On weak Landau–Ginzburg models for complete intersections in Grassmannians

V. V. Przyjalkowski and C. A. Shramov

A common way to construct Fano varieties in higher dimensions is to represent them as complete intersections in familiar varieties such as toric varieties or Grassmannians. The well-known Givental construction (see [5]) describes a Landau–Ginzburg (LG) model for a complete intersection in a toric variety as a complete intersection in a torus equipped with a function on the torus called the superpotential. In [1] and [2] LG models for complete intersections in Grassmannians. The method of Givental and Batyrev applied to a toric variety produces a regular function on a torus, that is, a Laurent polynomial.

Let $Y=X\cap Y_1\cap\dots\cap Y_l$ be a complete intersection in a smooth toric variety X with Picard number $\rho(X)=\rho$. Let variables u_i, $1\leqslant i\leqslant N$, correspond to the rays of the fan of X, and let $E_0,\dots,E_l\subset\{1,\dots,N\}$ be a nef-partition corresponding to Y. Denote by R_i the generators of the group of relations on rays of the fan of X, interpreted as monomials in the variables u_i. We fix parameters q_i corresponding to a symplectic form on Y. Then *Givental's LG model* is the variety

$$\Bigl\{R_i=q_i,\ \sum_{s\in E_i}u_s=1,\ 1\leqslant i\leqslant\rho,\ 1\leqslant s\leqslant l\Bigr\}$$

equipped with the superpotential $\sum_{s\in E_0}u_s$. *Givental's integral* for Y is the integral

$$I_Y=\int_{|u_i|=\varepsilon_i}\frac{1}{(2\pi i)^N}\,\frac{du_1}{u_1}\wedge\dots\wedge\frac{dx_N}{x_N}\,\frac{1}{\prod_{r=1}^{\rho}(R_r-q_r)\cdot\prod_{s=0}^{l}\bigl(1-\sum_{v\in E_s}u_v\bigr)}$$

for some positive numbers ε_i. We fix parameters q_i corresponding to an anticanonical form on Y.

An LG model for a complete intersection in a Grassmannian was described in a similar way in [1]. We present formulations for the case of Grassmannians of planes. Let $Y=G(2,k+2)\cap Y_1\cap\dots\cap Y_l$, $\deg Y_i=d_i$, be a Fano complete intersection. We define the following Laurent polynomials on $T=\operatorname{Spec}\mathbb{C}[a_{1,1}^{\pm1},\dots,a_{k,1}^{\pm1},a_{1,2}^{\pm1},\dots,a_{k,2}^{\pm1}]$:

$$F_0=a_{1,1},\quad F_i=\frac{a_{i+1,1}}{a_i,1}+\frac{a_{i+1,2}}{a_{i,k}}\,,\ \ 1\leqslant i\leqslant k-1,\quad F_k=\sum_{j=1}^{k}\frac{a_{j,2}}{a_{j,1}}\,,\ \ F_{k+1}=\frac{1}{a_{k,2}}\,.$$

Further, we define the Laurent polynomials E_s, $1\leqslant s\leqslant l$, as sums of d_s consecutive polynomials F_i, and the Laurent polynomial E_0 as the sum of all the polynomials F_i not used. Then the *standard LG model* for Y is the variety $\{E_i=1,\ 1\leqslant i\leqslant l\}\subset T$ with the superpotential E_0.

Theorem 1. *The standard LG model for a smooth Fano complete intersection Y in a Grassmannian of planes is birational to a torus. In particular, the superpotential can be given as a Laurent polynomial f_Y.*

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We expect that Theorem 1 can be generalized to the case of a complete intersection in an arbitrary Grassmannian.

Let f be a Laurent polynomial in m variables $x_1,\dots,x_m$. The integral

$$I_f(t)=\int_{|x_i|=\varepsilon_i}\frac{1}{(2\pi i)^m}\,\frac{dx_1}{x_1}\wedge\dots\wedge\frac{dx_m}{x_m}\,\frac{1}{1-tf}\in\mathbb{C}[[t]]$$

is called the *main period* for f, where the ε_i are arbitrary positive numbers. It can be computed in the following way: if ϕ_j is the constant term of f^j, then $I_f(t)=\sum\phi_j t^j$. This definition is justified by the following 'folklore' statement (see [8], Proposition 2, or [4], Theorem 3.2): If P is a Picard–Fuchs differential operator for a pencil of hypersurfaces that is determined by f in a torus, then $P[I_f(t)]=0$.

Proposition 2. $I_Y=I_{f_Y}$.

With any Fano variety V one can associate the *regularized constant term of the I-series* $\widetilde{I}_0^V=1+a_1t+a_2t^2+\cdots$ (see, for instance, [5]). It follows from [2], Proposition 3.5 in [3], and [7] that $\widetilde{I}_0^Y=I_Y$. The Laurent polynomial f_Y is called a *very weak LG model* for Y if $I_{f_Y}=\widetilde{I}_0^Y$.

Corollary 3. *Any smooth Fano complete intersection in a Grassmannian of planes has a very weak LG model.*

As an example, consider a smooth Fano fourfold Y of index 2 that is the intersection of the Grassmannian $\mathrm{Gr}(2,6)$ with 4 hyperplanes. The Laurent polynomial

$$f_Y=\frac{(a_4+a_3)(a_4+a_3+a_2)}{a_3a_2a_1}+\frac{a_4+a_3}{a_3a_2}+\frac{1}{a_3}+\frac{1}{a_4}+a_4+a_3+a_2+a_1$$

gives a very weak LG model for Y. Let $T=\operatorname{Spec}\mathbb{C}[a_1^{\pm1},a_2^{\pm1},a_3^{\pm1},a_4^{\pm1}]\cong(\mathbb{C}^*)^4$. We consider a 'naive' relative compactification of the family $f_Y\colon T\to\mathbb{A}^1$ given by an embedding of the torus T in the projective space $\mathbb{P}^4$ with homogeneous coordinates $a_0,\dots,a_4$. This is a family of compact singular Calabi–Yau threefolds. The total space of this family admits a crepant resolution of singularities $LG(Y)$. One can check that $LG(Y)$ is a family of Calabi–Yau threefolds with smooth generic fibre and 12 singular fibres. Furthermore, each of these singular fibres has exactly one singular point, and it is an ordinary double singularity. We expect that $LG(Y)$ satisfies the Homological Mirror Symmetry (HMS) conjecture. The structure of the singular fibres of $LG(Y)$ confirms this expectation. Indeed, by Corollary 10.3 in [6] there is a full exceptional collection of length 12 on Y. On the other hand, by the HMS conjecture the category $\mathscr{D}^b(Y)$ is equivalent to the Fukaya–Seidel category for a dual LG model.

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Viktor V. Przyjalkowski
Steklov Mathematical Institute
of Russian Academy of Sciences
E-mail: victorprz@mi.ras.ru

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Translated by THE AUTHORS

Constantin A. Shramov
Steklov Mathematical Institute
of Russian Academy of Sciences
E-mail: costya.shramov@gmail.com