); \qquad \gamma_\varepsilon(t) \to (x_\varepsilon^0, y_\varepsilon^0) \quad \text{as} \;\; t \to \pm\infty.$$

The corresponding separatrices of the system (3.1) split. We measure the rate of their splitting by the area $\mathcal{A}$ of a domain bounded by their segments (see §1 and Fig. 1). We put

$$P^\pm(\varepsilon) = \int_{-\infty}^{\infty} e^{\pm i\lambda}\big(\mathcal{H}^\pm(\gamma_\varepsilon(\lambda), \varepsilon) - \mathcal{H}^\pm(x_\varepsilon^0, y_\varepsilon^0, \varepsilon)\big)\, d\lambda. \tag{3.5}$$

Lemma 3.1. *The area $\mathcal{A}$ can be estimated as follows:*

$$\mathcal{A} = 4\sigma\varepsilon|P^+(\varepsilon)| + O(\sigma^2). \tag{3.6}$$

Proof of Lemma 3.1. We assume for the simplicity that γ_ε is projected one-to-one to the axis x. (Note that using slightly modified arguments, one can consider the general case as well.) The separatrix γ_ε can be defined by the generating function $S_0(x,\varepsilon)$, i.e. the equation of the curve γ_ε has the form $y = \partial S_0/\partial x$. The following equality holds:

$$\mathcal{H}^0(x, \partial S_0/\partial x, \varepsilon) = \mathcal{H}^0(x^0_\varepsilon, y^0_\varepsilon, \varepsilon). \tag{3.7}$$

For small nonzero σ the system (3.1)–(3.3) has a 2π-periodic hyperbolic solution, close to the equilibrium $(x^0_\varepsilon, y^0_\varepsilon)$. The corresponding asymptotic surface, consists in general of two pieces: the stable separatrix Γ^+, and the unstable one Γ^-. Their equations have the form

$$y = \partial S^\pm/\partial x, \quad S^\pm(x,t,\varepsilon,\sigma) = S_0(x) + \sigma S^\pm_1(x,\varepsilon) + \sigma^2 S^\pm_2(x,t,\varepsilon,\sigma). \tag{3.8}$$

The functions $S^\pm$ satisfy the Hamilton–Jacobi equations

$$\partial S^\pm/\partial t + H\big(x, \partial S^\pm/\partial x, t, \varepsilon, \sigma\big) = \varepsilon\mathcal{H}^0(x^0_\varepsilon, y^0_\varepsilon, \varepsilon) + \sigma f^\pm(t,\varepsilon,\sigma), \tag{3.9}$$

where $f^\pm$ are arbitrary functions. Expanding equation (3.9) in a power series in σ we obtain equality (3.7) as zero approximation. The first approximation yields:

$$\frac{\partial S^\pm_1}{\partial t} + \varepsilon\frac{\partial\mathcal{H}^0}{\partial y}\Big(x, \frac{\partial S_0}{\partial x}, \varepsilon\Big)\frac{\partial S^\pm_1}{\partial x} + \varepsilon\mathcal{H}^*\Big(x, \frac{\partial S_0}{\partial x}, t, \varepsilon\Big) = f^\pm(t,\varepsilon,0). \tag{3.10}$$

Since we are interested in the case of exponentially small σ, we may neglect the terms $\sigma^2 S^\pm_2$ in the generating functions (3.8).

Substitute into equation (3.10) the function $x_\varepsilon(t-s)$, which is the first component of the separatrix solution $\gamma_\varepsilon(t-s)$. We obtain the following equalities:

$$dS^\pm_1(x_\varepsilon(t-s), t, \varepsilon)/dt = -\varepsilon\mathcal{H}^*(\gamma(t-s), t, \varepsilon) + f^\pm(t,\varepsilon,0), \tag{3.11}$$

where the derivative d/dt is taken along solutions of equations (3.4). We put

$$f^\pm(t,\varepsilon,0) = \varepsilon\mathcal{H}^*(x^0_\varepsilon, y^0_\varepsilon, t, \varepsilon), \quad \widetilde{\mathcal{H}}^* = \mathcal{H}^* - f^\pm(t,\varepsilon,0). \tag{3.12}$$

Equations (3.11)–(3.12) imply the following relations:

$$\begin{aligned} S^+_1(x_\varepsilon(t-s), t, \varepsilon) &= \varepsilon\int_t^{+\infty} \widetilde{\mathcal{H}}^*(\gamma(r-s), r, \varepsilon)\, dr, \\ S^-_1(x_\varepsilon(t-s), t, \varepsilon) &= -\varepsilon\int_{-\infty}^t \widetilde{\mathcal{H}}^*(\gamma(r-s), r, \varepsilon)\, dr. \end{aligned} \tag{3.13}$$

Integrals in equalities (3.13) converge due to the condition $\widetilde{\mathcal{H}}^*(x^0_\varepsilon, y^0_\varepsilon, t, \varepsilon) = 0$.

The area $\mathcal{A}$ is bounded on the cylinder $\Pi = \{x \operatorname{mod} 2\pi, y;\ t = \text{const}\}$ by two segments of curves $\Gamma^\pm \cap \Pi$. These segments are situated between two neighboring intersection points (x',y') and (x'',y''). The points (x',y',t), $(x'',y'',t) \in \Gamma^+ \cup \Gamma^-$ have the following form:

$$(x',y',t) = \Big(x', \frac{\partial S^\pm}{\partial x}(x',t,\varepsilon), t\Big), \quad (x'',y'',t) = \Big(x'', \frac{\partial S^\pm}{\partial x}(x'',t,\varepsilon), t\Big).$$

The area $\mathcal{A}$ can be obtained by the formula

$$\begin{aligned}
\mathcal{A} &= \Big| \int_{x'}^{x''} \frac{\partial}{\partial x}\big(S^+(x,t,\varepsilon,\sigma) - S^-(x,t,\varepsilon,\sigma)\big)\, dx \Big| \\
&= \Big|\sigma \int_{x'}^{x''} \frac{\partial}{\partial x}\big(S_1^+(x,t,\varepsilon) - S_1^-(x,t,\varepsilon)\big)\, dx\Big| + O(\sigma^2) \\
&= \sigma\big|\big(S_1^+(x'',t,\varepsilon) - S_1^-(x'',t,\varepsilon)\big) - \big(S_1^+(x',t,\varepsilon) - S_1^-(x',t,\varepsilon)\big)\big| + O(\sigma^2).
\end{aligned}$$

Consider the function

$$P(s,\varepsilon) = \int_{-\infty}^{+\infty} \widetilde{\mathcal{H}}^*(\gamma(r-s), r, \varepsilon)\, dr = \int_{-\infty}^{+\infty} \widetilde{\mathcal{H}}^*(\gamma(r), r+s, \varepsilon)\, dr. \tag{3.14}$$

The function P is obviously 2π-periodic in s. Using relations (3.13) and (3.14), we get: $(S_1^- - S_1^+)(x(t-s),t) = -\varepsilon P(s,\varepsilon)$. Consequently,

$$\mathcal{A} = \sigma\varepsilon(P(s'',\varepsilon) - P(s',\varepsilon)) + O(\sigma^2),$$

where $x_\varepsilon(t-s') = x'$ and $x_\varepsilon(t-s'') = x''$. The points s' and s'' are neighboring roots of the equation $\partial P(s,\varepsilon)/\partial s = 0$, $s' < s''$. Indeed, the points s' and s'' satisfy the equation

$$\frac{\partial S^+}{\partial x}(x_\varepsilon(t-s),t,\varepsilon) - \frac{\partial S^-}{\partial x}(x_\varepsilon(t-s),t,\varepsilon) = 0$$

which can be transformed into the following one:

$$0 = \{\mathcal{H}^0, S_1^+ - S_1^-\}(\gamma_\varepsilon(t-s),t,\varepsilon) \equiv \varepsilon\partial P(s,\varepsilon)/\partial s.$$

In our case the function $\mathcal{H}^*$ satisfies relation (3.3). Hence,

$$\widetilde{\mathcal{H}}^* = e^{it}\big(\mathcal{H}^+(x,y,\varepsilon) - \mathcal{H}^+(x_\varepsilon^0, y_\varepsilon^0, \varepsilon)\big) + e^{-it}\big(\mathcal{H}^-(x,y,\varepsilon) - \mathcal{H}^-(x_\varepsilon^0, y_\varepsilon^0, \varepsilon)\big).$$

The function P takes the form $P(s,\varepsilon) = e^{is}P^+(\varepsilon) + e^{-is}P^-(\varepsilon)$ (see (3.5)). Since the function H is assumed to be real-analytic, the magnitudes P^+ and P^- are complex conjugate. We see that s' and s'' satisfy the equation $ie^{is}P^+(\varepsilon) - ie^{-is}P^-(\varepsilon) = 0$. Its solutions are $e^{is} = \pm\big(P^-(\varepsilon)/P^+(\varepsilon)\big)^{1/2}$. Hence $\mathcal{A} = 4\sigma\varepsilon|P^+(\varepsilon)| + O(\sigma^2)$. Lemma 3.1 is proved.

§4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The proof is based on some auxiliary statements (Theorems 2.1 and 4.1–4.4) which are proved in the following sections. Our main idea is to show that the system (2.3)–(2.4) has a solution in a complex neighborhood of the real separatrix γ for $0 \leqslant \delta \leqslant \hat{\delta} = \varepsilon^{-1}(\pi/2 - c_1)$, with $0 < c_1 < \pi/4$ and to apply Lemma 3.1 to the system with Hamiltonian $H(x, y, \hat{\delta})$.

First, we introduce some domains in the complex phase space $P_{\mathbb{C}} = \{x, y\}$ (see (2.14)):

$$D_1^\delta = \{|\mathrm{Im}\,\tau_*(x,y)| \leqslant c_2^{-1} + \pi/2 - \varepsilon\delta,\ c_3^{-1} \leqslant |\mathrm{Re}\,\tau_*(x,y)| \leqslant c_3,\ |h(x,y)| \leqslant c_2^{-1}\},$$
$$D_2 = \{|x-\pi| \leqslant c_4^{-1} \bmod 2\pi,\ |y| \leqslant c_4^{-1}\},$$
$$D_3^\delta = \{0 \leqslant \mathrm{Im}\,\tau_*(x,y) \leqslant \pi/2 - \varepsilon\delta - c_5\varepsilon,\ |\mathrm{Re}\,\tau_*(x,y)| \leqslant 2c_3^{-1},\ |h(x,y)| \leqslant c_6\varepsilon\}.$$

The large positive constants $c_2, \dots, c_6$ are chosen in such a way that the union $D_1 \cup D_2 \cup D_3$ contains the separatrix γ (see Fig. 2 presenting the projections of the domains D_1, D_2 and D_3 to the real phase space $P = \mathrm{Re}\, P_{\mathbb{C}}$).

Theorem 4.1. *There exists a solution of the system* (2.3)–(2.4) *for* $0 \leqslant \delta \leqslant \pi\varepsilon^{-1}/2$ *in the domains* D_1^δ*. Moreover, the following estimates are satisfied in* D_1^δ*:*

$$|H^{\pm}(x,y,\delta)| \leqslant Ae^{-\delta}, \tag{4.1}$$
$$|H^0(x,y,\delta) - \hat{H}^0(x,y)| \leqslant \varepsilon^2 A' \tag{4.2}$$

with some positive constants A, A'.

Theorem 4.2. *There exists a solution of the system* (2.3)–(2.4) *for* $0 \leqslant \delta \leqslant \pi\varepsilon^{-1}/2$ *in the domain* D_2 *and the estimates* (4.1)–(4.2) *are satisfied in* D_2.

Theorem 4.3. *For any constant* $c_1 > 0$ *there exists a solution of the system* (2.3)–(2.4) *for* $0 \leqslant \delta \leqslant (\pi/2 - c_1)/\varepsilon$ *in the domain* D_3^δ*. Moreover, if* δ *is of order* ε^{-1} *then the following inequalities hold in* D_3

$$|H_*^0(i\pi/2 - i\varepsilon\delta - i\varepsilon s, h, \delta) - h| \leqslant c_7\varepsilon^2,$$
$$|(H_*^0)_h(i\pi/2 - i\varepsilon\delta - i\varepsilon s, h, \delta) - 1| \leqslant c_7\varepsilon/(1 + \varepsilon^{-1}|\mathrm{Re}\, s|),$$

where $c_5 < Re\, s < \pi - 2\varepsilon\delta - 2c_5$ *and* $|Im\, s| < 2c_3^{-1}$.

Remark. Actually a solution of the system (2.3)–(2.4) exists in the vicinity of the set $\{(\tau_*, h) : \mathrm{Im}\,\tau_* = 0,\ h = 0\}$ for $0 \leqslant \delta \leqslant \pi\varepsilon^{-1}/2 - \mathrm{const}$, but we will not use this fact.

The followig theorem shows that the functions $\tilde{H}^+$ and H_*^+ are close in a certain sense when we write them in the same variables. (Recall that $\tilde{H}^+ = \tilde{H}^+(\tilde{\tau}, \tilde{h}, \delta)$ and $H_*^+ = H_*^+(\tau_*, h, \delta)$.)

Theorem 4.4. *Let* $\hat{\delta} = (\pi/2 - c_1)/\varepsilon$ *and* $t_\gamma = t + i\pi\varepsilon^{-1}/2 - i\hat{\delta} - ic_5$*. Then for any two functions* $\tilde{p}_1(t), \tilde{p}_2(t)$ *such that*

$$|\tilde{p}_1(t)| \leqslant c' \log(1 + |t|), \quad |\tilde{p}_2(t)| \leqslant c'', \qquad -c_3^{-1}/\varepsilon \leqslant t \leqslant c_3^{-1}/\varepsilon$$

the following estimate holds:

$$\begin{aligned}\int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} & e^{it}\Big[H_*^+\big(\varepsilon t_\gamma + \varepsilon^2\tilde{p}_1(t), \varepsilon^2\tilde{p}_2(t), \hat{\delta}\big) \\ &\qquad - \varepsilon^{-2}\tilde{H}^+\big(t_\gamma + \varepsilon\tilde{p}_1(t) - \varepsilon^{-1}T(\varepsilon^2\tilde{p}_2(t)), \varepsilon^4\tilde{p}_2(t), \hat{\delta}\big)\Big]\, dt \\ &= e^{-\hat{\delta}} O\big(\varepsilon^{-1}\log\varepsilon^{-1}\big).\end{aligned}$$

Proof of Theorem 1.1. The area $\mathcal{A}$ is defined by relations (3.5)–(3.6) with $\mathcal{H}^+ = H^+(x,y,\varepsilon,\hat\delta)e^{\hat\delta}$ and $\sigma = e^{-\hat\delta}$, $\hat\delta = (\pi/2 - c_1)/\varepsilon$. In order to evaluate the integral P^+, first, we find the separatrix solution γ_ε of the system with Hamiltonian $\varepsilon H^0(x,y,\varepsilon,\hat\delta)$.

The hyperbolic equilibrium position (x^e,y^e) of this system satisfies the equation $\operatorname{grad} H^0(x^e,y^e,\varepsilon,\hat\delta) = 0$. We consider of course the equilibrium close to the unperturbed one $(\pi,0)$. Using Theorem 4.2 we see that

$$(x^e,y^e) = (\pi,0) + O(\varepsilon^2). \tag{4.3}$$

The level of energy $h' = \varepsilon H^0(x^e,y^e,\hat\delta)$, corresponding to the separatrix γ_ε, has the form

$$h' = \varepsilon\hat H^0(x^e,y^e) + O(\varepsilon^3) = O(\varepsilon^3).$$

When establishing this relation we proceeded from the expansion (4.3) and the fact that the point $(\pi,0)$ is critical for the function $\hat H^0$. We put $H^+_e = H^+(x^e,y^e,\hat\delta)$. According to Theorem 4.2, $H^+_e \leqslant Ae^{-\hat\delta}$.

Equations of the separatrix γ_ε in the variables τ_*, h are as follows:

$$\dot\tau_* = \varepsilon(H^0_*)_h(\tau_*,h,\hat\delta), \quad H^0_*(\tau_*,h,\hat\delta) = \varepsilon^{-1}h' = O(\varepsilon^2).$$

Using estimates of Theorems 4.3, 4.2 and 4.1 for the functions H^0_*, H^0 and putting $\tau_*(0) = 0$, we see that the solution $\gamma_\varepsilon(t)$ is analytic for $0 \leqslant \operatorname{Im} t \leqslant \pi\varepsilon^{-1}/2 - \hat\delta - c_5$. Moreover, we get

$$\tau_*(t) = \varepsilon t + \varepsilon^2 p_1(t), \quad h(t) = \varepsilon^2 p_2(t), \tag{4.4}$$

where $|p_1(t)| \leqslant c_8 \log(1+|t|)$, $|p_2(t)| \leqslant c_8$ in the domain

$$\{t \in \mathbb{C} : |\operatorname{Re} t| \leqslant c_3/\varepsilon, 0 \leqslant \operatorname{Im} t \leqslant \pi\varepsilon^{-1}/2 - \hat\delta - c_5\}.$$

Let t_γ be determined by (2.19). Since the function $H^+(\gamma_\varepsilon(t_\gamma),\hat\delta) - H^+_e$ is defined and is of order $\mathrm{const}\cdot e^{-\hat\delta}$, we have:

$$\begin{aligned} P^+ &= \int_{-\infty}^{\infty} e^{\hat\delta + it}[H^+(\gamma_\varepsilon(t),\hat\delta) - H^+_e]\,dt \\ &= \int_{-\infty}^{\infty} e^{-\pi\varepsilon^{-1}/2 + 2\hat\delta + c_5 + it}[H^+(\gamma_\varepsilon(t_\gamma),\hat\delta) - H^+_e]\,dt \\ &= \int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} e^{-\pi\varepsilon^{-1}/2 + 2\hat\delta + c_5 + it} H^+(\gamma_\varepsilon(t_\gamma),\hat\delta)\,dt + e^{-\pi\varepsilon^{-1}/2 + \hat\delta}O(1). \end{aligned}$$

Using Theorem 4.4 we see that the difference $|P^+ - I_1|$, where

$$I_1 = \varepsilon^{-2}\int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} e^{-\pi\varepsilon^{-1}/2 + 2\hat\delta + c_5 + it}\tilde H^+\Big(t_\gamma + \varepsilon p_1(t_\gamma) - \varepsilon^{-1}T(\varepsilon^2 p_2(t_\gamma)), \varepsilon^4 p_2(t_\gamma), \hat\delta\Big)dt$$

does not exceed $e^{-\pi\varepsilon^{-1}/2 + \hat\delta}O(\varepsilon^{-1}\log\varepsilon^{-1})$. According to assertion (3) of Theorem 2.1 the difference $|I_2 - I_1|$, where

$$I_2 = \varepsilon^{-2}\int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} e^{-\pi\varepsilon^{-1}/2 + 2\hat\delta + c_5 + it}\tilde H^+(t - i\hat\delta - ic_5, 0, \hat\delta)\,dt.$$

does not exceed $e^{-\pi\varepsilon^{-1}/2+\hat{\delta}}O(\varepsilon^{-1}\log\varepsilon^{-1})$. According to assertion (4) of Theorem 2.1, $I_2 = \varepsilon^{-2}e^{-\pi\varepsilon^{-1}/2+\hat{\delta}}\big(J + O(\varepsilon)\big)$. As a result we have:

$$P^+ = \varepsilon^{-2}e^{-\pi\varepsilon^{-1}/2+\hat{\delta}}(J + O(\varepsilon\log\varepsilon^{-1})).$$

Using the estimate (3.6) we get:

$$\mathcal{A} = 4e^{-\hat{\delta}}\big|\varepsilon^{-1}e^{-\pi\varepsilon^{-1}/2+\hat{\delta}}(J + O(\varepsilon\log\varepsilon^{-1}))\big| = 4\varepsilon^{-1}e^{-\pi\varepsilon^{-1}/2}|J + O(\varepsilon\log\varepsilon^{-1})|.$$

Theorem 1.1 is proved.

§5. Majorant Method

Proofs of Theorems 2.1 and 4.1–4.4 are based on the majorant method.

Definition. Let the functions $f(z_1, \dots, z_m)$ and $g(z_1, \dots, z_m)$ be analytic at the point $(z_1, \dots, z_m) = (0, \dots, 0)$. We say that $f \ll g$ (or g is a majorant for f) if for any $n_j \in \{0, 1, \dots\}$, $j = 1, \dots, m$ the Taylor coefficients $f_{n_1,\dots,n_m}$ and $g_{n_1,\dots,n_m}$ satisfy the inequality $|f_{n_1,\dots,n_m}| \leqslant g_{n_1,\dots,n_m}$.

Below we use the following properties of the relation ($\ll$) without comments.

(i) If $f_1(z) \ll g_1(z)$ and $f_2(z) \ll g_2(z)$ then $f_1 + f_2 \ll g_1 + g_2$ and $f_1 f_2 \ll g_1 g_2$.

(ii) If $f(z) \ll g(z)$ then $df/dz \ll dg/dz$.

(iii) If for any value of the parameter $\lambda \in [a, b]$ we have $f(z, \lambda) \ll g(z, \lambda)$ then $\int_a^b f(z, \lambda)\, d\lambda \ll \int_a^b g(z, \lambda)\, d\lambda$.

In assertions (i)–(iii) we assume that z lies in a neighborhood of zero in $\mathbb{C}$. Generalizations to the case of vector complex variable z are obvious.

Other simple properties of the relation $\ll$ are contained in the following

Proposition 5.1.

(1) *Let $f(z_1, z_2)$ be analytic in the domain $\{|z_1| \leqslant c',\ |z_2| \leqslant c''\}$ and in this domain $|f| \leqslant M$. Then $f \ll c'c''M[(c' - z_1)(c'' - z_2)]^{-1}$.*

(2) *Let $f(z_1, z_2) \ll W^{-m}$, $g(z_1, z_2) \ll W^{-l}$, where $W = (c' - z_1)(c'' - z_2)$ and the variables z_1, z_2 be canonical conjugate. Then the Poisson bracket $\{f, g\} \ll 2mlW^{-1-m-l}$.*

(3) *For any two constants a and b such that $0 < a < b$*

$$(a - z)^{-1}(b - z)^{-1} \ll (b - a)^{-1}(a - z)^{-1}.$$

(4) *Let $a_1 > b_1 > 0$ and $a_2 > b_2 > 0$. Then $\phi_1(z_1, z_2) \ll \phi_2(z_1, z_2)$, where*

$$\phi_1 = [(a_1 - z_1)(a_2 - z_2) - b_1 b_2]^{-1}, \quad \phi_2 = [(a_1 - b_1 - z_1)(a_2 - b_2 - z_2)]^{-1}.$$

(5) *Suppose that $f(z_1, z_2) \ll g(z_1, z_2)$. Then $f(e^{ir}z_1, e^{is}z_2) \ll g(z_1, z_2)$ for any real r, s.*

(6) *For any $0 < b \leqslant a$ the estimate $a/(a - z) \ll b/(b - z)$ is valid.*

(7) *Let the variables τ and h be canonical conjugate and for some positive a, v the functions $\Phi^{\pm}(\tau, h)$ satisfy the relations*

$$\{\Phi^+, \Phi^-\} = 0; \quad \Phi^{\pm}(\tau - s, h) \ll a|s|^{-3}X^{-1}(s), \quad X(s) = (2v|s| - \tau)(v|s|^{-2} - h)$$

with $s \in \mathbb{C}$, $\operatorname{Im} s \geqslant u > 0$. Then for any $\lambda > 0$ and $s \in \mathbb{C}$ such that $\operatorname{Im} s - \lambda \geqslant u$ the following estimate holds:

$$\{\Phi^+(\tau - s + i\lambda, h), \Phi^-(\tau - s - i\lambda, h)\} \ll \frac{4\lambda v^{-1} a^2}{|s + i\lambda|^3 |s - i\lambda|^4 X_0^3(s - i\lambda, 2\lambda)},$$

where $X_0(\alpha, \beta) = (v|\alpha| - \tau)(v|\alpha + i\beta|^{-2} - h)$.

Proof of Proposition 5.1.

(1) Using the Cauchy formula we obtain the following estimate for the Taylor coefficient $f_{l,m}$: $|f_{l,m}| \leqslant M(c')^{-l}(c'')^{-m}$. Now assertion (1) follows from the equality

$$\frac{\partial^{m+l}}{\partial z_1^m \partial z_2^l}\Big|_{z_1=z_2=0} \frac{Mc'c''}{(c' - z_1)(c'' - z_2)} = \frac{l!m!M}{(c')^l (c'')^m}.$$

(2) This assertion follows from the estimate

$$\{f, g\} = \partial f/\partial z_2 \cdot \partial g/\partial z_1 - \partial g/\partial z_2 \cdot \partial f/\partial z_1 \ll 2mlW^{-m-l-1}.$$

(3) This estimate follows from the equality

$$\frac{1}{(b-a)(a-z)} = \frac{1}{(a-z)(b-z)} + \frac{1}{(b-a)(b-z)}.$$

(4) We note that all coefficients in the Taylor expansion of the functions ϕ_1 and ϕ_2 are positive. Now assertion (4) follows from the relation

$$\phi_2 = \phi_1 + [b_1(a_1 - b_1 - z_1)^{-1} + b_2(a_2 - b_2 - z_2)^{-1}]\phi_1 \gg 0.$$

(5) This assertion follows immediately from the equalities $|e^{ir}| = |e^{is}| = 1$.

(6) This assertion can be proved by straightforward computation.

(7) First, we put $\Phi_*^+ = \Phi^+(\tau - s + i\lambda, h) - \Phi^+(\tau - s - i\lambda, h)$. We have the estimate

$$\Phi_*^+ = \int_{-\lambda}^{\lambda} i\Phi_\tau^+(\tau - s + i\xi, h)\, d\xi \ll \int_{-\lambda}^{\lambda} \frac{a\, d\xi}{|s - i\xi|^3 (2v|s - i\xi| - \tau)^2 (v|s - i\xi|^{-2} - h)}$$
$$\ll \frac{2\lambda a}{|s - i\lambda|^3 (2v|s - i\lambda| - \tau)^2 (v|s + i\lambda|^{-2} - h)}.$$

Using assertion (3) of Proposition 5.1 we see that the last expression does not exceed ($\ll$)

$$\frac{2\lambda a}{|s - i\lambda|^3 (2v|s - i\lambda| - \tau) X_0(s - i\lambda, 2\lambda)} \ll \frac{2\lambda a v^{-1}}{|s - i\lambda|^4 X_0(s - i\lambda, 2\lambda)}.$$

Since $\Phi^-(\tau - s - i\lambda, h) \ll a|s + i\lambda|^{-3} X_0^{-1}(s - i\lambda, 2\lambda)$, we have:

$$\{\Phi^+(\tau - s + i\lambda, h), \Phi^-(\tau - s - i\lambda, h)\} = \{\Phi_*^+, \Phi^-(\tau - s - i\lambda, h)\}$$
$$\ll \frac{4\lambda a^2 v^{-1}}{|s + i\lambda|^3 |s - i\lambda|^4 X_0^3(s - i\lambda, 2\lambda)}.$$

Here we used assertion (2) of Proposition 5.1. Proposition 5.1 is proved.

Below we will use the following norm in $\mathbb{C}$:

$$||s|| = |\operatorname{Im} s| + |\operatorname{Re} s|.$$

We frequently prefer this norm because of the only reason: Proposition 5.3 is valid for the norm $||\cdot||$ and not valid in general for the ordinary one $|\cdot|$. Note that for any $z \in \mathbb{C}$ we have $|z| \leqslant ||z|| \leqslant \sqrt{2}|z|$.

Proposition 5.2. a). *Suppose that the functions* $\hat{H}^{\pm}_{*}$ *are defined by* (2.15) *and*

$$s \in \{c_3 \leqslant |\mathrm{Re}\, s| \leqslant c_3^{-1},\ |\mathrm{Im}\, s| \leqslant \pi/2\}.$$

Then for some positive κ_0 *and* N

$$\hat{H}^{\pm}_{*}(\tau_* - s, h) \ll N[(\kappa_0 - \tau_*)(\kappa_0 - h)]^{-1}.$$

b). *Suppose that* $||s|| \geqslant 1$. *Then the initial conditions* (2.18) *satisfy the estimate*

$$\tilde{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0) \ll \tilde{N}||s||^{-3}[(||s||/2 - \tilde{\tau})(||s||^{-2}/4 - \tilde{h})]^{-1}$$

for some positive $\tilde{N}$.

c). *There exist positive contants* $\check{N}_1, \check{N}_2, \check{w}$ *such that for any* $s \in \mathbb{C}$, $1 \leqslant \mathrm{Im}\, s \leqslant (\pi - c_1)/\varepsilon$, $|\mathrm{Re}\, s| \leqslant 2c_3^{-1}/\varepsilon$ *the functions* $\check{H}^{\pm}$ *defined by the equalities* $\check{H}^{\pm}(\tilde{\tau}, \tilde{h}, \delta) = \varepsilon^2 H^{\pm}(x, y, \delta)$ *satisfy the relations*

$$\check{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0) \ll \check{N}_1||s||^{-3}\check{Y}_0^{-1}(s), \tag{5.1}$$

$$\check{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0) - \tilde{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0) \ll \varepsilon\check{N}_2||s||^{-2}\check{Y}_0^{-1}(s), \tag{5.2}$$

$$\check{Y}_0(s) = (4\check{w}||s|| - \tilde{\tau})(2\check{w}||s||^{-2} - \tilde{h}). \tag{5.3}$$

Proof. **a)**. We put $\nu_*(\tau_*, h) = \nu(x, y)$. If positive $\kappa_0 < c_3$ is sufficiently small, the functions

$$\hat{H}^{\pm}(\tau_* - s, h) = B^{\pm}_{+}\nu^2(\tau_* - s, h) + B^{\pm}_{-}\nu^{-2}(\tau_* - s, h)$$

are analytic in the domain $\{|\tau_*| \leqslant \kappa_0,\ |h| \leqslant \kappa_0\}$ for any s satisfying conditions of assertion **a)**. Indeed, singularities of the functions $y(\tau_*, h)$ and $\nu(\tau_*, h)$ for small $|h|$ are close to the points $(\tau_*, h) = (i\pi/2 + i\pi k, 0)$, $k \in \mathbb{Z}$ or are situated in the domain $|\mathrm{Re}\, \tau_*| > 2c_3^{-1}$. Hence, we can use assertion (1) of Proposition 5.1.

b). We see from (2.18) that $\tilde{H}^{\pm}(\tilde{\tau}, \tilde{h}, 0) = \tilde{\tau}^{-2}\psi(\tilde{h}\tilde{\tau}^2)$, where the function $\psi(z)$, $z \in \mathbb{C}$ is analytic for $|z| \leqslant \pi^2$. For such z we have the estimate $|\psi(z)| \leqslant M$ with some constant M. Hence, the function $\tilde{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0)$ is analytic in the domain $\{|\tilde{\tau}| \leqslant |s|/2,\ |\tilde{h}| \leqslant \pi^2|s|^{-2} \cdot 4/9 < |s|^{-2}/4\}$. Moreover, in this domain

$$|\tilde{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0)| \leqslant \tilde{N}|s|^{-2}, \quad \tilde{N} = 4M.$$

Now assertion **b)** follows from assertion (1) of Proposition 5.1.

c). The estimates (5.1) and (5.2) for $s \in D' = \{1 \leqslant \mathrm{Im}\, s \leqslant \lambda^{-1}/\varepsilon, |\mathrm{Re}\, s| \leqslant \lambda^{-1}/\varepsilon\}$, where the constant $\lambda^{-1} > 0$ is sufficiently small, follow from the equality

$$\check{H}^{\pm}(\tilde{\tau}, \tilde{h}, 0) = \tilde{H}^{\pm}(\tilde{\tau}, \tilde{h}, 0) + \varepsilon\tilde{\tau}^{-1}F^{\pm}(\varepsilon\tilde{\tau}, \tilde{h}\tilde{\tau}^2)$$

(see (2.9)). Recall that the functions $F^{\pm}$ are analytic at the point (0,0).

For any $s \in D'' = \{1 \leqslant \mathrm{Im}\, s \leqslant (\pi - c_1/2)/\varepsilon,\ |\mathrm{Re}\, s| \leqslant 2c_3^{-1}/\varepsilon\} \setminus D'$ the functions $\tilde{H}^{\pm}(\tilde{\tau} - s, \tilde{h}, 0)$ and $\varepsilon(\tilde{\tau} - s)^{-1}F^{\pm}(\varepsilon(\tilde{\tau} - s), \tilde{h}(\tilde{\tau} - s)^2)$ are analytic in the domain $D''' = \{|\tilde{\tau}| \leqslant c'/\varepsilon, |\tilde{h}| \leqslant c''\varepsilon^2\}$. Their absolute values are bounded in D''' by $M\varepsilon^{-2}$, where the constant M does not depend on $s \in D''$. Obviously, there exists a constant $w > 0$ such that $c'/\varepsilon \geqslant 4w||s||$, $c''\varepsilon^2 \geqslant 2w||s||^{-2}$, $s \in D''$. This implies estimates (5.1) and (5.2) for $s \in D''$.

Proposition 5.3. *Suppose that $z \in \mathbb{C}$ and $\mu \in \mathbb{R}$ satisfy the inequalities $Im\, z > 0$, $\mu \geqslant 0$. Then for any nonnegative integer n*

$$\int_0^\mu \frac{\lambda^n \, d\lambda}{||z + i\lambda||^{n+2}} = \frac{\mu^{n+1}}{(n+1)||z||\,||z + i\mu||^{n+1}}.$$

Proof. Proposition 5.3 follows immediately from the equality

$$\int_0^\mu \frac{\lambda^n \, d\lambda}{(x + \lambda)^{n+2}} = \frac{\mu^{n+1}}{(n+1)x(x + \mu)^{n+1}}, \quad x = |\mathrm{Re}\, z| + \mathrm{Im}\, z > 0.$$

This equality can be easily proved by induction.

§6. Solution of the Averaging System in D_1.

In this section we prove Theorem 4.1. When studying the system (2.3)–(2.4) in D_1^δ it is convenient to use time–energy coordinates h, τ_*.

We fix the value $\mathrm{Re}\, s$ satisfying the inequality $c_3 \leqslant |\mathrm{Re}\, s| \leqslant c_3^{-1}$ and put

$$\begin{aligned} H_*^0(\tau_*, h, \delta) &= h + q^0(\tau_*, h, \delta), \qquad q^0(\tau_*, h, 0) = 0, \\ H_*^\pm(\tau_*, h, \delta) &= q^\pm(\tau_*, h, \delta) e^{-\delta}, \qquad q^\pm(\tau_*, h, 0) = \hat{q}^\pm(\tau_*, h), \end{aligned}$$

$H_*^0, H_*^\pm$ being defined by (2.15).

According to assertion **a)** of Proposition 5.2 we have the estimate

$$\hat{q}^\pm(\tau_* - s, h) \ll N Z_0^{-1}, \quad |\mathrm{Im}\, s| \leqslant \pi/2, \quad Z_0 = (\kappa_0 - \tau_*)(\kappa_0 - h). \tag{6.1}$$

The functions $q^0, q^\pm$ satisfy the equations

$$\begin{cases} q_\delta^0 = -2i\varepsilon e^{-2\delta}\{q^+, q^-\}, \\ q_\delta^\pm = \pm i\varepsilon q_{\tau_*}^\pm \pm i\varepsilon\{q^0, q^\pm\}. \end{cases} \tag{6.2}$$

Lemma 6.1. *Suppose that for some $\Delta \geqslant 0$ the estimates*

$$q^0(\tau_* - s, h, \Delta) \ll \varepsilon B Z^{-1}, \quad q^\pm(\tau_* - s, h, \Delta) \ll C Z^{-1}$$

hold, where $|Im\, s| \leqslant \alpha$ and $Z = (\kappa - \tau_*)(\kappa - h)$. We put $\rho = (\varepsilon^2 B^2 + 2e^{-2\Delta} C^2)^{1/2}$. Then for any r, $\mathrm{Im}\, s$ and μ satisfying the inequalities

$$(\gamma\mu\varepsilon\rho)^{1/4} \leqslant r < \kappa, \qquad |Im\, s| \leqslant \alpha - \varepsilon\mu, \quad 0 \leqslant \mu \leqslant \alpha/\varepsilon$$

the following estimates hold:

$$q^0(\tau_* - s, h, \Delta + \mu) \ll \varepsilon(B + \mu\gamma 2e^{-2\Delta} C^2 r^{-4})\tilde{Z}^{-1}, \tag{6.3}$$

$$q^\pm(\tau_* - s, h, \Delta + \mu) \ll C\tilde{Z}^{-1}. \tag{6.4}$$

Here $\tilde{Z} = (\kappa - r - \tau_*)(\kappa - r - h)$ and γ is the constant (A1.6).

Proof of Lemma 6.1 is contained in Appendix 1. Now we apply Lemma 6.1 successively step by step to estimate the solution of the system (6.2) on the intervals

$$\Delta_n \leqslant \delta \leqslant \Delta_{n+1}, \quad \Delta_n = 2^n - 1, \qquad n \in \mathbb{N}. \tag{6.5}$$

We put $\mu_n = \Delta_{n+1} - \Delta_n = 2^n$. At the beginning of n-th step we have the estimates

$$q^0(\tau_* - s, h, \Delta_n) \ll \varepsilon B_n Z_n^{-1}, \quad B_0 = 0; \quad q^\pm(\tau_* - s, h, \Delta_n) \ll C_n Z_n^{-1}, \quad C_0 = N,$$
$$Z_n = (\kappa_n - \tau_*)(\kappa_n - h), \quad \alpha_n = \pi/2 - \varepsilon\Delta_n.$$

According to Lemma 6.1 we can take

$$B_{n+1} = B_n + 2^n\gamma 2e^{-2\Delta_n} C_n^2 r_n^{-4}, \quad C_{n+1} = C_n = N, \tag{6.6}$$
$$\kappa_{n+1} = \kappa_n - r_n, \qquad \rho_n = (\varepsilon^2 B_n^2 + 2e^{-2\Delta_n} C_n^2)^{1/2}. \tag{6.7}$$

We choose

$$r_n = \kappa_0 (n+1)^{-2}/4. \tag{6.8}$$

Now we show that for any n satisfying the inequality

$$\Delta_n \leqslant \pi\varepsilon^{-1} \tag{6.9}$$

the following estimates hold:

$$\kappa_n = \kappa_0 \Big(1 - \sum_{k=0}^{n-1} \frac{1}{4}(k+1)^{-2}\Big) \geqslant \kappa_0/2, \tag{6.10}$$
$$B_n \leqslant \hat{B}, \quad r_n \geqslant (\mu_n \varepsilon\gamma\rho_n)^{1/4}, \tag{6.11}$$

where $\hat{B}$ does not depend on ε.

Indeed, relation (6.10) is evident because $\sum_{k=0}^\infty (k+1)^{-2} < 2$. Since $B_0 = 0$, we have:

$$B_{n+1} = \sum_{k=0}^n 2^k \gamma 2 e^{-2\Delta_k} N^2 r_k^{-4} \leqslant \gamma N^2 2^9 \kappa_0^{-4} \sum_{k=0}^\infty 2^k (k+1)^8 e^{-2\Delta_k} \equiv \hat{B} < \infty.$$

Hence, the first estimate (6.11) is proved. The second inequality (6.11) is verified in the following way. Using (6.7) we see that

$$\mu_n \varepsilon\gamma\rho_n \leqslant 2^n \varepsilon\gamma (\varepsilon^2 \hat{B} + 2e^{-2\Delta_n} N^2)^{1/2}.$$

The last expression is less than r_n^4 for $2n < \log_2 \varepsilon^{-1}$ due to the smallness of the multiplier $2^n\varepsilon$. For n satisfying (6.9) and the inequality $2n \geqslant \log_2 \varepsilon^{-1}$ the term $8e^{-2\Delta_n} N^2$ is negligible, hence the last inequality (6.11) is also valid.

We have proved that for any

$$0 \leqslant \delta \leqslant \pi\varepsilon^{-1}/2, \quad c_3 \leqslant |\operatorname{Re} s| \leqslant c_3^{-1}, \quad |\operatorname{Im} s| \leqslant \pi/2 - \varepsilon\delta$$

the following relations hold:

$$q^{\pm}(\tau_* - s, h, \delta) \ll N\hat{Z}^{-1}, \quad q^0(\tau_* - s, h, \delta) \ll \varepsilon\hat{B}\hat{Z}^{-1}, \tag{6.12}$$
$$\hat{Z} = (\kappa_0/2 - \tau_*)(\kappa_0/2 - h).$$

Now we verify the estimates (4.1)–(4.2) for any $0 \leqslant \delta \leqslant \pi\varepsilon^{-1}/2$. Inequality (4.1) follows immediately from the first estimate (6.12).

To prove estimate (4.2) we use the following relation:

$$H^0(\tau_*, h, \delta) - h = q^0(\tau_*, h, \delta) = -2i\varepsilon \int_0^\delta e^{-2\lambda}\{q^+, q^-\}(\tau_*, h, \lambda)\, d\lambda.$$

Since $\{q^+, q^-\}(\tau_*, h, 0) \equiv 0$, we see that $H^0(\tau_*, h, \delta) - h = I_1 + I_2$, where

$$I_1 = -2i\varepsilon \int_0^\delta e^{-2\lambda}\{q^+(\tau_*, h, \lambda) - q^+(\tau_*, h, 0), q^-(\tau_*, h, \lambda)\}\, d\lambda,$$
$$I_2 = -2i\varepsilon \int_0^\delta e^{-2\lambda}\{q^+(\tau_*, h, 0), q^-(\tau_*, h, \lambda) - q^-(\tau_*, h, 0)\}\, d\lambda.$$

Using Proposition 5.1 and estimates (6.12) we get:

$$\begin{aligned} &q^{\pm}(\tau_* - s_1, h, \lambda) - q^{\pm}(\tau_* - s_1, h, 0) \\ &\quad = \int_0^\lambda \big(\pm i\varepsilon q^{\pm}_{\tau_*} \pm i\varepsilon\{q^0, q^{\pm}\}\big)(\tau_* - s_1, h, u)\, du \\ &\quad \ll \varepsilon \int_0^\lambda \big(N\hat{Z}^{-2} + 2\varepsilon\hat{B}N\hat{Z}^{-3}\big)\, du = \varepsilon\lambda\big(N\hat{Z}^{-2} + 2\varepsilon\hat{B}N\hat{Z}^{-3}\big). \end{aligned}$$

Now we can estimate the integral I_1. We put $\hat{Z}_1 = (\kappa_0/4 - \tau_*)(\kappa_0/4 - h)$ and have:

$$\begin{aligned} I_1 &\ll 2\varepsilon \int_0^\delta e^{-2\lambda}\varepsilon\lambda \cdot 2\big(2N^2\hat{Z}^{-4} + 6\varepsilon\hat{B}N^2\hat{Z}^{-5}\big)\, d\lambda \\ &\ll 2\varepsilon^2\big(N^2\hat{Z}^{-4} + 3\varepsilon\hat{B}N^2\hat{Z}^{-5}\big) \ll 2\varepsilon^2\hat{Z}_1^{-1}\big(N^2(\kappa_0/4)^{-6} + 3\varepsilon\hat{B}N^2(\kappa_0/4)^{-8}\big) \\ &\ll 4\varepsilon^2\hat{Z}_1^{-1}N^2(\kappa_0/4)^{-6}. \end{aligned}$$

The integral I_2 can be estimated in the same way. Now inequality (4.2) follows from the estimate $I_1 + I_2 \ll 2^{15}N^2\kappa_0^{-6}\varepsilon^2\hat{Z}_1^{-1}$. Theorem 4.1 is proved.

§7. Solution of the averaging system in the domain D_2.

In this section we prove Theorem 4.2. First, we pass to the normal coordinates x_N, y_N for the Hamiltonian $\hat{H}^0(x, y)$ in a neighborhood of the point $(x, y) = (\pi, 0)$, [20]. These coordinates are connected with the original ones by the relations

$$x - \pi = (y_N + x_N)/\sqrt{2} + O(x_N^2 + y_N^2), \quad y = (-y_N + x_N)/\sqrt{2} + O(x_N^2 + y_N^2).$$

Below in this section we drop the subindices N for brief.

In the normal coordinates the Hamiltonian $\hat{H}^0$ has the form

$$\hat{H}^0 = \chi(xy) = \chi_1 xy + \chi_2 (xy)^2 + \dots, \qquad \chi_1 = 1.$$

Here χ is some real-analytic function in a neighborhood of zero. We put

$$H^0(x,y,\delta) = xy + p^0(x,y,\delta), \quad p^0(x,y,0) = \chi(xy) - xy,$$
$$H^\pm(x,y,\delta) = \big(p^\pm(x,y,\delta) + H^\pm(0,0,0)\big)e^{-\delta}, \quad p^\pm(x,y,0) = \hat{H}^\pm(x,y) - \hat{H}^\pm(0,0).$$

According to assertion (1) of Proposition 5.1 we have the estimate

$$p^0(x,y,0) \ll \varkappa_0^6 A_0 W_0^{-1}, \quad p^\pm(x,y,0) \ll \varkappa_0^3 A_* W_0^{-1}, \qquad W_0 = (\varkappa_0 - x)(\varkappa_0 - y),$$

where A_0 and A_* do not depend on $\varkappa_0$ if the values of the parameter $\varkappa_0 > 0$ are sufficiently small. The functions p^0, $p^\pm$ satisfy the equations

$$\begin{cases} p^0_\delta &= -2i\varepsilon e^{-2\delta}\{p^+, p^-\}, \\ p^\pm_\delta &= \pm i\varepsilon\{xy, p^\pm\} \pm i\varepsilon\{p^0, p^\pm\}. \end{cases} \tag{7.1}$$

Lemma 7.1. *For small $\varkappa_0 > 0$ there exists a solution of the system* (7.1) *for $0 \leqslant \delta \leqslant \pi\varepsilon^{-1}/2$ in the domain $\{|x| \leqslant \varkappa_0/2, |y| \leqslant \varkappa_0/2\}$. Moreover, for these values of δ the estimates* (4.1)–(4.2) *are satisfied.*

Lemma 7.1 implies Theorem 4.2. Proof of Lemma 7.1 is based on the following

Lemma 7.2. *Suppose that for some $\Delta \geqslant 0$ the estimates*

$$p^0(x,y,\Delta) \ll BW^{-1}, \quad p^\pm(x,y,\Delta) \ll CW^{-1}$$

hold, where $W = (\varkappa - x)(\varkappa - y)$. We put $\rho = (B^2 + 2C^2 e^{-2\Delta})^{1/2}$. Then for any $(\mu\varepsilon\gamma\rho)^{1/4} \leqslant r < \varkappa$ the following estimates are valid:

$$p^0(x,y,\Delta+\mu) \ll (B + 2e^{-2\Delta}C^2\mu\varepsilon\gamma r^{-4})\tilde{W}^{-1},$$
$$p^\pm(x,y,\Delta+\mu) \ll C\tilde{W}^{-1}.$$

Here $\tilde{W} = (\varkappa - r - x)(\varkappa - r - y)$ and γ is the constant (A1.6).

Proof of Lemma 7.2 is contained in Appendix 2. We apply Lemma 7.2 inductively to estimate solution of the system (7.1) on the intervals (6.5). At the beginning of n-th step we have the estimates

$$p^0(x,y,\Delta_n) \ll \varkappa_0^6 b_n W_n^{-1}, \quad b_0 = A_0; \qquad p^\pm(x,y,\Delta_n) \ll \varkappa_0^3 A_* W_n^{-1},$$
$$W_n = (\varkappa_n - x)(\varkappa_n - y).$$

According to Lemma 7.2 we can take

$$b_{n+1} = b_n + 2e^{-2\Delta} A_*^2 2^n \varepsilon\gamma r_n^{-4}, \quad \varkappa_{n+1} = \varkappa_n - r_n, \quad \rho_n = (\varkappa_0^{12} b_n^2 + \varkappa_0^6 2 A_*^2 e^{-2\Delta_n})^{1/2}.$$

We put $r_n = \varkappa_0 (n+1)^{-2}/8$. Then obviously $\varkappa_n \geqslant 3\varkappa_0/4$.

For any integer $n \geqslant 0$ satisfying the inequality

$$\Delta_n \leqslant 2\varepsilon^{-1/2} + 1 \tag{7.2}$$

the following estimates hold for sufficiently small constants $\varkappa_0 > 0$:

$$b_n \leqslant 2A_0, \quad r_n^4 \geqslant \mu_n \varepsilon\gamma\rho_n \tag{7.3}$$

Indeed,

$$b_n \leqslant A_0 + \sum_{k=0}^{\infty} 2e^{-2\Delta_k} A_*^2 2^{k+12} \varepsilon\gamma\varkappa_0^{-4}(k+1)^8 \leqslant 2A_0$$

if ε is small. On the other hand we have:

$$\begin{aligned} r_n^4 &= \varkappa_0^4 2^{-12}(n+1)^{-8}, \\ \mu_n\varepsilon\gamma\rho_n &\leqslant 2^n\varepsilon\gamma(\varkappa_0^{12}4A_0^2 + \varkappa_0^6 2A_*^2 e^{-2\Delta_n})^{1/2} \leqslant \varepsilon^{1/2}\gamma(\varkappa_0^{12}4A_0^2 + \varkappa_0^6 2A_*^2 e^{-2\Delta_n})^{1/2}. \end{aligned}$$

Since $n < \mathrm{const} \cdot \log\varepsilon$ we see that the second inequality (7.3) is valid.

Let n_* be maximal n satisfying (7.2). Obviously, $\Delta_{n_*} \geqslant \varepsilon^{-1/2}$. We have the estimates:

$$p^0(x,y,\Delta_{n_*}) \ll 2\varkappa_0^6 A_0 W_*^{-1}, \quad p^{\pm}(x,y,\Delta_{n_*}) \ll \varkappa_0^3 A_* W_*^{-1},$$

where $W_* = (3\varkappa_0/4 - x)(3\varkappa_0/4 - y)$. Now we apply Lemma 7.2 one more time. We put $r = \varkappa_0/4$, $\quad \mu = \pi\varepsilon^{-1}/2 - \Delta_{n_*}$. For any $\delta \in [0, \pi\varepsilon^{-1}/2]$ we have:

$$\begin{aligned} &p^0(x,y,\delta) \ll \varkappa_0^6\big(2A_0 + 2\exp(-2\Delta_{n_*})A_*^2(\pi/2)\gamma(\kappa_0/4)^{-4}\big)\tilde W_*^{-1} \ll 3\varkappa_0^6 A_0 \tilde W_*^{-1}, \\ &p^{\pm}(x,y,\delta) \ll \varkappa_0^3 A_* \tilde W_*^{-1} \end{aligned}$$

with $\tilde W_* = (\varkappa_0/2 - x)(\varkappa_0/2 - y)$. Now we can prove the estimates (4.1)–(4.2) following arguments of §6.

§8. Proof of Theorem 2.1

We put

$$\tilde H^0(\tilde\tau,\tilde h,\delta) = \tilde h + \tilde q^0(\tilde\tau,\tilde h,\delta), \quad \tilde H^{\pm}(\tilde\tau,\tilde h,\delta) = \tilde q^{\pm}(\tilde\tau,\tilde h,\delta)e^{-\delta}.$$

The system (2.17)–(2.18) takes the form

$$\begin{aligned} \tilde q^0_\delta = -2ie^{-2\delta}\{\tilde q^+,\tilde q^-\}^\sim, &\quad \tilde q^\pm_\delta = \pm i\tilde q^\pm_{\tilde\tau} \pm i\{\tilde q^0,\tilde q^\pm\}^\sim, \\ \tilde q^0(\tilde\tau,\tilde h,0) = 0, &\quad \tilde q^\pm(\tilde\tau,\tilde h,0) = -4\tilde h B^\pm_-\Big[1 - \cos\Big(\tilde\tau\sqrt{2\tilde h}\Big)\Big]^{-1}. \end{aligned} \tag{8.1}$$

Below we drop tildes at Poisson brackets for brief.

Lemma 8.1. *Suppose that for some positive* Δ, v *and for any* $s \in \mathbb{C}$ *with* $\operatorname{Im} s \geqslant u > 0$ *the following estimates are satisfied:*

$$\tilde{q}^0(\tilde{\tau} - s - i\Delta, \tilde{h}, \Delta) \ll B||s||\,||s + i\Delta||^{-4}||s + 2i\Delta||^{-2}Y^{-1}(s, 2\Delta), \tag{8.2}$$

$$\tilde{q}^{\pm}(\tilde{\tau} - s - i\Delta, \tilde{h}, \Delta) \ll C||s||^{-1}||s + 2i\Delta||^{-2}Y^{-1}(s, 2\Delta), \tag{8.3}$$

$$\begin{aligned}\tilde{q}^{\pm}(\tilde{\tau} - s - i\Delta, \tilde{h}, \Delta) &- \tilde{q}^{\pm}(\tilde{\tau} - s - i\Delta \pm i\Delta, \tilde{h}, 0)\\ &\ll C||s||^{-2}||s + 2i\Delta||^{-2}Y^{-1}(s, 2\Delta),\end{aligned} \tag{8.4}$$

$$Y(\alpha, \beta) = (v||\alpha|| - \tilde{\tau})(v||\alpha + i\beta||^{-2} - \tilde{h}), \quad C \geqslant \tilde{N}, \quad 0 < v \leqslant 1/4.$$

For a positive μ *let* $\Delta' = \Delta + \mu$ *and let* $b = \max\{B, C^2A^2, 1/16\}$*, where the constant* $A \geqslant 1$ *satisfies the inequalities*

$$A \geqslant \frac{2 \cdot 27||s + 2i\Delta'||^8(\Delta + 1)e^{-\Delta}}{v||s||^8}, \quad A \geqslant \frac{27||s + 2i\Delta'||^7e^{-\Delta}}{||s||^7}.$$

Then for any positive r *and* M*, where*

$$r \geqslant \tilde{\rho}, \quad \tilde{\rho} = \left[16b\gamma\mu\sigma/||s + 2i\Delta'||\right]^{1/4}, \quad \sigma = \max\{||s + 2i\Delta||^{-1}, e^{-\Delta/2}\},$$

the following estimates hold:

$$\tilde{q}^0(\tilde{\tau} - s - i\Delta', \tilde{h}, \Delta') \ll B'||s||\,||s + i\Delta'||^{-4}||s + 2i\Delta'||^{-2}\big(Y'(s, 2\Delta')\big)^{-1}, \tag{8.5}$$

$$\tilde{q}^{\pm}(\tilde{\tau} - s - i\Delta', \tilde{h}, \Delta') \ll C||s||^{-1}||s + 2i\Delta'||^{-2}\big(Y'(s, 2\Delta')\big)^{-1}, \tag{8.6}$$

$$\begin{aligned}\tilde{q}^{\pm}(\tilde{\tau} - s - i\Delta', \tilde{h}, \Delta') &- \tilde{q}^{\pm}(\tilde{\tau} - s - i\Delta' \pm i\Delta', \tilde{h}, 0)\\ &\ll C||s||^{-2}||s + 2i\Delta'||^{-2}\big(Y'(s, 2\Delta')\big)^{-1},\end{aligned} \tag{8.7}$$

$$\begin{aligned}\tilde{q}^{+}(\tilde{\tau} - s - i\Delta - iM, &\tilde{h}, \Delta + M) - \tilde{q}^{+}(\tilde{\tau} - s - i\Delta' \pm i\Delta', \tilde{h}, 0)\\ &\ll KC\Delta^{-1}||s||^{-2}||s + 2i(\Delta + M)||^{-2}\big(Y''(s, 2\Delta + 2M)\big)^{-1},\end{aligned} \tag{8.8}$$

where

$$B' = B + 16b\gamma\mu\sigma e^{-\Delta/2}A^{-1}r^{-4},$$
$$Y'(\alpha, \beta) = (v'||\alpha|| - \tilde{\tau})(v'||\alpha + i\beta||^{-2} - \tilde{h}), \quad v' = v - r,$$
$$Y''(\alpha, \beta) = (v||\alpha||/2 - \tilde{\tau})(v||\alpha + i\beta||^{-2}/2 - \tilde{h}), \quad K = 128b\gamma w^{-4},$$

the estimate (8.8) *is valid provided* $\Delta \geqslant K$ *and the constant* γ *is defined by* (*A*1.6)*.*

Remark 1. Suppose that conditions of Lemma 8.1 hold provided $s \in \mathbb{C}$ satisfies the additional inequalities $\operatorname{Im} s \leqslant U_1$, $|\operatorname{Re} s| \leqslant U_2$. Then the estimates (8.5)–(8.8) remain valid for complex s such that $u \leqslant \operatorname{Im} s \leqslant U_1 - 2\mu$, $|\operatorname{Re} s| \leqslant U_2$.

Remark 2. Consider instead of (8.1) the system

$$\tilde{q}^0_\delta = -6ie^{-2\delta}\{\tilde{q}^+, \tilde{q}^-\}^\sim, \quad \tilde{q}^\pm_\delta = \pm i\tilde{q}^\pm_{\tilde{\tau}} \pm 3i\{\tilde{q}^0, \tilde{q}^\pm\}^\sim$$

with the same initial conditions. Then solutions of this system satisfy the estimates (8.5)–(8.8), where γ is replaced by 3γ.

Assertions contained in Remarks 1–2 follow from proof of Lemma 8.1, see Appendix 3. We use these assertions when proving Theorems 4.3–4.4.

Now we use Lemma 8.1 to prove Theorem 2.1. We put

$$\Delta_m = 2^m - 1, \quad \mu_m = \Delta_{m+1} - \Delta_m = 2^m, \quad r_m = v_0(m+1)^{-2}/4.$$

At m-th step the solution is considered on the interval $[\Delta_m, \Delta_{m+1}]$. For $\Delta = \Delta_m$ the estimates (8.2)–(8.4) hold with

$$Y(s, 2\Delta) = Y_m(s, 2\Delta_m) = (v_m||s|| - \tilde{\tau})(v_m||s + 2i\Delta_m||^{-2} - \tilde{h});$$
$$A = A_m, \quad B = B_m, \quad C = C_m.$$

(Note that the same estimates are valid for any $0 \leqslant \Delta \leqslant \Delta_m$.)

We have $B_0 = 0$, $C_0 = \tilde{N}$, and $v_0 = 1/4$, see assertion **b** of Proposition 5.2. According to Lemma 8.1 we can put

$$B_{m+1} = B_m + 16 b_m \gamma \mu_m \sigma_m e^{-\Delta_m/2} A_m^{-1} r_m^{-4}, \quad C_{m+1} = C_m = \tilde{N},$$
$$r_m = v_0(m+1)^{-2}/4, \quad \tilde{\rho}_m = \big[16 b_m \gamma \mu_m \sigma_m / ||s + 2i\Delta_{m+1}||\big]^{1/4}.$$

We see that

$$v_{m+1} = v_0\Big(1 - \frac{1}{4}\sum_{k=1}^{m} \frac{1}{(k+1)^2}\Big) > v_0/2.$$

Now we prove that if

$$\operatorname{Im} s \geqslant \tilde{c} \qquad (\tilde{c} > 0 \text{ is sufficiently large}), \tag{8.9}$$

then the following inequalities hold:

$$A_m \leqslant \tilde{A}, \quad B_m \leqslant \tilde{B}, \quad r_m \geqslant \tilde{\rho}_m. \tag{8.10}$$

Here $\tilde{A}$ and $\tilde{B}$ are some positive constants.

Indeed we can put

$$A_m = \max\Big\{1, \quad \frac{4\cdot 27||s + 2i\Delta_{m+1}||^8(\Delta_m + 1)e^{-\Delta_m}}{v_0||s||^8}, \quad \frac{27||s + 2i\Delta_{m+1}||^7 e^{-\Delta_m}}{||s||^7}\Big\}.$$

For $||s|| \geqslant \text{const} > 0$ the sequence $\{A_m\}$ is obviously bounded.

The sequence $\{B_m\}$ can be estimated as follows: $B_m \leqslant \beta_m$, where

$$\beta_0 = 0, \quad \beta_{m+1} = \beta_m + 2^{12}(\beta_m + \tilde{N}^2\tilde{A}^2 + 1/16)\gamma\mu_m\sigma_m e^{-\Delta_m/2} v_0^{-4}(m+1)^8.$$

The sequence β_m obviously, converges.

Now we verify the third inequality (8.10). Since for nonnegative $n \in \mathbb{Z}$

$$\frac{\mu_m \sigma_m}{||s + 2i\Delta_{m+1}||} \leqslant 2\frac{2^m(||s + i(2^{m+1} - 2)||^{-1} + \exp(-2^{m-1} + 1/2))}{||s + i(2^{m+2} - 2)||} \leqslant \frac{6}{||s|| + 2^m},$$

we see that

$$\tilde{\rho}_m^4 \leqslant 16(\tilde{B} + \tilde{N}^2\tilde{A}^2 + 1/16)\gamma\mu_m\sigma_m/||s + 2i\Delta_{m+1}|| \leqslant \check{c}/(\hat{c} + 2^m). \tag{8.11}$$

with some constant $\check{c}$. On the other hand

$$r_m^4 = v_0^4 2^{-8}(m+1)^{-8}. \tag{8.12}$$

Now the inequality $r_m \geqslant \tilde{\rho}_m$ follows from the estimates (8.11)–(8.12) provided the ratio $\check{c}/\hat{c}$ is sufficiently small.

Thus we can apply Lemma 8.1 as many times as we want. After m step we see that the functions $q^{\pm}(\tilde{\tau}, \tilde{h}, \Delta)$ and $q^0(\tilde{\tau}, \tilde{h}, \Delta)$ exist for $\Delta \leqslant \Delta_{m+1}$ and satisfy the estimates (8.2)–(8.4) with $B = \tilde{B}$, $C = \tilde{N}$, and $Y = Y_{m+1}$ for $\operatorname{Im} s \geqslant \tilde{c}$. Hence, we have proved assertions (1) and (2) of Theorem 2.1.

Now we prove assertion (3). Since for $-c_3^{-1}/\varepsilon \leqslant t \leqslant c_3^{-1}/\varepsilon$ we have

$$t_\gamma + \varepsilon\tilde{p}_1(t) - \varepsilon^{-1}T(\varepsilon^2\tilde{p}_2(t)) = t - i\hat{\delta} - ic_5 + \phi(t), \quad |\phi(t)| \leqslant \text{const} \cdot \varepsilon \log \varepsilon^{-1},$$

we see that

$$\begin{aligned} &\left|\tilde{H}^+(t - i\hat{\delta} - ic_5, 0, \hat{\delta}) - \tilde{H}^+(t_\gamma + \varepsilon\tilde{p}_1(t) - \varepsilon^{-1}T(\varepsilon^2\tilde{p}_2(t)), \varepsilon^4\tilde{p}_2(t), \hat{\delta})\right| \\ &\quad \leqslant \text{const}\,[m_1(t)\varepsilon\log\varepsilon^{-1} + m_2(t)\varepsilon^4], \\ &\qquad m_1 = \max \tilde{H}^+_{\tilde{\tau}}(\tilde{\tau}, \tilde{h}, \hat{\delta}), \qquad m_2 = \max \tilde{H}^+_{\tilde{h}}(\tilde{\tau}, \tilde{h}, \hat{\delta}), \end{aligned}$$

where the maximums are taken for

$$\tilde{\tau} \in [t - i\hat{\delta} - ic_5, t - i\hat{\delta} - ic_5 + \phi(t)] \subset \mathbb{C}, \quad \tilde{h} \in [0, \varepsilon^4\tilde{p}_2(t)] \subset \mathbb{C}.$$

Using estimates (8.6) we get for $c_5 \geqslant \tilde{c}$:

$$m_1 \leqslant \text{const}\,(1 + |t|)^{-3}, \quad m_2 \leqslant \text{const}\,\hat{\delta}^2(1 + |t|)^{-2} \leqslant \text{const}\,\varepsilon^{-2}(1 + |t|)^{-2}.$$

Hence, the integral from assertion (3) of Theorem 2.1 does not exceed

$$\int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} \frac{\text{const}\, e^{-\hat{\delta}}\varepsilon\log\varepsilon^{-1}}{(1 + |t|)^2} dt \leqslant \text{const}\, e^{-\delta}\varepsilon\log\varepsilon^{-1}.$$

Assertion (4) of Theorem 2.1 can be proved as follows. The integral (2.19) exists according to assertions (1) and (2). It does not depend on Θ since change of Θ is equivalent to a shift of path of the integration. If δ is sufficiently large then according to the estimate (8.8) of Lemma 8.1, for any positive constant M we have:

$$\left|\tilde{H}^+(t - i\Theta - i\delta - iM, 0, \delta + M) - \tilde{H}^+(t - i\Theta - i\delta, 0, \delta)\right| \leqslant \text{const}\, e^{-\delta}\delta^{-1}(1 + |t|)^{-2}.$$

Therefore, the limit J exists by the Cauchy criterion and $\tilde{I}(\delta) = J + O(\delta^{-1})$. Theorem 2.1 is proved.

§9. Proof of Theorem 4.3

We consider the system (2.3)–(2.4) in the variables $\tilde\tau, \tilde h$. It has the form

$$\check H^0_\delta = -2i\{\check H^+, \check H^-\}^\sim, \quad \check H^\pm_\delta + \check H^\pm = \pm i\{\check H^0, \check H^\pm\}^\sim, \tag{9.1}$$

$$\check H^0(\tilde\tau, \tilde h, \delta) = \varepsilon^2 H^0_*(\tau_*, h, \delta), \quad \check H^\pm(\tilde\tau, \tilde h, \delta) = \varepsilon^2 H^\pm_*(\tau_*, h, \delta). \tag{9.2}$$

This system coincides with (2.17). The initial conditions satisfy the estimates (5.1) for $1 \leqslant \operatorname{Im} s \leqslant (\pi - c_1)/\varepsilon$, $|\operatorname{Re} s| \leqslant 2c_3^{-1}/\varepsilon$.

We see that the system (2.3)–(2.4) written in the variables $\tilde\tau, \tilde h$ is completely analogous to the system (2.17)–(2.18) for $0 \leqslant \delta \leqslant (\pi/2 - c_1)/\varepsilon$. Thus following arguments of §8 and §A3 and taking into account Remark 1 after Lemma 8.1 we obtain the following

Lemma 9.1. *The system* (9.1) *has a solution* $\check H^0(\tilde\tau, \tilde h, \delta), \check H^\pm(\tilde\tau, \tilde h, \delta)$ *for* $0 \leqslant \delta \leqslant (\pi/2 - c_1)/\varepsilon$. *Moreover, for* $s \in \mathbb{C}$ *satisfying the inequalities*

$$c' \leqslant \operatorname{Im} s \leqslant (\pi - c_1)/\varepsilon - 2\delta, \quad |\operatorname{Re} s| \leqslant 2c_3/\varepsilon, \quad |\tilde h| \leqslant c''|s + i\delta|^{-2}$$

the following estimates hold:

$$\begin{aligned} \check H^0(\tilde\tau - s - i\delta, \tilde h, \delta) &\leqslant \check B||s||\,||s + i\delta||^{-4}||s + 2i\delta||^{-2}\check Y^{-1}(s, 2\delta), \\ \check H^\pm(\tilde\tau - s - i\delta, \tilde h, \delta) &\leqslant \check C||s||^{-1}\,||s + 2i\delta||^{-2}\check Y^{-1}(s, 2\delta), \end{aligned} \tag{9.3}$$

$$\check Y(\alpha, \beta) = (\check w||\alpha|| - \tilde\tau)(\check w||\alpha + i\beta||^{-2} - \tilde h). \tag{9.4}$$

Now we write this solution in the coordinates τ_*, h. Using the relations

$$\tilde\tau = \varepsilon^{-1}\tau = \varepsilon^{-1}(\tau_* - T(h)), \quad \tilde h = \varepsilon^2 h,$$

where T is the analytic function defined in §2**b**, we get:

$$\begin{aligned} H^0_*(\tau_*, h, \delta) &= \varepsilon^{-2}\check H^0(\varepsilon^{-1}\tau_* - \varepsilon^{-1}T(h), \varepsilon^2 h, \delta), \\ H^\pm_*(\tau_*, h, \delta) &= \varepsilon^{-2}\check H^\pm(\varepsilon^{-1}\tau_* - \varepsilon^{-1}T(h), \varepsilon^2 h, \delta). \end{aligned}$$

Suppose that $(\tau_*, h) \in D^\delta_3$. Then $|T(h) - i\pi/2| \leqslant c_9|h|$. We put $s = -\varepsilon^{-1}\tau_* + \varepsilon^{-1}T(h) - i\delta$. Since we can assume that c_5 is much greater than c_6, we see that

$$c' \leqslant c_5 - c_9c_6 \leqslant \operatorname{Im} s \leqslant \pi/\varepsilon - 2\delta + c_9c_6 - c_5, \quad |\operatorname{Re} s| \leqslant 2c_3/\varepsilon + c_9c_6.$$

Now we can apply the estimates (9.3) with $\tilde\tau = 0$, $|\tilde h| = |\varepsilon^2 h| \leqslant \varepsilon^3 c_6$. We have:

$$H^0_*(\tau_*, h, \delta) \leqslant \frac{\varepsilon^{-2}\check B||s||}{||s + i\delta||^4||s + 2i\delta||^2 \cdot \check w||s|| \cdot \check w||s + 2i\delta||^{-2}/2} \leqslant \frac{2\varepsilon^{-2}\check B}{\check w^2||s + i\delta||^4}.$$

The last expression is of order ε^2 for $\delta \geqslant \operatorname{const}\varepsilon^{-1}$. Analogously

$$\begin{aligned} (H^0_*)_h(\tau_*, h, \delta) &= \big[-\varepsilon^{-3}T_{\tilde h}\check H^0_{\tilde\tau} + \check H^0_{\tilde h}\big](\varepsilon^{-1}\tau_* - \varepsilon^{-1}T(h), \varepsilon^2 h, \delta) \\ &\leqslant \frac{\varepsilon^{-3}c_9c_6\check B||s||}{||s + i\delta||^4||s + 2i\delta||^2 \cdot \check w^2||s||^2 \cdot \check w||s + 2i\delta||^{-2}/2} \\ &\quad + \frac{\check B||s||}{||s + i\delta||^4||s + 2i\delta||^2 \cdot \check w||s|| \cdot \check w^2||s + 2i\delta||^{-4}/4}. \end{aligned}$$

This sum does not exceed $\operatorname{const}\varepsilon/||s|| \leqslant \operatorname{const}\varepsilon/(1 + \varepsilon^{-1}|\operatorname{Re}\tau_*|)$. Theorem 4.3 is proved.

§10. Proof of Theorem 4.4

To prove Theorem 4.4 we estimate the differences

$$\xi^0(\tilde\tau,\tilde h,\delta)=\check H^0(\tilde\tau,\tilde h,\delta)-\tilde H^0(\tilde\tau,\tilde h,\delta),$$
$$\xi^\pm(\tilde\tau,\tilde h,\delta)=\check H^\pm(\tilde\tau,\tilde h,\delta)-\tilde H^\pm(\tilde\tau,\tilde h,\delta).$$

The functions $\tilde H^0,\tilde H^\pm$ and $\check H^0,\check H^\pm$ satisfy the systems (2.17)–(2.18) and (9.1)–(9.2). Hence, we have:

$$\xi^0_\delta=-2i\big(\{\tilde H^+,\xi^-\}^\sim+\{\xi^+,\tilde H^-\}^\sim+\{\xi^+,\xi^-\}^\sim\big),$$
$$\xi^\pm_\delta+\xi^\pm=\pm i\big(\{\tilde H^0,\xi^\pm\}^\sim+\{\xi^0,\tilde H^\pm\}^\sim+\{\xi^0,\xi^\pm\}^\sim\big).$$

Lemma 10.1. *For some constants β^0,β and any $s\in\mathbb{C}$ such that $c_5\leqslant Im\,s\leqslant(\pi-c_1)/\varepsilon$, $|Re\,s|\leqslant 2c_3^{-1}\varepsilon$ the following relations hold:*

$$\begin{aligned}\xi^0(\tilde\tau-s,\tilde h,\delta)&\ll\frac{\varepsilon\beta^0||s||}{||s+i\delta||^4||s+2i\delta||\hat Y(s,2\delta)},\\ \xi^\pm(\tilde\tau-s,\tilde h,\delta)&\ll\frac{\varepsilon\beta e^{-\delta}}{||s||||s+2i\delta||\hat Y(s,2\delta)}.\end{aligned}\tag{10.1}$$

Proof of Lemma 10.1. We put

$$\tilde H^0=h+\tilde q^0,\quad \tilde H^\pm=e^{-\delta}\tilde q^\pm,\quad \eta^0=\xi^0,\quad \eta^\pm=e^{-\delta}\xi^\pm.$$

The functions $\tilde q^0,\tilde q^\pm$ and $\eta^0,\eta^\pm$ satisfy the following equations:

$$\eta^0_\delta=-2ie^{-2\delta}\big(\{\tilde q^+,\eta^-\}^\sim+\{\eta^+,\tilde q^-\}^\sim+\{\eta^+,\eta^-\}^\sim\big),\tag{10.2}$$
$$\eta^\pm_\delta\mp i\eta^\pm_{\tilde\tau}=\pm i\big(\{\tilde q^0,\eta^\pm\}^\sim+\{\eta^0,\tilde q^\pm\}^\sim+\{\eta^0,\eta^\pm\}^\sim\big)\tag{10.3}$$

together with (8.1). According to Proposition 5.2 initial conditions for the system (10.2)–(10.3), (8.1) can be estimated as follows:

$$\tilde q^0(\tilde\tau,\tilde h,0)=0,\quad \tilde q^\pm(\tilde\tau-s,\tilde h,0)\ll\tilde N||s||^{-3}\check Y_0^{-1}(s),\tag{10.4}$$
$$\xi^0(\tilde\tau,\tilde h,0)=0,\quad \xi^\pm(\tilde\tau-s,\tilde h,0)\ll\varepsilon\tilde N_2||s||^{-2}\check Y_0^{-1}(s).\tag{10.5}$$

Note that for

$$s\in\{1\leqslant\operatorname{Im}s\leqslant\Lambda,\quad|\operatorname{Re}s|\leqslant\Lambda\}\tag{10.6}$$

the second estimate (10.5) can be rewritten in the form

$$\xi^\pm(\tilde\tau-s,\tilde h,0)\ll 2\varepsilon\tilde N_2\Lambda||s||^{-3}\check Y_0^{-1}(s).\tag{10.7}$$

Now consider one more system:

$$f^0_\delta=-6ie^{-2\delta}\{f^+,f^-\}^\sim,\quad f^\pm_\delta=\pm if^\pm_{\tilde\tau}\pm 3i\{f^0,f^\pm\}^\sim,\tag{10.8}$$
$$f^0(\tilde\tau,\tilde h,0)=0,\quad f^\pm(\tilde\tau-s,\tilde h,0)\ll L||s||^{-3}\check Y_0^{-1}(s),\tag{10.9}$$
$$\{f^+(\tilde\tau,\tilde h,0),f^-(\tilde\tau,\tilde h,0)\}=0,$$

where s satisfies (10.6). According to Remarks 1–2 after Lemma 8.1 the system (10.8)–(10.9) has a solution. Following arguments of §8 one can prove that the solution exists for $0 \leqslant \delta \leqslant \Lambda/2$ and satisfies the estimates

$$f^0(\tilde\tau - s - \delta, \tilde h, \delta) \ll B^* ||s||\, ||s + i\delta||^{-4} ||s + 2i\delta||^{-2} \check Y^{-1}(s, 2\delta),$$
$$f^\pm(\tilde\tau - s - \delta, \tilde h, \delta) \ll C^* ||s||^{-1} ||s + 2i\delta||^{-2} \check Y^{-1}(s, 2\delta),$$

where $s \in \{c_* \leqslant \operatorname{Im} s \leqslant \Lambda - 2\delta,\ |\operatorname{Re} s| \leqslant \Lambda\}$ and $\check Y$ is defined by (9.4). Our proofs are based on the majorant method. This method does not use specific properties of functions. Only majorant estimates play an essential role. Later we consider $\Lambda \leqslant \mathrm{const}/\varepsilon$. Hence, we can assume that $L \geqslant \max\{\tilde N, 2\varepsilon \tilde N_2 \Lambda\}$. We see that the system (10.2)–(10.3) is similar to (10.8) and majorants in (10.9) are greater ($\gg$) than majorants in (10.4), (10.7). Since the functions $\tilde q^0, \tilde q^\pm$ can be estimated by the same majorants as the functions $f^0, f^\pm$, we can assert that $\eta^0, \eta^\pm$ can be also estimated by the same majorants.

Since the right-hand sides of (10.2)–(10.3) are at least linear in $\eta^0, \eta^\pm$, we conclude that when decreasing the majorant parameter $\varepsilon\Lambda$ we will obtain (at least) proportional decrease of majorants for η^0 and $\eta^\pm$. Consequently, we have the estimates

$$\begin{gathered}\eta^0(\tilde\tau - s - \delta, \tilde h, \delta) \ll \varepsilon\Lambda B_1^* ||s||\, ||s + i\delta||^{-4} ||s + 2i\delta||^{-2} \check Y^{-1}(s, 2\delta),\\ \eta^\pm(\tilde\tau - s - \delta, \tilde h, \delta) \ll \varepsilon\Lambda C_1^* ||s||^{-1} ||s + 2i\delta||^{-2} \check Y^{-1}(s, 2\delta),\\ s \in \{c_* \leqslant \operatorname{Im} s \leqslant \Lambda - 2\delta, \quad |\operatorname{Re} s| \leqslant \Lambda\}.\end{gathered} \tag{10.10}$$

Choosing $\Lambda = \max\{\operatorname{Im} s + 2\delta, |\operatorname{Re} s|\}$, we have the inequality $2\Lambda \geqslant \operatorname{Im} s + 2\delta + |\operatorname{Re} s| = ||s + 2i\delta||$, which (together with (10.10)) implies Lemma 10.1.

It turns out that the estimate (10.1) for the function ξ^+ can be improved.

Lemma 10.2. *Suppose that the inequalities* (10.10) *hold with* $c_5 \geqslant 8(3\check B)^{1/2}$. *Then*

$$\begin{gathered}\xi^+(\tilde\tau - s, \tilde h, \delta) \ll \frac{\check c \varepsilon e^{-\delta}\big(1 + ||s||^{-2} \log(||s + i\delta||/||s||)\big)}{||s + 2i\delta||^2 Y'(s, 2\delta)},\\ Y'(\alpha, \beta) = (\check w ||s||/2 - \tilde\tau)(\check w ||\tilde s + 2i\delta||^{-2}/2 - \tilde h).\end{gathered} \tag{10.11}$$

We prove Lemma 10.2 in the end of this section. Now we show that Lemma 10.2 implies Theorem 4.4. Indeed, let us denote the integral from Theorem 4.4 by I. We have:

$$I = \int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} \varepsilon^{-2} e^{it} \big[\check H^+ - \tilde H^+\big]\big(t_\gamma + \varepsilon^2 \tilde p_1(t) - \varepsilon^{-1} T(\varepsilon^2 \tilde p_2(t)), \varepsilon^4 \tilde p_2(t), \hat\delta\big)\, dt.$$

According to properties of the functions $\tilde p_1$ and $\tilde p_2$ we get:

$$\begin{gathered}I = \int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon} \varepsilon^{-2} e^{it} \big[\check H^+ - \tilde H^+\big]\big(t - i\hat\delta - ic_5 + \phi_1(t), \phi_2(t), \hat\delta\big)\, dt,\\ |\phi_1(t)| \leqslant \mathrm{const}\, \varepsilon \log \varepsilon^{-1}, \quad |\phi_2(t)| \leqslant \mathrm{const}\, \varepsilon^4, \quad -c_3^{-1}\varepsilon \leqslant t \leqslant c_3^{-1}\varepsilon.\end{gathered}$$

Since $\check H^+ - \tilde H^+ = \xi^+$, integrand of I can be estimated by using (10.11). Putting $s = -t - ic_5 - \phi_1(t)$, $\tilde\tau = 0$, $\tilde h = \phi_2(t)$ we see that

$$\big|(\check H^+ - \tilde H^+)(-s - i\hat\delta, \phi_2(t), \hat\delta)\big| \leqslant \frac{\check c\varepsilon e^{-\hat\delta}\big[1 + ||s||^{-2}\log(||s+i\delta||/||s||)\big]}{||s+2i\delta||^2(\check w||s||/2)(\check w||s+2i\delta||^{-2}/2 - \phi_2(t))}$$
$$\leqslant \frac{16\check c\varepsilon e^{-\hat\delta}\big[1 + 2(|t|+c_5)^{-2}\log\varepsilon^{-1}\big]}{\check w^2(|t|+c_5)}.$$

Hence,

$$|I| \leqslant \frac{16\check c}{\varepsilon\check w^2}e^{-\hat\delta - L}\int_{-c_3^{-1}/\varepsilon}^{c_3^{-1}/\varepsilon}\Big(\frac{1}{|t|+c_5} + \frac{2\log\varepsilon^{-1}}{(|t|+c_5)^3}\Big)\,dt \leqslant \mathrm{const}\, e^{-\pi\varepsilon^{-1}/2}\varepsilon^{-1}\log\varepsilon^{-1}.$$

Theorem 4.4 is proved.

Now we prove Lemma 10.2. The function η^+ satisfies the equation

$$\eta^+_\delta = i\eta^+_{\tilde\tau} + i(\{\check q^0, \eta^+\} + \{\eta^0, \tilde q^+\}), \quad \check q^0 = \tilde q^0 + \eta^0. \tag{10.12}$$

Here and below in this section we drop tildes at Poisson brackets for brief. According to assertion **c** of Proposition 5.2 and Lemma 10.1 we have the following estimates:

$$\eta^+(\tilde\tau - s, \tilde h, 0) \ll \varepsilon\tilde N_2||s||^{-2}\check Y_0^{-1}(s,0), \tag{10.13}$$

$$\eta^0(\tilde\tau - s - i\delta, \tilde h, \delta) \ll \varepsilon\beta||s||\,||s+i\delta||^{-4}\,||s+2i\delta||^{-1}\check Y^{-1}(s,2\delta). \tag{10.14}$$

The functions $\check q^0, \tilde q^+$ can be estimated as follows:

$$\check q^0(\tilde\tau - s - i\delta, \tilde h, \delta) \ll \check B||s||\,||s+i\delta||^{-4}\,||s+2i\delta||^{-2}\check Y^{-1}(s,2\delta), \tag{10.15}$$

$$\tilde q^+(\tilde\tau - s - i\delta, \tilde h, \delta) \ll \check C||s||^{-1}\,||s+2i\delta||^{-2}\check Y^{-1}(s,2\delta). \tag{10.16}$$

We put $\eta^+ = \eta^{(0)+} + \eta^{(1)+} + \dots$, where

$$\eta^{(0)+}_\delta = i\eta^{(0)+}_{\tilde\tau} + i\{\eta^0, \tilde q^+\}, \quad \eta^{(m+1)+}_\delta = i\eta^{(m+1)+}_{\tilde\tau} + i\{\check q^0, \eta^{(m)+}\}, \qquad m \geqslant 0,$$
$$\eta^{(0)+}(\tilde\tau, \tilde h, 0) = \eta^+(\tilde\tau, \tilde h, 0), \quad \eta^{(m+1)+}(\tilde\tau, \tilde h, 0) = 0.$$

If the series for η^+ converges then it is a solution of the system (10.12). We have the following relations:

$$\begin{aligned}\eta^{(0)+}(\tilde\tau, \tilde h, \mu) &= \eta^+(\tilde\tau + i\mu, \tilde h, 0) + i\int_0^\mu \{\eta^0, \tilde q^+\}(\tilde\tau + i(\mu-\lambda), \tilde h, \lambda)\,d\lambda.\\ \eta^{(m+1)+}(\tilde\tau, \tilde h, \mu) &= i\int_0^\mu \{\check q^0, \eta^{(m)+}\}(\tilde\tau + i(\mu-\lambda), \tilde h, \lambda)\,d\lambda.\end{aligned} \tag{10.17}$$

First, we estimate the function $\eta^{(0)+}$. Using the relations (10.13)–(10.15), we see that $\eta^{(0)+}(\tilde\tau - s - i\mu, \tilde h, \mu)$ does not exceed ($\ll$)

$$\begin{aligned}&\frac{\varepsilon\tilde N_2}{||s||^2\check Y_0(s)} + \int_0^\mu \frac{\varepsilon\beta\check C\,d\lambda}{||s+i\lambda||^4||s+2i\lambda||^3\check Y^3(s,2\lambda)}\\ &\quad\ll \frac{\varepsilon\tilde N_2}{||s+2i\mu||^2\check Y(s,2\mu)} + \int_0^\mu \frac{8\varepsilon\beta\check C\,d\lambda}{||s+i\lambda||\,||s+2i\mu||^6\check Y^3(s,2\mu)}\\ &\quad\ll \frac{\varepsilon\tilde N_2}{||s+2i\mu||^2\check Y(s,2\mu)} + \frac{8\varepsilon\beta\check C\log(||s+i\mu||/||s||)}{||s+2i\mu||^6\check Y^3(s,2\mu)}.\end{aligned} \tag{10.18}$$

Here we used assertions (2) and (6) of Proposition 5.1.

Estimates of the functions $\eta^{(m)+}$, $m > 0$ are presented in the following

Proposition 10.1. *For any natural m the function $\eta^+(\tilde\tau - s - i\delta, \tilde h, \delta)$ does not exceed* ($\ll$)

$$\frac{\varepsilon(4\check B)^m\mu^m}{||s+i\mu||^m}\Big[\frac{2^m\tilde N_2}{||s+2i\mu||^{4m+2}\check Y^{2m+1}(s,2\mu)} + \frac{8\cdot 3^m\beta\check C}{||s+2i\mu||^{4m+6}\check Y^{2m+3}(s,2\mu)}\Big].$$

We prove Proposition 10.1 a few lines below and now we use it to estimate the function η^+. Putting $\check Y = \check Y(s,2\delta)$, we have:

$$\begin{aligned}\eta^+(\tilde\tau - s - i\delta, \tilde h, \delta) \ll &\frac{\varepsilon\tilde N_2}{||s+2i\delta||^2\check Y}\sum_{m=0}^\infty \frac{(8\check B\delta)^m}{||s+2i\delta||^m\,||s+2i\delta||^{4m}\check Y^{2m}}\\ &+\frac{8\varepsilon\beta\check C\log(||s+i\delta||/||s||)}{||s+2i\delta||^6\check Y^3}\\ &+\frac{8\varepsilon\beta\check C}{||s+2i\delta||^6\check Y^3}\sum_{m=0}^\infty\frac{(12\check B\delta)^m}{||s+i\delta||^m\,||s+2i\delta||^{4m}\check Y^{2m}}.\end{aligned}$$

We have the following estimates:

$$\frac{\delta}{||s+i\delta||}<1,\quad \log(||s+i\delta||/||s||)\leqslant 2\log\varepsilon^{-1},\quad \frac{1}{\check Y}\ll\frac{1}{Y'},\quad \frac{1}{\check Y Y'}\ll\frac{4||s+2i\delta||^2}{w^2||s||Y'}$$

$$\frac{1}{\check Y}\sum_{m=0}^\infty\frac{K^{2m}}{||s+2i\delta||^{4m}\check Y^{2m}}\ll\frac{1}{\check Y}\sum_{m=0}^\infty\frac{K^m}{||s+2i\delta||^{2m}\check Y^m}=\Big(\check Y-\frac{K}{||s+2i\delta||^2}\Big)^{-1}\ll\frac{1}{Y'}$$

provided $||s||>4K/\check w^2$. Using these estimates we get for $||s||\geqslant 4(12\check B)^{1/2}$:

$$\eta^+(\tilde\tau - s - i\delta, \tilde h, \delta)\ll\frac{\varepsilon\tilde N_2}{||s+2i\delta||^2Y'}+\frac{2^8\varepsilon\beta\check C\log\varepsilon^{-1}}{\check w^4||s+2i\delta||^2Y'}+\frac{2^7\varepsilon\beta\check C}{\check w^4||s+2i\delta||^2Y'}.$$

This relation implies Lemma 10.3.

Proof of Proposition 10.1. We put $\check Y_* = \check Y(s,2\mu)$. Using the relations (10.18), (10.16) and (10.19) we obtain that $\eta^{(1)+}(\tilde\tau - s - i\mu, \tilde h, \mu)$ does not exceed ($\ll$)

$$\begin{aligned}&\int_0^\mu\Big[\frac{\varepsilon\tilde N_2\check B||s||}{||s+i\lambda||^4\,||s+2i\lambda||^4\check Y^3(s,2\lambda)}+\frac{8\cdot 3\varepsilon\beta\check C\check B||s||\log(||s+i\lambda||/||s||)}{||s+i\lambda||^4\,||s+2i\lambda||^8\check Y^5(s,2\lambda)}\Big]d\lambda\\ &\quad\ll\int_0^\mu\Big[\frac{4\varepsilon\tilde N_2\check B||s||}{||s+i\lambda||^2\,||s+2i\mu||^6\check Y_*^3}+\frac{8\cdot 3\cdot 4\varepsilon\beta\check C\check B||s||\log(||s+i\lambda||/||s||)}{||s+i\lambda||^2\,||s+2i\lambda||^{10}\check Y_*^5}\Big]d\lambda\\ &\quad\ll\frac{4\varepsilon\tilde N_2\check B\mu}{||s+i\mu||\,||s+2i\mu||^6\check Y_*^3}+\frac{8\cdot 3\cdot 4\varepsilon\beta\check C\check B\mu}{||s+i\mu||^2\,||s+2i\mu||^{10}\check Y_*^5}.\end{aligned}$$

Here we used the inequality

$$\int_0^\mu\frac{\log(||s+i\lambda||/||s||)}{||s+i\lambda||^2}\,d\lambda\leqslant\frac{\mu}{||s||\,||s+\mu||}.$$

Now we apply induction with respect to m. Suppose that the proposition is valid for $n = j$. Then $\eta^{(j+1)+}(\tilde{\tau} - s - i\mu, \tilde{h}, \mu)$ does not exceed ($\ll$)

$$\begin{aligned}
&\int_0^\mu \frac{\varepsilon(4\check{B})^j\lambda^j}{||s+i\lambda||^j}\Big[\frac{(2j+1)\tilde{N}_2\check{B}2^j||s||}{||s+i\lambda||^4\,||s+2i\lambda||^{4j+4}\check{Y}^{2j+3}(s,2\lambda)}\\
&\qquad\qquad + \frac{8(2j+3)3^j\beta\check{B}\check{C}||s||}{||s+i\lambda||^4\,||s+2i\lambda||^{4j+8}\check{Y}^{2j+5}(s,2\lambda)}\Big]\,d\lambda\\
&\ll \varepsilon(4\check{B})^{j+1}||s||\int_0^\mu \frac{\lambda^j\,d\lambda}{||s+i\lambda||^{j+2}}\Big[\frac{(2j+1)\tilde{N}_2 2^j}{||s+2i\mu||^{4j+6}\check{Y}_*^{2j+3}}\\
&\qquad\qquad + \frac{8(2j+3)3^j\beta\check{C}}{||s+2i\mu||^{4j+10}\check{Y}_*^{2j+5}}\Big]\\
&\ll \frac{\varepsilon(4\check{B})^{j+1}\mu^{j+1}}{(j+1)||s+2i\mu||^{j+1}}\Big[\frac{(2j+1)\tilde{N}_2 2^j}{||s+2i\mu||^{4j+6}\check{Y}_*^{2j+3}} + \frac{8(2j+3)3^j\beta\check{C}}{||s+2i\mu||^{4j+10}\check{Y}_*^{2j+5}}\Big].
\end{aligned}$$

Here we used Proposition 5.3. Now Proposition 10.1 follows from the inequalities $(2j+1)/(j+1) < 2$ and $(2j+3)/(j+1) \leqslant 3$ for $j \geqslant 0$.

Appendix 1. Proof of Lemma 6.1

We put

$$q^0 = q^{(0)0} + q^{(1)0} + \cdots, \quad q^{\pm} = q^{(0)\pm} + q^{(1)\pm} + \cdots, \tag{A1.1}$$

where

$$\begin{aligned}
q^{(0)0}(\tau_*, h, \Delta) = q^0(\tau_*, h, \Delta), &\quad q^{(n)0}(\tau_*, h, \Delta) = 0,\\
q^{(0)\pm}(\tau_*, h, \Delta) = q^{\pm}(\tau_*, h, \Delta), &\quad q^{(n)\pm}(\tau_*, h, \Delta) = 0,
\end{aligned}$$

n is natural. Let the following equations be satisfied for any $n \in \mathbb{N}$:

$$\begin{cases} q_\delta^{(0)0} = 0,\\ q_\delta^{(0)\pm} = \pm i\varepsilon q_{\tau_*}^{(0)\pm}, \end{cases} \qquad \begin{cases} q_\delta^{(n+1)0} = -2i\varepsilon e^{-2\delta}\sum_{l+m=n}\{q^{(l)+}, q^{(m)-}\},\\ q_\delta^{(n+1)\pm} = \pm i\varepsilon q_{\tau_*}^{(n+1)\pm} \pm i\varepsilon\sum_{l+m=n}\{q^{(l)0}, q^{(m)\pm}\} \end{cases}$$

If the series (A1.1) converge, they are solutions of the system (6.2). The functions $q^{(n)0}, q^{(n)\pm}$ satisfy the relations

$$\begin{cases} q^{(0)0}(\tau_* - s, h, \Delta+\mu) = q^0(\tau_* - s, h, \Delta) \ll \varepsilon B Z^{-1},\\ q^{(0)\pm}(\tau_* - s, h, \Delta+\mu) = q^{\pm}(\tau_* - s \pm i\varepsilon\mu, h, \Delta) \ll C Z^{-1} \end{cases} \tag{A1.2}$$

with $c_3 \leqslant |\operatorname{Re} s| \leqslant c_3^{-1}$, $|\operatorname{Im} s| \leqslant \alpha - \varepsilon|\mu|$;

$$\begin{cases} q^{(n+1)0}(\tau_*, h, \Delta+\mu) = -2i\varepsilon \sum\limits_{l+m=n} \int_0^\mu e^{-2(\Delta+\lambda)}\{q^{(l)+}, q^{(m)-}\}(\tau_*, h, \Delta+\lambda)\,d\lambda,\\ q^{(n+1)\pm}(\tau_*, h, \Delta+\mu) = \pm i\varepsilon \sum\limits_{l+m=n} \int_0^\mu \{q^{(l)0}, q^{(m)\pm}\}(\tau_* \pm i\varepsilon(\mu-\lambda), h, \Delta+\lambda)\,d\lambda. \end{cases} \tag{A1.3}$$

Proposition A1.1. *For any* $n \in \mathbb{N}$; $0 \leqslant \mu \leqslant \alpha/\varepsilon$; $s \in \mathbb{C}$, $c_3 \leqslant |\mathrm{Re}\, s| \leqslant c_3^{-1}$, $|\mathrm{Im}\, s| \leqslant \alpha - \varepsilon\mu$ *the following estimates hold:*

$$q^{(n)0}(\tau_* - s, h, \Delta + \mu) \ll \frac{2e^{-2\Delta} C^2 (\mu\varepsilon\gamma)^n \rho^{n-1}}{(2n+1)^3 Z^{2n+1}}, \tag{A1.4}$$

$$q^{(n)\pm}(\tau_* - s, h, \Delta + \mu) \ll \frac{C(\mu\varepsilon\gamma\rho)^n}{(2n+1)^3 Z^{2n+1}}, \tag{A1.5}$$

where

$$\gamma = \max_M \sum_{u+v=M} \frac{2(2M+3)^3}{(M+1)(2u+1)^2(2v+1)^2}\,, \qquad M, u, v \in \{0, 1, \dots\}. \tag{A1.6}$$

Proof of Proposition A1.1 is contained in the end of this section. Note that $54 \leqslant \gamma < \infty$. Using Proposition A1.1 we obtain:

$$q^0(\tau_* - s, h, \Delta + \mu) \ll \varepsilon\Big(\frac{B}{Z} + \frac{2e^{-2\Delta}C^2\mu\gamma}{Z^3} \sum_{k=0}^{\infty} \frac{\nu^{4k}}{Z^{2k}}\Big),$$

where $\nu = (\mu\varepsilon\gamma\rho)^{1/4}$. The last sum does not exceed $\sum_{k=0}^{\infty} \nu^{2k} Z^{-k} = (1-\nu^2 Z^{-1})^{-1}$. Thus

$$q^0(\tau_* - s, h, \Delta + \mu) \ll \varepsilon\Big(\frac{B}{Z} + \frac{2e^{-2\Delta}C^2\mu\gamma}{Z^2(Z-\nu^2)}\Big) \ll \varepsilon\Big(\frac{B}{Z-r^2} + \frac{2e^{-2\Delta}C^2\mu\gamma}{Z^2(Z-r^2)}\Big),$$

because according to the assumptions of Theorem 4.1 we have $r \geqslant \nu$. Using assertions (3) and (4) of Proposition 5.1 we get: $Z^{-2}(Z-r^2)^{-1} \ll Z^{-2}\tilde{Z}^{-1} \ll r^{-4}\tilde{Z}^{-1}$. Hence,

$$q^0(\tau_* - s, h, \Delta + \mu) \ll \varepsilon(B + 2e^{-2\Delta}C^2\mu\gamma r^{-4})\tilde{Z}^{-1}.$$

Analogously

$$q^{\pm}(\tau_* - s, h, \Delta + \mu) \ll \frac{C}{Z} \sum_{l=0}^{\infty} \frac{\nu^{4l}}{Z^{2l}} \ll \frac{C}{\tilde{Z}}.$$

Lemma 6.1 is proved.

*Proof of Proposition A*1.1. We use induction with respect to n. The estimate (A1.5) is valid according to conditions of Lemma 8.1. Let the estimate (A1.5) holds for $n \leqslant k$. Using (A1.3) and Proposition 5.1 we have for $s \in \mathbb{C}$ such that $c_3 \leqslant |\mathrm{Re}\, s| \leqslant c_3^{-1}$ and $|\mathrm{Im}\, s| \leqslant \alpha - \varepsilon\mu$:

$$\begin{aligned} q^{(k+1)0}(\tau_* - s, h, \Delta + \mu) &\ll 2\varepsilon \int_0^{\mu} e^{-2\Delta} \sum_{l+m=k} \frac{2(2l+1)(2m+1)C^2(\lambda\varepsilon\gamma\rho)^k}{(2l+1)^3(2m+1)^3 Z^{2k+3}}\, d\lambda \\ &\ll \frac{2e^{-2\Delta}C^2(\mu\varepsilon)^{k+1}(\gamma\rho)^k}{(k+1)Z^{2k+3}} \sum_{l+m=k} \frac{2}{(2l+1)^2(2m+1)^2}. \end{aligned}$$

The last expression does not exceed ($\ll$) the right hand side of (A1.4) for $n = k+1$ due to the definition of γ.

Now let the estimates (A1.5) and (A1.4) hold for $n \leqslant k$. Then

$$
\begin{aligned}
&q^{(k+1)\pm}(\tau_* - s, h, \Delta + \mu)\\
&\quad = \pm i\varepsilon \int_0^\mu \Big(\sum_{l+m=k} \{q^{(l)0}, q^{(m)\pm}\} \Big)(\tau_* - s \pm i\varepsilon(\mu - \lambda), h, \Delta + \lambda)\, d\lambda\\
&\quad \ll \int_0^\mu \Big[\frac{2\varepsilon^2 BC(\lambda\varepsilon\gamma\rho)^n}{(2n+1)^2 Z^{2n+3}} + \varepsilon \sum_{l+m=k,\, l>0} \frac{4e^{-2\Delta} C^3 (\lambda\varepsilon\gamma)^n \rho^{n-1}}{(2l+1)^2(2m+1)^2 Z^{2n+3}} \Big]\, d\lambda\\
&\quad \ll \frac{C(\mu\varepsilon\rho)^{n+1}\gamma^n}{(k+1) Z^{2n+3}} \sum_{l+m=k} \frac{2}{(2l+1)^2(2m+1)^2}.
\end{aligned}
$$

The last expression does not exceed ($\ll$) the right hand side of (A1.5) for $n = k+1$. Proposition A1.1 is proved.

Appendix 2. Proof of Lemma 7.2

We put

$$
\begin{gathered}
p^0 = p^{(0)0} + p^{(1)0} + \cdots, \quad p^\pm = p^{(0)\pm} + p^{(1)\pm} + \cdots, \\
p^{(0)0}(x, y, \Delta) = p^0(x, y, \Delta), \quad p^{(n)0}(x, y, \Delta) = 0, \\
p^{(0)\pm}(x, y, \Delta) = p^\pm(x, y, \Delta), \quad p^{(n)\pm}(x, y, \Delta) = 0,
\end{gathered}
\tag{A2.1}
$$

n is natural. Let the following equations be satisfied for any $n \in \mathbb{N}$:

$$
\begin{cases} p_\delta^{(0)0} = 0, \\ p_\delta^{(0)\pm} = \pm i\varepsilon\{xy, p^{(0)\pm}\}, \end{cases}
\qquad
\begin{cases} p_\delta^{(n+1)0} = -2i\varepsilon e^{-2\delta} \sum_{l+m=n} \{p^{(l)+}, p^{(m)-}\}, \\ p_\delta^{(n+1)\pm} = \pm i\varepsilon\{xy, p^{(n+1)\pm}\} \\ \qquad\qquad \pm i\varepsilon \sum_{l+m=n} \{p^{(l)0}, p^{(m)\pm}\}. \end{cases}
$$

If the series (A2.1) converge, they are solutions of the system (7.2). The functions $p^{(n)0}, p^{(n)\pm}$ satisfy the relations

$$
\begin{cases} p^{(0)0}(x, y, \Delta + \mu) = p^0(x, y, \Delta) \ll BW^{-1}, \\ p^{(0)\pm}(x, y, \Delta + \mu) = p^\pm(e^{\pm i\varepsilon\delta} x, e^{\mp i\varepsilon\delta} y, \Delta) \ll CW^{-1} \end{cases}
\tag{A2.2}
$$

(see assertion (5) of Proposition 5.1);

$$
\begin{cases} p^{(n+1)0}(x, y, \Delta + \mu) = -2i\varepsilon \sum\limits_{l+m=n} \int_0^\mu e^{-2(\Delta+\lambda)} \{p^{(l)+}, p^{(m)-}\}(x, y, \Delta + \lambda)\, d\lambda, \\ p^{(n+1)\pm}(x, y, \Delta + \mu) = \pm i\varepsilon \sum\limits_{l+m=n} \int_0^\mu \{p^{(l)0}, p^{(m)\pm}\}(e^\pm_{\mu-\lambda} x, e^\mp_{\mu-\lambda} y, \Delta + \lambda)\, d\lambda. \end{cases}
\tag{A2.3}
$$

where $e^\pm_{\mu-\lambda} = e^{\pm i\varepsilon(\mu-\lambda)}$.

Proposition A2.1. *For any $n \in \mathbb{N}$ the following estimates hold:*

$$
p^{(n)0}(x, y, \Delta + \mu) \ll \frac{2e^{-2\Delta} C^2 (\mu\varepsilon\gamma)^n \rho^{n-1}}{(2n+1)^3 W^{2n+1}},
\tag{A2.4}
$$

$$
p^{(n)\pm}(x, y, \Delta + \mu) \ll \frac{C(\mu\varepsilon\gamma\rho)^n}{(2n+1)^3 W^{2n+1}},
\tag{A2.5}
$$

where γ is determined by (A1.6).

This proposition can be proved by induction analogously to Proposition A1.1. Using Proposition A2.1 we obtain:

$$p^0(x, y, \Delta + \mu) \ll \frac{B}{W} + \frac{2e^{-2\Delta} C^2 \mu\varepsilon\gamma}{W^3} \sum_{l=0}^{\infty} \frac{(\mu\varepsilon\gamma\rho)^l}{W^{2l}}.$$

The last sum does not exceed $\sum_{k=0}^{\infty} \nu^{2k} W^{-k} = (1 - \nu^2 W^{-1})^{-1}$, $\nu = (\mu\varepsilon\gamma\rho)^{1/4}$. Thus

$$p^0(x, y, \Delta + \mu) \ll \frac{B}{W} + \frac{2e^{-2\Delta} C^2 \mu\varepsilon\gamma}{(W - \nu^2) W^2} \ll \frac{B}{W - r^2} + \frac{2e^{-2\Delta} C^2 \mu\varepsilon\gamma}{(W - r^2) W^2}.$$

By using assertions (3) and (4) of Proposition 5.1 we get: $W^{-2}(W - r^2)^{-1} \ll W^{-2} \tilde{W}^{-1} \ll r^{-4} \tilde{W}^{-1}$. Hence,

$$p^0(x, y, \Delta + \mu) \ll (B + 2e^{-2\Delta} C^2 \mu\varepsilon\gamma r^{-4}) \tilde{W}^{-1}.$$

Analogously,

$$p^{\pm}(x, y, \Delta + \mu) \ll \frac{C}{W} \sum_{k=0}^{\infty} \frac{\nu^{2k}}{W^k} \ll \frac{C}{\tilde{W}}.$$

Appendix 3. Proof of Lemma 8.1

In this section we drop tildes over $q^0, q^{\pm}, \tau$ and h for brief. We put

$$\begin{gathered} q^0 = q^{(0)0} + q^{(1)0} + \cdots, \quad q^{\pm} = q^{(0)\pm} + q^{(1)\pm} + \cdots, \\ q^{(0)0}(\tau, h, \Delta) = q^0(\tau, h, \Delta), \quad q^{(n)0}(\tau, h, \Delta) = 0, \\ q^{(0)\pm}(\tau, h, \Delta) = q^{\pm}(\tau, h, \Delta), \quad q^{(n)\pm}(\tau, h, \Delta) = 0 \end{gathered} \tag{A3.1}$$

for $n \in \mathbb{N}$. Let the following equations be satisfied for $n \geqslant 0$:

$$\begin{cases} q_\delta^{(0)0} = 0, \\ q_\delta^{(0)\pm} = \pm i q_\tau^{(0)\pm}, \end{cases} \qquad \begin{cases} q_\delta^{(n+1)0} = -2ie^{-2\delta} \sum_{l+m=n} \{q^{(l)+}, q^{(m)-}\}, \\ q_\delta^{(n+1)\pm} = \pm i q_\tau^{(n+1)\pm} \pm i \sum_{l+m=n} \{q^{(l)0}, q^{(m)\pm}\}. \end{cases}$$

If the series (A3.1) converge, they are solutions of the system (2.17). Recall that we denote $\Delta' = \Delta + \mu$. The functions $\tilde{q}^{(n)0}, \tilde{q}^{(n)\pm}$ satisfy the relations

$$\begin{cases} q^{(0)0}(\tau - s - i\Delta', h, \Delta') = q^0(\tau - s - i\Delta', h, \Delta), \\ q^{(0)\pm}(\tau - s - i\Delta', h, \Delta') = q^{\pm}(\tau - s - i\Delta' \pm i\mu, h, \Delta), \end{cases} \tag{A3.2}$$

$$\begin{cases} q^{(n+1)0}(\tau - s - i\Delta', h, \Delta') \\ \qquad = -2i \sum\limits_{l+m=n} \int_0^{\mu} e^{-2(\Delta+\lambda)} \{q^{(l)+}, q^{(m)-}\}(\tau - s - i\Delta', h, \Delta + \lambda)\, d\lambda, \\ q^{(n+1)\pm}(\tau - s - i\Delta', h, \Delta') \\ \qquad = \pm i \sum\limits_{l+m=n} \int_0^{\mu} \{q^{(l)0}, q^{(m)\pm}\}(\tau - s - i\Delta' \pm i(\mu - \lambda), h, \Delta + \lambda)\, d\lambda. \end{cases} \tag{A3.3}$$

Proposition A3.1. *Let the constants C, b, γ, and A satisfy conditions of Lemma 8.1. Then the following estimates hold for $k > 0$:*

$$q^{(k)0}(\tau - s - i\Delta', h, \Delta') \ll \frac{(16b\gamma\mu\sigma)^k ||s||^{2k+2} e^{-\Delta/2}}{A(2k+1)^3 ||s+2i\Delta'||^{5k+6} Y^{2k+1}(s, 2\Delta')},$$
$$q^{(k)\pm}(\tau - s - i\Delta', h, \Delta') \ll \frac{C(16b\gamma\mu\sigma)^k ||s||^{2k-2}}{(2k+1)^3 |s+2i\Delta'|^{5k+2} Y^{2k+1}(s, 2\Delta')}.$$

We prove Proposition A3.1 in the end of this section. Now we have the estimate

$$q^{(0)0}(\tau - s - i\Delta', h, \Delta') \ll \frac{B||s+i\mu||}{||s+i\Delta'||^4 ||s+2i\Delta+i\mu||^2 Y(s+i\mu, 2\Delta)}$$
$$\ll \frac{B||s||}{||s+i\Delta'||^4 ||s+2i\Delta'||^2 Y(s+i\mu, 2\Delta)}$$

(Here we used assertion (6) of Proposition 5.1.) We put $Y = Y(s+i\mu, 2\Delta)$. According to the expansion (A3.1)

$$q^0(\tau - s - i\Delta', h, \Delta') \ll \frac{B||s||}{||s+i\Delta'||^4 ||s+2i\Delta'||^2 Y} + \sum_{k=1}^{\infty} \frac{(16b\gamma\mu\sigma)^k ||s||^{2k+2} e^{-\Delta/2}}{A||s+2i\Delta'||^{5k+6} Y^{2k+1}}$$
$$\ll ||s|| \cdot ||s+i\Delta'||^{-4} ||s+2i\Delta'||^{-2}$$
$$\times \left[\frac{B}{Y} + \frac{16b\gamma\mu\sigma||s||^3 e^{-\Delta/2}}{A||s+2i\Delta'||^5 Y^3} \sum_{l=0}^{\infty} \frac{\tilde{\rho}^{4l} ||s||^{2l}}{||s+2i\Delta'||^{4l} Y^{2l}}\right].$$

The last sum does not exceed ($\ll$)

$$\sum_{m=0}^{\infty} \frac{\tilde{\rho}^{2m} ||s||}{||s+2i\Delta'||^{2m} Y^m} = \left[1 - \frac{\tilde{\rho}^2 ||s||}{||s+2i\Delta'||^2 Y}\right]^{-1}.$$

Therefore,

$$q^0(\tau - s - i\Delta', h, \Delta') \ll ||s|| \cdot ||s+i\Delta'||^{-4} ||s+2i\Delta'||^{-2}$$
$$\times \left[\frac{B}{Y} + \frac{16b\gamma\mu\sigma||s||^3 e^{-\Delta/2}}{A||s+2i\Delta'||^5 Y^2 (Y - \tilde{\rho}^2 ||s|| \cdot ||s+2i\Delta'||^{-2})}\right].$$

According to assertions (4) and (3) of Proposition 5.1 we have:

$$\frac{||s||^3}{||s+2i\Delta'||^5 Y^2 (Y - \tilde{\rho}^2 ||s|| \cdot ||s+2i\Delta'||^{-2})}$$
$$\ll \frac{||s||^3}{||s+2i\Delta'||^5 Y^2 (Y - r^2 ||s|| \cdot ||s+2i\Delta'||^{-2})}$$
$$\ll ||s||^2 ||s+2i\Delta'||^{-4} Y^{-2} (Y')^{-1} \ll r^{-4} (Y')^{-1}.$$

Consequently,

$$q^0(\tau - s - i\Delta', h, \Delta') \ll \frac{||s||}{||s+i\Delta'||^4 ||s+2i\Delta'||^2 Y'} \left[B + \frac{16b\gamma\mu\sigma e^{-\Delta/2}}{Ar^4}\right].$$

The functions $q^{\pm}$ are estimated in the following way. Since

$$q^{(0)\pm}(\tau - s - i\Delta', h, \Delta') \ll \frac{C}{||s|| \cdot ||s + 2i\Delta + i\mu \pm i\mu||^2 Y(s + i\mu \pm i\mu, 2\Delta)} \ll \frac{C}{||s|| \cdot ||s + 2i\Delta'||^2 Y}$$

and $||s|| \geqslant 1$, we get:

$$q^{\pm}(\tau - s - i\Delta', h, \Delta') \ll \frac{C}{||s|| \cdot ||s + 2i\Delta'||^2 Y} \sum_{l=0}^{\infty} \frac{\tilde{\rho}^{4l} ||s||^{2l}}{||s + 2i\Delta'||^{4l} Y^{2l}} \ll \frac{C}{||s|| \cdot ||s + 2i\Delta'||^2 Y'}.$$

To estimate the differences $d^{\pm} = q^{\pm}(\tau - s - i\Delta', h, \Delta') - q^{\pm}(\tau - s - i\Delta' \pm i\Delta', h, 0)$ we note that due to conditions of Lemma 8.1

$$\begin{aligned} & q^{(0)\pm}(\tau - s - i\Delta', h, \Delta') - q^{\pm}(\tau - s - i\Delta' \pm i\Delta', h, 0) \\ & \quad = q^{\pm}(\tau - s - i\Delta' \pm i\mu, h, \Delta) - q^{\pm}(\tau - s - i\Delta' \pm i\Delta', h, 0) \\ & \quad \ll C||s||^{-2} ||s + 2i\Delta'||^{-2} Y^{-1}. \end{aligned}$$

Hence,

$$d^{\pm} \ll \frac{C}{||s||^2 ||s + 2i\Delta'||^2 Y} \sum_{k=0}^{\infty} \frac{\tilde{\rho}^{4k} ||s||^{2k}}{||s + 2i\Delta'||^{4k} Y^{2k}} \ll \frac{C}{||s||^2 ||s + 2i\Delta'||^2 Y'}.$$

Now we estimate the difference $d_1^+ = \tilde{q}^+(\tau - s - i\Delta - iM, \tilde{h}, \Delta + M) - \tilde{q}^+(\tau - s - i\Delta, \tilde{h}, \Delta)$. We have the following inequalities:

$$\frac{16b\gamma M\sigma}{||s + 2i(\Delta + M)||} \leqslant 16b\gamma\sigma \leqslant \frac{8b\gamma}{\Delta} \leqslant \frac{Kw^4}{16\Delta}.$$

Hence, putting $Y = Y(s, 2(\Delta + M))$, we get:

$$\begin{aligned} d_1^+ &\ll \sum_{k=1}^{\infty} \frac{C(16b\gamma M\sigma)^k ||s||^{2k-2}}{||s + 2i(\Delta + M)||^{5k+2} Y^{2k+1}} \ll \sum_{k=1}^{\infty} \frac{C(Kw^4/16)^k ||s||^{2k-2}}{\Delta^k ||s + 2i(\Delta + M)||^{4k+2} Y^{2k+1}} \\ &\ll \frac{CKw^4/16}{\Delta ||s||^2 ||s + 2i(\Delta + M)||^6 Y^3} \sum_{k=0}^{\infty} \frac{(w/2)^{2k} ||s||^k}{||s + 2i(\Delta + M)||^{2k} Y^k} \\ &\ll \frac{CKw^4/16}{\Delta ||s||^2 ||s + 2i(\Delta + M)||^6 Y^2 Y''(s, 2(\Delta + M))} \\ &\ll \frac{CK}{\Delta ||s||^2 ||s + 2i(\Delta + M)||^2 Y''(s, 2(\Delta + M))}. \end{aligned}$$

Lemma 8.1 is proved.

*Proof of Proposition A*3.1. First, we have the following estimates:

$$
\begin{aligned}
q^{(0)0}(\tau - s - i\Delta, h, \Delta+\lambda) &\equiv q^{(0)0}(\tau - s - i\Delta, h, \Delta) \\
&\ll \frac{B||s||}{||s+i\Delta||^4||s+2i\Delta||^2 Y(s,2\Delta)} \\
&\ll \frac{16b||s||}{||s+2i\Delta||^6 Y(s,2\Delta)}, \qquad (A3.4)\\
q^{(0)\pm}(\tau - s - i\Delta, h, \Delta+\lambda) &\equiv q^{(0)\pm}(\tau - s - i\Delta \pm i\lambda, h, \Delta) \\
&\ll \frac{C}{||s\mp i\lambda||\,||s+2i\Delta\mp i\lambda||^2 Y(s\mp i\lambda, 2\Delta)}. \qquad (A3.5)
\end{aligned}
$$

Now we prove Proposition A3.1 for $k=1$. We have:

$$
q^{(1)0}(\tau - s - i\Delta', h, \Delta') = -2i\int_0^\mu e^{-2\Delta-2\lambda}\{q^{(0)+}, q^{(0)-}\}(\tau - s - i\Delta', h, \Delta+\lambda)\,d\lambda.
$$

Hence, we can put $q^{(1)0}(\tau - s - i\Delta', h, \Delta') = I_1 + I_2 + I_3 + I_4$, where

$$
\begin{aligned}
I_1 &= -2i\int_0^\mu e^{-2\Delta-2\lambda}\{\check{q}^+, \check{q}^-\}\,d\lambda, \qquad I_2 = -2i\int_0^\mu e^{-2\Delta-2\lambda}\{Q^+, \check{q}^-\}\,d\lambda,\\
I_3 &= -2i\int_0^\mu e^{-2\Delta-2\lambda}\{\check{q}^+, Q^-\}\,d\lambda, \qquad I_4 = -2i\int_0^\mu e^{-2\Delta-2\lambda}\{Q^+, Q^-\}\,d\lambda,\\
\check{q}^\pm &= q^\pm(\tau - s - i\Delta' \pm (i\Delta + i\lambda), h, 0), \quad Q^\pm = q^{(0)\pm}(\tau - s - i\Delta', h, \Delta+\lambda) - \check{q}^\pm.
\end{aligned}
$$

Since

$$
q^\pm(\tau - s, h, 0) \ll \tilde{N}||s||^{-3}(2v_0||s|| - \tau)^{-1}(v_0||s||^{-2} - h)^{-1}
$$

(see Proposition 5.2) and $\{q^+(\tau,h,0), q^-(\tau,h,0)\}$, we can estimate the integral I_1 with the help of assertion (7) of Proposition 5.1. Putting

$$
Y_1 = Y(s + i\mu - i\lambda, 2\Delta + 2\lambda), \quad Y = Y(s, 2\Delta'), \qquad (A3.6)
$$

we get

$$
\begin{aligned}
I_1 &\ll \int_0^\mu \frac{2e^{-2\Delta-2\lambda}4(\Delta+\lambda)v_0^{-1}\tilde{N}^2\,d\lambda}{||s+i\Delta'+i\Delta+i\lambda||^3||s+i\Delta'-i\Delta-i\lambda||^4 Y_1^3}\\
&\ll \int_0^\mu \frac{e^{-2\Delta-2\lambda}8(\Delta+\lambda)v_0^{-1}\tilde{N}^2\,d\lambda}{||s+i\Delta'||^3||s||^4 Y^3} \ll \frac{4b\gamma\mu\sigma||s||^4 e^{-\Delta/2}}{27A||s+2i\Delta'||^{11}Y^3}
\end{aligned}
$$

because according to definition of A and the inequalities

$$
\sigma \geqslant e^{-\Delta/2}, \quad C \geqslant \tilde{N}, \quad (\Delta+\lambda)e^{-\Delta-2\lambda} \leqslant (\Delta+1)e^{-\Delta},
$$

we have:

$$
\frac{4b}{A} \geqslant 4C^2 A \geqslant \frac{8\cdot 27(\Delta+\lambda)\tilde{N}^2||s+2i\Delta'||^8 e^{-3\Delta/2-2\lambda}}{v\gamma\sigma||s||^8}.
$$

Using the relations

$$
\begin{aligned}
Q^{\pm} &= q^{\pm}(\tau - s - i\Delta' \pm i\lambda, h, \Delta) - q^{\pm}(\tau - s - i\Delta' \pm (i\Delta + i\lambda), h, 0) \\
&\ll \frac{C}{||s+i\mu \pm i\lambda||^2 ||s+i\mu \pm i\lambda + 2i\Delta||^2 Y(s+i\mu\pm i\lambda, 2\Delta)} \\
&\ll \frac{C}{||s||^2||s+2i\Delta'||^2 Y}, \\
\check{q}^{\pm} &\ll \frac{\tilde{N}}{||s+i\mu\pm i\lambda||^3 Y(s+i\mu\pm i\lambda, 0)} \ll \frac{C}{||s||\cdot||s+2i\Delta'||^2 Y},
\end{aligned}
$$

where $0 \leqslant \lambda \leqslant \mu$ and $Y = Y(s, 2\Delta')$, we have:

$$
I_2 \ll 2\int_0^{\mu} \frac{e^{-2\Delta-2\lambda} 2C^2\, d\lambda}{||s||^3||s+2i\Delta'||^4 Y^3} \ll \frac{b\gamma\mu\sigma||s||^4 e^{-\Delta/2}}{27A||s+2i\Delta'||^{11}Y^3}
$$

because

$$
\frac{16b}{4A} \geqslant 4C^2 A \geqslant \frac{4\cdot 27C^2||s+2i\Delta'||^7 e^{-3\Delta/2-2\lambda}}{\gamma\sigma||s|^7}.
$$

The same estimate obviously holds for I_3 and I_4. Hence, we have proved the required estimate for $q^{(1)0}$.

To estimate the functions $q^{(1)\pm}$ we use the relations $(A3.4)$–$(A3.5)$ and also assertion (6) of Proposition 5.1. We get:

$$
\begin{aligned}
q^{(1)+}(\tau - s - i\Delta', h, \Delta') &= i\int_0^{\mu} \{q^{(0)0}, q^{(0)+}\}(\tau - s - i\Delta' + i(\mu-\lambda), h, \Delta+\lambda)\, d\lambda \\
&\ll \int_0^{\mu} \frac{16bC||s+i\lambda||\, d\lambda}{||s+2i\Delta+i\lambda||^6||s||\cdot||s+2i\Delta||^2 Y(s+i\lambda, 2\Delta)Y(s,2\Delta)Y(s,2\Delta+i\lambda)} \\
&\ll \int_0^{\mu} \frac{16bC\, d\lambda}{||s+2i\Delta+i\lambda||^2||s+2i\Delta'||^6 Y^3} \\
&\ll \frac{16bC\mu}{||s+2i\Delta||\,||s+2i\Delta+i\mu||\,||s+2i\Delta'||^6 Y^3} \ll \frac{16bC\gamma\mu\sigma}{27||s+2i\Delta'||^7 Y^3}
\end{aligned}
$$

because $\gamma \geqslant 54$. The corresponding estimate for $q^{(1)-}$ can be obtained analogously. Hence, Proposition A3.1 is proved for $k = 1$.

Now we use induction in k. Suppose that Proposition A3.1 is valid for $k \leqslant n$. Then using relations $(A3.3)$–$(A3.5)$ and the assumption of induction we have:

$$
\begin{aligned}
&q^{(n+1)0}(\tau - s - i\Delta', h, \Delta') \\
&\qquad \ll \int_0^{\mu} \sum_{l+m=n} \frac{2e^{-2\Delta-2\lambda}2C^2(16b\gamma\lambda\sigma)^n||s+i\mu-i\lambda||^{2n-3}\, d\lambda}{(2l+1)^2(2m+1)^2||s+2i\Delta+i\mu+i\lambda||^{5n+4}Y_1^{2n+3}},
\end{aligned} \tag{A3.7}
$$

where Y_1 is determined by $(A3.6)$. Here we used again assertion (2) of Proposition 5.1. Note that for $l, m \neq 0$ we actually get in $(A3.7)$ the term $||s+i\mu-i\lambda||^{2n-4}$ instead of $||s+i\mu-i\lambda||^{2n-3}$ and use the inequality $||s+i\mu-i\lambda|| \geqslant 1$. According to assertion (6) of Proposition 5.1 the right hand side of $(A3.7)$ does not exceed ($\ll$)

$$
\int_0^{\mu} \sum_{l+m=n} \frac{e^{-2\Delta-2\lambda}4C^2(16b\gamma\lambda\sigma)^n||s||^{2n-3}\, d\lambda}{(2l+1)^2(2m+1)^2||s+2i\Delta+i\mu+i\lambda||^{n-2}||s+2i\Delta'||^{4n+6}Y^{2n+3}}.
$$

Since

$$e^{-\Delta-2\lambda} \leqslant \frac{A||s||^7}{||s+2i\Delta+i\mu+i\lambda||^4||s+2i\Delta'||^3}, \quad e^{-\Delta/2} \leqslant \sigma$$

the expression $(A3.7)$ does not exceed $(\ll)$

$$\frac{4C^2A(16b\gamma)^n\sigma^{n+1}||s||^{2n+4}e^{-\Delta/2}}{||s+2i\Delta'||^{4n+9}Y^{2n+3}} \int_0^\mu \frac{\lambda^n\,d\lambda}{||s+2i\Delta+i\mu+i\lambda||^{n+2}} \times \sum_{l+m=n} \frac{1}{(2l+1)^2(2m+1)^2}. \tag{A3.8}$$

Using Proposition 5.3, definition of γ and the inequality $8C^2A \leqslant 16b/A$ we obtain that the expression $(A3.8)$ does not exceed $(\ll)$

$$\frac{(16b\mu\sigma)^{n+1}\gamma^n||s||^{2n+4}e^{-\Delta/2}}{2A(n+1)||s+2i\Delta+i\mu||\,||s+2i\Delta'||^{5n+10}Y^{2n+3}} \sum_{l+m=n} \frac{1}{(2l+1)^2(2m+1)^2}$$
$$\ll \frac{(16b\gamma\mu\sigma)^{n+1}||s||^{2n+4}e^{-\Delta/2}}{A(2n+3)^3||s+2i\Delta'||^{5n+11}Y^{2n+3}}.$$

Now we estimate $q^{(n+1)+}$:

$$\begin{aligned}
&q^{(n+1)+}(\tau-s-i\Delta',h,\Delta')\\
&\quad= i\int_0^\mu \Big[\{q^{(0)0},q^{(n)+}\}+\{q^{(n)0},q^{(0)+}\}\\
&\qquad+\sum_{\substack{l+m=n,\\ l>0,m>0}} \{q^{(l)0},q^{(m)+}\}\Big](\tau-s-i\Delta-i\lambda,h,\Delta+\lambda)\,d\lambda\\
&\quad\ll \int_0^\mu \Big[\frac{2\cdot 16bC(16b\gamma\lambda\sigma)^n||s||^{2n-1}}{(2n+1)^2||s+2i\Delta+2i\lambda||^{5n+8}Y^{2n+3}(s,2\Delta+2\lambda)}\\
&\qquad+\frac{2C(16b)^{n+1}(\gamma\lambda\sigma)^n||s||^{2n+1}e^{-\Delta/2}}{(2n+1)^2||s+2i\Delta+2i\lambda||^{5n+8}Y^{2n+3}(s,2\Delta+2\lambda)}\\
&\qquad+\sum_{\substack{l+m=n,\\ l>0,m>0}} \frac{2C(16b)^{n+1}(\gamma\lambda\sigma)^n||s||^{2n}e^{-\Delta/2}}{(2l+1)^2(2m+1)^2||s+2i\Delta+2i\lambda||^{5n+8}Y^{2n+3}(s,2\Delta+2\lambda)}\Big]\,d\lambda\\
&\quad\ll \int_0^\mu \sum_{l+m=n} \frac{2C(16b)^{n+1}(\gamma\lambda\sigma)^n||s||^{2n}\max\{1,||s||e^{-\Delta/2}\}\,d\lambda}{(2l+1)^2(2m+1)^2||s+2i\Delta+2i\lambda||^{n+2}||s+2i\Delta'||^{4n+6}Y^{2n+3}}\\
&\quad\ll \frac{C(16b\gamma\mu\sigma)^{n+1}||s||^{2n}}{(2n+3)^3||s+2i\Delta'||^{5n+7}Y^{2n+3}}.
\end{aligned}$$

The function $q^{(n+1)-}$ can be estimated analogously. Proposition A3.1 is proved.

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